Signal Analysis: Wavelets, Filter Banks, Time-Frequency Transforms and Applications. Alfred Mertins

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Chapter 2

Integral Signal Representations

The integral transform is one of the most important tools in signal theory. The best known example is the Fourier transform, but there are many other transforms of interest. In the following, we will first discuss the basic concepts of integral transforms. Then we will study the Fourier, Hartley, and Hilbert transforms. Finally, we will focus on real bandpass processes and their representation by means of their complex envelope.

2.1 Integral Transforms

The basic idea of an integral representation is to describe a signal $x(t)$ via its density $\hat{x}(s)$ with respect to an arbitrary *kernel* $\varphi(t,s)$:

$$x(t) = \int_S \hat{x}(s)\ \varphi(t,s)\ ds, \qquad t \in T. \tag{2.1}$$

Analogous to the reciprocal basis in discrete signal representations (see Section 3.3) a *reciproal kernel* $\theta(s,t)$ may be found such that the density $\hat{x}(s)$ can be calculated in the form

$$\hat{x}(s) = \int_T x(t)\ \theta(s,t)\ dt, \quad s \in S. \tag{2.2}$$

Contrary to discrete representations, we do not demand that the kernels $\varphi(t,s)$ and $\theta(s,t)$ be integrable with respect to t.

From (2.2) and (2.1), we obtain

$$\begin{aligned} x(t) &= \int_S \int_T x(\tau)\ \theta(s,\tau)\ d\tau\ \ \varphi(t,s)\ ds \\ &= \int_T x(\tau) \int_S \theta(s,\tau)\ \varphi(t,s)\ ds\ d\tau. \end{aligned} \tag{2.3}$$

In order to state the condition for the validity of (2.3) in a relatively simple form the so-called *Dirac impulse* $\delta(t)$ is required. By this we mean a generalized function with the property

$$x(t) = \int_{-\infty}^{\infty} \delta(t-\tau)\ x(\tau)\ d\tau, \qquad x \in L_1(\mathbb{R}). \tag{2.4}$$

The Dirac impulse can be viewed as the limit of a family of functions $g_\alpha(t)$ that has the following property for all signals $x(t)$ continuous at the origin:

$$\int_{-\infty}^{\infty} g_\alpha(t)\ dt = 1, \qquad \lim_{\alpha\to 0} \int_{-\infty}^{\infty} g_\alpha(t)\ x(t)\ dt = x(0). \tag{2.5}$$

An example is the Gaussian function

$$g_\alpha(t) = \frac{1}{\sqrt{2\pi\alpha}}\ e^{-\frac{t^2}{2\alpha}}, \qquad \alpha > 0. \tag{2.6}$$

Considering the Fourier transform of the Gaussian function, that is

$$\begin{aligned} G_\alpha(\omega) &= \int_{-\infty}^{\infty} g_\alpha(t)\ e^{-j\omega t}\ dt \\ &= e^{-\frac{\omega^2}{2\alpha}}, \end{aligned} \tag{2.7}$$

we find that it approximates the constant one for $\alpha \to 0$, that is $G_\alpha(\omega) \approx 1$, $\omega \in \mathbb{R}$. For the Dirac impulse the correspondence $\delta(t) \longleftrightarrow 1$ is introduced so that (2.4) can be expressed as $X(\omega) = 1\,X(\omega)$ in the frequency domain.

Equations (2.3) and (2.4) show that the kernel and the reciprocal kernel must satisfy

$$\int_S \theta(s,\tau)\ \varphi(t,s)\ ds = \delta(t-\tau). \tag{2.8}$$

By substituting (2.1) into (2.2) we obtain

$$\begin{aligned} \hat{x}(s) &= \int_T \int_S \hat{x}(\sigma)\ \varphi(t,\sigma)\ d\sigma\ \ \theta(s,t)\ dt \\ &= \int_S \hat{x}(\sigma) \int_T \varphi(t,\sigma)\ \theta(s,t)\ dt\ d\sigma, \end{aligned} \tag{2.9}$$

which implies that

$$\int_T \varphi(t,\sigma)\ \theta(s,t)\ dt = \delta(s-\sigma). \tag{2.10}$$

Equations (2.8) and (2.10) correspond to the relationship $\langle \boldsymbol{\varphi}_i, \boldsymbol{\theta}_j \rangle = \delta_{ij}$ for the discrete case (see Chapter 3).

Self-Reciprocal Kernels. A special category is that of self-reciprocal kernels. They correspond to orthonormal bases in the discrete case and satisfy

$$\varphi(t,s) = \theta^*(s,t). \tag{2.11}$$

Transforms that contain a self-reciprocal kernel are also called *unitary*, because they yield $\|\hat{\boldsymbol{x}}\| = \|\boldsymbol{x}\|$.

The Discrete Representation as a Special Case. The discrete representation via series expansion, which is discussed in detail in the next chapter, can be regarded as a special case of the integral representation. In order to explain this relationship, let us consider the discrete set

$$\varphi_i(t) = \varphi(t,s_i), \qquad i = 1,2,3,\dots. \tag{2.12}$$

For signals $x(t) \in \operatorname{span}\{\varphi(t,s_i);\ i=1,2,\dots\}$ we may write

$$x(t) = \sum_i \alpha_i\ \varphi_i(t) = \sum_i \alpha_i\ \varphi(t,s_i). \tag{2.13}$$

Insertion into (2.2) yields

$$\begin{aligned} \hat{x}(s) &= \int_T x(t)\theta(s,t)\ dt \\ &= \int_T \sum_i \alpha_i \varphi(t,s_i)\theta(s,t)\ dt \\ &= \sum_i \alpha_i \int_T \varphi(t,s_i)\theta(s,t)\ dt. \end{aligned} \tag{2.14}$$

The comparison with (2.10) shows that in the case of a discrete representation the density $\hat{x}(s)$ concentrates on the values s_i:

$$\hat{x}(s) = \sum_i \alpha_i\ \delta(s-s_i). \tag{2.15}$$

Parseval's Relation. Let the signals $x(t)$ and $y(t)$ be square integrable, $\boldsymbol{x}, \boldsymbol{y} \in L_2(T)$. For the densities let

$$\begin{aligned} \hat{x}(s) &= \int_T x(t)\ \theta(s,t)\ dt, \\ \hat{y}(s) &= \int_T y(t)\ \theta(s,t)\ dt, \end{aligned} \tag{2.16}$$

where $\theta(s,t)$ is a self-reciprocal kernel satisfying

$$\begin{aligned} \int_S \theta(s,t)\ \theta^*(s,\tau)\ ds &= \int_S \theta(s,t)\ \varphi(\tau,s)\ ds \\ &= \delta(t-\tau). \end{aligned} \tag{2.17}$$

Now the inner products

$$\begin{aligned} \langle \hat{\boldsymbol{x}}, \hat{\boldsymbol{y}} \rangle &= \int_S \hat{x}(s)\ \hat{y}^*(s)\ ds, \\ \langle \boldsymbol{x}, \boldsymbol{y} \rangle &= \int_T x(t)\ y^*(t)\ dt \end{aligned} \tag{2.18}$$

are introduced. Substituting (2.16) into (2.18) yields

$$\langle \hat{\boldsymbol{x}}, \hat{\boldsymbol{y}} \rangle = \int_S \int_T \int_T x(\tau)\ \theta(s,\tau)\ y^*(t)\ \theta^*(s,t)\ d\tau\ dt\ ds. \tag{2.19}$$

Because of (2.17), (2.19) becomes

$$\begin{aligned} \langle \hat{\boldsymbol{x}}, \hat{\boldsymbol{y}} \rangle &= \int_T x(\tau) \int_T y^*(t)\ \delta(t-\tau)\ dt\ d\tau \\ &= \int_T x(\tau)\ y^*(\tau)\ d\tau. \end{aligned} \tag{2.20}$$

From (2.20) and (2.18) we conclude that

$$\langle \hat{\boldsymbol{x}}, \hat{\boldsymbol{y}} \rangle = \langle \boldsymbol{x}, \boldsymbol{y} \rangle\,. \tag{2.21}$$

Equation (2.21) is known as *Parseval's relation.* For $y(t) = x(t)$ we obtain

$$\langle \hat{\boldsymbol{x}}, \hat{\boldsymbol{x}} \rangle = \langle \boldsymbol{x}, \boldsymbol{x} \rangle \qquad \rightarrow \qquad ||\hat{\boldsymbol{x}}|| = ||\boldsymbol{x}||\,. \tag{2.22}$$

2.2 The Fourier Transform

We assume a real or complex-valued, continuous-time signal $x(t)$ which is absolutely integrable ($\boldsymbol{x} \in L_1(\mathbb{R})$). For such signals the Fourier transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\ e^{-j\omega t}\ dt \tag{2.23}$$

exists. Here, $\omega = 2\pi f$, and f is the frequency in Hertz.

The Fourier transform $X(\omega)$ of a signal $\boldsymbol{x} \in L_1(\mathbb{R})$ has the following properties:

1. $\boldsymbol{X} \in L_\infty(\mathbb{R})$ with $\|\boldsymbol{X}\|_\infty \leq \|\boldsymbol{x}\|_1$.
2. $\boldsymbol{X}$ is continuous.
3. If the derivative $x'(t)$ exists and if it is absolutely integrable, then
$$\int_{-\infty}^{\infty} x'(t)\ e^{-j\omega t}\ dt = j\omega\ X(\omega). \tag{2.24}$$
4. For $\omega \to \infty$ and $\omega \to -\infty$ we have $X(\omega) \to 0$.

If $X(\omega)$ is absolutely integrable, $x(t)$ can be reconstructed from $X(\omega)$ via the inverse Fourier transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\ e^{j\omega t}\ d\omega \tag{2.25}$$

for all t where $x(t)$ is continuous.

The kernel used is

$$\varphi(t,\omega) = \frac{1}{2\pi} e^{j\omega t}, \qquad T = (-\infty, \infty), \tag{2.26}$$

and for the reciprocal kernel we have[1]

$$\theta(\omega, t) = e^{-j\omega t}, \qquad S = (-\infty, \infty). \tag{2.27}$$

In the following we will use the notation $x(t) \longleftrightarrow X(\omega)$ in order to indicate a Fourier transform pair.

We will now briefly recall the most important properties of the Fourier transform. Most proofs are easily obtained from the definition of the Fourier transform itself. More elaborate discussions can be found in [114, 22].

[1]A self-reciprocal kernel is obtained either in the form $\varphi(t,\omega) = \exp(j\omega t)/\sqrt{2\pi}$ or by integrating over frequency f, not over $\omega = 2\pi f$: $\varphi(t, f) = \exp(j2\pi f t)$.

Linearity. It directly follows from (2.23) that

$$\alpha\, x(t) + \beta\, y(t) \longleftrightarrow \alpha\, X(\omega) + \beta\, Y(\omega). \tag{2.28}$$

Symmetry. Let $x(t) \longleftrightarrow X(\omega)$ be a Fourier transform pair. Then

$$X(t) \longleftrightarrow 2\pi\, x(-\omega). \tag{2.29}$$

Scaling. For any real α, we have

$$x(\alpha t) \longleftrightarrow \frac{1}{|\alpha|} X\left(\frac{\omega}{\alpha}\right). \tag{2.30}$$

Shifting. For any real t_0, we have

$$x(t - t_0) \longleftrightarrow e^{-j\omega t_0}\, X(\omega). \tag{2.31}$$

Accordingly,

$$e^{j\omega_0 t} x(t) \longleftrightarrow X(\omega - \omega_0). \tag{2.32}$$

Modulation. For any real ω_0, we have

$$\cos\omega_0 t\; x(t) \longleftrightarrow \frac{1}{2}\, X(\omega - \omega_0) + \frac{1}{2}\, X(\omega + \omega_0). \tag{2.33}$$

Conjugation. The correspondence for conjugate functions is

$$x^*(t) \longleftrightarrow X^*(-\omega). \tag{2.34}$$

Thus, the Fourier transform of real signals $x(t) = x^*(t)$ is symmetric: $X^*(\omega) = X(-\omega)$.

Derivatives. The generalization of (2.24) is

$$\frac{d^n}{dt^n}\, x(t) \longleftrightarrow (j\omega)^n\, X(\omega). \tag{2.35}$$

Accordingly,

$$(-jt)^n\, x(t) \longleftrightarrow \frac{d^n}{d\omega^n}\, X(\omega). \tag{2.36}$$

Convolution. A convolution in the time domain results in a multiplication in the frequency domain.

$$x(t) \;*\; y(t) \longleftrightarrow X(\omega)\, Y(\omega). \tag{2.37}$$

Accordingly,

$$x(t)\ y(t) \longleftrightarrow \frac{1}{2\pi}\ X(\omega)\ *\ Y(\omega). \tag{2.38}$$

Moments. The nth moment of $x(t)$ given by

$$m_n = \int_{-\infty}^{\infty} t^n\ x(t)\ dt, \qquad n = 0, 1, 2 \ldots \tag{2.39}$$

and the nth derivative of $X(\omega)$ at the origin are related as

$$(-j)^n\ m_n = \frac{d^n}{d\omega^n}\ X(\omega)\Big|_{\omega\,=\,0} \tag{2.40}$$

Parseval's Relation. According to Parseval's relation, inner products of two signals can be calculated in the time as well as the frequency domain. For signals $x(t)$ and $y(t)$ and their Fourier transforms $X(\omega)$ and $Y(\omega)$, respectively, we have

$$\int_{-\infty}^{\infty} x(t)\ y^*(t)\ dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\ Y^*(\omega)\ d\omega. \tag{2.41}$$

This property is easily obtained from (2.21) by using the fact that the scaled kernel $(2\pi)^{-\frac{1}{2}} e^{j\omega t}$ is self-reciprocal.

Using the notation of inner products, Parseval's relation may also be written as

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \frac{1}{2\pi}\ \langle \boldsymbol{X}\,, \boldsymbol{Y}\,\rangle\,. \tag{2.42}$$

From (2.41) with $x(t) = y(t)$ we see that the signal energy can be calculated in the time and frequency domains:

$$\begin{aligned} E_x &= \int_{-\infty}^{\infty} |x(t)|^2\ dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\ d\omega. \end{aligned} \tag{2.43}$$

This relationship is known as *Parseval's theorem.* In vector notation it can be written as

$$\langle \boldsymbol{x}, \boldsymbol{x} \rangle = \frac{1}{2\pi}\ \langle \boldsymbol{X}\,, \boldsymbol{X}\,\rangle\,. \tag{2.44}$$

2.3 The Hartley Transform

In 1942 Hartley proposed a real-valued transform closely related to the Fourier transform [67]. It maps a real-valued signal into a real-valued frequency function using only real arithmetic. The kernel of the Hartley transform is the so-called *cosine-and-sine* (cas) function, given by

$$\operatorname{cas} \omega t = \cos \omega t + \sin \omega t. \tag{2.45}$$

This kernel can be seen as a real-valued version of $e^{j\omega t} = \cos \omega t + j \sin \omega t$, the kernel of the Fourier transform. The forward and inverse Hartley transforms are given by

$$X_H(\omega) = \int_{-\infty}^{\infty} x(t) \operatorname{cas} \omega t \, dt \tag{2.46}$$

and

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_H(\omega) \operatorname{cas} \omega t \, d\omega, \tag{2.47}$$

where both the signal $x(t)$ and the transform $X_H(\omega)$ are real-valued.

In the literature, one also finds a more symmetric version based on the self-reciprocal kernel $(2\pi)^{-\frac{1}{2}} \operatorname{cas} \omega t$. However, we use the non-symmetric form in order to simplify the relationship between the Hartley and Fourier transforms.

The Relationship between the Hartley and Fourier Transforms. Let us consider the even and odd parts of the Hartley transform, given by

$$X_H^e(\omega) = \frac{X_H(\omega) + X_H(-\omega)}{2} = \int_{-\infty}^{\infty} x(t) \cos \omega t \, dt \tag{2.48}$$

and

$$X_H^o(\omega) = \frac{X_H(\omega) - X_H(-\omega)}{2} = \int_{-\infty}^{\infty} x(t) \sin \omega t \, dt. \tag{2.49}$$

The Fourier transform may be written as

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t) \, e^{-j\omega t} \, dt \\ &= \int_{-\infty}^{\infty} x(t) \cos \omega t \, dt - j \int_{-\infty}^{\infty} x(t) \sin \omega t \, dt \\ &= X_H^e(\omega) - j X_H^o(\omega) \\ &= \frac{X_H(\omega) + X_H(-\omega)}{2} - j \, \frac{X_H(\omega) - X_H(-\omega)}{2}. \end{aligned} \tag{2.50}$$

Thus,

$$\begin{aligned} \Re\{X(\omega)\} &= X_H^e(\omega), \\ \Im\{X(\omega)\} &= -X_H^o(\omega). \end{aligned} \tag{2.51}$$

The Hartley transform can be written in terms of the Fourier transform as

$$X_H(\omega) = \Re\{X(\omega)\} - \Im\{X(\omega)\}. \tag{2.52}$$

Due to their close relationship the Hartley and Fourier transforms share many properties. However, some properties are entirely different. In the following we summarize the most important ones.

Linearity. It directly follows from the definition of the Hartley transform that

$$\alpha\, x(t) + \beta\, y(t) \longleftrightarrow \alpha\, X_H(\omega) + \beta\, Y_H(\omega). \tag{2.53}$$

Scaling. For any real α, we have

$$x(\alpha t) \longleftrightarrow \frac{1}{|\alpha|} X_H\left(\frac{\omega}{\alpha}\right). \tag{2.54}$$

Proof.

$$\int_{-\infty}^{\infty} x(\alpha t)\ \mathrm{cas}\,\omega t\ dt = \frac{1}{|\alpha|}\int_{-\infty}^{\infty} x(\xi)\ \mathrm{cas}\left(\frac{\omega\xi}{\alpha}\right)\ d\xi = \frac{1}{|\alpha|}\ X_H\left(\frac{\omega}{\alpha}\right).$$

Time Inversion. From (2.54) with $\alpha = -1$ we get

$$x(-t) \longleftrightarrow X_H(-\omega). \tag{2.55}$$

Shifting. For any real t_0, we have

$$x(t - t_0) \longleftrightarrow \cos\omega t_0\ X_H(\omega) + \sin\omega t_0\ X_H(-\omega). \tag{2.56}$$

Proof. We may write

$$\int_{-\infty}^{\infty} x(t - t_0)\ \mathrm{cas}\,\omega t\ dt = \int_{-\infty}^{\infty} x(\xi)\ \mathrm{cas}\,(\omega[\xi + t_0])\ d\xi.$$

Expanding the integral on the right-hand side using the property

$$\mathrm{cas}\,(\alpha + \beta) = [\cos\alpha\ + \sin\alpha]\ \cos\beta + [\cos\alpha\ - \sin\alpha]\ \sin\beta$$

yields (2.56). $\square$

Modulation. For any real ω_0, we have

$$\cos\omega_0 t\ x(t) \longleftrightarrow \frac{1}{2} X_H(\omega - \omega_0) + \frac{1}{2} X_H(\omega + \omega_0). \tag{2.57}$$

Proof. Using the property

$$\cos\alpha\ \mathrm{cas}\,\beta = \frac{1}{2}\,\mathrm{cas}\,(\alpha - \beta)\ + \frac{1}{2}\,\mathrm{cas}\,(\alpha + \beta),$$

we get

$$\begin{aligned}
&\int_{-\infty}^{\infty} x(t)\ \cos\omega_0 t\ \mathrm{cas}\,\omega t\ dt \\
&\quad = \frac{1}{2}\int_{-\infty}^{\infty} x(t)\,\mathrm{cas}\,([\omega - \omega_0]t)\ dt + \frac{1}{2}\int_{-\infty}^{\infty} x(t)\,\mathrm{cas}\,([\omega + \omega_0]t)\ dt \\
&\quad = \frac{1}{2} X_H(\omega - \omega_0) + \frac{1}{2} X_H(\omega + \omega_0).
\end{aligned}$$

□

Derivatives. For the nth derivative of a signal $x(t)$ the correspondence is

$$\frac{d^n}{dt^n}\ x(t) \longleftrightarrow \omega^n\ \left[\cos\left(\frac{n\pi}{2}\right)\ X_H(\omega) - \sin\left(\frac{n\pi}{2}\right)\ X_H(-\omega)\right]. \tag{2.58}$$

Proof. Let $y(t) = \frac{d^n}{dt^n}\ x(t)$. The Fourier transform is $Y(\omega) = (j\omega)^n\ X(\omega)$. By writing j^n as $j^n = \cos(\frac{n\pi}{2}) + j\ \sin(\frac{n\pi}{2})$, we get

$$\begin{aligned}
Y(\omega) &= \omega^n\ \left[\cos\left(\tfrac{n\pi}{2}\right) + j\ \sin\left(\tfrac{n\pi}{2}\right)\right]\ X(\omega) \\
&= \omega^n\ \left[\cos\left(\tfrac{n\pi}{2}\right)\Re\{X(\omega)\} - \sin\left(\tfrac{n\pi}{2}\right)\Im\{X(\omega)\}\right] \\
&\quad + j\ \omega^n\ \left[\cos\left(\tfrac{n\pi}{2}\right)\Im\{X(\omega)\} + \sin\left(\tfrac{n\pi}{2}\right)\Re\{X(\omega)\}\right].
\end{aligned}$$

For the Hartley transform, this means

$$\begin{aligned}
Y_H(\omega) &= \omega^n\ \Big[\cos\left(\tfrac{n\pi}{2}\right)\ X_H^e(\omega) - \sin\left(\tfrac{n\pi}{2}\right)\ X_H^o(\omega) \\
&\qquad + \cos\left(\tfrac{n\pi}{2}\right)\ X_H^o(\omega) + \sin\left(\tfrac{n\pi}{2}\right)\ X_H^e(\omega)\Big].
\end{aligned}$$

Rearranging this expression, based on (2.48) and (2.49), yields (2.58). □

Convolution. We consider a convolution in time of two signals $x(t)$ and $y(t)$. The Hartley transforms are $X_H(\omega)$ and $Y_H(\omega)$, respectively. The correspondence is

$$\begin{aligned} x(t) * y(t) \longleftrightarrow \frac{1}{2}[& X_H(\omega)Y_H(\omega) + X_H(-\omega)Y_H(\omega) \\ & +X_H(\omega)Y_H(-\omega) - X_H(-\omega)Y_H(-\omega)]. \end{aligned} \tag{2.59}$$

The expression becomes less complex for signals with certain symmetries. For example, if $x(t)$ has even symmetry, then $x(t) * y(t) \longleftrightarrow X_H(\omega)\, Y_H(\omega)$. If $x(t)$ is odd, then $x(t) * y(t) \longleftrightarrow X_H(\omega)\, Y_H(-\omega)$.

Proof.

$$\begin{aligned} \int_{-\infty}^{\infty} [x(t) * y(t)] \operatorname{cas}\omega t\; dt &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\; d\tau \right] \operatorname{cas}\omega t\; dt \\ &= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} y(t-\tau) \operatorname{cas}\omega t\; dt \right] d\tau \\ &= \int_{-\infty}^{\infty} x(\tau) \left[\cos\omega\tau\, Y_H(\omega) + \sin\omega\tau\, Y_H(-\omega) \right] d\tau. \end{aligned}$$

To derive the last line, we made use of the shift theorem. Using (2.48) and (2.49) we finally get (2.59). □

Multiplication. The correspondence for a multiplication in time is

$$\begin{aligned} x(t)\, y(t) \longleftrightarrow \frac{1}{4\pi} [& X_H(\omega) * Y_H(\omega) + X_H(-\omega) * Y_H(\omega) \\ & +X_H(\omega) * Y_H(-\omega) - X_H(-\omega) * Y_H(-\omega)]. \end{aligned} \tag{2.60}$$

Proof. In the Fourier domain, we have

$$\begin{aligned} & 2\pi\, x(t)\, y(t) \\ & \quad \updownarrow \\ X(\omega) * Y(\omega) \quad &= \quad [\, \Re\{X(\omega)\} * \Re\{Y(\omega)\} - \Im\{X(\omega)\} * \Im\{Y(\omega)\}] \\ & \qquad +j\ [\Im\{X(\omega)\} * \Re\{Y(\omega)\} + \Re\{X(\omega)\} * \Im\{Y(\omega)\}\,]. \end{aligned}$$

For the Hartley transform this means

$$2\pi\, x(t)\, y(t)$$
$$\updownarrow$$
$$X_H^e(\omega) * Y_H^e(\omega) - X_H^o(\omega) * Y_H^o(\omega) + X_H^o(\omega) * Y_H^e(\omega) + X_H^e(\omega) * Y_H^o(\omega).$$

Writing this expression in terms of $X_H(\omega)$ and $Y_H(\omega)$ yields (2.60). □

Parseval's Relation. For signals $x(t)$ and $y(t)$ and their Hartley transforms $X_H(\omega)$ and $Y_H(\omega)$, respectively, we have

$$\int_{-\infty}^{\infty} x(t)\ y(t)\ dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_H(\omega)\ Y_H(\omega)\ d\omega. \tag{2.61}$$

Similarly, the signal energy can be calculated in the time and in the frequency domains:

$$\begin{aligned} E_x &= \int_{-\infty}^{\infty} x^2(t)\ dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_H^2(\omega)\ d\omega. \end{aligned} \tag{2.62}$$

These properties are easily obtained from the results in Section 2.1 by using the fact that the kernel $(2\pi)^{-\frac{1}{2}} \operatorname{cas} \omega t$ is self-reciprocal.

Energy Density and Phase. In practice, one of the reasons to compute the Fourier transform of a signal $x(t)$ is to derive the energy density $S_{xx}^E(\omega) = |X(\omega)|^2$ and the phase $\angle X(\omega)$. In terms of the Hartley transform the energy density becomes

$$\begin{aligned} S_{xx}^E(\omega) &= |\Re\{X(\omega)\}|^2 + |\Im\{X(\omega)\}|^2 \\ &= [X_H^e(\omega)]^2 + [X_H^o(\omega)]^2 \\ &= \frac{X_H^2(\omega) + X_H^2(-\omega)}{2}. \end{aligned} \tag{2.63}$$

The phase can be written as

$$\begin{aligned} \angle X(\omega) &= \tan^{-1} \frac{\Im\{X(\omega)\}}{\Re\{X(\omega)\}} \\ &= \tan^{-1} \frac{X_H(-\omega) - X_H(\omega)}{X_H(\omega) + X_H(-\omega)}. \end{aligned} \tag{2.64}$$

2.4 The Hilbert Transform

2.4.1 Definition

Choosing the kernel

$$\varphi(t-s) = \frac{-1}{\pi(t-s)}, \tag{2.65}$$

we obtain the Hilbert transform. For the reciprocal kernel $\theta(s-t)$ we use the notation $\hat{h}(s-t)$ throughout the following discussion. It is

$$\hat{h}(s-t) = \frac{1}{\pi(s-t)} = \varphi(t-s). \tag{2.66}$$

With $\hat{x}(s)$ denoting the Hilbert transform of $x(t)$ we obtain the following transform pair:

$$\begin{aligned} x(t) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \hat{x}(s)\frac{-1}{t-s}\,ds \\ &\updownarrow \\ \hat{x}(s) &= \frac{1}{\pi}\int_{-\infty}^{\infty} x(t)\frac{1}{s-t}\,dt. \end{aligned} \tag{2.67}$$

Here, the integration has to be carried out according to the Cauchy principal value:

$$\int_{-\infty}^{\infty} := \lim_{\varepsilon\to 0}\left(\int_{-\infty}^{s-\varepsilon} + \int_{s+\varepsilon}^{\infty}\right), \quad \varepsilon > 0. \tag{2.68}$$

The Fourier transforms of $\varphi(t)$ and $\hat{h}(t)$ are:

$$\Phi(\omega) = j\ \operatorname{sgn}(\omega) \ \text{ with } \ \Phi(0) = 0, \tag{2.69}$$

$$\hat{H}(\omega) = -j\ \operatorname{sgn}(\omega) \ \text{ with } \ \hat{H}(0) = 0. \tag{2.70}$$

In the spectral domain we then have:

$$X(\omega) = \Phi(\omega)\ \hat{X}(\omega) = j\ \operatorname{sgn}(\omega)\ \hat{X}(\omega) \tag{2.71}$$

$$\hat{X}(\omega) = \hat{H}(\omega)\ X(\omega) = -j\ \operatorname{sgn}(\omega)\ X(\omega). \tag{2.72}$$

We observe that the spectrum of the Hilbert transform $\hat{x}(s)$ equals the spectrum of $x(t)$, except for the prefactor $-j\ \operatorname{sgn}(\omega)$. Furthermore, we see that, because of $\Phi(0) = \hat{H}(0) = 0$, the transform pair (2.67) is valid only for signals $x(t)$ with zero mean value. The Hilbert transform of a signal with non-zero mean has zero mean.

2.4.2 Some Properties of the Hilbert Transform

1. Since the kernel of the Hilbert transform is self-reciprocal we have

$$\langle \hat{\boldsymbol{x}}, \hat{\boldsymbol{y}} \rangle = \langle \boldsymbol{x}, \boldsymbol{y} \rangle . \tag{2.73}$$

2. A real-valued signal $x(t)$ is orthogonal to its Hilbert transform $\hat{x}(t)$:

$$\langle \boldsymbol{x}, \hat{\boldsymbol{x}} \rangle = 0. \tag{2.74}$$

We prove this by making use of Parseval's relation:

$$\begin{aligned} 2\pi \langle \boldsymbol{x}, \hat{\boldsymbol{x}} \rangle &= \langle \boldsymbol{X}, \hat{\boldsymbol{X}} \rangle \\ &= \int_{-\infty}^{\infty} X(\omega) \left[\hat{X}(\omega)\right]^* d\omega \\ &= \int_{-\infty}^{\infty} X(\omega) \ [-j \ \operatorname{sgn}(\omega)]^* \ X^*(\omega) \ d\omega \\ &= j \int_{-\infty}^{\infty} |X(\omega)|^2 \ \operatorname{sgn}(\omega) \ d\omega \\ &= 0. \end{aligned} \tag{2.75}$$

3. From (2.67) and (2.70) we conclude that applying the Hilbert transform twice leads to a sign change of the signal, provided that the signal has zero mean value.

2.5 Representation of Bandpass Signals

A bandpass signal is understood as a signal whose spectrum concentrates in a region $\pm[\omega_0 - B, \omega_0 + B]$ where $\omega_0 \geq B > 0$. See Figure 2.1 for an example of a bandpass spectrum.

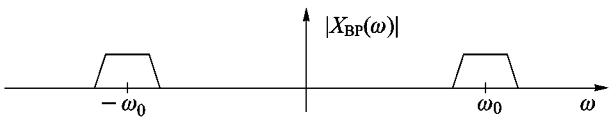

Figure 2.1. Example of a bandpass spectrum.

2.5.1 Analytic Signal and Complex Envelope

The Hilbert transform allows us to transfer a *real bandpass signal* $x_{\text{BP}}(t)$ into a *complex lowpass signal* $x_{\text{LP}}(t)$. For that purpose, we first form the so-called *analytic signal* $x^{+}_{\text{BP}}(t)$, first introduced in [61]:

$$x^{+}_{\text{BP}}(t) = x_{\text{BP}}(t) + j\, \hat{x}_{\text{BP}}(t). \tag{2.76}$$

Here, $\hat{x}_{\text{BP}}(t)$ is the Hilbert transform of $x_{\text{BP}}(t)$.

The Fourier transform of the analytic signal is

$$X^{+}_{\text{BP}}(\omega) = X_{\text{BP}}(\omega) + j\, \hat{X}_{\text{BP}}(\omega) = \begin{cases} 2\, X_{\text{BP}}(\omega) & \text{for} \quad \omega > 0, \\ X_{\text{BP}}(\omega) & \text{for} \quad \omega = 0, \\ 0 & \text{for} \quad \omega < 0. \end{cases} \tag{2.77}$$

This means that the analytic signal has spectral components for positive frequencies only.

In a second step, the complex-valued analytic signal can be shifted into the baseband:

$$x_{\text{LP}}(t) = x^{+}_{\text{BP}}(t)\; e^{-j\omega_0 t}. \tag{2.78}$$

Here, the frequency ω_0 is assumed to be the center frequency of the bandpass spectrum, as shown in Figure 2.1. Figure 2.2 illustrates the procedure of obtaining the complex envelope. We observe that it is not necessary to realize an ideal Hilbert transform with system function $\hat{H}(\omega) = -j\,\text{sgn}(\omega)$ in order to carry out this transform.

The signal $x_{\text{LP}}(t)$ is called the *complex envelope* of the bandpass signal $x_{\text{BP}}(t)$. The reason for this naming convention is outlined below.

In order to recover a real bandpass signal $x_{\text{BP}}(t)$ from its complex envelope $x_{\text{LP}}(t)$, we make use of the fact that

$$\begin{aligned} x_{\text{BP}}(t) &= \Re\{x^{+}(t)\} \\ &= \Re\{x_{\text{LP}}(t)\; e^{j\omega_0 t}\} \\ &= u(t)\, \cos\omega_0 t \;-\; v(t)\, \sin\omega_0 t \end{aligned} \tag{2.79}$$

for

$$\begin{aligned} u(t) &= \Re\{x_{\text{LP}}(t)\}, \\ v(t) &= \Im\{x_{\text{LP}}(t)\}, \\ x_{\text{LP}}(t) &= u(t) + j\, v(t). \end{aligned} \tag{2.80}$$

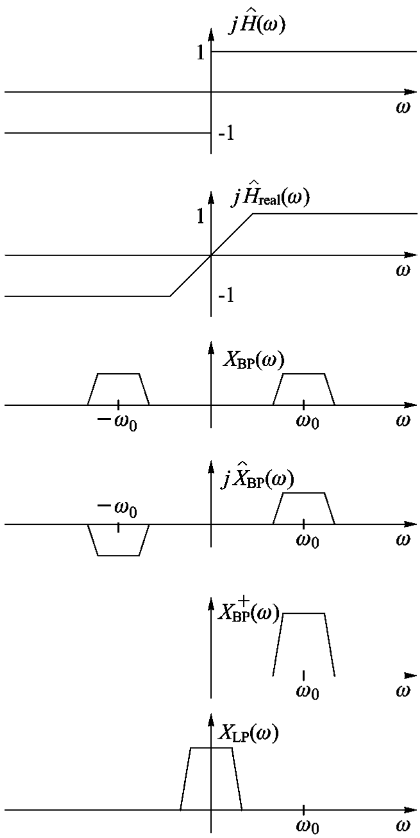

Figure 2.2. Producing the complex envelope of a real bandpass signal.

Another form of representing $x_{\text{BP}}(t)$ is obtained by describing the complex envelope with polar coordinates:

$$x_{\text{LP}}(t) = |x_{\text{LP}}(t)|\; e^{j\theta(t)} \tag{2.81}$$

with

$$|x_{\text{LP}}(t)| = \sqrt{u^2(t) + v^2(t)}, \qquad \tan\theta(t) = \frac{v(t)}{u(t)}. \tag{2.82}$$

From (2.79) we then conclude for the bandpass signal:

$$x_{\text{BP}}(t) = |x_{\text{LP}}(t)|\; \cos(\omega_0 t + \theta(t)). \tag{2.83}$$

We see that $|x_{\text{LP}}(t)|$ can be interpreted as the *envelope of the bandpass signal* (see Figure 2.3). Accordingly, $x_{\text{LP}}(t)$ is called the complex envelope, and the

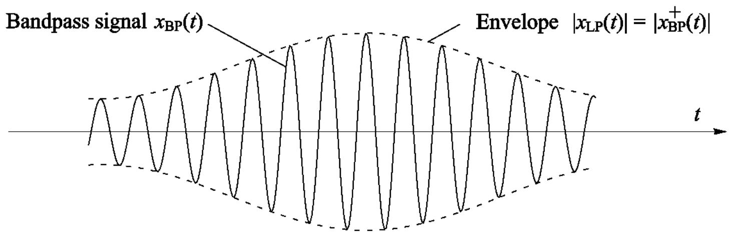

Figure 2.3. Bandpass signal and envelope.

analytic signal is called the *pre-envelope*. The real part $u(t)$ is referred to as the *in-phase component*, and the imaginary part $v(t)$ is called the *quadrature component*.

Equation (2.83) shows that bandpass signals can in general be regarded as amplitude and phase modulated signals. For $\theta(t) = \theta_0$ we have a pure amplitude modulation.

It should be mentioned that the spectrum of a complex envelope is always limited to $-\omega_0$ at the lower bound:

$$X_{\text{LP}}(\omega) \equiv 0 \quad \text{for} \quad \omega < -\omega_0. \tag{2.84}$$

This property immediately results from the fact that an analytic signal contains only positive frequencies.

Application in Communications. In communications we often start with a lowpass complex envelope $x_{\text{LP}}(t)$ and wish to transmit it as a real bandpass signal $x_{\text{BP}}(t)$. Here, the real bandpass signal $x_{\text{BP}}(t)$ is produced from $x_{\text{LP}}(t)$ according to (2.79). In the receiver, $x_{\text{LP}}(t)$ is finally reconstructed as described above. However, one important requirement must be met, which will be discussed below.

The real bandpass signal

$$x_{\text{BP}}(t) = u(t)\ \cos\omega_0 t \tag{2.85}$$

is considered. Here, $u(t)$ is a given real lowpass signal. In order to reconstruct $u(t)$ from $x_{\text{BP}}(t)$, we have to add the imaginary signal $ju(t)\sin\omega_0 t$ to the bandpass signal:

$$x^{(p)}(t) := u(t)\ [\cos\omega_0 t + j\sin\omega_0 t] = u(t)\ e^{j\omega_0 t}. \tag{2.86}$$

Through subsequent modulation we recover the original lowpass signal:

$$u(t) = x^{(p)}(t)\ e^{-j\omega_0 t}. \tag{2.87}$$

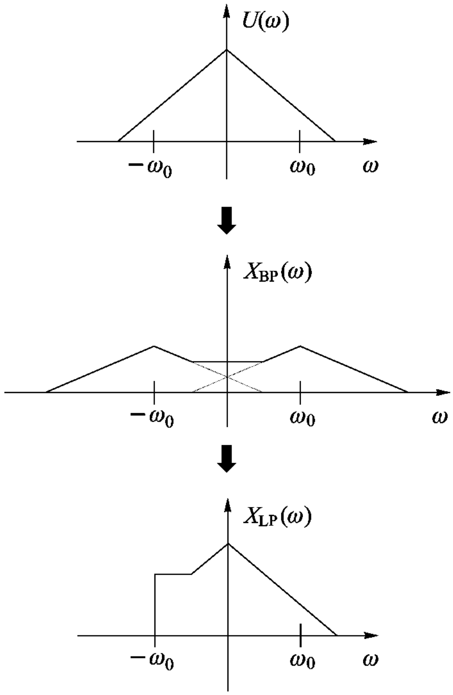

Figure 2.4. Complex envelope for the case that condition (2.88) is violated.

The problem, however, is to generate $u(t) \sin \omega_0 t$ from $u(t) \cos \omega_0 t$ in the receiver. We now assume that $u(t)\ e^{j\omega_0 t}$ is analytic, which means that

$$U(\omega) \equiv 0 \text{ for } \omega < -\omega_0. \tag{2.88}$$

As can easily be verified, under condition (2.88) the Hilbert transform of the bandpass signal is given by

$$\hat{x}(t) = u(t)\ \sin \omega_0 t. \tag{2.89}$$

Thus, under condition (2.88) the required signal $x^{(p)}(t)$ equals the analytic signal $x^{+}_{\text{BP}}(t)$, and the complex envelope $x_{\text{LP}}(t)$ is identical to the given $u(t)$. The complex envelope describes the bandpass signal unambiguously, that is, $x_{\text{BP}}(t)$ can always be reconstructed from $x_{\text{LP}}(t)$; the reverse, however, is only possible if condition (2.88) is met. This is illustrated in Figure 2.4.

Bandpass Filtering and Generating the Complex Envelope. In practice, generating a complex envelope usually involves the task of filtering the real bandpass signal $x_{\text{BP}}(t)$ out of a more broadband signal $x(t)$. This means that $x_{\mathrm{BP}}(t) = x(t) * g(t)$ has to be computed, where $g(t)$ is the impulse response of a real bandpass.

The analytic bandpass $g^+(t)$ associated with $g(t)$ has the system function

$$G^+(\omega) = G(\omega)\ [1 + j\,\hat{H}(\omega)]. \tag{2.90}$$

Using the analytic bandpass, the analytic signal can be calculated as

$$\begin{array}{rcl} x^+_{\mathrm{BP}}(t) & = & x(t) * g^+(t) \\ & \updownarrow & \\ X^+_{\mathrm{BP}}(\omega) & = & X(\omega)\ G^+(\omega). \end{array} \tag{2.91}$$

For the complex envelope, we have

$$\begin{array}{rcl} x_{\mathrm{LP}}(t) & = & [x(t) * g^+(t)]\ e^{-j\omega_0 t} \\ & \updownarrow & \\ X_{\mathrm{LP}}(\omega) & = & X(\omega + \omega_0)\ G^+(\omega + \omega_0). \end{array} \tag{2.92}$$

If we finally describe the analytic bandpass by means of the complex envelope of the real bandpass

$$\begin{array}{rcl} g^+(t) & = & g_{\mathrm{LP}}(t)\ e^{j\omega_0 t} \\ & \updownarrow & \\ G^+(\omega) & = & G_{\mathrm{LP}}(\omega - \omega_0), \end{array} \tag{2.93}$$

this leads to

$$X_{\mathrm{LP}}(\omega) = X(\omega + \omega_0)\ G_{\mathrm{LP}}(\omega). \tag{2.94}$$

We find that $X_{\mathrm{LP}}(\omega)$ is also obtained by modulating the real bandpass signal with $e^{-j\omega_0 t}$ and by lowpass filtering the resulting signal. See Figure 2.5 for an illustration.

The equivalent lowpass $G_{\mathrm{LP}}(\omega)$ usually has a complex impulse response. Only if the symmetry condition $G_{\mathrm{LP}}(\omega) = G^*_{\mathrm{LP}}(-\omega)$ is satisfied, the result is a real lowpass, and the realization effort is reduced. This requirement means that $|G(\omega)|$ must have even symmetry around ω_0 and the phase response of $G(\omega)$ must be anti-symmetric. In this case we also speak of a *symmetric bandpass.*

Realization of Bandpass Filters by Means of Equivalent Lowpass Filters. We consider a signal $y(t) = x(t) * g(t)$, where $x(t)$, $y(t)$, and $g(t)$ are

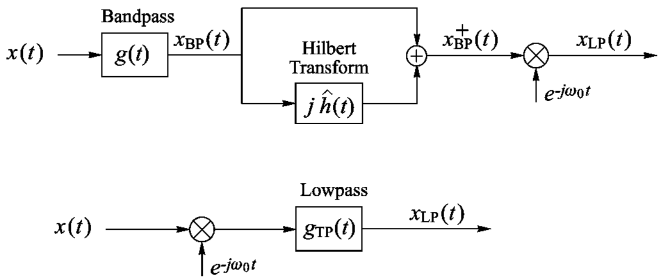

Figure 2.5. Generating the complex envelope of a real bandpass signal.

real-valued. The signal $x(t)$ is now described by means of its complex envelope with respect to an arbitrary positive center frequency ω_0:

$$x(t) = \Re\{x_{\text{LP}}(t)\; e^{j\omega_0 t}\}. \tag{2.95}$$

For the spectrum we have

$$X(\omega) = \frac{1}{2}\, X_{\text{LP}}(\omega - \omega_0) + \frac{1}{2}\, X^*_{\text{LP}}(-\omega - \omega_0). \tag{2.96}$$

Correspondingly, the system function of the filter can be written as

$$G(\omega) = \frac{1}{2}\, G_{\text{LP}}(\omega - \omega_0) + \frac{1}{2}\, G^*_{\text{LP}}(-\omega - \omega_0). \tag{2.97}$$

For the spectrum of the output signal we have

$$\begin{aligned}
Y(\omega) &= X(\omega)\; G(\omega) \\
&= \tfrac{1}{4}\; X_{\text{LP}}(\omega - \omega_0)\; G_{\text{LP}}(\omega - \omega_0) \\
&\quad + \tfrac{1}{4}\; X^*_{\text{LP}}(-\omega - \omega_0)\; G^*_{\text{LP}}(-\omega - \omega_0) \\
&\quad + \tfrac{1}{4}\; X_{\text{LP}}(\omega - \omega_0)\; G^*_{\text{LP}}(-\omega - \omega_0) \\
&\quad + \tfrac{1}{4}\; X^*_{\text{LP}}(-\omega - \omega_0)\; G_{\text{LP}}(\omega - \omega_0).
\end{aligned} \tag{2.98}$$

The last two terms vanish since $G_{\text{LP}}(\omega) = 0$ for $\omega < -\omega_0$ and $X_{\text{LP}}(\omega) = 0$ for $\omega < -\omega_0$:

$$\begin{aligned}
Y(\omega) &= \tfrac{1}{4}\; X_{\text{LP}}(\omega - \omega_0)\; G_{\text{LP}}(\omega - \omega_0) \\
&\quad + \tfrac{1}{4}\; X^*_{\text{LP}}(-\omega - \omega_0)\; G^*_{\text{LP}}(-\omega - \omega_0) \\
&= \tfrac{1}{2}\; Y_{\text{LP}}(\omega - \omega_0) + \tfrac{1}{2}\; Y^*_{\text{LP}}(-\omega - \omega_0).
\end{aligned} \tag{2.99}$$

Altogether this yields

$$Y_{\rm LP}(\omega) = \frac{1}{2}\, X_{\rm LP}(\omega)\; G_{\rm LP}(\omega). \tag{2.100}$$

This means that a real convolution in the bandpass domain can be replaced by a complex convolution in the lowpass domain:

$$y(t) = x(t) * g(t) \qquad \rightarrow \qquad y_{\rm LP}(t) = \frac{1}{2}\, x_{\rm LP}(t) * g_{\rm LP}(t). \tag{2.101}$$

Note that the prefactor $1/2$ must be taken into account. This prefactor did not appear in the combination of bandpass filtering and generating the complex envelope discussed above. As before, a real filter $g_{\rm LP}(t)$ is obtained if $G(\omega)$ is symmetric with respect to ω_0.

Inner Products. We consider the inner product of two analytic signals

$$x^+(t) = x(t) + j\,\hat{x}(t) \qquad \text{and} \qquad y^+(t) = y(t) + j\,\hat{y}(t),$$

where $x(t)$ and $y(t)$ are real-valued. We have

$$\langle \boldsymbol{x}^+, \boldsymbol{y}^+ \rangle = \langle \boldsymbol{x}, \boldsymbol{y} \rangle + \langle \hat{\boldsymbol{x}}, \hat{\boldsymbol{y}} \rangle + j\,\langle \hat{\boldsymbol{x}}, \boldsymbol{y} \rangle + j\,\langle \boldsymbol{x}, \hat{\boldsymbol{y}} \rangle\,. \tag{2.102}$$

Observing (2.73), we get for the real part

$$\Re\{\langle \boldsymbol{x}^+, \boldsymbol{y}^+ \rangle\} = 2\,\langle \boldsymbol{x}, \boldsymbol{y} \rangle\,. \tag{2.103}$$

If we describe $x(t)$ and $y(t)$ by means of their complex envelope with respect to the same center frequency, we get

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \frac{1}{2}\,\Re\{\langle \boldsymbol{x}_{\rm LP}, \boldsymbol{y}_{\rm LP} \rangle\}. \tag{2.104}$$

For the implementation of correlation operations this means that correlations of deterministic bandpass signals can be computed in the bandpass domain as well as in the equivalent lowpass domain.

Group and Phase Delay. The group and phase delay of a system $C(\omega)$ are defined as

$$\tau_g(\omega) = -\,\frac{d\varphi(\omega)}{d\omega} \tag{2.105}$$

and

$$\tau_p(\omega) = -\,\frac{\varphi(\omega)}{\omega} \tag{2.106}$$

where

$$C(\omega) = |C(\omega)|\; e^{j\varphi(\omega)}. \tag{2.107}$$

In order to explain this, let us assume that $C(\omega)$ is a narrowband bandpass with $B << \omega_0$. The system function of the associated analytic bandpass may be written as

$$C_{\mathrm{BP}}^{+}(\omega) = \begin{cases} |C(\omega)| e^{j\varphi(\omega)}, & |\omega - \omega_0| \le B/2 \\ 0, & \text{otherwise.} \end{cases} \tag{2.108}$$

Because of $B << \omega_0$, $C_{\mathrm{BP}}^{+}(\omega)$ may be approximated as

$$C_{\mathrm{BP}}^{+}(\omega) \approx \begin{cases} |C(\omega_0)| e^{j[\varphi(\omega_0) + (\omega - \omega_0)\,(\frac{d\varphi(\omega)}{d\omega}|_{\omega=\omega_0})]}, & |\omega - \omega_0| \le B/2 \\ 0, & \text{otherwise.} \end{cases} \tag{2.109}$$

For the complex envelope $C_{\mathrm{LP}}(\omega) = C_{\mathrm{BP}}(\omega + \omega_0)$ it follows that

$$C_{\mathrm{LP}}(\omega) \approx |C(\omega_0)|\; e^{-j\omega_0 \tau_p(\omega_0)}\; e^{-j\omega \tau_g(\omega_0)}, \qquad \omega \le B/2, \tag{2.110}$$

with τ_g and τ_p according to (2.105) and (2.106). If we now look at the input-output relation (2.100) we get

$$Y_{\mathrm{LP}}(\omega) \approx \frac{1}{2}|C(\omega_0)|\; e^{-j\omega_0 \tau_p(\omega_0)}\; e^{-j\omega \tau_g(\omega_0)}\; X_{\mathrm{LP}}(\omega). \tag{2.111}$$

Hence, in the time domain

$$y_{\mathrm{LP}}(t) \approx \frac{1}{2}|C(\omega_0)| e^{-j\omega_0 \tau_p(\omega_0)}\; x_{\mathrm{LP}}(t - \tau_g(\omega_0)), \tag{2.112}$$

which means that the narrowband system $C(\omega)$ provides a phase shift by $\tau_p(\omega_0)$ and a time delay by $\tau_g(\omega_0)$.

2.5.2 Stationary Bandpass Processes

In communications we must assume that noise interferes with bandpass signals that are to be transmitted. Therefore the question arises of which statistical properties the complex envelope of a stationary bandpass process has. We assume a real-valued, zero mean, wide-sense stationary bandpass process $x(t)$. The autocorrelation function of the process is given by

$$r_{xx}(\tau) = r_{xx}(-\tau) = E\left\{x(t)\; x(t+\tau)\right\}. \tag{2.113}$$

Now we consider the transformed process $\hat{x}(t)$. For the power spectral density of the transformed process, $S_{\hat{x}\hat{x}}(\omega)$, we conclude from (1.105):

$$S_{\hat{x}\hat{x}}(\omega) = \underbrace{|\hat{H}(\omega)|^2}_{1 \text{ for } \omega \ne 0} \cdot \underbrace{S_{xx}(\omega)}_{0 \text{ for } \omega = 0} = S_{xx}(\omega), \tag{2.114}$$

where $\hat{h}(t) \longleftrightarrow \hat{H}(\omega)$. Thus, the process $\hat{x}(t)$ has the same power spectral density, and consequently the same autocorrelation function, as the process $x(t)$:

$$r_{\hat{x}\hat{x}}(\tau) = r_{xx}(\tau). \tag{2.115}$$

For the cross power spectral densities $S_{x\hat{x}}(\omega)$ and $S_{\hat{x}x}(\omega)$ we get according to (1.102):

$$\begin{aligned} S_{x\hat{x}}(\omega) &= \hat{H}(\omega)\; S_{xx}(\omega), \\ S_{\hat{x}x}(\omega) &= \hat{H}^*(\omega)\; S_{xx}(\omega). \end{aligned} \tag{2.116}$$

Hence, for the cross correlation functions:

$$\begin{aligned} r_{x\hat{x}}(\tau) &= \hat{r}_{xx}(\tau), \\ r_{\hat{x}x}(\tau) &= r_{x\hat{x}}(-\tau) = \hat{r}_{xx}(-\tau) = -\hat{r}_{xx}(\tau). \end{aligned} \tag{2.117}$$

Now we form the analytic process $x^+(t)$:

$$x^+(t) = x(t) + j\,\hat{x}(t). \tag{2.118}$$

For the autocorrelation function we have

$$\begin{aligned} r_{x^+x^+}(\tau) &= E\left\{[x(t) + j\,\hat{x}(t)]^*\; [x(t+\tau) + j\,\hat{x}(t+\tau)]\right\} \\ &= r_{xx}(\tau) + j\; r_{x\hat{x}}(\tau) - j\; r_{\hat{x}x}(\tau) + r_{\hat{x}\hat{x}}(\tau) \\ &= 2\; r_{xx}(\tau) + 2j\; \hat{r}_{xx}(\tau). \end{aligned} \tag{2.119}$$

This means that the autocorrelation function of the analytic process is an analytic signal itself. The power spectral density is

$$S_{x^+x^+}(\omega) = \begin{cases} 4\; S_{xx}(\omega) & \text{for } \omega > 0, \\ 0 & \text{for } \omega < 0. \end{cases} \tag{2.120}$$

Finally, we consider the complex process $x_{\rm LP}(t)$ derived from the analytic process

$$\begin{aligned} x_{\rm LP}(t) &= x^+(t)\; e^{-j\omega_0 t} \\ &= u(t) + j\; v(t). \end{aligned} \tag{2.121}$$

For the real part $u(t)$ we have

$$\begin{aligned} u(t) &= \Re\{[x(t) + j\,\hat{x}(t)]\; e^{-j\omega_0 t}\} \\ &= x(t)\;\cos\omega_0 t + \hat{x}(t)\;\sin\omega_0 t \\ &= \tfrac{1}{2}\;[x^+(t)\; e^{-j\omega_0 t} + [x^+(t)]^*\; e^{j\omega_0 t}], \end{aligned} \tag{2.122}$$

and for its autocorrelation function follows

$$\begin{aligned}E\{u(t)u(t+\tau)\} &= \tfrac{1}{4}\, E\Big\{x^+(t)\; x^+(t+\tau)\; e^{-j\omega_0(2t+\tau)} \\ &\qquad + x^+(t)\; [x^+(t+\tau)]^*\; e^{j\omega_0\tau} \\ &\qquad + [x^+(t)]^*\; x^+(t+\tau)\; e^{-j\omega_0\tau} \\ &\qquad + [x^+(t)]^*\; [x^+(t+\tau)]^*\; e^{j\omega_0(2t+\tau)}\Big\}.\end{aligned} \tag{2.123}$$

In (2.123) two complex exponential functions dependent on t are included whose prefactors reduce to zero:

$$\begin{aligned}E\left\{[x^+(t)]^*\; [x^+(t+\tau)]^*\right\}^* &= E\left\{x^+(t)\; x^+(t+\tau)\right\} \\ &= E\left\{(x(t)+j\,\hat{x}(t))\; (x(t+\tau)+j\,\hat{x}(t+\tau))\right\} \\ &= \underbrace{r_{xx}(\tau)-r_{\hat{x}\hat{x}}(\tau)}_{0}+\underbrace{jr_{x\hat{x}}(\tau)+jr_{\hat{x}x}(\tau)}_{0} \\ &= 0.\end{aligned} \tag{2.124}$$

What remains is

$$\begin{aligned}r_{uu}(\tau) &= E\left\{u(t)\; u(t+\tau)\right\} \\ &= \tfrac{1}{4}\; \left[[r_{x^+x^+}(\tau)]^*\; e^{j\omega_0\tau} + r_{x^+x^+}(\tau)\; e^{-j\omega_0\tau}\right] \\ &= r_{xx}(\tau)\;\cos\omega_0\tau + \hat{r}_{xx}(\tau)\;\sin\omega_0\tau.\end{aligned} \tag{2.125}$$

In a similar way we obtain

$$r_{vv}(\tau) = r_{uu}(\tau) \tag{2.126}$$

for the autocorrelation function of the imaginary part of the complex envelope. The cross correlation function between the real and the imaginary part is given by

$$\begin{aligned}r_{uv}(\tau) &= -r_{vu}(\tau) \\ &= \hat{r}_{xx}(\tau)\;\cos\omega_0\tau - r_{xx}(\tau)\;\sin\omega_0\tau.\end{aligned} \tag{2.127}$$

From (2.125) – (2.127) we conclude that the autocorrelation function of the complex envelope equals the modulated autocorrelation function of the analytic signal:

$$\begin{aligned} r_{x_{\mathrm{LP}}x_{\mathrm{LP}}}(\tau) &= E\{[u(t) - j\ v(t)]\ [u(t+\tau) + j\ v(t+\tau)]\} \\ &= 2\, r_{uu}(\tau) + 2j\, r_{uv}(\tau) \qquad (2.128) \\ &= 2\ [r_{xx}(\tau) + j\, \hat{r}_{xx}(\tau)]\ e^{-j\omega_0\tau}. \end{aligned}$$

Correspondingly, we get for the power spectral density:

$$\begin{aligned} S_{x_{\mathrm{LP}}x_{\mathrm{LP}}}(\omega) &= S_{x^+x^+}(\omega + \omega_0) \\ &= \begin{cases} 4\ S_{xx}(\omega+\omega_0) & \text{for } \omega + \omega_0 > 0, \\ 0 & \text{for } \omega + \omega_0 < 0. \end{cases} \end{aligned} \qquad (2.129)$$

We notice that the complex envelope is a wide-sense stationary process with specific properties:

- The autocorrelation function of the real part equals that of the imaginary part.

- The cross correlation function between the real and imaginary part is antisymmetric with respect to τ. In particular, we have

$$r_{uv}(0) = r_{vu}(0) = 0.$$

In the special case of a symmetric bandpass process, we have

$$S_{x_{\mathrm{LP}}x_{\mathrm{LP}}}(\omega) = S_{x_{\mathrm{LP}}x_{\mathrm{LP}}}(-\omega). \qquad (2.130)$$

Hence, we see that the autocorrelation function of $x_{\mathrm{LP}}(t)$ is real-valued. It also means that the cross correlation between the real and imaginary part vanishes:

$$r_{uv}(\tau) = 0, \quad \forall\, \tau. \qquad (2.131)$$