Detection, Estimation, and Modulation Theory, Part III:
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5

Discussion: Detection of Gaussian Signals

In Chapters 2 through 4 we studied the problem of detecting Gaussian signals in the presence of Gaussian noise. In this chapter we first discuss some related topics. We then summarize the major results of our detection theory discussion.

5.1 RELATED TOPICS

5.1.1 *M*-ary Detection: Gaussian Signals in Noise

All of our discussion in Chapters 2 through 4 dealt with the binary detection problem. In this section we discuss briefly the M-hypothesis problem. The general Gaussian M-ary problem is

$$r(t) = r_i(t), \qquad T_b \le t \le T_f : H_i, \qquad i = 1, \ldots, M, \tag{1}$$

where

$$E[r_i(t) \mid H_i] = m_i(t) \tag{2}$$

and

$$E[(r_i(t) - m_i(t))(r_i(u) - m_i(u)) \mid H_i] = K_{H_i}(t, u). \tag{3}$$

Most of the ideas from the binary case carry over to the M-ary case with suitable modifications. As an illustration we consider a special case of the general problem.

The problem of interest is described by the following model. The received waveforms on the M hypotheses are

$$r(t) = s_i(t) + n(t), \qquad T_b \le t \le T_f : H_i, \qquad i = 1, 2, \ldots, M. \tag{4}$$

The additive noise $n(t)$ is a sample function from a zero-mean Gaussian process with covariance function $K_n(t, u)$. A white noise term is not necessarily present. The signal processes are sample functions from Gaussian processes and are statistically independent of the noise process. The signal processes are characterized by

$$E[s_i(t)] = m_i(t), \qquad T_b \le t \le T_f, \qquad i = 1, \dots, M \tag{5}$$

and

$$E\{[s_i(t) - m_i(t)][s_i(u) - m_i(u)]\} = K_{s_i}(t, u), \qquad T_b \le t, u \le T_f, \quad i = 1, \dots, M. \tag{6}$$

The a-priori probability of the ith hypothesis is P_i and the criterion is minimum $\Pr(\epsilon)$. We assume that each pair of hypotheses would lead to a nonsingular binary test. The derivation of the optimum receiver is similar to the derivation for the binary case, and so we shall simply state the results. The reader can consult [1]–[3] or Problem 5.1.1 for details of the derivation.

To perform the likelihood ratio test we compute a set of M sufficient statistics, which we denote by l_i, $i = 1, \dots, M$. The first component of the ith sufficient statistic is

$$l_{R_i} = \int_{T_b}^{T_f} r(t) h_i(t, u) r(u)\, dt\, du \tag{7}$$

where $h_i(t, u)$ is specified by the integral equation

$$\iint_{T_b}^{T_f} K_n(t, x) h_i(x, y)[K_n(y, u) + K_{s_i}(y, u)]\, dx\, dy = K_{s_i}(t, u), \qquad T_b \le t, u \le T_f. \tag{8}$$

The second component of the ith sufficient statistic is

$$l_{D_i} = \int_{T_b}^{T_f} r(t)\left[g_i(t) - \int_{T_b}^{T_f} h_i(t, u) m_i(u)\, du \right] dt, \tag{9}$$

where $g_i(t)$ is specified by the integral equation

$$\int_{T_b}^{T_f} K_n(t, u) g_i(u)\, du = m_i(t), \qquad T_b \le t \le T_f. \tag{10}$$

The bias component of the ith sufficient statistic is

$$l_{B_i} = -\tfrac{1}{2} \sum_{k=1}^{\infty} \ln(1 + \lambda_{ik}^*), \tag{11}$$

where the λ_{ik}^* are the eigenvalues of the kernel,

$$K_{s_i}^*(t, u) = \iint_{T_b}^{T_f} K_n^{[-\frac{1}{2}]}(t, x) K_{s_i}(x, y) K_n^{[-\frac{1}{2}]}(u, y)\, dx\, dy. \tag{12}$$

The complete ith sufficient statistic is

$$l_i = l_{R_i} + l_{D_i} + l_{B_i}, \qquad i = 1, \ldots, M. \tag{13}$$

The test consists of computing

$$l_i + \ln P_i, \qquad i = 1, \ldots, M \tag{14}$$

and choosing the largest.

A special case of (4) that occurs frequently is

$$K_n(t, u) = \frac{N_0}{2}\,\delta(t-u), \tag{15}$$

$$m_i(t) = 0, \qquad i = 1, \ldots, M. \tag{16}$$

Then $h_i(t, u)$ satisfies the equation

$$\frac{N_0}{2}\,h_i(t, u) + \int_{T_b}^{T_f} h_i(t, y)K_{s_i}(y, u)\,dy = K_{s_i}(t, u), \qquad T_b \le t, u \le T_f,$$
$$i = 1, \ldots, M. \tag{17}$$

All of the canonical realizations in Chapter 2 are valid for this case. The bias term is

$$l_{B_i} = -\frac{1}{N_0}\int_{T_b}^{T_f} \xi_{Pi}\left(t \,\Big|\, s_i(\cdot), \frac{N_0}{2}\right) dt, \tag{18}$$

where $\xi_{Pi}(t \mid s_i(\cdot),)$ is defined as in (2.137).

The performance calculation for the general M-ary case is difficult. We would anticipate this because, even in the known signal case, exact M-ary performance calculations are usually not feasible.

An important problem in which we can get accurate bounds is that of digital communication over a Rayleigh channel using M-orthogonal signals. The binary version of this problem was discussed in Examples 3 and 4 of Chapter 4 (see pages 111–117). We now indicate the results for the M-ary problem.

The transmitted signal on the ith hypothesis is

$$s_t(t) = \sqrt{\frac{2E_t}{T}}\sin\omega_i t \qquad T_b \le t \le T_f{:}H_i \tag{19}$$

the signal passes over a fluctuating Rayleigh channel. The received waveform on the ith hypothesis is

$$r(t) = s_i(t) + w(t), \qquad T_b \le t \le T_f{:}H_i. \tag{20}$$

The ith signal $s_i(t)$ is a sample function of a bandpass process centered at ω_i, whose spectrum is symmetric around ω_i.† The signal processes

† The symmetric assumption is included to keep the notation simple. After we introduce complex notation in Chapter 9, we can handle the asymmetric case easily.

are essentially disjoint in frequency. The additive noise is a sample function from a zero-mean white Gaussian random process with spectral height $N_0/2$. The low-pass spectra of the signal processes are identical. We denote them by $S_{s_l}(\omega)$. The total received power in the signal is

$$P_r = 2\sigma_b^{\,2}\int_{-\infty}^{\infty} S_{s_l}(\omega)\,\frac{d\omega}{2\pi}\,. \tag{21}$$

Kennedy [4] and Viterbi [5] have studied the performance for this case. Our discussion follows the latter's. Starting from a general result in [6], one can show that

$$\Pr(\epsilon) \le \exp[\rho TR]\,\frac{\left[D_{\mathscr{F}}\left(\dfrac{2}{N_0}\right)\right]^{\rho}}{\left[D_{\mathscr{F}}\left(\dfrac{2\rho}{N_0(1+\rho)}\right)\right]^{1+\rho}}, \qquad 0 \le \rho \le 1, \tag{22}$$

where $D_{\mathscr{F}}(z)$ is the Fredholm determinant of the low-pass process, T is the length of interval, and

$$R = \ln M \tag{23}$$

is the transmission rate in nats per second. The parameter ρ is used to optimize the bound. When the observation time is long, we can use (I-3.182) to obtain

$$\ln D_{\mathscr{F}}(z) = T\int_{-\infty}^{\infty} \ln\left(1 + zS_{s_l}(\omega)\right)\frac{d\omega}{2\pi}\,. \tag{24}$$

We now define

$$E_0(\rho) \triangleq (1+\rho)\int_{-\infty}^{\infty} \ln\left[1 + \frac{\rho}{1+\rho}\,\frac{2S_{s_l}(\omega)}{N_0}\right]\frac{d\omega}{2\pi} - \rho\int_{-\infty}^{\infty} \ln\left[1 + \frac{2S_{s_l}(\omega)}{N_0}\right]\frac{d\omega}{2\pi} \tag{25}$$

and

$$E(R) = \max_{0\le\rho\le 1}\,[E_0(\rho) - \rho R]. \tag{26}$$

Comparing (25) and (4.21), we observe that

$$-\frac{T}{2}\,\frac{E_0(\rho)}{1+\rho}\bigg|_{\rho=(1-s)/s} = \mu_\infty(s) \tag{27}$$

and, in particular,

$$-TE_0(1) = 4\mu_\infty(\tfrac{1}{2}) = 2\mu_{\mathrm{BP},\infty}(\tfrac{1}{2}) = \mu_{\mathrm{BS,BP},\infty}(\tfrac{1}{2}). \tag{28}$$

Using (24)–(26), (22) reduces to

$$\Pr(\epsilon) \leq e^{-TE(R)} \tag{29}$$

The final step is to perform the maximization indicated in (26). The result is obtained as follows:

1. If

$$\dot{E}_0(\rho) \triangleq \frac{\partial E_0(\rho)}{\partial \rho} = R \tag{30}$$

has a solution for $0 \leq \rho \leq 1$, we denote it as ρ_m. Then

$$E(R) = E_0(\rho_m) - \rho_m \dot{E}_0(\rho_m), \tag{31}$$

$$R = \dot{E}_0(\rho_m), \tag{32}$$

and

$$\dot{E}_0(1) \leq R \leq C_{OR} = \dot{E}_0(0). \tag{33}$$

2. If (30) does not have a solution in the allowable range of ρ, the maximum is at $\rho = 1$, and

$$E(R) = E_0(1) - R, \tag{34}$$

$$0 < R < \dot{E}_0(1). \tag{35}$$

The results in (31) and (34) provide the exponents in the $\Pr(\epsilon)$ expression. In the problems, we include a number of examples to illustrate the application of these results.

This concludes our brief discussion of the M-ary problem. For a large class of processes we can find the optimum receiver, but, except for orthogonal signal processes, the performance evaluation is usually difficult.

5.1.2 Suboptimum Receivers

We have been able to find the optimum receiver to implement the likelihood ratio test for a large class of Gaussian signal processes. Frequently, the filters in the receiver are time-varying and may be difficult to implement. This motivates the search for suboptimum receivers, which are simpler to implement than the optimum receiver but perform almost as well as the optimum receiver. To illustrate this idea we consider a simple example.

Example. Consider the simple binary detection example discussed on page 104. The received waveforms on the two hypotheses are

$$
\begin{aligned}
r(t) &= s(t) + w(t), \qquad T_i \leq t \leq T_f : H_1, \\
r(t) &= w(t), \qquad T_i \leq t \leq T_f : H_0,
\end{aligned}
\tag{36}
$$

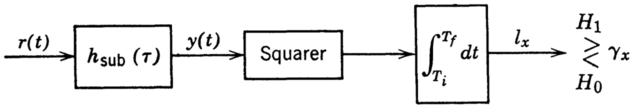

Fig. 5.1 Suboptimum receiver.

where $w(t)$ is a white Gaussian process with spectral height $N_0/2$ and $s(t)$ is a Gaussian process with spectrum

$$S_s(\omega) = \frac{2kP}{\omega^2 + k^2}. \tag{37}$$

We saw that the optimum receiver could be implemented as a cascade of a time-varying filter, a square-law device, and an integrator. The difficulty arises in implementing the time-varying filter.

A receiver that is simpler to implement is shown in Fig. 5.1. The structure is the same as the optimum receiver, but the linear filter is time-invariant,

$$h_{\text{sub}}(\tau) = e^{-\beta\tau}u_{-1}(\tau), \qquad -\infty < \tau < \infty. \tag{38}$$

We choose β to optimize the performance. From our results in Section 4.1 (page 104) we know that if

$$\beta = k\left(1 + \frac{4P}{kN_0}\right)^{1/2}, \tag{39}$$

then the suboptimum receiver will be essentially optimum for long observation times. For arbitrary observation times, some other choice of β might give better performance. Thus, the problem of interest is to choose β to maximize the performance.

With this example as motivation, we consider the general question of suboptimum receivers. The choice of the structure for the suboptimum receiver is strongly dependent on the particular problem. Usually one takes the structure of the optimum receiver as a starting point, tries various modifications, and analyzes the resulting performance. In this section we discuss the performance of suboptimum receivers.

To motivate our development, we first recall the performance results for the optimum receiver. The optimum receiver computes l, the logarithm of the likelihood ratio, and compares it with a threshold. The error probabilities are

$$P_F = \Pr(\epsilon|H_0) = \int_{\gamma}^{\infty} p_{l|H_0}(L \mid H_0)\, dL \tag{40}$$

and

$$P_M = \Pr(\epsilon|H_1) = \int_{-\infty}^{\gamma} p_{l|H_1}(L \mid H_0)\, dL. \tag{41}$$

All of our performance discussion in the Gaussian signal problem has been based on $\mu(s)$, which is defined as

$$\mu(s) = \ln M_{l|H_0}(s) = \ln \int_{-\infty}^{\infty} e^{sL} p_{l|H_0}(L \mid H_0)\, dL, \tag{42}$$

the logarithm of the moment-generating function of l, given that H_0 is true. Since l is the logarithm of the likelihood ratio, we can also write $\mu(s)$ in terms of $M_{l|H_1}(s)$,

$$\mu(s) = \ln M_{l|H_1}(s - 1) \tag{43}$$

(see pages I-118–I-119). Thus we can express both P_M and P_F in terms of $\mu(s)$.

A suboptimum receiver computes a test statistic l_x and compares it with a threshold γ_x in order to make a decision. The statistic l_x is *not* equivalent to l and generally is used because it is easier to compute. For suboptimum receivers, the probability densities of l_x on H_1 and H_0 are not uniquely related, and so we can no longer express P_M and P_F in terms of a single function. This forces us to introduce two functions analogous to $\mu(s)$ and makes the performance calculations more involved.

To analyze the suboptimum receiver, we go through a development parallel to that in Sections I-2.7 and III-2.2. Because the derivation is straightforward, we merely state the results. We define

$$\mu_0(s) \triangleq \ln M_{l_x|H_0}(s) = \int_{-\infty}^{\infty} e^{sL} p_{l_x|H_0}(L \mid H_0)\, dL \tag{44}$$

and

$$\mu_1(s) \triangleq \ln M_{l_x|H_1}(s) = \int_{-\infty}^{\infty} e^{sL} p_{l_x|H_1}(L \mid H_1)\, dL. \tag{45}$$

The Chernoff bounds are

$$\Pr(\epsilon \mid H_0) \le \exp[\mu_0(s_0) - s_0\gamma], \qquad s_0 > 0, \tag{46}$$

$$\Pr(\epsilon \mid H_1) \le \exp[\mu_1(s_1) - s_1\gamma], \qquad s_1 < 0, \tag{47}$$

where

$$\dot{\mu}_0(s_0) = \gamma, \qquad s_0 > 0 \tag{48}$$

and

$$\dot{\mu}_1(s_1) = \gamma, \qquad s_1 < 0. \tag{49}$$

The equations (48) and (49) will have a unique solution if

$$E[l_x \mid H_0] \le \gamma \le E[l_x \mid H_1]. \tag{50}$$

The first-order asymptotic approximations are

$$\Pr(\epsilon \mid H_0) \simeq \operatorname{erfc}_*\left(s_0\sqrt{\ddot{\mu}_0(s_0)}\right) \exp\left[\mu_0(s_0) - s_0\dot{\mu}_0(s_0) + \frac{s_0^2}{2}\ddot{\mu}_0(s_0)\right] \tag{51}$$

and

$$\Pr(\epsilon \mid H_1) \simeq \operatorname{erfc}_*\left(-s_1\sqrt{\ddot{\mu}_1(s_1)}\right) \exp\left[\mu_1(s_1) - s_1\dot{\mu}_1(s_1) + \frac{s_1^2}{2}\ddot{\mu}_1(s_1)\right], \tag{52}$$

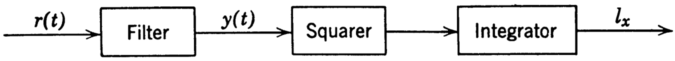

Fig. 5.2 General filter-squarer-integrator (*FSI*) receiver.

where s_0 and s_1 satisfy (48) and (49). Equations (51) and (52) are analogous to (179) and (188). Results similar to (181) and (189) follow easily.

The results in (44)–(52) are applicable to an arbitrary detection problem. To apply them to the general Gaussian problem, we must be able to evaluate $\mu_0(s)$ and $\mu_1(s)$ efficiently. The best technique for evaluating $\mu_0(s)$ and $\mu_1(s)$ will depend on the structure of the suboptimum receiver. We demonstrate the technique for the general filter-squarer-integrator (FSI) receiver shown in Fig. 5.2. The filter may be time-varying. For this structure the techniques that we developed in Section 2.2 (pages 35–44) are still valid. We illustrate the procedure by finding an expression for $\mu_1(s)$.

Calculation of $\mu_1(s)$ for an FSI Receiver. To find $\mu_1(s)$, we expand $y(t)$, the input to the squarer under H_1, in a Karhunen-Loève expansion. Thus

$$y_1(t) \triangleq \sum_{i=1}^{\infty} y_i \phi_{1i}(t), \qquad T_i \le t \le T_f : H_1, \tag{53}$$

where the $\phi_{1i}(t)$ are the eigenfunctions of $y(t)$ on H_1. The corresponding eigenvalues are λ_{1i}. We assume that the eigenvalues are ordered in magnitude so that λ_{1i} is the largest. From (45),

$$\begin{aligned}\mu_1(s) &= \ln \{E[e^{sl_x} \mid H_1]\} \\ &= \ln \left\{ E\left[\exp\left(s \sum_{i=1}^{\infty} y_i^2 \right) \middle| H_1 \right] \right\} \\ &= -\tfrac{1}{2} \sum_{i=1}^{\infty} \ln(1 - 2s\lambda_{1i}), \qquad s < \frac{1}{2\lambda_{1i}}.\end{aligned} \tag{54}$$

The expectation is a special case of Problem I-4.4.2. The sum can be written as a Fredholm determinant,†

$$\mu_1(s) = -\tfrac{1}{2} \ln D_{\mathscr{F}|H_1}(-2s), \qquad s < \frac{1}{2\lambda_{1i}}. \tag{55}$$

A similar result follows for $\mu_0(s)$,

$$\mu_0(s) = -\tfrac{1}{2} \ln D_{\mathscr{F}|H_0}(-2s), \qquad s < \frac{1}{2\lambda_{0i}}. \tag{56}$$

We now have $\mu_0(s)$ and $\mu_1(s)$ expressed in terms of Fredholm determinants. The final step is to evaluate these functions. Three cases in which we can evaluate $\ln D_{\mathscr{F}|H_i}(\cdot)$ are the following:

1. Stationary processes, long observation time.
2. Separable kernels.
3. State-representable processes.

† This result is due to Kac and Siegert [8] (e.g., [9, Chapter 9]).

The procedure for the first two cases is clear. In the third case, we can use the algorithms in section 2.2.1 or section 2.2.3 to evaluate $\mu_0(s)$ and $\mu_1(s)$. The important point to remember is that the state equation that we use to compute $\mu_1(s)$ corresponds to the system that produces $y_1(t)$ when driven by white noise. Similarly, the state equation that we use to compute $\mu_0(s)$ corresponds to the system that produces $y_0(t)$ when driven by white noise.

In this section we have developed the performance expressions needed to analyze suboptimum receivers. Because the results are straightforward modifications of our earlier results, our discussion was brief. The analysis based on these results is important in the implementation of practical receiver configurations. A number of interesting examples are developed in the problems. In Chapter 11, we encounter suboptimum receivers again and discuss them in more detail.

5.1.3 Adaptive Receivers

A complete discussion of adaptive receivers would take us too far afield. On the other hand, several simple observations are useful.

All of our discussion of communication systems assumed that we made a decision on each baud. This decision was independent of those made on previous bauds. If the channel process is correlated over several bauds, one should be able to exploit this correlation in order to reduce the probability of error. Since the optimum "single-baud" receiver is an estimator-correlator, a logical approach is to perform a continuous channel estimation and use this to adjust the receiver filters and gains. An easy way to perform the channel estimation is through the use of decision-directed feedback. Here we assume that all past decisions are correct in order to perform the channel estimation. As long as most of the decisions are correct, this reduces the channel estimation problem to that of a "known" signal into an unknown channel. Decision feedback schemes for simple channels have been studied by Proakis and Drouilhet [10]. More complicated systems have been studied by Glaser [11], Jakowitz, Shuey, and White [12], Scudder [13, 14], Boyd [15], and Austin [16]. Another procedure to exploit the correlation of the channel process would be to devote part of the available energy to send a known signal to measure the channel.

There has been a great deal of work done on adaptive systems. In almost all cases, the receivers are so complicated and difficult to analyze that one cannot make many useful general statements. We do feel the reader should recognize that many of these systems are logical extrapolations from the general Gaussian problem we have studied. References that deal with various types of adaptive systems include [17]–[30].

5.1.4 Non-Gaussian Processes

All of our results have dealt with Gaussian random processes. When the processes involved are non-Gaussian, the problems are appreciably more difficult. We shall divide our comments on these problems into four categories:

1. Processes derived from Gaussian processes.
2. Structured non-Gaussian processes.
3. Unspecified non-Gaussian processes.
4. Analysis of fixed receivers.

We shall explain the descriptions in the course of our discussion.

Processes Derived from Gaussian Processes. We have emphasized cases in which the received waveform is conditionally Gaussian. A related class of problems comprises those in which $r(t)$ is a sample function of a process that can be derived from a Gaussian process. A common case is one in which either the mean-value function or the covariance function contains a random parameter set. In this case, we might have $m(t, \boldsymbol{\theta}_m)$ and $K_r(t, u:\boldsymbol{\theta}_k)$. If the probability densities of $\boldsymbol{\theta}_m$ and $\boldsymbol{\theta}_k$ are known, the parameters are integrated out in an obvious manner (conceptually, at least). Whether we can actually carry out the integration depends on how the parameters enter into the expression.

If either $\boldsymbol{\theta}_m$ or $\boldsymbol{\theta}_k$ is a nonrandom variable, we can check to see if a uniformly most powerful test exists. If it does not, a generalized likelihood ratio test may be appropriate.

Structural Non-Gaussian Processes. The key to the simplicity in Gaussian problems is that we can completely characterize the process by its mean-value function and covariance function. We would expect that whenever the processes involved could be completely characterized in a reasonably simple manner, one could find the optimum receiver. An important example of such a processes is the Poisson process. References [31]–[35] discuss this problem. A second important example is Markov processes (e.g., [2-21]–[2-24]).

Unspecified Non-Gaussian Processes. In this case we would like to make some general statements about the optimum receiver without restricting the process to have a particular structure. One result of this type is available in the LEC case that we studied in Section 4.3. Middleton [36], [37], derives the LEC receiver without requiring that the signal process be Gaussian. (See [39] for a different series expansion approach.) A second important result concerning unspecified Gaussian processes is given in [38]. Here, Kailath extends the realizable estimator-correlator receiver to include non-Gaussian processes.

Analysis of Fixed Receivers. In this case, we consider a fixed receiver structure and analyze its performance in the presence of non-Gaussian signals and noise. Suitable examples of this type of analysis are contained in [40], [41].

These four topics illustrate some of the issues involved in the study of non-Gaussian processes. The selection was intended to be representative, not exhaustive.

5.1.5 Vector Gaussian Processes

We have not discussed the case in which the received signal is a vector random process. The formal extension of our results to this case is straightforward. In fact, all of the necessary equations have been developed in the problem sections of Chapters 2–4. The important issues in the vector case are the solution of the equations specifying the optimum receiver and its performance and the interpretation of the results in the context of particular physical situations. In the subsequent volume [42], we shall study the vector problem in the context of array processing in sonar and seismic systems. At that time, we shall discuss the above issues in detail.

5.2 SUMMARY OF DETECTION THEORY

In Chapters 2 through 5 we have studied the detection of Gaussian signals in Gaussian noise in detail. The motivation of this detailed study is to provide an adequate background for actually solving problems we encounter when modeling physical situations.

In Chapter 2 we considered the simple binary problem. The first step was to develop the likelihood ratio test. We saw that the likelihood ratio contained three components. The first was obtained by a nonlinear operation on the received waveform and arose because of the randomness in the signal. The second was obtained by a linear operation on the received waveform and was due to the deterministic part of the received signal. This component was familiar from our earlier work. The third component was the bias term, which had to be evaluated in order to conduct a Bayes test.

We next turned our attention to the problem of realizing the nonlinear operation needed to generate l_R. Four canonical realizations were developed:

1. The estimator-correlator receiver.
2. The filter-correlator receiver.
3. The filter-squarer receiver.
4. The optimum realizable filter receiver.

The last realization was particularly appealing when the process had finite state-variable representation. In this case we could use all of the effective state-variable procedures that we developed in Section I-6.3 actually to find the receiver.

A more difficult issue was the performance of the optimum receiver. As we might expect from our earlier work, an exact performance calculation is not feasible in many cases. By building on our earlier work on bounds and approximate expressions in Section I-2.7, we developed performance results for this problem. The key to the results was the $\mu(s)$ function defined in (2.148). We were able to express this in terms of both a realizable filtering error and the logarithm of the Fredholm determinant. We have effective computational procedures to evaluate each of these functions.

We next turned to the general binary problem in Chapter 3, where the received waveform could contain a nonwhite component on each hypothesis. The procedures were similar to the simple binary case. A key result was (3.33), whose solution was the kernel of the nonlinear part of the receiver. The modifications of the various canonical realizations were straightforward, and the performance bounds were extended. A new issue that we encountered was that of singularity. We first derived simple upper and lower bounds on the probability of error in terms of $\mu(\frac{1}{2})$. We then showed that a necessary and sufficient condition for a nonsingular test was that $\mu(\frac{1}{2})$ be finite. This condition was then expressed in terms of a square-integrability requirement on a kernel. As before, singularity was never an issue when the same white noise component was assumed to be present on both hypotheses.

In Chapter 4 we considered three special cases that led to particularly simple solutions. In Section 4.1 we looked at the stationary-process–long-time-interval case. This assumption enabled us to neglect homogeneous solutions in the integral equation specifying the kernel and allowed us to solve this equation using Fourier transform techniques. Several practical examples were considered. The separable kernel case was studied in Section 4.2. We saw that this was a suitable model for pulsed radars with slowly fluctuating targets, ionospheric communications over resolvable multipath channels, and frequency-diversity systems. The solution for this case was straightforward. Finally, in Section 4.3, we studied the low-energy-coherence case, which occurs frequently in passive sonar and radar astronomy problems. The energy in the signal process is spread over a large number of coordinates so that each eigenvalue is small when compared to the white noise level. This smallness enabled us to obtain a series solution to the integral equation. In this particular case we found that the output signal-to-noise ratio (d^2) is an accurate performance measure. In addition to these three special cases, we had previously developed a complete solution for the case in which the processes have a finite state representation. A large portion of the physical situations that we encounter can be satisfactorily approximated by one of these cases.

In Section 5.1 we extended our results to the M-ary problem. The optimum receiver is a straightforward extension of our earlier results, but the performance calculation for the general problem is difficult. A reasonably simple bound for the case of M-orthogonal processes was presented. In Section 5.2 we derived performance expressions for suboptimum receivers.

Our discussion of the detection problem has been lengthy and, in several instances, quite detailed. The purpose is to give the reader a thorough understanding of the techniques involved in solving actual problems. In addition to the references we have cited earlier, the reader may we wish to consult [43]–[48] for further reading in this area. In the next two chapters we consider the parameter estimation problem that was described in Chapter 1.

5.3 PROBLEMS

P.5.1 Related Topics

M-ARY DETECTION

Problem 5.1.1. Consider the model described in (4)–(6). Assume that $n(t)$ contains a white noise component with spectral height $N_0/2$. Assume that

$$m_i(t) = 0, \qquad i = 1, \dots, M.$$

1. Derive (7)–(8) and (11)–(14).
2. Draw a block diagram of the optimum receiver.

Problem 5.1.2. Generalize the model in Problem 5.1.1 to include nonzero means and unequal costs. Derive the optimum Bayes receiver.

Problem 5.1.3. Consider the model in (15)–(17).

Assume that

$$K_{s_i}(t, u) = iK_s(t - u), \qquad i = 1, \ldots, M$$

that

$$S_s(\omega) = \frac{2k}{\omega^2 + k^2},$$

and that the SPLOT condition is valid. The hypotheses are equally likely.

1. Draw a block diagram of the optimum receiver.
2. Consider the case in which $M = 3$. Derive a bound on the Pr (ϵ).

Problem 5.1.4. Consider the communication system using M-orthogonal signals that is described in (19)–(21). On pages I-263–I-264, we derived a bound on the Pr (ϵ) in an M-ary system in terms of the Pr (ϵ) in a binary system.

1. Extend this technique to the current problem of interest.
2. Compare the bound in part 1 with the bound given by (22)–(35). For what values of R is the bound in part 1 useful?

Problem 5.1.5. Consider the problem of detecting one of M-orthogonal bandpass processes in the presence of white noise. Assume that each process has N eigenvalues.

1. We can immediately reduce the problem to one with MN dimensions. Denote this resulting vector as $\mathbf{R}$. Compute $p_{\mathbf{r}|H_m}(\mathbf{R} \mid H_m)$.
2. In [6], Gallager derived the basic formula for a bound on the error probability,

$$\Pr(\epsilon \mid H_m) \leq \int_{-\infty}^{\infty} \cdots \int d\mathbf{R}\,[p_{\mathbf{r}|H_m}(\mathbf{R} \mid H_m)]^{1/(1+\rho)} \times \left[\sum_{k \neq m}^{M} [p_{\mathbf{r}|H_k}(\mathbf{R} \mid H_k)]^{1/(1+\rho)}\right]^{\rho}, \qquad \rho \geq 0. \tag{P.1}$$

Use (P.1) to derive (22). (*Hint:* Use the fact that $E[x^\rho] \leq (E[x])^\rho$, $0 \leq \rho \leq 1$.)

Problem 5.1.6.

1. Verify the results in (29)–(35).
2. One can show that C_{or} is the capacity of the channel for this type of communication system (i.e., we require M-orthogonal signals and use rectangular signal envelopes). Assume

$$S_{s_l}(\omega) = \frac{2k}{\omega^2 + k^2} \tag{P.1}$$

and

$$P_r \triangleq 2\sigma_b^2. \tag{P.2}$$

Plot C_{or} as a function of

$$\Lambda_B = \frac{\pi P_r}{kN_0}. \tag{P.3}$$

3. Repeat for

$$S_{s_l}(\omega) = \begin{cases} \dfrac{\pi}{k}, & |\omega| \leq k, \\ 0, & |\omega| > k. \end{cases} \tag{P.4}$$

Problem 5.1.7. The error exponent, $E(R)$, is defined by (31) and (34).

1. Plot $E(R)$ as a function of N_0R/P_r for the bandlimited message spectrum in (P.4) of Problem 5.1.6.
2. Plot $E(R)$ as a function of N_0R/P_r for the one-pole message spectrum in (P.1) of Problem 5.1.6.

Problem 5.1.8. Assume that we want to signal at low rates so that

$$E(R) \simeq E(0).$$

1. Consider the one-pole message spectrum in (P.1) of Problem 5.1.6. Plot $E(0)/(P_r/N_0)$ as a function of Λ_B. What value of Λ_B maximizes $E(0)/(P_r/N_0)$?
2. Repeat part 1 for the ideal bandlimited message spectrum in (P.4) of Problem 5.1.6.
3. Compare the results in parts 1 and 2 with those in (4.60) and (4.69).

Problem 5.1.9. Assume that we want to signal at the rate

$$R = \frac{1}{10} C_\infty = \frac{1}{10} \frac{P_r}{N_0}.$$

We want to maximize $E(R)/(P_r/N_0)$ by choosing Λ_B.

1. Carry out this maximization for the one-pole spectrum.
2. Carry out this maximization for the ideal bandlimited spectrum.
3. Compare your results with those in Problem 5.1.8.

Problem 5.1.10. Define

$$E_n^*(R) = \max_{\Lambda_B} \left[\frac{E(R)}{P_r/N_0}\right].$$

1. Find $E_n^*(R)$ as R varies from 0 to C_∞ for the ideal bandlimited spectrum.
2. Repeat part 1 for the one-pole spectrum.

Problem 5.1.11 [5]. Assume that each signal process has a non-zero mean. Specifically,

$$E\{[s_i(t)\sqrt{2}\cos(\omega_i t)]_{\mathrm{LP}}\} = m(t),$$
$$E\{[s_i(t)\sqrt{2}\sin(\omega t_i)]_{\mathrm{LP}}\} = 0.$$

Show that the effect of the non-zero mean is to add a term to $E_0(\rho)$ in (25), which is

$$E_{0m}(\rho) = \frac{\rho}{2N_0(1+\rho)} \int_{-\infty}^{\infty} |M(j\omega)|^2 \left[1 + \left(\frac{\rho}{1+\rho}\right)\frac{2S_{s_l}(\omega)}{N_0}\right]^{-1} \frac{d\omega}{2\pi}.$$

Problem 5.1.12 [7]. Consider the special case of Problem 5.1.11 in which

$$S_{s_l}(\omega) = 0.$$

1. Prove

$$\frac{E(R)}{C_\infty} = \begin{cases} \frac{1}{2} - \dfrac{R}{C_\infty}, & 0 \le \dfrac{R}{C_\infty} \le \frac{1}{4}, \\ \left(1 - \sqrt{\dfrac{R}{C_\infty}}\right)^2, & \frac{1}{4} \le \dfrac{R}{C_\infty} < 1, \end{cases}$$

where

$$C_\infty = \frac{P_r}{N_0} = \frac{1}{N_0}\int_{-\infty}^{\infty} m^2(t)\,dt.$$

2. Discuss the significance of this result.

SUBOPTIMUM RECEIVERS

Problem 5.1.13. Consider the definitions of $\mu_0(s)$ and $\mu_1(s)$ given in (44) and (45).

1. Derive the Chernoff bounds in (46)–(49).
2. Derive the approximate error expressions in (51) and (52).

Problem 5.1.14. Consider the simple binary detection problem described on page 151 and the filter-squarer-integrator receiver in Fig. 5.1. The filter is time-invariant with transfer function

$$H(j\omega) = \frac{1}{j\omega + \beta}.$$

The message spectrum is given in (37).

1. Write the state equations that are needed to evaluate $\mu_1(s)$ and $\mu_0(s)$.
2. Assume that the long-time-interval approximation is valid. Find $\mu_{1\infty}(s)$ and $\mu_{0\infty}(s)$. Verify that the value of β in (39) is optimum.

Problem 5.1.15.

1. Repeat part 1 of Problem 5.1.14 for the case in which $s(t)$ is a Wiener process,

$$s(0) = 0, \qquad t \geq 0,$$
$$E[s^2(t)] = \sigma^2 t, \qquad t \geq 0.$$

2. Find the optimum value of β for long observation times.

Problem 5.1.16. Consider the binary symmetric communication problem whose model was given in Section 3.4.3. The quantities $r_{c_1}(t)$, $r_{s_1}(t)$, $r_{c_0}(t)$, and $r_{s_0}(t)$ were defined in Fig. 3.9. We operate on each of these waveforms as shown in Fig. 3.10. Instead of the optimum filter $h_{fr}(t, u)$, we use some arbitrary filter $h_{\text{sub}}(\tau)$ in each path. Denote the output of the top branch as l_1 and the output of the bottom branch as l_0. Define

$$l_x = l_1 - l_0.$$

The bias terms are both zero and

$$\ln \eta \triangleq \gamma_x = 0.$$

Define

$$\mu_{1_j}(s) = \ln E[e^{sl_1} \mid H_j], \qquad j = 0, 1,$$

and

$$\mu_{0_j}(s) = \ln E[e^{sl_0} \mid H_j], \qquad j = 0, 1.$$

1 Prove

$$\mu_{\text{BS},1}(s) = \mu_{11}(s) + \mu_{01}(-s),$$
$$\mu_{\text{BS},0}(s) = \mu_{00}(-s) + \mu_{10}(s) = \mu_{\text{BS},1}(-s).$$

2. Prove

$$\Pr(\epsilon) < \tfrac{1}{2}\exp\left(\mu_{\text{BS},1}(s_m)\right),$$

where

$$\dot{\mu}_{\text{BS},1}(s)\big|_{s=s_m} = 0.$$

3. Prove

$$\Pr(\epsilon) \simeq \frac{1}{2(2\pi\ddot{\mu}_{BS,1}(s_m))^{1/2} s_m(1-s_m)} \exp \mu_{BS,1}(s_m).$$

4. Express $\mu_{BS,1}(s)$ in terms of Fredholm determinants.

Problem 5.1.17. Consider the binary communication system described in Problem 5.1.16. Assume that $s_1(t)$ is a sample function of a stationary process whose low-pass equivalent spectrum is $S_{\tilde{s}_l}(\omega)$ and $h(t, \tau)$ is a time-invariant filter with a rational transfer function. Assume that the SPLOT condition is valid.

1. Find an expression for $\mu_{BS,1\infty}(s)$ in terms of $S_{\tilde{s}_l}(\omega)$, $H(j\omega)$, and N_0.
2. Verify that $\mu_{BS,1\infty}(s)$ reduces to $\mu_{OPT,\infty}(s)$ when $H(j\omega)$ is chosen optimally.
3. Plot $\mu_{BS,1\infty}(s)$ for the case in which

$$S_{\tilde{s}_l}(\omega) = \frac{kP_r}{\omega^2 + k^2}$$

and

$$H(j\omega) = \frac{1}{j\omega + \beta}.$$

Find s_m.

Problem 5.1.18 (continuation). Consider the binary communication system discussed in Problems 5.1.16 and 5.1.17. We are interested in the case discussed in part 3 of Problem 5.1.17.

One of the problems in designing the optimum receiver is that P_r may be unknown or may vary slowly. Assume that we think that

$$P_r = P_{rn}$$

and design the optimum receiver.

1. Evaluate $\mu_{BS,1\infty}(s_m)$ and $\mu_{OPT,\infty}(\frac{1}{2})$ for this receiver when

$$\Lambda_1 \triangleq \frac{2P_{rn}}{kN_0} = 100$$

2. Now assume that

$$0.1P_{rn} \le P_r \le 10P_{rn}.$$

Plot $\mu_{BS,1\infty}(s_m)$. The receiver design is fixed.

3. Assume that the receiver is redesigned for each P_r. Compare $\mu_{OPT,\infty}(\frac{1}{2})$ with $\mu_{BS,1\infty}(s_m)$.

Problem 5.1.19. The LEC receiver was derived in Section 4.3 and was shown in Fig. 4.21. This receiver is sometimes used when the LEC condition is not satisfied.

1. Derive an approximate expression for the performance of this receiver.
2. Assume that $s(t)$ has a finite-dimensional state representation. Find a state equation for $\mu_1(s)$ and $\mu_0(s)$.
3. Assume that the SPLOT condition is valid. Find a simple expression for $\mu_1(s)$ and $\mu_0(s)$.

REFERENCES

[1] R. Price, "Optimum Detection of Random Signals in Noise with Applications to Scatter-Multipath Communication. I," IRE Trans. Information Theory **IT-2,** 125–135 (Dec. 1956).

[2] G. Turin, "Communication through Noisy, Random-Multipath Channels," 1956 IRE Convention Record, Pt. 4, pp. 154–156.

[3] D. Middleton, *Introduction to Statistical Communication Theory*, McGraw-Hill, New York, 1960.

[4] R. S. Kennedy, *Fading Dispersive Communication Channels*, Wiley, New York, 1969.

[5] A. J. Viterbi, "Performance of an *M*-ary Orthogonal Communication System Using Stationary Stochastic Signals," IEEE Trans. Information Theory **IT-13,** No. 3, 414–422 (July 1967).

[6] R. G. Gallager, "A Simple Derivation of the Coding Theorem and Some Applications," IEEE Trans. Information Theory **IT-11,** 3–18 (Jan. 1965).

[7] J. M. Wozencraft and I. M. Jacobs, *Principles of Communications Engineering*, Wiley, New York, 1965.

[8] M. Kac and A. J. F. Siegert, "On the Theory of Noise in Radio-Receivers with Square-Law Detectors," J. Appl. Phys. **18,** 383–397 (April 1947).

[9] W. B. Davenport and W. L. Root, *An Introduction to the Theory of Random Signals and Noise*, McGraw-Hill, New York, 1958.

[10] J. G. Proakis and P. R. Drouilhet, "Performance of Coherent Detection Systems Using Decision-Directed Channel Measurement," IEEE Trans., PGCS, 54 (March 1964).

[11] E. M. Glaser, "Signal Detection by Adaptive Filters," IRE Trans. **IT-7,** No. 2, 87–98 (April 1961).

[12] C.V. Jakowitz, R. L. Shuey, and G. M. White, "Adaptive Waveform Recognition," TR 60-RL-2353E, General Electric Research Laboratory, Schenectady, September 1960.

[13] H. J. Scudder, "Adaptive Communication Receivers," IEEE Trans. **IT-11,** No. 2, 167–174 (April 1965).

[14] H. J. Scudder, "The Design of Some Synchronous and Asynchronous Adaptive Receivers," First IEEE Ann. Communication Conv., Boulder, Colo. 759 (June 1965).

[15] D. Boyd, "Several Adaptive Detection Systems," M.Sc. Thesis, Department of Electrical Engineering, Massachusetts Institute of Technology, July 1965.

[16] M. E. Austin, "Decision-Feedback Equalization for Digital Communication over Dispersive Channels," Sc.D. Thesis, Massachusetts Institute of Technology, May 1967.

[17] C. S. Weaver, "Adaptive Communication Filtering," IEEE Trans. Information Theory **IT-8,** S169–S178 (Sept. 1962).

[18] A. M. Breipohl and A. H. Koschmann, "A Communication System with Adaptive Decoding," Third Symposium on Discrete Adaptive Processes, 1964 National Electronics Conference, Chicago, October 19–21, pp. 72–85.

[19] J. G. Lawton, "Investigations of Adaptive Detection Techniques," Cornell Aeronautical Laboratory Report, No. RM-1744-S-2, Buffalo, N.Y., November 1964.

[20] R. W. Lucky, "Techniques for Adaptive Equalization of Digital Communication System," Bell. Syst. Tech. J. 255–286 (Feb. 1966).

[21] D. T. Magill, "Optimal Adaptive Estimation of Sampled Stochastic Processes,"

Stanford Electronics Laboratories, Tech. Rept. 6302-2, Stanford University, December 1963.

[22] K. Steiglitz and J. B. Thomas, "A Class of Adaptive Matched Digital Filters," Third Symposium on Discrete Adaptive Processes, 1964 National Electronics Conference, Chicago, October 19–21, pp. 102–115.

[23] H. L. Groginsky, L. R. Wilson, and D. Middleton, "Adaptive Detection of Statistical Signals in Noise," IEEE Trans. Information Theory **IT-12** (July 1966), pages 337–348.

[24] L. W. Nolte, "An Adaptive Realization of the Optimum Receiver for a Sporadic-Recurrent Waveform in Noise," First IEEE Ann. Communication Conv., Boulder. Colo. 593–598 (June 1965).

[25] R. Esposito, D. Middleton, and J. A. Mullen, "Advantages of Amplitude and Phase Adaptivity in the Detection of Signals Subject to Slow Rayleigh Fading," IEEE Trans, Information Theory **IT-11,** 473–482 (Oct. 1965).

[26] J. C. Hancock and P. A. Wintz, "An Adaptive Receiver Approach to the Time Synchronization Problem," IEEE Trans Commun. Syst. **COM-13,** 90–96 (March 1965).

[27] M. J. DiToro, "A New Method of High-Speed Adaptive Serial Communication through Any Time-Variable and Dispersive Transmission Medium," First IEEE Ann. Communication Conv., Boulder, Colo. 763–767 (June 1965).

[28] L. D. Davisson, "A Theory of Adaptive Filtering," IEEE Trans. Information Theory **IT-12,** 97–102 (April 1966).

[29] H. J. Scudder, "Probability of Error of Some Adaptive Pattern-Recognition Machines," IEEE Trans. Information Theory **IT-11,** 363–371 (July 1965).

[30] Y. C. Ho and R. C. Lee, "Identification of Linear Dynamic Systems," Third Symposium on Discrete Adaptive Processes, 1964 National Electronics Conference, Chicago, October 19–21, pp. 86–101.

[31] B. Reiffen and H. Sherman, "An Optimum Demodulator for Poisson Processes: Photon Sources Detectors," Proc. IEEE **51,** 1316–1320 (Oct. 1963).

[32] K. Abend, "Optimum Photon Detection," IEEE Trans. Information Theory **IT-12,** 64–65 (Jan. 1966).

[33] T. F. Curran and M. Ross, "Optimum Detection Thresholds in Optical Communication," Proc. IEEE **53,** 1770–1771 (Nov. 1965).

[34] C. W. Helstrom, "The Detection and Resolution of Optical Signals," IEEE Trans. Information Theory **IT-10,** 275–287 (Oct. 1964).

[35] I. Bar-David, "Communication under the Poisson Regime," IEEE Trans. Information Theory **IT-15,** No. 1, 31–37 (Jan. 1969).

[36] D. Middleton, "Canonically Optimum Threshold Detection," IEEE Trans. Information Theory **IT-12,** No. 2, 230–243 (April 1966).

[37] D. Middleton, *Topics in Communication Theory*, McGraw-Hill, New York, 1960.

[38] T. Kailath, "A General Likelihood-Ratio Formula for Random Signals in Gaussian Noise," IEEE Trans. Information Theory, **IT-15,** No. 3, 350–361 (May 1969).

[39] S. C. Schwartz, "A Series Technique for the Optimum Detection of Stochastic Signals in Noise," IEEE Trans. Information Theory **IT-15,** No. 3, 362–370 (May 1969).

[40] P. A. Bello, "Bounds on the Error Probability of FSK and PSK Receivers Due to Non-Gaussian Noise in Fading Channels," IEEE Trans. Information Theory **IT-12,** No. 3, 315–326 (July 1966).

[41] P. A. Bello, "Error Probabilities Due to Atmospheric Noise and Flat Fading in HF Ionospheric Communications Systems," IEEE Trans. Commun. Tech., **COM-13,** 266–279 (Sept. 1965).

[42] H. L. Van Trees, *Array Processing*, Wiley, New York, 1971.
[43] T. Kailath, "Correlation Detection of Signals Perturbed by a Random Channel," IRE Trans. Information Theory **IT-6,** 361–366 (June 1960).
[44] P. Bello, "Some Results on the Problem of Discriminating between Two Gaussian Processes," IRE Trans. Information Theory **IT-7,** No. 4, 224–233 (Oct. 1961).
[45] R. C. Davis, "The Detectability of Random Signals in the Presence of Noise," IRE Trans. Information Theory **IT-3**, 52–62, 1957.
[46] D. Slepian, "Some Comments on the Detection of Gaussian Signals in Gaussian Noise, IRE Trans. Information Theory **IT-4,** 65–68, 1958.
[47] T. T. Kadota, "Optimum Reception of Binary Sure and Gaussian Signals," Bell Syst. Tech. J. **44,** No. 8, 1621–1658 (Oct. 1965).
[48] D. Middleton and R. Esposito, "Simultaneous Optimum Detection and Estimation of Signals in Noise," IEEE Trans. Information Theory **IT-14,** No. 3, 434–445 (May 1968).