Signal Analysis: Wavelets, Filter Banks, Time-Frequency Transforms and Applications. Alfred Mertins

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Chapter 1

Signals and Signal Spaces

The goal of this chapter is to give a brief overview of methods for characterizing signals and for describing their properties. We will start with a discussion of signal spaces such as Hilbert spaces, normed and metric spaces. Then, the energy density and correlation function of deterministic signals will be discussed. The remainder of this chapter is dedicated to random signals, which are encountered in almost all areas of signal processing. Here, basic concepts such as stationarity, autocorrelation, and power spectral density will be discussed.

1.1 Signal Spaces

1.1.1 Energy and Power Signals

Let us consider a deterministic continuous-time signal $x(t)$, which may be real or complex-valued. If the energy of the signal defined by

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt \tag{1.1}$$

is finite, we call it an *energy signal.* If the energy is infinite, but the mean power

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt \tag{1.2}$$

is finite, we call $x(t)$ a *power signal*. Most signals encountered in technical applications belong to these two classes.

A second important classification of signals is their assignment to the signal spaces $L_p(a,b)$, where a and b are the interval limits within which the signal is considered. By $L_p(a,b)$ with $1 \leq p < \infty$ we understand that class of signals x for which the integral

$$\int_a^b |x(t)|^p \, dt$$

to be evaluated in the Lebesgue sense is finite. If the interval limits a and b are expanded to infinity, we also write $L_p(\infty)$ or $L_p(\mathbb{R})$. According to this classification, energy signals defined on the real axis are elements of the space $L_2(\mathbb{R})$.

1.1.2 Normed Spaces

When considering normed signal spaces, we understand signals as vectors that are elements of a linear vector space X. The norm of a vector $\boldsymbol{x}$ can somehow be understood as the length of $\boldsymbol{x}$. The notation of the norm is $\|\boldsymbol{x}\|$.

Norms must satisfy the following three axioms, where α is an arbitrary real or complex-valued scalar, and $\mathbf{0}$ is the null vector:

$$\text{(i)} \qquad \|\boldsymbol{x}\| \geq 0, \quad \|\boldsymbol{x}\| = 0 \text{ if and only if } \boldsymbol{x} = \mathbf{0}, \tag{1.3}$$

$$\text{(ii)} \qquad \|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|, \tag{1.4}$$

$$\text{(iii)} \qquad \|\alpha \boldsymbol{x}\| = |\alpha| \, \|\boldsymbol{x}\|. \tag{1.5}$$

Norms for Continuous-Time Signals. The most common norms for continuous-time signals are the L_p norms:

$$\|\boldsymbol{x}\|_{L_p} = \left[\int_a^b |x(t)|^p \, dt \right]^{1/p}, \quad 1 \leq p < \infty \tag{1.6}$$

For $p \to \infty$, the norm (1.6) becomes $\|\boldsymbol{x}\|_{L_\infty} = \operatorname*{ess\,sup}_{a \leq t \leq b} |x(t)|$.

For $p = 2$ we obtain the well-known *Euclidean norm*:

$$\|\boldsymbol{x}\|_{L_2} = \sqrt{\int_a^b |x(t)|^2 \, dt}, \qquad \boldsymbol{x} \in L_2(a,b). \tag{1.7}$$

Thus, the signal energy according to (1.1) can also be expressed in the form

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \|\boldsymbol{x}\|_{L_2}^2, \qquad \boldsymbol{x} \in L_2(\mathbb{R}). \tag{1.8}$$

Norms for Discrete-Time Signals. The spaces $\ell_p(n_1, n_2)$ are the discrete-time equivalent to the spaces $L_p(a, b)$. They are normed as follows:

$$\|\boldsymbol{x}\|_{\ell_p} = \left[\sum_{n=n_1}^{n_2} |x(n)|^p\right]^{1/p}, \quad 1 \leq p < \infty. \tag{1.9}$$

For $p \to \infty$, (1.9) becomes $\|\boldsymbol{x}\|_{\ell_\infty} = \sup_{n=n_1}^{n_2} |x(n)|$.

For $p = 2$ we obtain

$$\|\boldsymbol{x}\|_{\ell_2} = \sqrt{\sum_{n=n_1}^{n_2} |x(n)|^2}, \qquad \boldsymbol{x} \in \ell_2(n_1, n_2). \tag{1.10}$$

Thus, the energy of a discrete-time signal $x(n)$, $n \in \mathbb{Z}$ can be expressed as:

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \|\boldsymbol{x}\|_{\ell_2}^2, \qquad \boldsymbol{x} \in \ell_2(-\infty, \infty). \tag{1.11}$$

1.1.3 Metric Spaces

A function that assigns a real number to two elements x and y of a non-empty set X is called a metric on X if it satisfies the following axioms:

$$\text{(i)} \qquad d(x, y) \geq 0, \quad d(x, y) = 0 \text{ if and only if } x = y, \tag{1.12}$$

$$\text{(ii)} \qquad d(x, y) = d(y, x), \tag{1.13}$$

$$\text{(iii)} \qquad d(x, z) \leq d(x, y) + d(y, z). \tag{1.14}$$

The metric $d(x, y)$ can be understood as the distance between x and y.

A normed space is also a metric space. Here, the metric induced by the norm is the norm of the difference vector:

$$d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|. \tag{1.15}$$

Proof (norm $\to$ metric). For $d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|$ the validity of (1.12) immediately follows from (1.3). With $\alpha = -1$, (1.5) leads to $\|\boldsymbol{x} - \boldsymbol{y}\| = \|\boldsymbol{y} - \boldsymbol{x}\|$, and (1.13) is also satisfied. For two vectors $\boldsymbol{x} = \boldsymbol{a} - \boldsymbol{b}$ and $\boldsymbol{y} = \boldsymbol{b} - \boldsymbol{c}$ the following holds according to (1.4):

$$\|\boldsymbol{a} - \boldsymbol{c}\| = \|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\| = \|\boldsymbol{a} - \boldsymbol{b}\| + \|\boldsymbol{b} - \boldsymbol{c}\|.$$

Thus, $d(\boldsymbol{a}, \boldsymbol{c}) \leq d(\boldsymbol{a}, \boldsymbol{b}) + d(\boldsymbol{b}, \boldsymbol{c})$, which means that also (1.14) is satisfied. □

An example is the *Euclidean metric* induced by the Euclidean norm:

$$d(\boldsymbol{x},\boldsymbol{y}) = \left[\int_a^b |x(t)-y(t)|^2 dt\right]^{1/2}, \qquad \boldsymbol{x},\boldsymbol{y} \in L_2(a,b). \tag{1.16}$$

Accordingly, the following distance between discrete-time signals can be stated:

$$d(\boldsymbol{x},\boldsymbol{y}) = \left[\sum_{n=n_1}^{n_2} |x(n)-y(n)|^2\right]^{1/2}, \qquad \boldsymbol{x},\boldsymbol{y} \in \ell_2(n_1,n_2). \tag{1.17}$$

Nevertheless, we also find metrics which are not associated with a norm. An example is the Hamming distance

$$d(\boldsymbol{x},\boldsymbol{y}) = \sum_{k=1}^{n} [(x_k + y_k) \bmod 2],$$

which states the number of positions where two binary code words $\boldsymbol{x} = [x_1, x_2, \ldots, x_n]$ and $\boldsymbol{y} = [y_1, y_2, \ldots, y_n]$ with $x_i, y_i \in \{0,1\}$ differ (the space of the code words is not a linear vector space).

Note. The normed spaces L_p and ℓ_p are so-called *Banach spaces*, which means that they are normed linear spaces which are complete with regard to their metric $d(\boldsymbol{x},\boldsymbol{y}) = \|\boldsymbol{x}-\boldsymbol{y}\|$. A space is complete if any Cauchy sequence of the elements of the space converges within the space. That is, if $\|\boldsymbol{x}_n - \boldsymbol{x}_m\| \to 0$ as n and $m \to \infty$, while the limit of $\boldsymbol{x}_n$ for $n \to \infty$ lies in the space.

1.1.4 Inner Product Spaces

The signal spaces most frequently considered are the spaces $L_2(a,b)$ and $\ell_2(n_1,n_2)$; for these spaces inner products can be stated. An inner product assigns a complex number to two signals $x(t)$ and $y(t)$, or $x(n)$ and $y(n)$, respectively. The notation is $\langle \boldsymbol{x},\boldsymbol{y}\rangle$. An inner product must satisfy the following axioms:

$$\text{(i)} \quad \langle \boldsymbol{x},\boldsymbol{y}\rangle = \langle \boldsymbol{y},\boldsymbol{x}\rangle^* \tag{1.18}$$

$$\text{(ii)} \quad \langle \alpha\boldsymbol{x} + \beta\boldsymbol{y},\boldsymbol{z}\rangle = \alpha\langle \boldsymbol{x},\boldsymbol{z}\rangle + \beta\langle \boldsymbol{y},\boldsymbol{z}\rangle \tag{1.19}$$

$$\text{(iii)} \quad \langle \boldsymbol{x},\boldsymbol{x}\rangle \geq 0, \quad \langle \boldsymbol{x},\boldsymbol{x}\rangle = 0 \text{ if and only if } \boldsymbol{x} = \boldsymbol{0}. \tag{1.20}$$

Here, α and β are scalars with $\alpha, \beta \in \mathbb{C}$, and $\boldsymbol{0}$ is the null vector.

Examples of inner products are

$$\langle \boldsymbol{x},\boldsymbol{y}\rangle = \int_a^b x(t)\, y^*(t)\, dt, \quad \boldsymbol{x},\boldsymbol{y} \in L_2(a,b) \tag{1.21}$$

and

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{n=n_1}^{n_2} x(n)\ y^*(n), \quad \boldsymbol{x}, \boldsymbol{y} \in \ell_2(n_1, n_2). \tag{1.22}$$

The inner product (1.22) may also be written as

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{y}^H \boldsymbol{x}, \quad \boldsymbol{x}, \boldsymbol{y} \in \ell_2(n_1, n_2), \tag{1.23}$$

where the vectors are understood as column vectors:[1]

$$\begin{aligned} \boldsymbol{x} &= [x(n_1), x(n_1+1), \ldots, x(n_2)]^T, \\ \boldsymbol{y} &= [y(n_1), y(n_1+1), \ldots, y(n_2)]^T. \end{aligned} \tag{1.24}$$

More general definitions of inner products include weighting functions or weighting matrices. An inner product of two continuous-time signals $x(t)$ and $y(t)$ including weighting can be defined as

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \int_a^b g(t)\ x(t)\ y^*(t)\ dt, \tag{1.25}$$

where $g(t)$ is a real weighting function with $g(t) > 0,\ a \leq t \leq b$.

The general definition of inner products of discrete-time signals is

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{y}^H \boldsymbol{G}\, \boldsymbol{x}, \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^N, \tag{1.26}$$

where $\boldsymbol{G}$ is a real-valued, Hermitian, positive definite weighting matrix. This means that $\boldsymbol{G}^H = \boldsymbol{G}^T = \boldsymbol{G}$, and all eigenvalues λ_i of $\boldsymbol{G}$ must be larger than zero. As can easily be verified, the inner products (1.25) and (1.26) meet conditions (1.18) – (1.20).

The mathematical rules for inner products basically correspond to those for ordinary products of scalars. However, the order in which the vectors occur must be observed: (1.18) shows that changing the order leads to a conjugation of the result.

As equation (1.19) indicates, a scalar prefactor of the left argument may directly precede the inner product: $\langle \alpha \boldsymbol{x}, \boldsymbol{y} \rangle = \alpha \langle \boldsymbol{x}, \boldsymbol{y} \rangle$. If we want a prefactor

[1] The superscript T denotes transposition. The elements of $\boldsymbol{x}$ and $\boldsymbol{y}$ may be real or complex-valued. A superscript H, as in (1.23), means transposition and complex conjugation. A vector $\boldsymbol{x}^H$ is also referred to as the *Hermitian* of $\boldsymbol{x}$. If a vector is to be conjugated but not to be transposed, we write $\boldsymbol{x}^*$ such that $\boldsymbol{x}^H = [\boldsymbol{x}^*]^T$.

of the right argument to precede the inner product, it must be conjugated, since (1.18) and (1.19) lead to

$$\langle \boldsymbol{x}, \alpha \boldsymbol{y} \rangle = \langle \alpha \boldsymbol{y}, \boldsymbol{x} \rangle^* = [\alpha \langle \boldsymbol{y}, \boldsymbol{x} \rangle]^* = \alpha^* \langle \boldsymbol{x}, \boldsymbol{y} \rangle . \tag{1.27}$$

Due to (1.18), an inner product $\langle \boldsymbol{x}, \boldsymbol{x} \rangle$ is always real: $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = \Re\{\langle \boldsymbol{x}, \boldsymbol{x} \rangle\}$.

By defining an inner product we obtain a norm and also a metric. The norm induced by the inner product is

$$\|\boldsymbol{x}\| = \langle \boldsymbol{x}, \boldsymbol{x} \rangle^{1/2} . \tag{1.28}$$

We will prove this in the following along with the *Schwarz inequality*, which states

$$| \langle \boldsymbol{x}, \boldsymbol{y} \rangle | \leq \|\boldsymbol{x}\| \ \|\boldsymbol{y}\| . \tag{1.29}$$

Equality in (1.29) is given only if $\boldsymbol{x}$ and $\boldsymbol{y}$ are linearly dependent, that is, if one vector is a multiple of the other.

Proof (inner product → norm). From (1.20) it follows immediately that (1.3) is satisfied. For the norm of $\alpha \boldsymbol{x}$, we conclude from (1.18) and (1.19)

$$\|\alpha \boldsymbol{x}\| = \langle \alpha \boldsymbol{x}, \alpha \boldsymbol{x} \rangle^{1/2} = [\ |\alpha|^2 \ \langle \boldsymbol{x}, \boldsymbol{x} \rangle \ ]^{1/2} = |\alpha| \ \langle \boldsymbol{x}, \boldsymbol{x} \rangle^{1/2} = |\alpha| \ \|\boldsymbol{x}\| .$$

Thus, (1.5) is also proved.

Now the expression $\|\boldsymbol{x} + \boldsymbol{y}\|^2$ will be considered. We have

$$\begin{aligned} \|\boldsymbol{x} + \boldsymbol{y}\|^2 &= \langle \boldsymbol{x} + \boldsymbol{y}, \boldsymbol{x} + \boldsymbol{y} \rangle \\ &= \langle \boldsymbol{x}, \boldsymbol{x} \rangle + \langle \boldsymbol{x}, \boldsymbol{y} \rangle + \langle \boldsymbol{y}, \boldsymbol{x} \rangle + \langle \boldsymbol{y}, \boldsymbol{y} \rangle \\ &= \langle \boldsymbol{x}, \boldsymbol{x} \rangle + 2\Re\{\langle \boldsymbol{x}, \boldsymbol{y} \rangle\} + \langle \boldsymbol{y}, \boldsymbol{y} \rangle \\ &\leq \langle \boldsymbol{x}, \boldsymbol{x} \rangle + 2 \, | \langle \boldsymbol{x}, \boldsymbol{y} \rangle | + \langle \boldsymbol{y}, \boldsymbol{y} \rangle . \end{aligned}$$

Assuming the Schwarz inequality is correct, we conclude

$$\|\boldsymbol{x} + \boldsymbol{y}\|^2 \leq \|\boldsymbol{x}\|^2 + 2 \ \|\boldsymbol{x}\| \ \|\boldsymbol{y}\| + \|\boldsymbol{y}\|^2 = (\|\boldsymbol{x}\| + \|\boldsymbol{y}\|)^2 .$$

This shows that also (1.4) holds. □

Proof of the Schwarz inequality. The validity of the equality sign in the Schwarz inequality (1.29) for linearly dependent vectors can easily be proved

by substituting $\boldsymbol{x} = \alpha\boldsymbol{y}$ or $\boldsymbol{y} = \alpha\boldsymbol{x}$, $\alpha \in \mathbb{C}$, into (1.29) and rearranging the expression obtained, observing (1.28). For example, for $\boldsymbol{x} = \alpha\boldsymbol{y}$ we have

$$|\langle \boldsymbol{x}, \boldsymbol{y} \rangle| = |\langle \alpha\boldsymbol{y}, \boldsymbol{y} \rangle| = |\alpha| \ \langle \boldsymbol{y}, \boldsymbol{y} \rangle = |\alpha| \ \|\boldsymbol{y}\|^2 = \|\alpha\boldsymbol{y}\| \ \|\boldsymbol{y}\| = \|\boldsymbol{x}\| \ \|\boldsymbol{y}\| .$$

In order to prove the Schwarz inequality for linearly independent vectors, some vector $\boldsymbol{z} = \boldsymbol{x} + \alpha\boldsymbol{y}$ will be considered. On the basis of (1.18) – (1.20) we have

$$\begin{aligned} 0 &\leq \langle \boldsymbol{z}, \boldsymbol{z} \rangle \\ &= \langle \boldsymbol{x} + \alpha\boldsymbol{y}, \boldsymbol{x} + \alpha\boldsymbol{y} \rangle \\ &= \langle \boldsymbol{x}, \boldsymbol{x} + \alpha\boldsymbol{y} \rangle + \langle \alpha\boldsymbol{y}, \boldsymbol{x} + \alpha\boldsymbol{y} \rangle \\ &= \langle \boldsymbol{x}, \boldsymbol{x} \rangle + \alpha^* \langle \boldsymbol{x}, \boldsymbol{y} \rangle + \alpha \langle \boldsymbol{y}, \boldsymbol{x} \rangle + \alpha\alpha^* \langle \boldsymbol{y}, \boldsymbol{y} \rangle . \end{aligned} \tag{1.30}$$

This also holds for the special α (assumption: $\boldsymbol{y} \neq \boldsymbol{0}$)

$$\alpha = -\frac{\langle \boldsymbol{x}, \boldsymbol{y} \rangle}{\langle \boldsymbol{y}, \boldsymbol{y} \rangle},$$

and we get

$$0 \leq \langle \boldsymbol{x}, \boldsymbol{x} \rangle - \frac{\langle \boldsymbol{x}, \boldsymbol{y} \rangle^* \langle \boldsymbol{x}, \boldsymbol{y} \rangle}{\langle \boldsymbol{y}, \boldsymbol{y} \rangle} - \frac{\langle \boldsymbol{x}, \boldsymbol{y} \rangle \langle \boldsymbol{y}, \boldsymbol{x} \rangle}{\langle \boldsymbol{y}, \boldsymbol{y} \rangle} + \frac{\langle \boldsymbol{x}, \boldsymbol{y} \rangle \langle \boldsymbol{x}, \boldsymbol{y} \rangle^* \langle \boldsymbol{y}, \boldsymbol{y} \rangle}{\langle \boldsymbol{y}, \boldsymbol{y} \rangle \cdot \langle \boldsymbol{y}, \boldsymbol{y} \rangle}.$$

The second and the fourth term cancel,

$$0 \leq \langle \boldsymbol{x}, \boldsymbol{x} \rangle - \frac{\langle \boldsymbol{x}, \boldsymbol{y} \rangle \langle \boldsymbol{y}, \boldsymbol{x} \rangle}{\langle \boldsymbol{y}, \boldsymbol{y} \rangle} = \langle \boldsymbol{x}, \boldsymbol{x} \rangle - \frac{|\langle \boldsymbol{x}, \boldsymbol{y} \rangle|^2}{\langle \boldsymbol{y}, \boldsymbol{y} \rangle}, \tag{1.31}$$

and we obtain

$$|\langle \boldsymbol{x}, \boldsymbol{y} \rangle|^2 \leq \langle \boldsymbol{x}, \boldsymbol{x} \rangle \cdot \langle \boldsymbol{y}, \boldsymbol{y} \rangle . \tag{1.32}$$

Comparing (1.32) with (1.28) and (1.29) confirms the Schwarz inequality. □

Equation (1.28) shows that the inner products given in (1.21) and (1.22) lead to the norms (1.7) and (1.10).

Finally, let us remark that a linear space with an inner product which is *complete* with respect to the induced metric is called a *Hilbert space.*

1.2 Energy Density and Correlation

1.2.1 Continuous-Time Signals

Let us reconsider (1.1):

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt. \tag{1.33}$$

According to Parseval's theorem, we may also write

$$E_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega, \tag{1.34}$$

where $X(\omega)$ is the Fourier transform of $x(t)$.[2] The quantity $|x(t)|^2$ in (1.33) represents the distribution of signal energy with respect to time t; accordingly, $|X(\omega)|^2$ in (1.34) can be viewed as the distribution of energy with respect to frequency ω. Therefore $|X(\omega)|^2$ is called the *energy density spectrum* of $x(t)$. We use the following notation

$$S_{xx}^E(\omega) = |X(\omega)|^2. \tag{1.35}$$

The energy density spectrum $S_{xx}^E(\omega)$ can also be regarded as the Fourier transform of the so-called *autocorrelation function*

$$r_{xx}^E(\tau) = \int_{-\infty}^{\infty} x^*(t) \; x(t+\tau) \; dt = x^*(-\tau) \; * \; x(\tau). \tag{1.36}$$

We have

$$S_{xx}^E(\omega) = \int_{-\infty}^{\infty} r_{xx}^E(\tau) \; e^{-j\omega\tau} \; d\tau. \tag{1.37}$$

The correspondence is denoted as $S_{xx}^E(\omega) \longleftrightarrow r_{xx}^E(\tau)$.

The autocorrelation function is a measure indicating the similarity between an energy signal $x(t)$ and its time-shifted variant $x_\tau(t) = x(t+\tau)$. This can be seen from

$$\begin{aligned} d(\boldsymbol{x}, \boldsymbol{x}_\tau)^2 &= \|\boldsymbol{x} - \boldsymbol{x}_\tau\|^2 \\ &= \langle \boldsymbol{x}, \boldsymbol{x} \rangle - \langle \boldsymbol{x}, \boldsymbol{x}_\tau \rangle - \langle \boldsymbol{x}_\tau, \boldsymbol{x} \rangle + \langle \boldsymbol{x}_\tau, \boldsymbol{x}_\tau \rangle \\ &= 2\,\|\boldsymbol{x}\|^2 - 2\,\Re\{\langle \boldsymbol{x}_\tau, \boldsymbol{x} \rangle\} \\ &= 2\,\|\boldsymbol{x}\|^2 - 2\,\Re\{r_{xx}^E(\tau)\}. \end{aligned} \tag{1.38}$$

With increasing correlation the distance decreases.

[2] In this section, we freely use the properties of the Fourier transform. For more detail on the Fourier transform and Parseval's theorem, see Section 2.2.

Similarly, the *cross correlation function*

$$r_{xy}^E(\tau) = \int_{-\infty}^{\infty} y(t+\tau)\ x^*(t)\ dt \tag{1.39}$$

and the corresponding *cross energy density spectrum*

$$S_{xy}^E(\omega) = \int_{-\infty}^{\infty} r_{xy}^E(\tau)\ e^{-j\omega\tau}\ d\tau, \tag{1.40}$$

that is

$$S_{xy}^E(\omega) \longleftrightarrow r_{xy}^E(\tau), \tag{1.41}$$

are introduced, where $r_{xy}^E(\tau)$ may be viewed as a measure of the similarity between the two signals $x(t)$ and $y_\tau(t) = y(t+\tau)$.

1.2.2 Discrete-Time Signals

All previous considerations are applicable to discrete-time signals $x(n)$ as well. The signals $x(n)$ may be real or complex-valued. As in the continuous-time case, we start the discussion with the energy of the signal:

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2. \tag{1.42}$$

According to Parseval's relation for the discrete-time Fourier transform, we may alternatively compute E_x from $X(e^{j\omega})$:[3]

$$E_x = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2\ d\omega. \tag{1.43}$$

The term $|X(e^{j\omega})|^2$ in (1.43) is called the *energy density spectrum* of the discrete-time signal. We use the notation

$$S_{xx}^E(e^{j\omega}) = |X(e^{j\omega})|^2. \tag{1.44}$$

The energy density spectrum $S_{xx}^E(e^{j\omega})$ is the discrete-time Fourier transform of the *autocorrelation sequence*

$$r_{xx}^E(m) = \sum_{n=-\infty}^{\infty} x^*(n)\ x(n+m). \tag{1.45}$$

[3]See Section 4.2 for more detail on the discrete-time Fourier transform.

We have

$$\begin{aligned} S_{xx}^{E}(e^{j\omega}) &= \sum_{m=-\infty}^{\infty} r_{xx}^{E}(m)\ e^{-j\omega m} \\ &\updownarrow \\ r_{xx}^{E}(m) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{xx}^{E}(e^{j\omega})\ e^{j\omega m}\ d\omega. \end{aligned} \tag{1.46}$$

Note that the energy density may also be viewed as the product $\tilde{X}(z)X(z)$, evaluated on the unit circle ($z = e^{j\omega}$), where $X(z)$ is the z-transform of $x(n)$.

The definition of the *cross correlation sequence* is

$$r_{xy}^{E}(m) = \sum_{n=-\infty}^{\infty} y(n+m)\ x^{*}(n). \tag{1.47}$$

For the corresponding *cross energy density spectrum* the following holds:

$$S_{xy}^{E}(e^{j\omega}) = \sum_{m=-\infty}^{\infty} r_{xy}^{E}(m)\ e^{-j\omega m}, \tag{1.48}$$

that is

$$S_{xy}^{E}(e^{j\omega}) \longleftrightarrow r_{xy}^{E}(m). \tag{1.49}$$

1.3 Random Signals

Random signals are encountered in all areas of signal processing. For example, they appear as disturbances in the transmission of signals. Even the transmitted and consequently also the received signals in telecommunications are of random nature, because only random signals carry information. In pattern recognition, the patterns that are to be distinguished are modeled as random processes. In speech, audio, and image coding, the signals to be compressed are modeled as such.

First of all, one distinguishes between *random variables* and *random processes*. A random variable is obtained by assigning a real or complex number to each feature m_i from a feature set M. The features (or events) occur randomly. Note that the features themselves may also be non-numeric.

If one assigns a function ${}^{i}x(t)$ to each feature m_i, then the totality of all possible functions is called a *stochastic process*. The features occur randomly whereas the assignment $m_i \rightarrow {}^{i}x(t)$ is deterministic. A function ${}^{i}x(t)$ is called the *realization* of the stochastic process $x(t)$. See Figure 1.1 for an illustration.

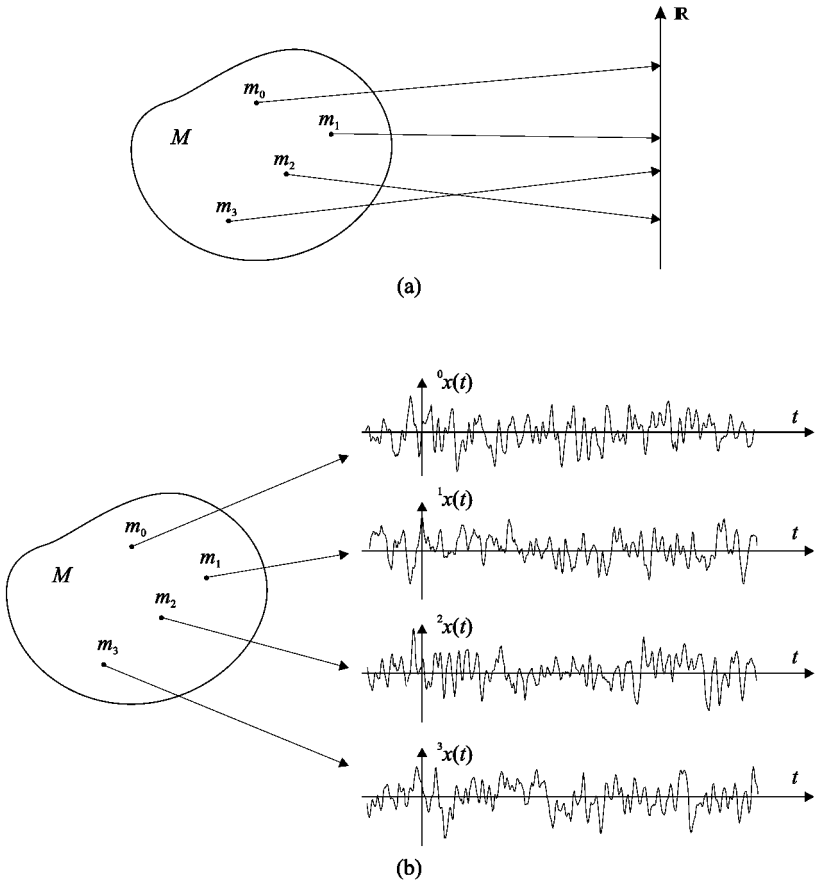

Figure 1.1. Random variables (a) and random processes (b).

1.3.1 Properties of Random Variables

The properties of a real random variable x are thoroughly characterized by its *cumulative distribution function* $F_x(\alpha)$ and also by its *probability density function* (pdf) $p_x(\alpha)$. The distribution states the probability P with which the value of the random variable x is smaller than or equal to a given value α:

$$F_x(\alpha) = P(x \leq \alpha). \tag{1.50}$$

Here, the *axioms of probability* hold, which state that

$$\lim_{\alpha \to -\infty} F_x(\alpha) = 0, \quad \lim_{\alpha \to \infty} F_x(\alpha) = 1, \quad F_x(\alpha_1) \leq F_x(\alpha_2) \quad \text{for} \quad \alpha_1 \leq \alpha_2. \tag{1.51}$$

Given the distribution, we obtain the pdf by differentiation:

$$p_x(\alpha) = \frac{d}{d\alpha} F_x(\alpha). \tag{1.52}$$

Since the distribution is a non-decreasing function, we have

$$p_x(\alpha) \geq 0. \tag{1.53}$$

Joint Probability Density. The joint probability density $p_{x_1,x_2}(\xi_1, \xi_2)$ of two random variables x_1 and x_2 is given by

$$p_{x_1,x_2}(\xi_1, \xi_2) = p_{x_1}(\xi_1)\ p_{x_2|x_1}(\xi_2|\xi_1), \tag{1.54}$$

where $p_{x_2|x_1}(\xi_2|\xi_1)$ is a conditional probability density (density of x_2 provided x_1 has taken on the value ξ_1). If the variables x_1 and x_2 are statistically independent of one another, (1.54) reduces to

$$p_{x_1,x_2}(\xi_1, \xi_2) = p_{x_1}(\xi_1)\ p_{x_2}(\xi_2). \tag{1.55}$$

The pdf of a complex random variable is defined as the joint density of its real and imaginary part:

$$p_x(\xi) = p_u(\xi_1)\ p_{v|u}(\xi_2|\xi_1), \qquad u = \Re\{x\},\ v = \Im\{x\},\ \xi = \xi_1 + j\xi_2. \tag{1.56}$$

Moments. The properties of a random variable are often described by its moments

$$m_x^{(n)} = E\{|x|^n\}. \tag{1.57}$$

Herein, $E\{\cdot\}$ denotes the *expected value* (statistical average). An expected value $E\{g(x)\}$, where $g(x)$ is an arbitrary function of the random variable x, can be calculated from the density as

$$E\{g(x)\} = \int_{-\infty}^{\infty} g(\xi)\ p_x(\xi)\ d\xi. \tag{1.58}$$

For $g(x) = x$ we obtain the mean value (first moment):

$$m_x = E\{x\} = \int_{-\infty}^{\infty} \xi\ p_x(\xi)\ d\xi. \tag{1.59}$$

For $g(x) = |x|^2$ we obtain the average power (second moment):

$$s_x^2 = E\{|x|^2\} = \int_{-\infty}^{\infty} |\xi|^2\ p_x(\xi)\ d\xi. \tag{1.60}$$

The *variance* (second central moment) is calculated with $g(x) = |x - m_x|^2$ as

$$\sigma_x^2 = E\left\{|x - m_x|^2\right\} = \int_{-\infty}^{\infty} |\xi - m_x|^2 \; p_x(\xi) \; d\xi. \tag{1.61}$$

The following holds:

$$\sigma_x^2 = s_x^2 - m_x^2. \tag{1.62}$$

Characteristic Function. The characteristic function of a random variable x is defined as

$$\Phi_x(\nu) = \int_{-\infty}^{\infty} e^{j\nu x} \; p_x(\nu) \; d\nu, \tag{1.63}$$

which means that, apart from the sign of the argument, it is the Fourier transform of the pdf. According to the moment theorem of the Fourier transform (see Section 2.2), the moments of the random variable can also be computed from the characteristic function as

$$m_x^{(n)} = (-j)^n \; \frac{d^n \Phi_x(\nu)}{d\nu^n}\bigg|_{\nu\,=\,0} \tag{1.64}$$

1.3.2 Random Processes

The starting point for the following considerations is a stochastic process $x(t)$, from which the random variables $x_{t_1}, x_{t_2}, \ldots, x_{t_n}$ with $x_{t_k} = x(t_k)$ are taken at times $t_1 < t_2 < \ldots < t_n$, $n \in \mathbb{Z}$. The properties of these random variables are characterized by their joint pdf $p_{x_{t_1}, x_{t_2}, \ldots, x_{t_n}}(\alpha_1, \alpha_2, \ldots, \alpha_n)$. Then a second set of random variables is taken from the process $x(t)$, applying a time shift τ: $x_{t_1+\tau}, x_{t_2+\tau}, \ldots, x_{t_n+\tau}$ with $x_{t_k+\tau} = x(t_k + \tau)$. If the joint densities of both sets are equal for all time shifts τ and all n, that is, if we have

$$p_{x_{t_1}, x_{t_2}, \ldots, x_{t_n}}(\alpha_1, \alpha_2, \ldots, \alpha_n) = p_{x_{t_1+\tau}, x_{t_2+\tau}, \ldots, x_{t_n+\tau}}(\alpha_1, \alpha_2, \ldots, \alpha_n), \quad \forall\, n, \tau, \tag{1.65}$$

then we speak of a *strictly stationary process*, otherwise we call the process *non-stationary*.

Autocorrelation and Autocovariance Functions of Non-Stationary Processes. The autocorrelation function of a general random process is defined as a second-order moment:

$$\begin{aligned} r_{xx}(t_1, t_2) &= E\left\{x^*(t_2) \; x(t_1)\right\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \xi_1 \; \xi_2 \; p_{x_1, x_2}(\xi_1, \xi_2) \; d\xi_1 \; d\xi_2, \end{aligned} \tag{1.66}$$

where $x_1 = x(t_1)$ and $x_2 = x^*(t_2)$.

Basically, the autocorrelation function indicates how similar the process is at times t_1 and t_2, since for the expected Euclidean distance we have

$$E\left\{|x_1 - x_2|^2\right\} = E\left\{|x_1|^2\right\} + E\left\{|x_2|^2\right\} - 2\,\Re\{r_{xx}(t_1, t_2)\}.$$

The *autocovariance function* of a random process is defined as

$$\begin{aligned} c_{xx}(t_1, t_2) &= E\left\{\left[x^*(t_2) - m^*_{t_2}\right]\left[x(t_1) - m_{t_1}\right]\right\} \\ &= r_{xx}(t_1, t_2) - m^*_{t_2} m_{t_1}, \end{aligned} \tag{1.67}$$

where m_{t_k} denotes the expected value at time t_k; i.e.

$$m_{t_k} = E\left\{x(t_k)\right\}. \tag{1.68}$$

Wide-Sense Stationary Processes. There are processes whose mean value is constant and whose autocorrelation function is a function of $t_1 - t_2$. Such processes are referred to as "*wide-sense stationary*", even if they are non-stationary according to the above definition.

Cyclo-Stationary Process. If a process is non-stationary according to the definition stated above, but if the properties repeat periodically, then we speak of a *cyclo-stationary process.*

Autocorrelation and Autocovariance Functions of Wide-Sense Stationary Processes. In the following we assume wide-sense stationarity, so that the first and second moments are independent of the respective time. Because of the stationarity we must assume that the process realizations are not absolutely integrable, and that their Fourier transforms do not exist. Since in the field of telecommunications one also encounters complex-valued processes when describing real bandpass processes in the complex baseband, we shall continue by looking at complex-valued processes. For wide-sense stationary processes the *autocorrelation function* (acf) depends only on the time shift between the respective times; it is given by

$$r_{xx}(\tau) = E\left\{x^*(t)\; x(t+\tau)\right\}. \tag{1.69}$$

For $x_1 = x(t+\tau)$ and $x_2 = x^*(t)$, the expected value $E\{\cdot\}$ can be written as

$$r_{xx}(\tau) = E\left\{x_1\; x_2\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \xi_1\; \xi_2\; p_{x_1,x_2}(\xi_1, \xi_2)\; d\xi_1\; d\xi_2. \tag{1.70}$$

The maximum of the autocorrelation function is located at $\tau = 0$, where its value equals the mean square value:

$$r_{xx}(0) = s_x^2 = E\{|x|^2\} = \int_{-\infty}^{\infty} |\xi|^2 \, p_x(\xi) \, d\xi. \tag{1.71}$$

Furthermore we have $r_{xx}(-\tau) = r_{xx}^*(\tau)$.

When subtracting the mean

$$m_x = E\{x(t)\} \tag{1.72}$$

prior computing the autocorrelation function, we get the autocovariance function

$$\begin{aligned} c_{xx}(\tau) &= E\{[x^*(t) - m_x^*]\,[x(t+\tau) - m_x]\} \\ &= r_{xx}(\tau) - |m_x|^2. \end{aligned} \tag{1.73}$$

Power Spectral Density. The *power spectral density*, or *power density spectrum*, describes the distribution of power with respect to frequency. It is defined as the Fourier transform of the autocorrelation function:

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} r_{xx}(\tau)\, e^{-j\omega\tau} \, d\tau \tag{1.74}$$

$$\updownarrow$$

$$r_{xx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{j\omega\tau} \, d\omega. \tag{1.75}$$

This definition is based on the *Wiener–Khintchine theorem*, which states that the physically meaningful power spectral density given by

$$S_{xx}(\omega) = \lim_{T\to\infty} \frac{1}{T} E\{|X_T(\omega)|^2\} \tag{1.76}$$

with

$$X_T(\omega) \;\longleftrightarrow\; x(t)\ \mathrm{rect}(\frac{t}{T}),$$

and

$$\mathrm{rect}(t) = \begin{cases} 1, & \text{for } |t| \le 0.5 \\ 0, & \text{otherwise} \end{cases}$$

is identical to the power spectral density given in (1.74).

Taking (1.75) for $\tau = 0$, we obtain

$$s_x^2 = r_{xx}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) \, d\omega. \tag{1.77}$$

Cross Correlation and Cross Power Spectral Density. The *cross correlation* between two wide-sense stationary random processes $x(t)$ and $y(t)$ is defined as

$$r_{xy}(\tau) = E\{x^*(t)\ y(t+\tau)\}. \tag{1.78}$$

The Fourier transform of $r_{xy}(\tau)$ is the *cross power spectral density*, denoted as $S_{xy}(\omega)$. Thus, we have the correspondence

$$\begin{aligned} S_{xy}(\omega) &= \int_{-\infty}^{\infty} r_{xy}(\tau)\ e^{-j\omega\tau}\ d\tau \\ &\updownarrow \\ r_{xy}(\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xy}(\omega) e^{j\omega\tau}\ d\omega. \end{aligned} \tag{1.79}$$

Discrete-Time Signals. The following definitions for discrete-time signals basically correspond to those for continuous-time signals; the correlation and covariance functions, however, become correlation and covariance sequences. For the *autocorrelation sequence* we have

$$r_{xx}(m) = E\{x^*(n)\ x(n+m)\}. \tag{1.80}$$

The *autocovariance sequence* is defined as

$$\begin{aligned} c_{xx}(m) &= E\{[x^*(n) - m_x^*]\ [x(n+m) - m_x]\} \\ &= r_{xx}(m) - |m_x|^2, \end{aligned} \tag{1.81}$$

where

$$m_x = E\{x(n)\}. \tag{1.82}$$

The discrete-time Fourier transform of the autocorrelation sequence is the *power spectral density* (Wiener–Khintchine theorem). We have

$$S_{xx}(e^{j\omega}) = \sum_{m=-\infty}^{\infty} r_{xx}(m)\ e^{-j\omega m} \tag{1.83}$$

$$\updownarrow$$

$$r_{xx}(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{xx}(e^{j\omega})\ e^{j\omega m}\ d\omega. \tag{1.84}$$

The definition of the *cross correlation sequence* is

$$r_{xy}(m) = E\{x^*(n)\ y(n+m)\}, \tag{1.85}$$

where the Fourier transform of $r_{xy}(m)$ is the *cross power spectral density* $S_{xy}(e^{j\omega})$:

$$\begin{aligned} r_{xy}(m) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{xy}(e^{j\omega})\ e^{j\omega m}\ d\omega \\ &\updownarrow \\ S_{xy}(e^{j\omega}) &= \sum_{m=-\infty}^{\infty} r_{xy}(m)\ e^{-j\omega m}. \end{aligned} \tag{1.86}$$

A *cross covariance sequence* can be defined as

$$\begin{aligned} c_{xy}(m) &= E\{[x^*(n) - m_x^*]\ [y(n+m) - m_y]\} \\ &= r_{xy}(m) - m_x^*\, m_y \end{aligned} \tag{1.87}$$

with

$$m_x = E\{x(n)\}, \quad m_y = E\{y(n)\}. \tag{1.88}$$

Correlation Matrices. *Auto* and *cross correlation matrices* are frequently required. We use the following definitions

$$\begin{aligned} \boldsymbol{R}_{xx} &= E\{\boldsymbol{x}\,\boldsymbol{x}^H\}, \\ \boldsymbol{R}_{xy} &= E\{\boldsymbol{y}\,\boldsymbol{x}^H\}, \end{aligned} \tag{1.89}$$

where

$$\begin{aligned} \boldsymbol{x} &= [x(n), x(n+1), \ldots, x(n+N_x-1)]^T, \\ \boldsymbol{y} &= [y(n), y(n+1), \ldots, y(n+N_y-1)]^T. \end{aligned} \tag{1.90}$$

The terms $\boldsymbol{x}\boldsymbol{x}^H$ and $\boldsymbol{y}\boldsymbol{x}^H$ are *dyadic products.*

For the sake of completeness it shall be noted that the autocorrelation matrix $\boldsymbol{R}_{xx}$ of a stationary process $x(n)$ has the following *Toeplitz structure*:

$$\boldsymbol{R}_{xx} = \begin{bmatrix} r_{xx}(0) & r_{xx}(-1) & \ldots & r_{xx}(-N_x+1) \\ r_{xx}(1) & r_{xx}(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & r_{xx}(-1) \\ r_{xx}(N_x-1) & \ldots & r_{xx}(1) & r_{xx}(0) \end{bmatrix}. \tag{1.91}$$

Here, the property

$$r_{xx}(-i) = r_{xx}^*(i), \tag{1.92}$$

which is concluded from (1.80) by taking stationarity into consideration, has been used.

If two processes $x(n)$ and $y(n)$ are pairwise stationary, we have

$$r_{xy}(-i) = r_{yx}^*(i), \tag{1.93}$$

and the cross correlation matrix $\boldsymbol{R}_{xy} = E\{\boldsymbol{y}\,\boldsymbol{x}^H\}$ has the following structure:

$$\boldsymbol{R}_{xy} = \begin{bmatrix} r_{xy}(0) & r_{xy}(-1) & \dots & r_{xy}(-N_x+1) \\ r_{xy}(1) & r_{xy}(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & r_{xy}(-1) \\ r_{xy}(N_y-1) & \dots & r_{xy}(1) & r_{xy}(0) \end{bmatrix}. \tag{1.94}$$

Auto and *cross-covariance matrices* can be defined in an analog way by replacing the entries $r_{xy}(m)$ through $c_{xy}(m)$.

Ergodic Processes. Usually, the autocorrelation function is calculated according to (1.70) by taking the ensemble average. An exception to this rule is the *ergodic process*, where the ensemble average can be replaced by a temporal average. For the autocorrelation function of an ergodic continuous-time process we have

$$r_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} {}^i x^*(t)\ {}^i x(t+\tau)\ dt, \tag{1.95}$$

where ${}^i x(t)$ is an arbitrary realization of the stochastic process. Accordingly, we get

$$r_{xx}(m) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} {}^i x^*(n)\ {}^i x(n+m) \tag{1.96}$$

for discrete-time signals.

Continuous-Time White Noise Process. A wide-sense stationary continuous-time noise process $x(t)$ is said to be white if its power spectral density is a constant:

$$S_{xx}(\omega) = \sigma^2. \tag{1.97}$$

The autocorrelation function of the process is a Dirac impulse with weight σ^2:

$$r_{xx}(\tau) = \sigma^2 \, \delta(\tau). \tag{1.98}$$

Since the power of such a process is infinite it is not realizable. However, the white noise process is a convenient model process which is often used for describing properties of real-world systems.

Continuous-Time Gaussian White Noise Process. We consider a real-valued wide-sense stationary stochastic process $x(t)$ and try to represent it on the interval $[-a, a]$ via a series expansion[4] with an arbitrary real-valued orthonormal basis $\varphi_i(t)$ for $L_2(-a, a)$. The basis satisfies

$$\int_{-a}^{a} \varphi_i(t) \, \varphi_j(t) \, dt = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

If the coefficients of the series expansion given by

$$\alpha_i = \int_{-a}^{a} \varphi_i(t) \, x(t) \, dt$$

are Gaussian random variables with

$$E\{\alpha_i^2\} = \sigma^2 \qquad \forall \, i$$

we call $x(t)$ a Gaussian white noise process.

Bandlimited White Noise Process. A bandlimited white noise process is a white noise process whose power spectral density is constant within a certain frequency band and zero outside this band. See Figure 1.2 for an illustration.

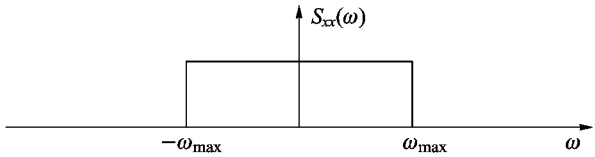

Figure 1.2. Bandlimited white noise process.

Discrete-Time White Noise Process. A discrete-time white noise process has the power spectral density

$$S_{xx}(e^{j\omega}) = \sigma^2 \tag{1.99}$$

[4]Series expansions are discussed in detail in Chapter 3.

and the autocorrelation sequence

$$r_{xx}(m) = \sigma^2 \, \delta_{m0}. \tag{1.100}$$

1.3.3 Transmission of Stochastic Processes through Linear Systems

Continuous-Time Processes. We assume a linear time-invariant system with the impulse response $h(t)$, which is excited by a stationary process $x(t)$. The cross correlation function between the input process $x(t)$ and the output process $y(t)$ is given by

$$\begin{aligned} r_{xy}(\tau) &= E\{x^*(t)\ y(t+\tau)\} \\ &= \int_{-\infty}^{\infty} E\{x^*(t)\ x(t+\tau-\lambda)\}\, h(\lambda) d\lambda \\ &= r_{xx}(\tau) * h(\tau). \end{aligned} \tag{1.101}$$

The cross power spectral density is obtained by taking the Fourier transform of (1.101):

$$S_{xy}(\omega) = S_{xx}(\omega)\ H(\omega). \tag{1.102}$$

Calculating the autocorrelation function of the output signal is done as follows:

$$\begin{aligned} r_{yy}(\tau) &= E\{y^*(t)\ y(t+\tau)\} \\ &= \iint E\{x^*(t-\alpha)\ x(t+\tau-\beta)\}\ h^*(\alpha)\, h(\beta)\, d\alpha\, d\beta \\ &= \iint r_{xx}(\tau+\alpha-\beta)\ h^*(\alpha)\ h(\beta)\, d\alpha\, d\beta \\ &= \int r_{xx}(\tau-\lambda) \int h^*(\alpha) h(\alpha+\lambda)\ d\alpha\, d\lambda \\ &= \int r_{xx}(\tau-\lambda)\ r_{hh}^E(\lambda)\ d\lambda. \end{aligned} \tag{1.103}$$

Thus, we obtain the following relationship:

$$r_{yy}(\tau) = r_{xx}(\tau) * r_{hh}^{E}(\tau). \tag{1.104}$$

Taking the Fourier transform of (1.104), we obtain the power spectral density of the output signal:

$$S_{yy}(\omega) = S_{xx}(\omega) \ |H(\omega)|^2 . \tag{1.105}$$

We observe that the phase of $H(\omega)$ has no influence on $S_{yy}(\omega)$. Consequently, only the magnitude frequency response of $H(\omega)$ can be determined from $S_{xx}(\omega)$ and $S_{yy}(\omega)$.

Discrete-Time Processes. The results for continuous-time signals and systems can be directly applied to the discrete-time case, where a system with impulse response $h(n)$ is excited by a process $x(n)$, yielding the output process $y(n)$. The cross correlation sequence between input and output is

$$r_{xy}(m) = r_{xx}(m) * h(m). \tag{1.106}$$

The cross power spectral density becomes

$$S_{xy}(e^{j\omega}) = S_{xx}(e^{j\omega}) \ H(e^{j\omega}). \tag{1.107}$$

For the autocorrelation sequence and the power spectral density at the output we get

$$r_{yy}(m) = r_{xx}(m) * r_{hh}^{E}(m) \tag{1.108}$$

and

$$S_{yy}(e^{j\omega}) = S_{xx}(e^{j\omega}) \ \left|H(e^{j\omega})\right|^2 . \tag{1.109}$$

As before, the phase of $H(e^{j\omega})$ has no influence on $S_{yy}(e^{j\omega})$.

Here we cease discussion of the transmission of stochastic processes through linear systems, but we will return to this topic in Section 5 of Chapter 2, where we will study the representation of stationary bandpass processes by means of their complex envelope.