Detection, Estimation, and Modulation Theory, Part III: Radar–Sonar Signal Processing and Gaussian Signals in Noise. Harry L. Van Trees

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Special Categories of Estimation Problems

As in the detection problem, there are several categories of processes for which we can obtain a reasonably complete solution for the estimator. In this chapter we discuss four categories:

1. Stationary processes, long observation time (7.1).
2. Finite-state processes (7.2).
3. Separable kernel processes (7.3).
4. Low-energy coherence (7.4).

We exploit the similarity to the detection problem whenever possible and use the results from Chapter 4 extensively. For algebraic simplicity, we assume that $m(t, A)$ is zero throughout the chapter.

In Section 7.5, we consider some related topics. In Section 7.6, we summarize the results of our estimation theory discussion.

7.1 STATIONARY PROCESSES: LONG OBSERVATION TIME

The model of interest is

$$r(t) = s(t, A) + w(t), \qquad T_i \le t \le T_f. \tag{1}$$

We assume that $s(t, A)$ is a sample function from a zero-mean, stationary Gaussian random process with covariance function

$$K_s(t, u:A) \triangleq K_s(t - u:A). \tag{2}$$

The additive noise $w(t)$ is a sample function from an independent, zero-mean, white Gaussian process with spectral height $N_0/2$. Thus,

$$K_r(t, u:A) = K_s(t - u:A) + \frac{N_0}{2}\,\delta(t - u). \tag{3}$$

The power density spectrum of $r(t)$ is

$$S_r(\omega:A) = S_s(\omega:A) + \frac{N_0}{2}. \tag{4}$$

In addition, we assume that

$$T \triangleq T_f - T_i \tag{5}$$

is large enough that we can neglect transient effects at the ends of the interval. (Recall the discussion on pages 99–101.)

In this section we discuss the simplifications that result when the SPLOT condition is valid. In Section 7.1.1, we develop some general results and introduce the amplitude estimation problem. In Section 7.1.2, we study the performance of truncated estimators (we define the term at that point). In Section 7.1.3, we discuss suboptimum receivers. Finally, in Section 7.1.4, we summarize our results.

7.1.1 General Results

We want to find simple expressions for $l_R(A)$, $l_B(A)$, the MAP and ML equations, and the lower bound on the variance. Using (6.17), (6.18), (4), and the same procedure as on pages 100–101, we have

$$l_R(A) = \frac{1}{N_0} \iint_{T_i}^{T_f} r(t)h(t-u:A)r(u)\,dt\,du, \tag{6}$$

where $h(\tau:A)$ is a time-invariant filter with the transfer function

$$\boxed{H(j\omega:A) = \frac{S_s(\omega:A)}{S_s(\omega:A) + N_0/2}.} \tag{7}$$

The filter in (7) is unrealizable and corresponds to Canonical Realization No. 1 in the detection theory problem. A simple realization can be obtained by factoring $H(j\omega:A)$.

$$\boxed{H_{fr\infty}(j\omega:A) \triangleq \left[\frac{S_s(\omega:A)}{S_s(\omega:A) + N_0/2}\right]^+.} \tag{8}$$

Then

$$l_R(A) = \frac{1}{N_0} \int_{T_i}^{T_f} dt \left[\int_{T_i}^{t} h_{fr\infty}(t-z:A)r(z)\,dz\right]^2. \tag{9}$$

This is a filter-squarer-integrator realization and is analogous to Canonical Realization No. 3. Notice that $h_{fr\infty}(\tau:A)$ is a realizable filter.

The bias term follows easily from the asymptotic mean-square-error expression. Using (4.16) and (6.25), we obtain

$$l_B(A) = -\frac{T}{2}\int_{-\infty}^{\infty} \ln\left(1 + \frac{2S_s(\omega:A)}{N_0}\right)\frac{d\omega}{2\pi}. \tag{10}$$

From (6.27) we construct

$$l(A) = l_R(A) + l_B(A) \tag{11}$$

and choose the value of A where the maximum occurs. The general receiver structure, using (9) and (10), is shown in Fig. 7.1.

The ML equation is obtained by substituting (6) and (10) into (11), differentiating, and equating the result to zero for $A = \hat{a}_0$. Normally, we refer to the solution of maximum likelihood equation as $\hat{a}_{ml}$. However, the ML equation only provides a necessary condition and we must check to see that the maximum is interior to χ_a and that $\hat{a}_0$ is the absolute maximum. In several examples that we shall consider, the maximum can be at the endpoint of χ_a. Therefore, we must be careful to check the conditions. The solution $A = \hat{a}_0$ can be interpreted as a candidate for $\hat{a}_{ml}$. Carrying out the indicated steps, we obtain

$$\left[-\frac{T}{2}\int_{-\infty}^{\infty}\left(\frac{\frac{2}{N_0}\frac{\partial S_s(\omega:A)}{\partial A}}{1 + \frac{2S_s(\omega:A)}{N_0}}\right)\frac{d\omega}{2\pi} + \frac{1}{N_0}\iint_{T_i}^{T_f} r(t)\,\frac{\partial h(t-u:A)}{\partial A}\,r(u)\,dt\,du\right]_{A=\hat{a}_0} = 0, \tag{12}$$

where

$$\frac{\partial h(\tau:A)}{\partial A} = \int_{-\infty}^{\infty}\left[\frac{(N_0/2)[\partial S_s(\omega:A)/\partial A]}{(S_s(\omega:A) + N_0/2)^2}\right]e^{j\omega\tau}\,\frac{d\omega}{2\pi}. \tag{13}$$

To find the Cramèr-Rao bound, we take the asymptotic version of (6.61).

$$J^{(2)}(A) = \frac{1}{N_0}\iint_{T_i}^{T_f} dt\,du\,\frac{\partial K_r(t-u:A)}{\partial A}\,\frac{\partial h(t-u:A)}{\partial A}$$

$$= \frac{2T}{N_0}\int_0^T\left(1 - \frac{\tau}{T}\right)\frac{\partial K_r(\tau:A)}{\partial A}\,\frac{\partial h(\tau:A)}{\partial A}\,d\tau. \tag{14}$$

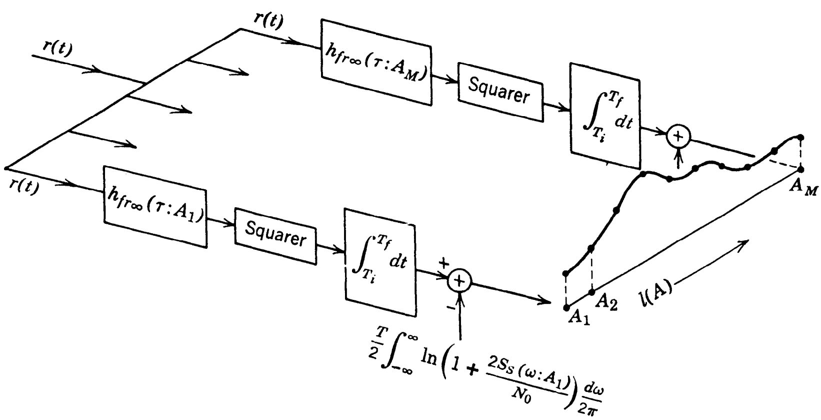

Fig. 7.1 Generation of $l(A)$ for stationary-process, long-observation-time case.

For large T, this can be written in the frequency domain as

$$J^{(2)}(A) = \frac{T}{N_0}\int_{-\infty}^{\infty} \frac{\partial S_r(\omega:A)}{\partial A}\,\frac{\partial H(j\omega:A)}{\partial A}\,\frac{d\omega}{2\pi}. \tag{15}$$

Using (13) in (15), we obtain

$$\boxed{\begin{aligned} J^{(2)}(A) &= \frac{T}{2}\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\left[\frac{[\partial S_s(\omega:A)]/\partial A}{S_s(\omega:A) + N_0/2}\right]^2 \\ &= \frac{T}{2}\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\left[\frac{\partial}{\partial A}\ln\left[1 + \frac{2S_s(\omega:A)}{N_0}\right]\right]^2. \end{aligned}} \tag{16}$$

From (6.44) we have

$$E\{[\hat{a} - A]^2\} \geq [J^{(2)}(A)]^{-1} \tag{17}$$

for any unbiased estimate.

To illustrate these results, we consider a series of amplitude estimation problems. The examples are important because they illustrate the difficulties that arise when we try to solve a particular problem and how we can resolve these difficulties.

Example 1. The first problem is that of estimating the amplitude of the spectrum of a stationary random process corrupted by additive white noise. The signal spectrum is

$$S_s(\omega:A) = AS(\omega), \tag{18}$$

where $S(\omega)$ is known. The parameter A lies in the range $[A_\alpha, A_\beta]$ and is nonrandom. Substituting (18) into (7) gives

$$H(j\omega;A) = \frac{AS(\omega).}{AS(\omega) + N_0/2} \tag{19}$$

From (8) we have

$$H_{fr\infty}(j\omega:A) = \sqrt{A}\left[\frac{S(\omega)}{AS(\omega) + N_0/2}\right]^+, \tag{20}$$

and from (10),

$$l_B(A) = -\frac{T}{2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left[\ln\left(1 + \frac{2AS(\omega)}{N_0}\right)\right]. \tag{21}$$

We construct

$$l(A) = \frac{1}{N_0}\iint_{T_i}^{T_f} r(t)h(t - u:A)r(u)\,dt\,du + l_B(A) \tag{22}$$

as a function of A and choose that value of A where it is a maximum. The resulting receiver is shown in Fig. 7.2.

To obtain the ML equation we substitute (18) into (12). The result is

$$\left\{\frac{1}{N_0}\iint_{T_i}^{T_f} r(t)\frac{\partial h(t - u:A)}{\partial A}r(u)\,dt\,du - \frac{T}{2}\int_{-\infty}^{\infty}\frac{2S(\omega)/N_0}{1 + (2AS(\omega))/N_0}\frac{d\omega}{2\pi}\right\}_{A=\hat{a}_0} = 0, \tag{23}$$

where

$$\frac{\partial H(j\omega:A)}{\partial A} = \frac{N_0S(\omega)/2}{(N_0/2 + AS(\omega))^2}. \tag{24}$$

In general, we cannot solve (23) explicitly, and so we must still implement the receiver using a set of parallel processors. If the resulting estimate is unbiased, its normalized variance is bounded by

$$\boxed{\frac{\text{Var}[\hat{a} - A]}{A^2} \geq \left\{\frac{T}{2}\int_{-\infty}^{\infty}\left(\frac{S(\omega)}{N_0/2A^2 + S(\omega)}\right)^2\right\}^{-1}.} \tag{25}$$

We now examine the results in Example 1 in more detail for various special cases.

Example 2. We assume that $S(\omega)$ is strictly bandlimited:

$$S(\omega) = 0, \qquad |\omega| > 2\pi W. \tag{26}$$

We can always approximate $H_{fr\infty}(j\omega)$ arbitrarily closely and use the receiver in Fig. 7.2, but there are two limiting cases that lead to simpler realizations.

The first limiting case is when the signal spectrum is much larger than $N_0/2$. This case is sometimes referred to the high-*input* signal-to-noise ratio case:

$$AS(\omega) \gg \frac{2}{N_0}, \qquad |\omega| \leq 2\pi W. \tag{27}$$

To exploit this, we expand the terms in (24) and (23) in a power series in $(N_0/2AS(\omega)$ to obtain

$$\frac{N_0S(\omega)/2}{(N_0/2 + AS(\omega)^2} = \frac{N_0}{2A^2S(\omega)}\left\{1 - 2\left(\frac{N_0}{2AS(\omega)}\right) + \cdots\right\} \tag{28}$$

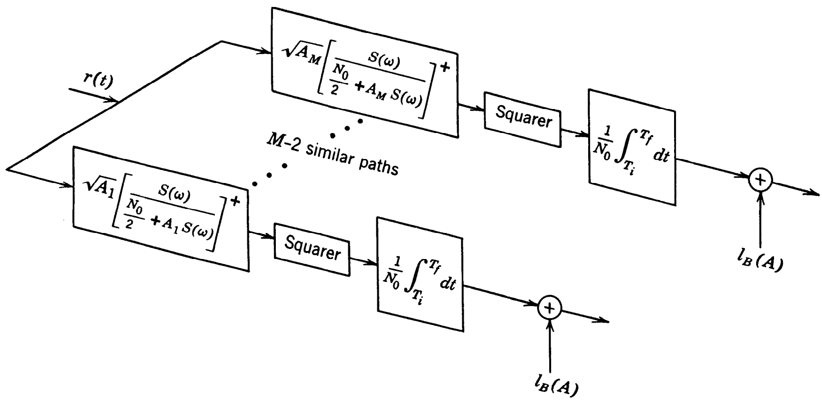

Fig. 7.2 Generation of ln $\Lambda(A)$: Canonical Realization No. 3.

and

$$-\frac{T}{2}\int_{-\infty}^{\infty}\frac{2S(\omega)/N_0}{1+[2AS(\omega)/N_0]}\frac{d\omega}{2\pi}=-\frac{T}{2A}\int_{-2\pi W}^{2\pi W}\left(1-\frac{N_0}{2AS(\omega)}+\cdots\right)\frac{d\omega}{2\pi}$$

$$=-\frac{WT}{A}+\frac{TN_0}{4A^2}\int_{-2\pi W}^{2\pi W}\frac{1}{2S(\omega)}\frac{d\omega}{2\pi}+\cdots. \tag{29}$$

Using (28) and (29) in (22) and neglecting powers of $1/S(\omega)$ greater than 1, we obtain

$$\hat{a}_0=\frac{1}{2WT}\left\{\left[\iint_{T_i}^{T_f} r(t)h_\alpha(t-u)r(u)\,dt\,du\right]-\frac{TN_0}{2}\int_{-2\pi W}^{2\pi W}\frac{1}{S(\omega)}\frac{d\omega}{2\pi}\right\}, \tag{30}$$

where

$$H_\alpha(j\omega)=\begin{cases}\dfrac{1}{S(\omega)}, & |\omega|\le 2\pi W,\\ 0, & \text{elsewhere.}\end{cases} \tag{31}$$

Now we see why we were careful to denote the estimate in (30) as $\hat{a}_0$ rather than $\hat{a}_{ml}$. The reason follows by looking at the right side of (30). The first term is a random variable whose value is always non-negative. The second term is a negative bias. Therefore $\hat{a}_0$ can assume negative values. Because the parameter A is the spectral amplitude, it must be non-negative. This means that the lower limit on the range of A, which we denoted by A_α, must be non-negative. For algebraic simplicity we assume that $A_\alpha = 0$. Therefore, whenever $\hat{a}_0$ is negative, we choose zero for the maximum likelihood estimate,

$$\hat{a}_{ml}=\begin{cases}\hat{a}_0, & \text{if } \hat{a}_0\ge 0,\\ 0, & \text{if } \hat{a}_0<0.\end{cases} \tag{32}$$

Notice that this result is consistent with our original discussion of the ML equation on page I-65. We re-emphasize that the ML equation provides a necessary condition on $\hat{a}_{ml}$ only when the maximum of the likelihood function is interior to the range of A. We shall find that in most cases of interest the probability that $\hat{a}_0$ will be negative is small. In the next section we discuss this issue quantitatively.

It is easy to verify that $\hat{a}_0$ is unbiased.

$$E[\hat{a}_0] = \frac{1}{2WT}\left\{E\left[\iint_{T_i}^{T_f} r(t)h_a(t-u)r(u)\,dt\,du\right] - \frac{TN_0}{2}\int_{-2\pi W}^{2\pi W}\frac{1}{S(\omega)}\frac{d\omega}{2\pi}\right\}$$

$$= \frac{1}{2WT}\left\{T\int_{-2\pi W}^{2\pi W} H_a(j\omega)S_r(\omega)\frac{d\omega}{2\pi} - \frac{TN_0}{2}\int_{-2\pi W}^{2\pi W}\frac{1}{S(\omega)}\frac{d\omega}{2\pi}\right\}$$

$$= \frac{1}{2WT}\left\{T\int_{-2\pi W}^{2\pi W}\frac{AS(\omega)+[N_0/2]}{S(\omega)}\frac{d\omega}{2\pi} - \frac{TN_0}{2}\int_{-2\pi W}^{2\pi W}\frac{1}{S(\omega)}\frac{d\omega}{2\pi}\right\} = A. \tag{33}$$

Looking at (32), we see that (33) implies that $\hat{a}_{ml}$ is biased. This means that we cannot use the Cramèr-Rao bound in (17). Moreover, since it is difficult to find the bias as a function of A, we cannot modify the bound in an obvious manner (i.e., we cannot use the results of Problem 6.3.1). Since this issue arises in many problems, we digress and develop a technique for analyzing it.

7.1.2 Performance of Truncated Estimates

The reason $\hat{a}_{ml}$ is biased is that we have truncated $\hat{a}_0$ at zero. We study the effect of this truncation for the receiver shown in Fig. 7.3. This receiver is a generalization of the receiver in Example 2. We use the notation

$$\hat{a}_0 = G\{l - B\}, \tag{34}$$

where

$$l = \int_{T_i}^{T_f} r(t)h_\alpha(t-u)r(u)\,dt\,du. \tag{35}$$

Equivalently,

$$l = \int_{T_i}^{T_f} dt\left[\int_{T_i}^{T_f} h_\alpha^{[1/2]}(t-z)r(z)\,dz\right]^2, \tag{36}$$

where $h_\alpha^{[1/2]}(t-z)$ is the functional square root of $h_\alpha(t-u)$. The constants G and B denote "gain" and "bias," respectively.

In Example 2,

$$G = \frac{1}{2WT} \tag{37}$$

and

$$B = -\frac{TN_0}{2}\int_{-2\pi W}^{2\pi W}\frac{1}{S(\omega)}\frac{d\omega}{2\pi}. \tag{38}$$

In our initial discussion, we leave G and B as parameters. Later, we consider the specific values in (37) and (38). Notice that $\hat{a}_0$ will satisfy (30) only when the values in (37) and (38) are used. We shall assume that

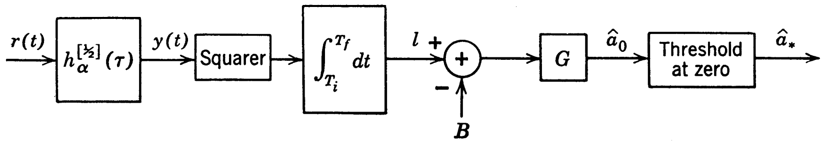

Fig. 7.3 Filter-squarer-integrator receiver to generate $\hat{a}_*$.

G and B are adjusted so that $\hat{a}_0$ is an *unbiased* estimate of A. We denote the truncated output as $\hat{a}_*$. In Example 2, $\hat{a}_*$ equals $\hat{a}_{ml}$, but for an arbitrary $h_\alpha(\tau)$ they will be different. Notice that we can compute the variance of $\hat{a}_0$ exactly (see Problem 7.1.1) for any $h_\alpha(\tau)$.

A typical probability density for l is shown in Fig. 7.4. Notice that A is a nonrandom parameter and $p_{l|a}(L \mid A)$ is our usual notation. We have shaded the region of L where $\hat{a}_0$ will be truncated. If the probability that l will lie in the shaded region is small, $\hat{a}_*$ will have a small bias. We would anticipate that if the probability of l being in the shaded region is large, the mean-square estimation error will be large enough to make the estimation procedure unsatisfactory. We now put these statements on a quantitative basis. Specifically, we compute three quantities:

1. An upper bound on $\Pr[l < B]$.
2. An upper bound on the bias of $\hat{a}_*$.
3. A lower bound on $E[(\hat{a}_* - A)^2]$.

The general expressions that we shall derive are valid for an arbitrary receiver of the form shown in Fig. 7.4 with the restriction

$$E[l] \geq B. \tag{39}$$

The general form of the results, as well as the specific answers, is important.

Upper Bound on $\boldsymbol{Pr[l < B]}$. Looking at Fig. 7.4, we see that the problem of interest is similar to the computation of P_M for a suboptimum receiver

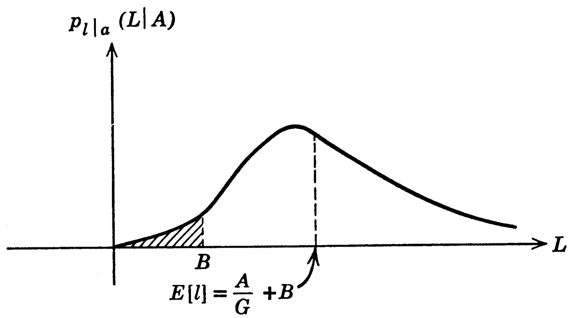

Fig. 7.4 Typical probability density for l.

that we solved in Section 5.1.2.† Using the asymptotic form of the Fredholm determinant in (5.55), we have

$$\mu(s, A) = -\frac{T}{2}\int_{-\infty}^{\infty} \ln\left(1 - 2sS_y(\omega{:}A)\right)\frac{d\omega}{2\pi}, \qquad s \leq 0. \tag{40}$$

The waveform $y(t)$ is the input to the squarer, and $S_y(\omega)$ is its power density spectrum. The Chernoff bound is

$$\Pr[l < B] \leq \exp[\mu(s_1, A) - s_1 B], \tag{41}$$

where s_1 is chosen so that

$$\left.\frac{\partial\mu(s, A)}{\partial s}\right|_{s=s_1} \triangleq \left.\dot{\mu}(s, A)\right|_{s=s_1} = B. \tag{42}$$

[Recall the result in (I-2.461) and notice the change of sign in s.] Since

$$B \leq E[l], \tag{43}$$

(42) will have a unique solution. Notice that this result is valid for any FSI receiver subject to the constraint on the bias in (43) [i.e., different $h_\alpha^{[1/2]}(\tau)$ could be used]. To illustrate the calculation, we consider a special case of Example 2.

Example 2 (continuation). We consider the receiver in Example 2. In (26) we assumed that the signal spectrum, $S(\omega)$, was strictly bandlimited to W cycles per second. We now consider the special case in which $S(\omega)$ is constant over this frequency range. Thus,

$$S(\omega) = \begin{cases} \dfrac{1}{2W}, & |\omega| \leq 2\pi W, \\ 0, & \text{elsewhere.} \end{cases} \tag{44}$$

Then

$$S_y(\omega{:}A) = \begin{cases} (2W)\left[A\dfrac{1}{2W} + \dfrac{N_0}{2}\right] = A\left[1 + \dfrac{N_0 W}{A}\right], & |\omega| < 2\pi W, \\ 0, & \text{elsewhere,} \end{cases} \tag{45}$$

and

$$B = (2WT)(N_0 W). \tag{46}$$

Using (45) in (40) gives

$$\mu(s, A) = -WT\ln\left[1 - 2sA\left(1 + \frac{N_0 W}{A}\right)\right]. \tag{47}$$

To find s_1, we differentiate (47) and equate the result to B. The result is

$$s_1 = -\left[2\left(1 + \frac{N_0 W}{A}\right)N_0 W\right]^{-1}. \tag{48}$$

† We suggest that the reader review Section 5.1.2 and Problem 5.1.13 before reading this discussion.

Substituting (48) into (47) and (41) gives

$$\Pr[l < B] \leq \left(1 + \frac{A}{N_0 W}\right)^{-WT} \exp\left[\frac{WT}{(1 + N_0 W/A)}\right]. \tag{49}$$

We see that the bound depends on A/N_0W, which is the signal-to-noise ratio in the message bandwidth, and on WT, which is one-half the time-bandwidth product of the signal. We have plotted the bound in Fig. 7.5 as a function of WT for various values of A/N_0W. In most cases of interest, the probability that $\hat{a}_0$ will be negative is negligible. For example, if

$$\frac{A}{N_0 W} = 10 \tag{50}$$

and

$$WT = 5, \tag{51}$$

the probability is less than 10^{-3} that $\hat{a}_0$ will be negative.

We used the Chernoff bound in our discussion. If it is desired, one can obtain a better approximation to the probability by using the approximate formula

$$\Pr[l < B] \simeq \frac{1}{\sqrt{2\pi s_1^2 \ddot{\mu}(s_1, A)}} \exp[\mu(s_1, A) - s_1 B] \tag{52}$$

(see Problem 5.1.13). In most cases, this additional refinement is not necessary. The next step is to bound the bias of $\hat{a}_*$.

Upper Bound on Bias of $\hat{a}_*$. We can compute a bound on the bias of $\hat{a}_*$ by using techniques similar to those used to derive (41). Recall that $\hat{a}_0$ is an *unbiased* estimate of A and can be written as

$$\hat{a}_0 = Gl - GB. \tag{53}$$

Therefore,

$$E[\hat{a}_0] = \int_{-\infty}^{\infty} G(L - B) p_{l|a}(L|A)\, dL = A. \tag{54}$$

Dividing the integration region into two intervals, we have

$$\int_0^B G(L - B) p_{l|a}(L|A)\, dL + \int_B^{\infty} G(L - B) p_{l|a}(L \mid A)\, dL = A. \tag{55}$$

The second integral is $E[\hat{a}_*]$. Thus, the bias of $\hat{a}_*$ is

$$\mathscr{B}(\hat{a}_*) \triangleq E[\hat{a}_*] - A = -\int_0^B G(L - B) p_{l|a}(L \mid A)\, dL. \tag{56}$$

The next step is to bound the term on the right side of (56). We develop two bounds. The first is quite simple and is adequate for most cases. The second requires a little more computation, but gives a better result.

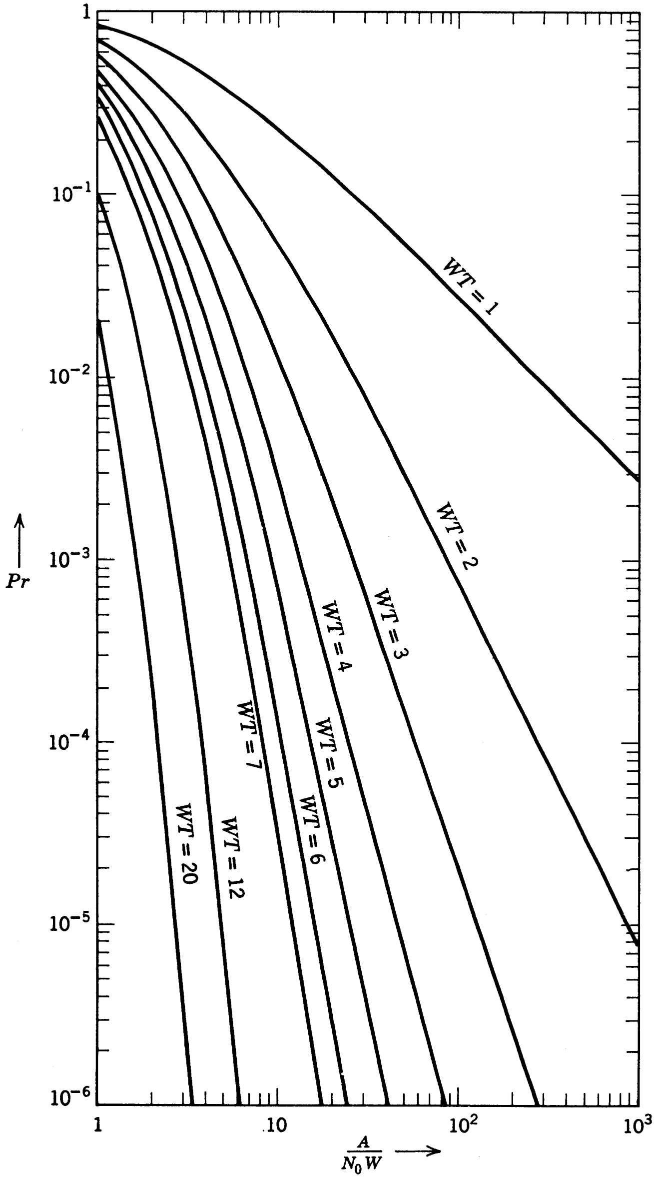

Fig. 7.5 Bound on probability that $\hat{a}_0$ will be negative.

Bound No. 1. The first bound is

$$\begin{aligned}\mathscr{B}(\hat{a}_*) &= -\int_0^B G(L - B)p_{l|a}(L \mid A)\,dL\\ &\le BG\int_0^B p_{l|a}(L \mid A)\,dL\\ &= BG \Pr[l \le B]\\ &\le BG\exp[\mu(s_1, A) - s_1 B],\end{aligned} \tag{57}$$

where s_1 satisfies (42). The result in (57) can be normalized to give

$$\frac{\mathscr{B}(\hat{a}_*)}{A} \le \frac{BG}{A}\exp[\mu(s_1, A) - s_1 B]. \tag{58}$$

For the spectrum in (44), this reduces to

$$\frac{\mathscr{B}(\hat{a}_*)}{A} \le \left(\frac{N_0 W}{A}\right)\left(1 + \frac{A}{N_0 W}\right)^{-WT}\exp\left[\frac{WT}{(1 + N_0 W)}\right]. \tag{59}$$

After deriving the second bound, we shall plot the result in (59). Notice that (59) can be plotted directly from Fig. 7.5 by multiplying each value by N_0W/A.

Bound No. 2 [2]. We can obtain a better bound by tilting the density. We define

$$p_{l_s}(L) = \exp[sL - \mu(s, A)]\,p_{l|a}(L|A), \qquad s \le 0. \tag{60}$$

[Recall (I-2.450).] Using (60) in (57) gives

$$\begin{aligned}\mathscr{B}(\hat{a}_*) &= G\int_0^B (B - L)\exp[\mu(s, A) - sL]\,p_{l_s}(L)\,dL\\ &= G\exp[\mu(s, A) - sB]\int_0^B (B - L)\exp[s(B - L)]p_{l_s}(L)\,dL\\ &\le G\exp[\mu(s, A) - sB]\{\max_{0\le L\le B}[(B - L)\exp[s(B - L)]\}\\ &\quad\times\int_0^B p_{l_s}(L)\,dL \qquad s \le 0\end{aligned} \tag{61}$$

We now upperbound the integral by unity and let

$$Z = B - L \tag{62}$$

in the term in the braces. Thus,

$$\mathscr{B}(\hat{a}_*) \le G\exp[\mu(s, A) - sB]\{\max_{0\le Z\le B}[Ze^{sZ}]\}, \qquad s \le 0. \tag{63}$$

The term in the braces is maximized by

$$Z = Z_m \triangleq \min\left[B, -\frac{1}{s}\right], \qquad s \le 0. \tag{64}$$

Using (64) in (63) gives

$$\mathscr{B}(\hat{a}_*) \le GZ_m\exp[\mu(s, A) - s(B - Z_m)], \qquad s \le 0. \tag{65}$$

We now minimize the bound as a function of s. The minimum is specified by

$$\left(\dot{\mu}(s, A) - \frac{1}{s}\right)\Bigg|_{s=s_2} = B, \qquad -\frac{1}{B} \leq s_2 \leq 0. \tag{66}$$

We can demonstrate that (66) has a solution in the allowable range (see Problem 7.1.2). Thus,

$$\frac{\mathscr{B}(\hat{a}_*)}{A} \leq \frac{-G}{As_2} \exp\{\mu(s_2, A) - s_2 B - 1\}. \tag{67}$$

For the spectrum in (44),

$$As_2 = -\frac{C_2}{2C_1}\left[1 + \sqrt{1 + \frac{4C_1}{C_2^2}}\right], \tag{68}$$

where

$$C_1 \triangleq 4WT\frac{N_0 W}{A}\left(1 + \frac{N_0 W}{A}\right) \tag{69a}$$

and

$$C_2 \triangleq 2(1 + WT)\left(1 + \frac{N_0 W}{A}\right) - 2WT\frac{N_0 W}{A} \tag{69b}$$

(see Problem 7.1.3). In Fig. 7.6, we plot the bounds given by (59) and (67) for the case in which $WT = 5$. We see that the second bound is about an order of magnitude better than the first bound in this case, and that the bias is negligible. Similar results can be obtained for other WT products. From Fig. 7.5, we see that the bias is negligible in most cases of interest. Just as on page 198, we can obtain a better approximation to the bias by using a formula similar to (52) (see Problem 7.1.4). The next step is computing a bound on the mean-square error.

Mean-square-error Bound. The mean-square error using $\hat{a}_*$ is

$$\begin{aligned}\xi_* &\triangleq E[(\hat{a}_* - A)^2] \\ &= E[(\hat{a}_* - \hat{a}_0 + \hat{a}_0 - A)^2] \\ &= E[\hat{a}_* - \hat{a}_0)^2] + 2E[(\hat{a}_* - \hat{a}_0)(\hat{a}_0 - A)] + E[(\hat{a}_0 - A)^2].\end{aligned} \tag{70}$$

Observe that

$$\hat{a}_* = \hat{a}_0 \quad \text{for} \quad \hat{a}_0 \geq 0 \tag{71}$$

and

$$\hat{a}_* = 0 \quad \text{for} \quad \hat{a}_0 < 0. \tag{72}$$

Thus, (70) can be written as

$$\begin{aligned}\xi_* &= \int_0^B (\hat{a}_0(L))^2 p_{l|a}(L|A)\, dL + 2\int_0^B \hat{a}_0(L)[A - \hat{a}_0(L)]p_{l|a}(L|A)\, dL + \xi_{\hat{a}_0} \\ &= \int_0^B [2\hat{a}_0(L)A - (\hat{a}_0(L))^2]p_{l|a}(L|A)\, dL + \xi_{\hat{a}_0}.\end{aligned} \tag{73}$$

Recalling that

$$\hat{a}_0(L) = G(L - B), \tag{74}$$

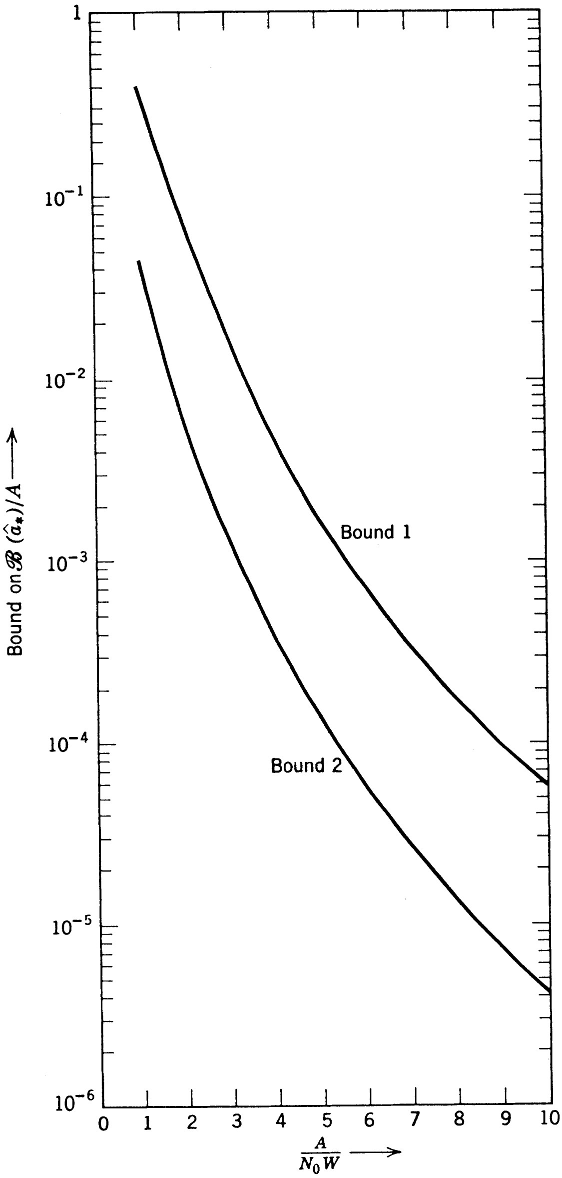

Fig. 7.6 Comparison of bounds on normalized bias [$WT = 5$].

we have

$$\xi_* = -\left\{\int_0^B [2AG(B-L) + G^2(B-L)^2] p_{l|a}(L|A)\, dL\right\} + \xi_{\hat{a}_0} \quad (75)$$

The term in the braces is always positive. Thus,

$$\xi_{\hat{a}_0} - \Delta\xi \leq \xi_* \leq \xi_{\hat{a}_0}, \quad (76)$$

where

$$\Delta\xi \triangleq \int_0^B [2AG(B-L) + G^2(B-L)^2] p_{l|a}(L|A)\, dL. \quad (77)$$

We can now proceed as on pages 199–201 to find a bound on $\Delta\xi$. A simple bound is

$$\Delta\xi \leq [2AGB + G^2B^2] \Pr(l < B). \quad (78)$$

A tighter bound can be found in the same manner in which (67) was derived (see Problem 7.1.5). In many cases, the simple bound in (78) is adequate. For example, for the numerical values in (50) and (51),

$$\frac{\Delta\xi}{A^2} \leq 1.25 \times 10^{-4}. \quad (79)$$

Notice that we must also find $\xi_{\hat{a}_0}$. As we indicated on page 195, we can always find $\xi_{\hat{a}_0}$ exactly (see Problem 7.1.1.).

We have developed techniques for evaluating the bias and mean-square error of a truncated estimate. A detailed discussion was included because both the results and techniques are important. In most of our subsequent discussion, we shall neglect the truncation effect and assume that the performances of $\hat{a}_*$ and $\hat{a}_0$ are essentially the same. The results of this section enable us to check that assumption in any particular problem. We now return to Example 2 and complete our discussion.

Example 2 (continuation). If we assume that the bias is negligible, the performance $\hat{a}_{ml}$ can be approximated by the performance of $\hat{a}_0$. The Cramèr-Rao bound on the variance of $\hat{a}_0$ is given by (18). When $2AS(\omega)/N_0$ is large, the bound is independent of the spectrum

$$\frac{\text{Var}[\hat{a}_0 - A]}{A^2} \geq \frac{2}{2WT}. \quad (80)$$

We can show that the actual variance approaches this bound as $N_0/2AS(\omega) \to 0$ Two observations are useful:

1. In the small-noise case, the *normalized* variance is independent of the *spectral height* and *spectral shape*. This is analogous to the classical estimation problem of Chapter I-2. There, we estimated the variance of a Gaussian random variable x. We saw that the *normalized* variance of the estimate was independent of the actual variance σ_x^2.

2. We recall from our discussion in Chapter I-3 that if a stationary process is bandlimited to W and observed over an interval of length T, there are $N = 2WT$ significant eigenvalues. Thus, (80) can be written as

$$\frac{\mathrm{Var}\,[\hat{a}_0 - A]}{A^2} \geq \frac{2}{N}, \tag{81}$$

which is identical with the corresponding classical estimation results.

We have used Example 2 as a vehicle for studying the performance of a biased ML estimate in detail. There were two reasons for this detailed study.

1. We encounter biased estimates frequently in estimating random process parameters. It is necessary to have a quantitative estimate of the effect of this bias. Fortunately, the effect is negligible in many problems of interest. Our bounds enable us to determine when we can neglect the bias and when we must include it.

2. The basic analytic techniques used in the study are useful in other estimation problems. Notice that we used the Chernoff bound in (41). If the probability is non-negligible and we want a better estimate of its exact value, we can use the approximate expression in (52).

We now consider another special case of the amplitude estimation problem.

Example 3. Low-input Signal-to-Noise Ratio. The other limiting case corresponds to a low-input signal-to-noise ratio. This case is analogous to the LEC case that we encountered in detection problems (see page 131). Assuming that

$$AS(\omega) \ll \frac{N_0}{2}, \tag{82}$$

then (22) can be expanded in a series. Carrying out the expansion and retaining the first term gives

$$\hat{a}_0 \simeq \left[T\int_{-\infty}^{\infty} S^2(\omega)\,\frac{d\omega}{2\pi}\right]^{-1}\left\{\iint_{T_i}^{T_f} r(t)h_\beta(t-u)r(u)\,dt\,du - \frac{N_0 T}{2}\int_{-\infty}^{\infty} S(\omega)\,\frac{d\omega}{2\pi}\right\}, \tag{83}$$

where

$$H_\beta(j\omega) = S(\omega). \tag{84}$$

Notice that it is not necessary to assume that $S(\omega)$ is bandlimited. The approximate ML estimate is

$$\hat{a}_{ml} = \begin{cases} \hat{a}_0, & \hat{a}_0 \geq 0, \\ 0, & \hat{a}_0 < 0. \end{cases} \tag{85}$$

All of the general derivations concerning bias and mean-square error are valid for this case.

For the flat bandlimited spectrum in (44), the probability that $\hat{a}_0$ will be negative is bounded by

$$\Pr[\hat{a}_0 < 0] \le \exp\left[-WT\left(\frac{A}{WN_0}\right)\right]. \tag{86}$$

The restriction in (82) implies that

$$\frac{A}{WN_0} \ll 1. \tag{87}$$

Thus, WT must be very large in order for the probability in (86) to be negligible.

The Cramèr-Rao bound on the variance of any unbiased estimate is

$$\frac{\text{Var}[\hat{a} - A]}{A^2} \ge \frac{(N_0/2)^2}{(A^2T/2)\int_{-2\pi W}^{2\pi W} S^2(\omega)(d\omega/2\pi)}. \tag{88}$$

For the flat bandlimited spectrum, (88) reduces to

$$\frac{\text{Var}[\hat{a} - A]}{A^2} \ge \left(\frac{WN_0}{A}\right)^2 \frac{1}{WT}. \tag{89}$$

We see that WT must be large in order for this bound to be small. When this is true, we can show that the variance of $\hat{a}_0$ approaches this bound. Under these conditions the probability in (86) is negligible, so that $\hat{a}_{ml}$ equals $\hat{a}_0$ on almost all realizations of the experiment. In many cases, the probability in (86) will not be negligible, and so we use the results in (34)–(78) to evaluate the performance. This analysis is carried out in Problem 7.1.6.

In these two limiting cases of high and low signal-to-noise ratio that we studied in Examples 2 and 3, the receiver assumed the simple form shown in Fig. 7.7. In the high signal-to-noise ratio case,

$$H_C(j\omega) = \frac{1}{S(\omega)} \quad \text{[an inverse filter]}, \tag{90}$$

with

$$B = \frac{TN_0}{2}\int_{-2\pi W}^{2\pi W} \frac{1}{S(\omega)}\frac{d\omega}{2\pi} \tag{91}$$

and

$$G = \frac{1}{2WT}. \tag{92}$$

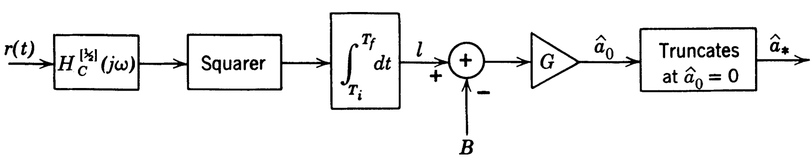

Fig. 7.7 Amplitude estimator.

In the low signal-to-noise ratio case,

$$H_C(j\omega) = S(\omega), \tag{93}$$

with

$$B = \frac{TN_0}{2}\int_{-2\pi W}^{2\pi W} S(\omega)\,\frac{d\omega}{2\pi} \tag{94}$$

and

$$G = \left[T\int_{-\infty}^{\infty} S^2(\omega)\,\frac{d\omega}{2\pi}\right]^{-1} \tag{95}$$

A receiver of the form in Fig. 7.7 is commonly referred to as a radiometer in the radio astronomy field [3]. It is, of course, a form of filter-squarer-integrator receiver that we have seen previously in this chapter.

The obvious advantage of the structure in Fig. 7.7 is that it generates the estimate by passing $r(t)$ through a single processing sequence. By contrast, in the general case we had to build M processors, as shown in Fig. 7.1. In view of the simplicity of the filter-squarer-integrator receiver, we consider briefly a suboptimum receiver that uses the structure in Fig. 7.7 but allows us to choose $H_C(j\omega)$, B, and G.

7.1.3 Suboptimum Receivers

The receiver of interest is shown in Fig. 7.7. Looking at (19), we see that a logical parametric form of $H_C(j\omega)$ is

$$H_C(j\omega) = \frac{S(\omega)}{(N_0/2 + CS(\omega))^2}, \tag{96}$$

where C is a constant that we shall choose. Observe that we achieve the two limiting cases of high and low signal-to-noise ratio by letting C equal infinity and zero, respectively.

We choose B and G so that $\hat{a}_0$ will be an unbiased estimate for all values of A. This requires

$$B = \int_{-\infty}^{\infty} \frac{N_0 S(\omega)/2}{(N_0/2 + CS(\omega))^2}\,\frac{d\omega}{2\pi} \tag{97}$$

and

$$G = \left\{T\int_{-\infty}^{\infty} \frac{S^2(\omega)}{(N_0/2 + CS(\omega))^2}\,\frac{d\omega}{2\pi}\right\}^{-1}. \tag{98}$$

The only remaining parameter is C. In order to choose C for the general case, we first compute the mean-square error. This is a straightforward calculation (e.g., Problem 7.1.7) whose result is

$$\xi_{\hat{a}_0} \triangleq E[\hat{a}_0 - A)^2] = \frac{GT}{2}\int_{-\infty}^{\infty} \frac{S^2(\omega)[AS(\omega) + N_0/2]^2}{(N_0/2 + CS(\omega))^4}\,\frac{d\omega}{2\pi}. \tag{99}$$

If we plot (99) as a function of A, we find that it is a minimum when $C = A$. This result is exactly what we would expect but does not tell us how to choose C, because A is the unknown parameter. Frequently, we know the range of A. If we know that

$$A_\alpha \leq A \leq A_\beta. \tag{100}$$

then we choose a value of C in $[A_\alpha, A_\beta]$ according to some criterion. Some possible criteria will be discussed in the context of an example.

Notice that we must still investigate the bias of $\hat{a}_*$ using the techniques developed above. In the regions of the most interest (i.e., good performance) the bias is negligible and we may assume $\hat{a}_* = \hat{a}_0$ on almost all experiments.

In the next example we investigate the performance of our suboptimum receiver for a particular message spectrum.

Example 4. In several previous examples we used an ideal bandlimited spectrum. For that spectrum, the $H_C(j\omega)$ as specified in (96) is always an ideal low-pass filter. In this example we let

$$S(\omega:A) = \frac{2kA}{\omega^2 + k^2}, \qquad -\infty < \omega < \infty. \tag{101}$$

Now $H_C(j\omega)$ will change as C changes. The lower bound on the variance of any unbiased estimate is obtained by using (101) in (25). The result is (see Problem 7.1.8)

$$\frac{\text{Var}\,[\hat{a} - A]}{A^2} \geq \frac{8}{kT} \frac{(1 + \Lambda(A))^{3/2}}{\Lambda^2(A)}, \tag{102}$$

where $\Lambda(A)$ is the signal-to-noise ratio in the message bandwidth

$$\Lambda(A) = \frac{4A}{kN_0}. \tag{103}$$

We use the suboptimum receiver shown in Fig. 7.7. The normalized variance of $\hat{a}_0$ is obtained by substituting (101) into (99). The result is

$$\frac{\text{Var}\,[\hat{a}_0 - A]}{A^2} = \frac{1}{(kT)(\Lambda^2(A))} [1 + c_1(A)\Lambda(A)]^{3/2}[5c_2^{\,2}(A) + 2c_2^{\,2}(A) + 1], \tag{104}$$

where

$$c_1(A) = \frac{C}{A} \tag{105}$$

and

$$c_2(A) = \frac{1 + \Lambda(A)}{1 + c_1(A)\Lambda(A)}. \tag{106}$$

When $c_1(A)$ equals unity, the variance in (104) reduces to that in (102). In Fig. 7.8, we have plotted the variance using the suboptimum estimator for the case in which $kT = 100$. The value at $c_1(A) = 1$ is the lower bound on any unbiased estimate. For these parameter values we see that we could use the suboptimum receiver over a decade range

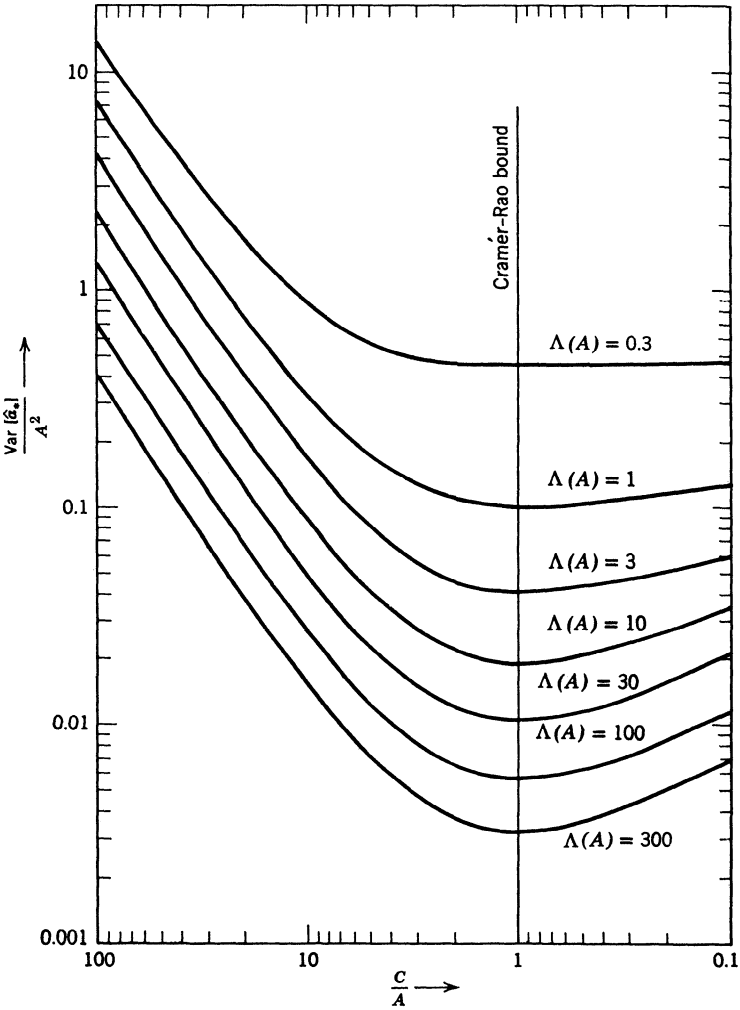

Fig. 7.8 Performance of suboptimum receiver (from [2]).

($A_\beta = 10A_\alpha$) at the cost of about a 50% increase in the error at the endpoints. Two observations are appropriate:

1. We must compute the bias of $\hat{a}_*$. If it is negligible, the mean-square error given in (102) can be used. If not, we evaluate (41) and (78). (See Problem 7.1.10.)
2. If the increase in variance because of the suboptimum receiver is too large, there are several possibilities. The first is to return the original receiver in Fig. 7.1. The second is to use several suboptimum receivers in parallel to cover the parameter range. The third is to estimate A sequentially. We record $r(t)$ and process it once, using a suboptimum receiver to obtain an estimate that we denote as $\hat{a}_1$. We then let $C = \hat{a}_1$ in the

suboptimum receiver and reprocess $r(t)$ to obtain an estimate $\hat{a}_2$. Repeating the procedure will lead us to the desired estimate. The difficulty with the third procedure is proving when it converges to the correct estimate.

This completes our discussion of suboptimum amplitude estimators. We now summarize the results of our discussion.

7.1.4 Summary

In this section we have studied the stationary-process, long-observation-time case in detail. We chose this case for our detailed study because it occurs frequently in practice. When the SPLOT condition is valid, the expressions needed to generate $\ln \Lambda(A)$ and evaluate the Cramèr-Rao bound can be found easily.

Our discussion concentrated on the amplitude estimation problem because it illustrated a number of important issues. Other parameter estimation problems are discussed in Section 7.7.

The procedure in each case is similar:

1. The ML estimate is the value of A that maximizes (11). In the general case, one must generate this expression as a function of A and find the absolute maximum. The utility of this estimation procedure rests on being able to find a practical method of generating this function (or a good approximation).
2. In some special cases (this usually will correspond physically to either a low or high input signal-to-noise ratio), approximations can be made that lead to a unique solution for $\hat{a}_{ml}$.
3. If the estimate is unbiased, one can find a lower bound on the variance of the estimate using (16). Usually the variance of $\hat{a}_{ml}$ approaches this bound when the error is small. If the estimate is biased, we must modify our results to include the effect of the bias. If the bias can be found as a function of A, then the appropriate bound follows. Usually $\mathscr{B}(A)$ cannot be found exactly, and we use approximation techniques similar to those developed in Section 7.1.2.

The procedure is easy to outline, but the amount of work required to carry it out will depend on how the parameter is imbedded in the process.

In the next sections we discuss three other categories of estimation problems. Many of the issues that we have encountered here arise in the cases that we shall discuss in the next three sections. Because we have treated them carefully here, we leave many details to the reader in the ensuing discussions.

7.2 FINITE-STATE PROCESSES

In our discussion up to this point we have estimated a parameter of a random process, $s(t, A)$. The statistics of the process depended on A through the mean-value function $m(t, A)$ and the covariance function $K_s(t, u{:}A)$. We assume that $m(t, A)$ is zero for simplicity. Instead of characterizing $s(t, A)$ by its covariance function, we can characterize it by the differential equations

$$\dot{\mathbf{x}}(t, A) = \mathbf{F}(t, A)\mathbf{x}(t, A) + \mathbf{G}(t, A)\mathbf{u}(t), \qquad t \geq T_i \tag{107}$$

and

$$s(t, A) = \mathbf{C}(t, A)\mathbf{x}(t, A), \qquad t \geq T_i, \tag{108}$$

where

$$E[\mathbf{u}(t)\mathbf{u}^T(\tau)] = \mathbf{Q}(A)\,\delta(t - \tau), \tag{109}$$

$$E[\mathbf{x}(T_i)] = \mathbf{0}, \tag{110}$$

and

$$E[\mathbf{x}(T_i)\mathbf{x}^T(T_i)] = \mathbf{P}_0(A). \tag{111}$$

Two observations are useful:

1. The parameter A may appear in $\mathbf{F}(t, A)$, $\mathbf{G}(t, A)$, $\mathbf{C}(t, A)$, $\mathbf{Q}(A)$, and $\mathbf{P}_0(A)$ in the general case. Notice that there is only a single parameter. In most problems of interest only one or two of these functions will depend on A.
2. In the model of Section 6.1 we assumed that $s(t, A)$ was conditionally Gaussian. Thus the *linear* state equation in (107) is completely general if $s(t, A)$ is state-representable. By using the techniques described in Chapter II-7, we could study parameter estimation for Markovian non-Gaussian processes, but this is beyond the scope of our discussion.

For the zero-mean case the likelihood function is

$$l(A) = l_R(A) + l_B(A). \tag{112}$$

From (6.32),

$$\boxed{l_R(A) = \frac{2}{N_0}\int_{T_i}^{T_f} r(t)\hat{s}_r(t, A)\,dt - \frac{1}{N_0}\int_{T_i}^{T_f} \hat{s}_r^{\,2}(t, A)\,dt,} \tag{113}$$

and from (6.25),

$$l_B(A) = -\frac{1}{N_0}\int_{T_i}^{T_f} \xi_P(t{:}A)\,dt. \tag{114}$$

The function $\hat{s}_r(t, A)$ is the realizable MMSE estimate of $s(t, A)$, assuming that A is known. From Chapter I-6, we know that it is specified by the differential equations

$$\dot{\hat{\mathbf{x}}}(t, A) = \mathbf{F}(t, A)\hat{\mathbf{x}}(t, A) + \boldsymbol{\xi}_P(t, A)\mathbf{C}^T(t, A)\frac{2}{N_0}[r(t) - \mathbf{C}(t, A)\hat{\mathbf{x}}(t, A)], \tag{115}$$

$$\dot{\boldsymbol{\xi}}_P(t, A) = \mathbf{F}(t, A)\boldsymbol{\xi}_P(t, A) + \boldsymbol{\xi}_P(t, A)\mathbf{F}^T(t, A) - \boldsymbol{\xi}_P(t, A)\mathbf{C}^T(t, A)\frac{2}{N_0}\mathbf{C}(t, A)\boldsymbol{\xi}_P(t, A) + \mathbf{G}(t, A)\mathbf{Q}(A)\mathbf{G}^T(t, A), \tag{116}$$

and

$$\hat{s}_r(t, A) = \mathbf{C}(t, A)\hat{\mathbf{x}}(t, A). \tag{117}$$

with appropriate initial conditions. The function $\xi_P(t{:}A)$ is the minimum mean-square error in estimating $s(t, A)$, assuming that A is known. In almost all cases, we must generate $l(A)$ for a set of A_i that span the allowable range of A and choose the value at which $l(A_i)$ is the largest. This realization is shown in Fig. 7.9.

In order to bound the variance of any unbiased estimate, we use the bound in (6.44), (6.46), and (6.60). For finite-state processes, the form in (6.60) is straightforward. The expression is

$$J^{(2)}(A) = \frac{4}{N_0}\left[\frac{\partial^2}{\partial A_1^{\,2}}\left\{\int_{T_i}^{T_f} \xi_P\left(t \mid \sqrt{\tfrac{1}{2}}\, s(\cdot, A_1) + \sqrt{\tfrac{1}{2}}\, s(\cdot, A), \frac{N_0}{2}\right) dt - \frac{1}{2}\int_{T_i}^{T_f} \xi_P\left(t \mid s(\cdot, A_1), \frac{N_0}{2}\right) dt\right\}\right]_{A_1=A}. \tag{118}$$

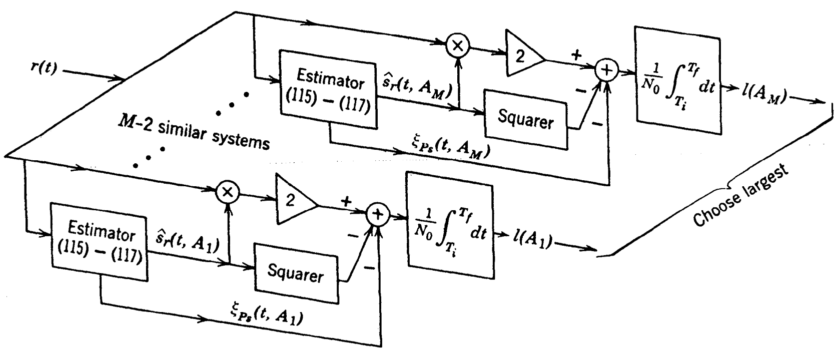

Fig. 7.9 Likelihood function generation for finite-state processes.

The error expression in the integrand of the second term follows from (116) as

$$\xi_P\left(t \mid s(\cdot, A_1), \frac{N_0}{2}\right) = \mathbf{C}(t, A_1)\boldsymbol{\xi}_P(t, A_1)\mathbf{C}^T(t, A_1). \tag{119}$$

To compute the error expression in the first term, we go through a similar analysis for the composite process. Notice that the composite process is the sum of two statistically independent processes with different values of A. As in the analogous detection theory problem, there is an alternative form for $J^{(2)}(A)$ that is sometimes simpler to compute (see pages 179–181). The details are developed in Problem 7.2.1.

Several examples of estimating a parameter of a state-representable process are developed in the problems. We now consider separable kernel processes.

7.3 SEPARABLE KERNELS

In Chapter 4 we discussed several physical situations that led to detection theory problems with separable kernels. Most of these situations have a counterpart in the estimation theory context. In view of this similarity, we simply define the problems and work a simple example. The signal covariance function is $K_s(t, u{:}A)$. If we can write it as

$$K_s(t, u{:}A) = \sum_{i=1}^{K} \lambda_i(A)\phi_i(t, A)\phi_i(u, A), \qquad T_i \le t, u \le T_f \tag{120}$$

for some finite K and for every value of A in the range $[A_\alpha, A_\beta]$, we have a separable-kernel estimation problem. Notice that both the eigenvalues and eigenfunctions may depend on A. To illustrate some of the implications of separability, we consider a simple amplitude estimation problem.

Example 5.† The received waveform is

$$r(t) = s(t, A) + w(t), \qquad T_i \le t \le T_f. \tag{121}$$

The signal is zero-mean with covariance function

$$K_s(t, u{:}A) = AK_s(t, u), \qquad T_i \le t, u \le T_f. \tag{122}$$

We assume that $K_s(t, u)$ is separable:

$$K_s(t, u) = \sum_{i=1}^{K} \lambda_i\phi_i(t)\phi_i(u), \qquad T_i \le t, u \le T_f. \tag{123}$$

† This particular problem has been studied by Hofstetter [7].

The likelihood function is

$$\ln \Lambda(A) = \frac{1}{2} \sum_{i=1}^{K} \ln \left(1 + \frac{2A\lambda_i}{N_0}\right) + \frac{1}{N_0} \sum_{i=1}^{K} \left(\frac{A\lambda_i}{N_0/2 + A\lambda_i}\right) r_i^2, \tag{124}$$

where

$$r_i = \int_{T_i}^{T_f} r(t)\phi_i(t)\, dt. \tag{125}$$

To find $\hat{a}_{ml}$, we must find the value of A where $\ln \Lambda(A)$ has its maximum. If $\hat{a}_{ml}$ is unbiased, its variance is bounded by

$$\frac{\text{Var}\,[(\hat{a} - A)]}{A^2} \geq 2\left[\sum_{i=1}^{K} \frac{\lambda_i^2}{(N_0/2A + \lambda_i)^2}\right]^{-1} \tag{126}$$

If the maximum is interior to the range of A and $\ln \Lambda(A)$ has a continuous first derivative, then a necessary condition is obtained by differentiating (124),

$$\left\{-\frac{1}{2} \sum_{i=1}^{K} \frac{\lambda_i}{N_0/2 + \lambda_i A} + \frac{1}{2} \sum_{i=1}^{K} \frac{\lambda_i}{(N_0/2 + A\lambda_i)^2} r_i^2\right\}_{A=\hat{a}_0} = 0. \tag{127}$$

For arbitrary values of N_0, A, and λ_i this condition is not too useful. There are three cases in which a simple result is obtained for the estimate:

1. The K eigenvalues are all equal.
2. All of the λ_i are much greater than $N_0/2A$.
3. All of the λ_i are much less than $N_0/2A$.

The last two cases are analogous to the limiting cases that we discussed in Section 7.1, and so we relegate them to the problems (see Problems 7.3.1 and 7.3.2).

In the first case,

$$\lambda_i = \lambda_c, \qquad i = 1, 2, \ldots, K, \tag{128}$$

and (127) reduces to

$$\hat{a}_0 = \frac{1}{\lambda_c}\left[\frac{1}{K} \sum_{i=1}^{K} r_i^2 - \frac{N_0}{2}\right]. \tag{129}$$

Since $\hat{a}_0$ can assume negative values, we have

$$\hat{a}_{ml} = \begin{cases} \hat{a}_0, & \hat{a}_0 \geq 0, \\ 0\,, & \hat{a}_0 < 0. \end{cases} \tag{130}$$

In this particular case we can compute $p_{\hat{a}_0}(A_0)$ exactly. It is a chi-square density (page I-109) shifted by $N_0/2\lambda_c$. We can also compute the bias and variance exactly. For moderate K ($K > 8$) the approximate expressions in Section 7.1.2 give an accurate answer. The various expressions are derived in the problems.

We should also observe that in the equal eigenvalue case, $\hat{a}_0$ is an *efficient* unbiased estimate of A. In other words, its variance satisfies (126) with an equality sign. This can be verified by computing Var $[\hat{a}_0 - A]$ directly.

This example illustrates the simplest type of separable kernel problem. In the general case we have to use the parallel processing configuration in

Fig. 6.1. Notice that *each* path will contain K filter-squarers. Thus, if there are M paths, the complete structure will contain MK filter-squarers. In view of this complexity, we usually try to find a suboptimum receiver whose performance is close to the optimum receiver. The design of this suboptimum receiver will depend on how the parameter enters into the covariance function. Several typical cases are given in the problems.

7.4 LOW-ENERGY-COHERENCE CASE

In Section 4.3 of the detection theory discussion, we saw that when the largest eigenvalue was less than $N_0/2$ we could obtain an iterative solution for $h(t, u)$. We can proceed in exactly the same manner for the estimation problem. The only difference is that the largest eigenvalue may depend on A. From (6.16) we have

$$\ln \Lambda(A) = l_R(A) + l_B(A)$$
$$= \frac{1}{N_0}\sum_{i=1}^{\infty}\left(\frac{\lambda_i(A)}{\lambda_i(A) + N_0/2}\right)R_i^{\,2} - \frac{1}{2}\sum_{i=1}^{\infty}\ln\left(1 + \frac{2\lambda_i(A)}{N_0}\right). \tag{131}$$

Assuming that

$$\lambda_i(A) < \frac{N_0}{2} \tag{132}$$

for all A, we can expand each term in the sums in a convergent power series in $[2\lambda_i(A)]/N_0$. The result is

$$l_R(A) = \frac{1}{2}\left(\frac{2}{N_0}\right)^2 \sum_{i=1}^{\infty}\lambda_i(A)\left[1 - \frac{2}{N_0}\lambda_i(A) + \left(\frac{2}{N_0}\right)^2(\lambda_i(A))^2\cdots\right]r_i^{\,2} \tag{133}$$

and

$$l_B(A) = -\frac{1}{2}\left(\frac{2}{N_0}\right)\sum_{i=1}^{\infty}\left[\lambda_i(A) - \frac{1}{2}\left(\frac{2}{N_0}\right)(\lambda_i(A))^2 + \frac{1}{3}\left(\frac{2}{N_0}\right)^2(\lambda_i(A))^3\cdots\right]. \tag{134}$$

In the LEC case we have

$$\lambda_i(A) \ll \frac{N_0}{2} \tag{135}$$

for all A. When this inequality holds, we construct the approximate likelihood function by retaining the first term and the average value of the second term in (133) and the first two terms in (134). (See discussion on page 133.) This gives

$$\ln \Lambda(A) = \frac{1}{2}\left(\frac{2}{N_0}\right)^2 \iint_{T_i}^{T_f} r(t) K_s(t, u:A) r(u)\, dt\, du - \frac{1}{2}\left(\frac{2}{N_0}\right) \int_{T_i}^{T_f} K_s(t, t:A)\, dt - \frac{1}{4}\left(\frac{2}{N_0}\right)^2 \int_{T_i}^{T_f} K_s^2(t, u:A)\, dt\, du. \tag{136}$$

To find $\hat{a}_{ml}$ we must construct $\ln \Lambda(A)$ as a function of A and choose the value of A where it is a maximum.

The lower bound on the variance of any unbiased estimate is

$$\operatorname{Var}[\hat{a} - A] \geq \frac{\left(\frac{N_0}{2}\right)^2}{\frac{1}{2} \iint_{T_i}^{T_f} \left[\frac{\partial K_s(t, u:A)}{\partial A}\right]^2 dt\, du} \tag{137}$$

If A is the value of a random parameter, we obtain the MAP estimate by adding $\ln p_a(A)$ to (136) and finding the maximum of the over-all function. To illustrate the simplicity caused by the LEC condition, we consider two simple examples.

Example 6. In this example, we want to estimate the amplitude of the correlation function of a random process,

$$K_r(t, u:A) = AK(t, u) + \frac{N_0}{2}\delta(t - u), \tag{138}$$

where $K(t, u)$ is a known covariance function. Assuming that the LEC condition is satisfied, we may use (136) to obtain

$$\ln \Lambda(A) = +\frac{1}{2}\left(\frac{2}{N_0}\right)^2 \left\{ A \iint_{T_i}^{T_f} r(t) K(t, u) r(u)\, dt\, du - \frac{N_0}{2} A \int_{T_i}^{T_f} K(t, t)\, dt - \frac{A^2}{2} \iint_{T_i}^{T_f} [K(t, u)]^2\, dt\, du \right\}. \tag{139}$$

† This particular problem has been solved by Price [1] and Middleton [4].

Differentiating and equating to zero, we obtain

$$\hat{a}_0 = \frac{\displaystyle\iint_{T_i}^{T_f} r(t)K(t, u)r(u)\, dt\, du - \frac{N_0}{2}\int_{T_i}^{T_f} K(t, t)\, dt}{\displaystyle\iint_{T_i}^{T_f} [K(t, u)]^2\, dt\, du} \tag{140}$$

As before,

$$\hat{a}_{ml} = \begin{cases} \hat{a}_0, & \hat{a}_0 \geq 0, \\ 0, & \hat{a}_0 < 0. \end{cases} \tag{141}$$

We can obtain an upper bound on the bias of $\hat{a}_{ml}$ by using the techniques on pages 198–201. The lower bound on the variance on any unbiased estimate is

$$\frac{\text{Var}\,[\hat{a} - A]}{A^2} \geq \frac{2(N_0/2)^2}{A^2 \displaystyle\int_{T_i}^{T_f} [K(t, u)]^2\, dt\, du}. \tag{142}$$

We see that the right side of (142) is the reciprocal of d^2 in the LEC detection problem [see (4.148)]. We might expect this relation in view of the results in Chapter I-4 for amplitude estimation in the known signal case.

All our examples have considered the estimation of nonrandom parameters. We indicated that the modification to include random parameters was straightforward. In the next example we illustrate the modifications.

Example 7. We assume that the covariance function of the received signal is given by (138). Now we model A as the value of a random variable a. In general the probability density is not known, and so we choose a density with several free parameters. We then vary the parameters in order to match the available experimental evidence. As we discussed on page I-142, we frequently choose a reproducing density for the a-priori density because of the resulting computational simplicity. For this example, a suitable a-priori density is the Gamma probability density,

$$p_a(A) = \begin{cases} \dfrac{\lambda^{n+1}}{n!} A^n e^{-\lambda A}, & A \geq 0, \\ 0, & A < 0, \end{cases} \tag{143}$$

where λ is a positive constant and n is a positive integer that we choose based on our a-priori information about a. In our subsequent discussion we assume that n and λ are known. We want to find $\hat{a}_{\text{map}}$. To do this we construct the function

$$f(A) = \ln \Lambda(A) + \ln p_a(A) \tag{144}$$

and find its maximum. Substituting (139) and (143) into (144) and collecting terms, we have

$$f(A) = -c_1 A^2 + f(r(t))A + n \ln (\lambda A) + \ln \lambda, \qquad A \geq 0, \tag{145}$$

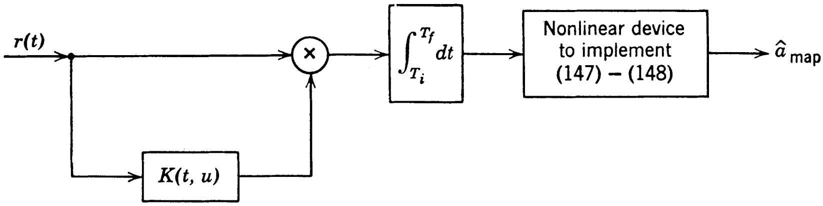

Fig. 7.10 Realization of *MAP* amplitude estimator under *LEC* conditions.

where

$$c_1 = \left(\frac{1}{N_0}\right)^2 \iint_{T_i}^{T_f} [K(t, u)]^2 \, dt \, du \tag{146}$$

and

$$f(r(t)) = \frac{1}{2}\left(\frac{2}{N_0}\right)^2 \left[\iint_{T_i}^{T_f} r(t)K(t, u)r(u) \, dt \, du - \frac{N_0}{2} \int_{T_i}^{T_f} K(t, t) \, dt \right] - \lambda \tag{147}$$

Differentiating $f(A)$ with respect to A, equating the result to zero, and solving the resulting quadratic equation, we have

$$\hat{a}_{\text{map}} = \begin{cases} \dfrac{f(r(t))}{4c_1}\left[\left(1 + \dfrac{8c_1 n}{[f(r(t))]^2 \lambda}\right)^{1/2} + 1\right], & f(r(t)) \geq 0, \\ -\dfrac{f(r(t))}{4c_1}\left[\left(1 + \dfrac{8c_1 n}{[f(r(t))]^2 \lambda}\right)^{1/2} - 1\right], & f(r(t)) < 0. \end{cases} \tag{148}$$

The second derivative of $f(A)$ is always negative and $f(0) = -\infty$, so that this is a unique maximum. The receiver is shown in Fig. 7.10. We see that the receiver carries out two operations in cascade. The first section computes

$$l_*(r(t)) = \iint_{T_i}^{T_f} r(t)K(t, u)r(u) \, dt \, du. \tag{149}$$

The quadratic operation is familiar from our earlier work. The second section is a nonlinear, no-memory device that implements (147) and (148). The calculation of the performance is difficult because of the nonlinear, no-memory operation, which is not a quadratic operation.

Several other examples of parameter estimation under LEC conditions are developed in the problems. This completes our discussion of special categories of estimation problems. In the next section, we discuss some related topics.

7.5 RELATED TOPICS

In this section we discuss two topics that are related to the parameter estimation problem we have been studying. In Section 7.5.1, we discuss the multiple-parameter estimation problem. In Section 7.5.2, we discuss generalized likelihood ratio tests.

7.5.1 Multiple-Parameter Estimation

As we would expect from our earlier work (e.g., Section I-4.6), the basic results for the single-parameter case can be extended easily to include the multiple-parameter case. We state the model and some of the more important results. The received waveform is

$$r(t) = s(t, \mathbf{A}) + w(t), \qquad T_i \leq t \leq T_f, \tag{150}$$

where $\mathbf{A}$ is an M-dimensional parameter vector. The signal $s(t, \mathbf{A})$ is a sample function from a Gaussian random process whose statistics depend on $\mathbf{A}$,

$$E[s(t, \mathbf{A})] = m(t, \mathbf{A}), \qquad T_i \leq t \leq T_f, \tag{151}$$

and

$$E[(s(t, \mathbf{A}) - m(t, \mathbf{A}))(s(u, \mathbf{A}) - m(u, \mathbf{A}))] = K_s(t, u:\mathbf{A}), \quad T_i \leq t, u \leq T_f. \tag{152}$$

The additive noise $w(t)$ is a sample function from a zero-mean white Gaussian process with spectral height $N_0/2$. We are interested both in the case in which $\mathbf{A}$ is a nonrandom vector and in the case in which $\mathbf{A}$ is a value of a random vector.

Nonrandom Parameters. We assume that A is a unknown nonrandom vector that lies in the set $\chi_{\mathbf{a}}$. The likelihood function is

$$\begin{aligned}\ln \Lambda(\mathbf{A}) = &\frac{1}{N_0}\int_{T_i}^{T_f} dt \int_{T_i}^{T_f} du\; r(t)h(t, u:\mathbf{A})r(u) \\ &+ \int_{T_i}^{T_f} dt \int_{T_i}^{T_f} du\; r(t)Q_r(t, u:\mathbf{A})m(u, \mathbf{A}) \\ &- \frac{1}{2}\int_{T_i}^{T_f} dt \int_{T_i}^{T_f} du\; m(t, \mathbf{A})Q_r(t, u:\mathbf{A})m(u, \mathbf{A}) \\ &- \frac{1}{2}\sum_{i=1}^{\infty} \ln\left(1 + \frac{2}{N_0}\lambda_i(\mathbf{A})\right), \qquad \mathbf{A} \in \chi_{\mathbf{a}}.\end{aligned} \tag{153}$$

The ML estimate, $\hat{a}_{ml}$, is that value of A where this function is a maximum. In general, we must construct this function for some set of $\mathbf{A}_i$ that span the set χ_a and choose the value of $\mathbf{A}_i$ where $\ln \Lambda(\mathbf{A}_i)$ is the largest. If the maximum of $\ln \Lambda(\mathbf{A})$ is interior to χ_a and $\ln \Lambda(\mathbf{A})$ has a continuous first derivative, a necessary condition is given by the M likelihood equations,

$$\frac{\partial \ln \Lambda(\mathbf{A})}{\partial A_i}\bigg|_{\mathbf{A}=\hat{\mathbf{a}}_{ml}} = \Bigg\{\frac{1}{2}\iint_{T_i}^{T_f} K_r(t, u:\mathbf{A})\frac{\partial Q_r(t, u:\mathbf{A})}{\partial A_i}\,dt\,du$$

$$+\iint_{T_i}^{T_f} \frac{\partial m(t, \mathbf{A})}{\partial A_i} Q_r(t, u:\mathbf{A})[r(u) - m(u, \mathbf{A})]\,dt\,du$$

$$-\frac{1}{2}\iint_{T_i}^{T_f} [r(t) - m(t, \mathbf{A})]\frac{\partial Q_r(t, u:\mathbf{A})}{\partial A_i}$$

$$\times [r(u) - m(u, \mathbf{A})]\,dt\,du\Bigg\}\bigg|_{\mathbf{A}=\hat{\mathbf{a}}_{ml}} = 0,$$

$$i = 1, 2, \ldots, M. \quad (154)$$

The elements in the information matrix are

$$J_{ij}(\mathbf{A}) = \iint_{T_i}^{T_f} \left[\frac{\partial m(t, \mathbf{A})}{\partial A_i} Q_r(t, u:\mathbf{A})\frac{\partial m(u, \mathbf{A})}{\partial A_j} - \frac{1}{2}\frac{\partial K_r(t, u:\mathbf{A})}{\partial A_i}\frac{\partial Q_r(t, u:\mathbf{A})}{\partial A_j}\right] dt\,du. \quad (155)$$

The information matrix is used in exactly the same manner as in Section I-4.6. The first term is analogous to $d^2(\mathbf{A})$ and can be computed in a straightforward manner. The second term, $J_{ij}^{(2)}(\mathbf{A})$, is the one that requires some work.

We can also express $J_{ij}^{(2)}(A)$ in terms of the derivative of the Bhattacharyya distance.

$$B(\mathbf{A}_1, \mathbf{A}) = -\mu(\tfrac{1}{2}, \mathbf{A}_1, \mathbf{A}) \triangleq -\ln \int_{-\infty}^{\infty} p_{r|a}^{1/2}(\mathbf{R} \mid \mathbf{A}_1)p_{r|a}^{1/2}(\mathbf{R} \mid \mathbf{A})\,d\mathbf{R}. \quad (156)$$

Then, using a derivation similar to (6.47)–(6.60), we obtain

$$J_{ij}^{(2)}(\mathbf{A}) = 4\left(\frac{\partial^2 B(\mathbf{A}_1, \mathbf{A})}{\partial A_{1i}\,\partial A_{1j}}\bigg|_{\mathbf{A}_1=\mathbf{A}}\right). \quad (157)$$

The expression for $B(\mathbf{A}_1, \mathbf{A})$ is an obvious modification of (6.59). This formula provides an effective procedure for computing $J_{ij}^{(2)}(\mathbf{A})$ numerically.

(Notice that the numerical calculation of the second derivative must be done carefully.)

For stationary processes and long time intervals, the second term in (155) has a simple form,

$$J_{ij}^{(2)}(\mathbf{A}) = \frac{T}{2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left[\frac{\partial}{\partial A_i}\left(\ln\left[S_s(\omega, \mathbf{A}) + \frac{N_0}{2}\right]\right)\right] \times \left[\frac{\partial}{\partial A_j}\left(\ln\left[S_s(\omega, \mathbf{A}) + \frac{N_0}{2}\right]\right)\right]. \quad (158)$$

The results in (153)–(158) indicate how the single-parameter formulas are modified to study the multiple-parameter problem. Just as in the single parameter case, the realization of the estimator depends on the specific problem.

Random Parameters. For the general random parameter case, the results are obtained by appropriately modifying those in the preceding section. A specific case of interest in Chapter II-8 is the case in which the parameters to be estimated are independent, zero-mean Gaussian random variables with variances $\sigma_{a_i}^2$. The MAP equations are

$$\hat{a}_i = \sigma_{a_i}^2\left\{\int_{T_i}^{T_f}dt\int_{T_i}^{T_f}du\left[\tfrac{1}{2}K_r(t, u:\mathbf{A})\frac{\partial Q_r(t, u:\mathbf{A})}{\partial A_i} + [r(u) - m(u, \mathbf{A})] \times \left[\frac{\partial m(t, \mathbf{A})}{\partial A_i}Q_r(t, u:\mathbf{A}) - \tfrac{1}{2}[r(t) - m(t, \mathbf{A})]\frac{\partial Q_r(t, u:\mathbf{A})}{\partial A_i}\right]\right]_{\mathbf{A}=\hat{\mathbf{a}}_{map}}\right\},$$

$$i = 1, 2, \ldots, M. \quad (159)$$

The terms in the information matrix are

$$J_{ij} = \frac{\delta_{ij}}{\sigma_{a_i}^2} + E_{\mathbf{a}}[J_{ij}^{(1)}(\mathbf{a}) + J_{ij}^{(2)}(\mathbf{a})], \quad (160)$$

where $J_{ij}^{(1)}(\mathbf{a})$ and $J_{ij}^{(2)}(\mathbf{a})$ are the two terms in (155). Notice that J_{ij} contains an average over $p_{\mathbf{a}}(\mathbf{A})$, so that the final result does not depend on $\mathbf{A}$.

Several joint parameter estimation examples are developed in the problems.

7.5.2 Composite-hypothesis Tests

In some detection problems the signals contain unknown nonrandom parameters. We can write the received waveforms on the two hypotheses as

$$r(t) = s_1(t, \boldsymbol{\theta}) + w(t), \qquad T_i \le t \le T_f: H_1,$$

$$r(t) = s_0(t, \boldsymbol{\theta}) + w(t), \qquad T_i \le t \le T_f: H_0, \quad (161)$$

where $s_1(t, \boldsymbol{\theta})$ and $s_0(t, \boldsymbol{\theta})$ are conditionally Gaussian processes. This model is a generalization of the classical model in Section I-2.5. Usually a uniformly most powerful test does not exist, and so we use a generalized likelihood ratio test. This test is just a generalization of that in (I-2.305). We use the procedures of Chapters 6 and 7 to find $\hat{\boldsymbol{\theta}}_{ml}$. We then use this value as if it were correct in the likelihood ratio test of Chapter 4. Although the extension is conceptually straightforward, the actual calculations are usually quite complicated. It is difficult to make any general statements about the performance. Some typical examples are developed in the problems.

7.6 SUMMARY OF ESTIMATION THEORY

In Chapters 6 and 7, we have studied the problem of estimating the parameters of a Gaussian random process in the presence of additive Gaussian noise. As in earlier estimation problems, the first step was to construct the log likelihood function. For our model,

$$\ln \Lambda(A) = \frac{1}{N_0} \iint_{T_i}^{T_f} r(t)h(t, u:A)r(u)\, dt\, du + \int_{T_i}^{T_f} r(t)g(t, A)\, dt - \frac{1}{N_0} \int_{T_i}^{T_f} \xi_P(t:A)\, dt - \frac{1}{2} \int_{T_i}^{T_f} m(t, A)g(t, A)\, dt. \tag{161}$$

In order to find the maximum likelihood estimate, we construct $\ln \Lambda(A)$ as a function of A. In practice it is usually necessary to construct a discrete approximation by computing $\ln \Lambda(A_i)$ for a set of values than span χ_a.

In order to analyze the performance, we derived a lower bound of the variance of any unbiased estimate. The bound is

$$\operatorname{Var}[\hat{a} - A] \geq \left\{ \iint_{T_i}^{T_f} \frac{\partial m(t, A)}{\partial A} Q_r(t, u:A) \frac{\partial m(u, A)}{\partial A}\, dt\, du - \frac{1}{2} \iint_{T_i}^{T_f} \frac{\partial K_r(t, u:A)}{\partial A} \frac{\partial Q_r(t, u:A)}{\partial A}\, dt\, du \right\}^{-1} \tag{162}$$

for any unbiased estimate. In most problems we must evaluate the bound using numerical techniques. Since the estimates of the process parameters are usually not efficient, the bound in (162) may not give an accurate indication of the actual variance. In addition, the estimate may have a bias that we cannot evaluate, so that we cannot use (162) or the generalization of it derived in Problem 6.3.1. We pointed out that other bounds, such as the Barankin bound and the Kotelnikov bound, were available but did not discuss them in detail.

The discussion in Chapter 6 provided the general theory needed to study the parameter estimation problem. An equally important topic was the application of this theory to a particular problem in order actually to obtain the estimate and evaluate its performance.

In Chapter 7 we illustrate the transition from theory to practice for a particular problem. In Section 7.1 we studied the problem of estimating the mean-square value of a stationary random process. After finding expressions for the likelihood function, we considered some limiting cases in which we could generate $\hat{a}_{ml}$ easily. We encountered the issue of a truncated estimate and developed new techniques for computing the bias and mean-square error. Finally, we looked at some suboptimum estimation procedures. This section illustrated a number of the issues that one encounters and must resolve in a typical estimation problem. In Sections 7.2–7.4, we considered finite-state processes, separable kernel processes, and the low-energy-coherence problem, respectively. In all of these special cases we could solve the necessary integral equation and generate $\ln \Lambda(A)$ explicitly. It is important to emphasize that, even after we have solved the integral equation, we usually still have to construct $\ln \Lambda(A_i)$ for a set of values that span χ_a. Thus, the estimation problem is appreciably more complicated than the detection problem.

We have indicated some typical estimation problems in the text and in the problem section. References dealing with various facets of parameter estimation include [5]–[15].

This chapter completes our work on detection and estimation of Gaussian processes. Having already studied the modulation theory problem in Chapter II-8, we have now completed the hierarchy of problems that we outlined in Chapter I-1. The remainder of the book deals with the application of this theory to the radar-sonar problem.

7.7 PROBLEMS

P.7.1 Stationary Processes: Long Observation Time

Problem 7.1.1. Consider the FSI estimator in Fig. 7.3. The filter $h_{\alpha}^{1/2}(\tau)$ and the parameters G and B are arbitrary subject to the constraint

$$E[\hat{a}_0] = A. \tag{P.1}$$

1. Derive an exact expression for

$$\xi_{\hat{a}_0} \triangleq E[(\hat{a}_0 - A)^2].$$

2. Consider the condition in (26) and (27) and assume that (31), (37), and (38) are satisfied. Denote the Cramér-Rao bound in (25) as ξ_{CR}. Prove

$$\lim_{(N_0/2AS(\omega))\to 0} [\xi_{\hat{a}_0}/A^2 - \xi_{\text{CR}}] = 0.$$

Problem 7.1.2. The function $\mu(s, A)$ is defined in (40), and B satisfies (43). Prove that (66) has a solution in the allowable range. A possible procedure is the following:

(i) Evaluate $\dot{\mu}(0, A)$ and $\dot{\mu}(-\infty, A)$.
(ii) Prove that this guarantees a solution to (66) for some $s < 0$.
(iii) Use the fact that $\ddot{\mu}(s, A) > 0$ to prove the desired result.

Problem 7.1.3. Assume

$$S(\omega) = \begin{cases} \dfrac{1}{2W}, & |\omega| \le 2\pi W, \\ 0, & \text{elsewhere.} \end{cases}$$

Then, from (47),

$$\mu(s, A) = -WT \ln\left[1 - 2sA\left(1 + \frac{N_0 W}{A}\right)\right].$$

1. Solve (66) for s_2.
2. Verify the results in (68) and (69).

Problem 7.1.4.

1. Modify the derivation of bounds 1 and 2 to incorporate (52).
2. Compare your results with those in Fig. 7.6.

Problem 7.1.5 [2].

1. Consider the expression for ξ_* given in (73). Use the same procedure as in (60)–(69) to obtain a bound on ξ_*.
2. Modify the derivation in part 1 to incorporate (52).

Problem 7.1.6. Recall the result in part 1 of Problem 7.1.1 and consider the receiver in Example 3.

1. Show that $\xi_{\hat{a}_0}$ approaches the Cramér-Rao bound as $WT \to \infty$.
2. Investigate the bias and the mean-square error in the ML estimate.

Problem 7.1.7. Recall the result in part 1 of Problem 7.1.1. Substitute (96)–(98) into this formula and verify that (99) is true.

Problem 7.1.8. Consider the model in Example 4. Verify that the result in (102) is true.

Problem 7.1.9. Consider the result in (99). Assume

$$S(\omega) = \begin{cases} \dfrac{1}{2W}, & |\omega| \le 2\pi W, \\ 0, & \text{elsewhere.} \end{cases}$$

1. Evaluate (99) to obtain $\xi_{\hat{a}_0}$ as a function of C.
2. Check the two limiting cases of Examples 2 and 3 as $C \to \infty$ and $C \to 0$.
3. Plot $\xi_{\hat{a}_0}$ as a function of C for various values of A/N_0W and WT.

Problem 7.1.10. Carry out the details of the bias calculation for the model in Example 4.

Problem 7.1.11. Assume that $s(t, A)$ is a Wiener process,

$$E[s^2(t)] = At,$$

and that the SPLOT assumption may be used.

1. Evaluate the Cramér-Rao bound by using (25).
2. Consider the suboptimum receiver described by (96)–(98). Evaluate $\xi_{\hat{a}_0}$ in (99). Plot $\xi_{\hat{a}_0}$ as a function of C.
3. Calculate a bound on Pr $[\hat{a}_0 < 0]$.

Problem 7.1.12. Consider the model in (1)–(5). Assume that

$$K_s(\tau:A) = e^{-A|\tau|}, \qquad -\infty < \tau < \infty,$$

where A is a nonrandom positive parameter.

1. Draw a block diagram of a receiver to find $\hat{a}_{ml}$.
2. Evaluate the Cramér-Rao bound in (16).

Problem 7.1.13. Consider the model in (1)–(5). Assume that $s(t, A)$ is a bandpass process whose spectrum is

$$S_s(\omega:A) = S_{s,\mathrm{LP}}(-\omega - A) + S_{s,\mathrm{LP}}(\omega - A),$$

where $S_{s,\mathrm{LP}}(\omega)$ is a known low-pass spectrum.

1. Draw a block diagram of a receiver to find $\hat{a}_{ml}$.
2. Evaluate the Cramér-Rao bound in (16).
3. Specialize the result in part 2 to the case in which

$$S_{s,\mathrm{LP}}(\omega) = \frac{2kP}{\omega^2 + k^2}.$$

Problem 7.1.14 [5]. Suppose that

$$s(t, A) = c_1[s(t) + c_2 s(t - A)],$$

where c_1 and c_2 are known constants and $s(t)$ is a sample function from a stationary random process. Evaluate the Cramér-Rao bound in (16).

Problem 7.1.15. In the text we assumed that $s(t, A)$ was a zero-mean process. In this problem we remove that restriction.

1. Derive the SPLOT version of (6.16)–(6.26).
2. Derive the SPLOT version of (6.45) and (6.46).

Problem 7.1.16. Consider the system shown in Fig. P.7.1. The input $s(t)$ is a sample function of a zero-mean Gaussian random process with spectrum $S(\omega)$. The additive noise $w(t)$ is a sample function of a white Gaussian random process with spectral height $N_0/2$. We observe $r(t)$, $T_i \le t \le T_f$, and want to find $\hat{a}_{ml}$.

1. Draw a block diagram of the optimum receiver.
2. Write an expression for the Cramér-Rao bound. Denote the variance given by this bound as ξ_{CR}.

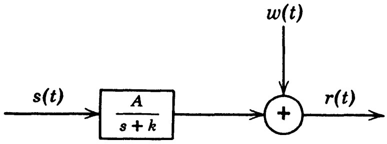

Fig. P.7.1

3. Constrain

$$\int_{-\infty}^{\infty} S(\omega)\frac{d\omega}{2\pi} = P.$$

Choose $S(\omega)$ to minimize ξ_{CR}.

Problem 7.1.17. The received waveform is

$$r(t) = s(t, A) + n_c(t), \qquad T_i \le t \le T_f. \tag{P.1}$$

The additive noise $n_c(t)$ is a sample function of a zero-mean, finite-power, stationary Gaussian noise process with spectrum $S_c(\omega)$. Notice that there is no white noise component.

1. Derive an expression for the Cramér-Rao bound.

2. Discuss the question of singular estimation problems. In particular, consider the case in which

$$S(\omega, A) = AS(\omega). \tag{P.2}$$

3. Assume that (P.1) and (P.2) hold and

$$\int_{-\infty}^{\infty} S_c(\omega)\frac{d\omega}{2\pi} = P_c. \tag{P.3}$$

Choose $S_c(\omega)$ to maximize the Cramér-Rao bound.

Problem 7.1.18. The received waveform is

$$\mathbf{r}(t) = \mathbf{s}(t, A) + \mathbf{w}(t), \qquad T_i \le t \le T_f.$$

Assume that the SPLOT conditions are valid.

1. Derive an expression for $\ln \Lambda(A)$.

2. Derive an expression for the Cramér-Rao bound.

Problem 7.1.19. Consider the two-element receiver shown in Fig. P.7.2. The signal $s(t)$ is a sample function of a zero-mean stationary Gaussian random process with spectrum $S(\omega)$. It propagates along a line that is α radians from the y-axis. The received signals at the two elements are

$$r_1(t) = s(t) + w_1(t), \qquad T_i \le t \le T_f,$$

$$r_2(t) = s\left(t - \frac{L \sin \alpha}{c}\right) + w_2(t), \qquad T_i \le t \le T_f,$$

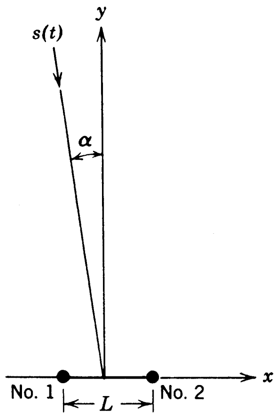

Fig. P.7.2

where c is the velocity of propagation. The additive noises $w_1(t)$ and $w_2(t)$ are statistically independent white noises with spectral height $N_0/2$. Assume that $|\alpha| \le \pi/8$ and that $S(\omega)$ is known.

1. Draw a block diagram of a receiver to generate $\hat{\alpha}_{ml}$.
2. Write an expression for the Cramér-Rao bound.
3. Evaluate the bound in part 2 for the case in which

$$S(\omega) = \begin{cases} \dfrac{P}{2W}, & |\omega| \le 2\pi W, \\ 0, & \text{elsewhere.} \end{cases}$$

4. Discuss various suboptimum receiver configurations and their performance.
5. One procedure for estimating α is to compute

$$\varphi(\tau) = \frac{1}{T_f - T_i} \int_{T_i}^{T_f} r_1(t) r_2(t - \tau)\, dt$$

as a function of τ and find the value of τ where $\varphi(\tau)$ has its maximum. Denote this point as $\hat{\tau}_*$. Then define

$$\hat{\alpha}_* \triangleq \frac{c\hat{\tau}_*}{L}.$$

Discuss the rationale for this procedure. Compare its performance with the Cramér-Rao bound.

Problem 7.1.20. Consider the problem of estimating the height of the spectrum of a random process $s(t)$ at a particular frequency. A typical estimator is shown in Fig. P.7.3.

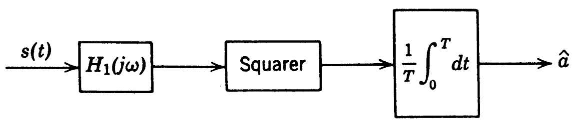

Fig. P.7.3

Denote the particular point in the spectrum that we want to estimate as

$$A \triangleq S(\omega_1). \tag{P.1}$$

The filter $H_1(j\omega)$ is an ideal bandpass filter centered at ω_1, as shown in Fig. P.7.4.

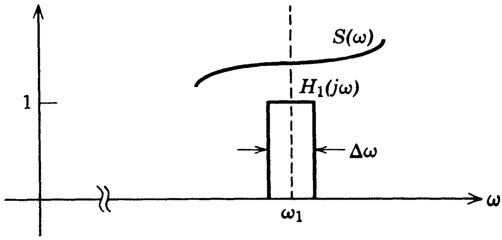

Fig. P.7.4

1. Compute the bias in $\hat{a}$. Under what conditions will this bias be negligible?
2. Assume that the bias is negligible. Compute the normalized variance as a function of the various parameters.
3. Demonstrate a choice of parameters such that the normalized variance goes to zero as $T \to \infty$
4. Demonstrate a choice of parameters such that the normalized variance goes to two as $T \to \infty$.

Comment: This problem illustrates some of the issues in power spectrum estimation. The interested reader should consult [6.1]–[6.3] for a detailed discussion.

Problem 7.1.21. Assume that

$$S(\omega{:}A) = AS(\omega).$$

Denote the expression in the Cramér-Rao bound of (25) as ξ_{CR}.

1. Evaluate ξ_{CR} for the case

$$S(\omega) = \frac{2nP}{k} \frac{\sin(\pi/2n)}{(\omega/k)^{2n} + 1}.$$

2. Assume

$$\int_{-\infty}^{\infty} S(\omega) \frac{d\omega}{2\pi} = P.$$

Find the spectrum $S(\omega)$ that minimizes ξ_{CR}.

Problem 7.1.22. Assume that

$$S(\omega{:}A) = \frac{2AP}{k} \frac{\sin(\pi/2A)}{(\omega/k)^{2A} + 1},$$

where A is a positive integer.

1. Find a receiver to generate $\hat{a}_{ml}$.
2. Find a lower bound on the variance of any unbiased estimate of A.

Problem 7.1.23. Consider the estimation problem described in Problem 7.1.13. Assume

$$A_\alpha < A < A_\beta,$$

$$S(\omega) = \begin{cases} \dfrac{P}{2W}, & |\omega| \le 2\pi W, \\ 0, & \text{elsewhere}, \end{cases}$$

and

$$W = \frac{A_\alpha - A_\beta}{M}.$$

1. Draw a block diagram of a receiver to generate $\hat{a}_{ml}$.
2. Is the Cramér-Rao bound valid in this problem?
3. Use the techniques on pages I-278–I-284 and the results of Section 5.1 to evaluate the receiver performance. How can you analyze the weak noise performance (local errors)?

P.7.2 Finite-state Processes

Problem 7.2.1. The signal process is a Wiener process,

$$E[s^2(t)] = At.$$

We want to find $\hat{a}_{ml}$.

1. Draw a block diagram of the receiver. Specify all components (including initial conditions) completely.
2. Verify that

$$\xi_P(t, A) = \left(\frac{AN_0}{2}\right)^{1/2} \left[\frac{1 - \exp\left[-2(2A/N_0)^{1/2} t\right]}{1 + \exp\left[-2(2A/N_0)^{1/2} t\right]}\right].$$

3. Use the result in part 2 to compute $J^{(2)}(A)$ in (118).
4. Plot $J^{(2)}(A)/A^2$ as a function of $2AT^2/N_0$.

Problem 7.2.2. Assume

$$s(t, A) = As(t),$$

where $s(t)$ has a known finite-dimensional state representation. Consider the FSI receiver in Fig. 7.4. Assume that $h_\alpha^{[1/2]}(\tau)$ has a finite-dimensional state representation. Derive a differential equation specifying $\xi_{\hat{a}_0}$.

Problem 7.2.3. Consider the model in (107)–(111). Assume that $\mathbf{F}(t, A)$, $\mathbf{G}(t\ A)$, $\mathbf{Q}(A)$, and $\mathbf{P}_0(A)$ are not functions of A.

$$\mathbf{C}(t, A) = f(t - A)\mathbf{C},$$

where $f(t)$ is only nonzero in $[\alpha < t < \beta]$ and $\alpha - A$ and $\beta - A$ are in $[T_i, T_f]$. We want to make a maximum-likelihood estimate of A.

1. Draw a block diagram of the optimum receiver.
2. Write an expression for the Cramér-Rao bound.

Problem 7.2.4. Consider the model in (107)–(111). Assume

$$F(t, A) = -k,$$

$$G(t, A) = 1,$$

$$C(t, A) = 1,$$

$$Q(A) = 2kP,$$

and

$$P_0(A) = A.$$

We want to make a maximum-likelihood estimate of A.

1. Draw a block diagram of a receiver to generate $\hat{a}_{ml}$.
2. Discuss various suboptimum configurations. (*Hint:* What segment of the received waveform contains most of the information about A?)
3. Write an expression for the Cramér-Rao bound.

P.7.3 Separable Kernels

Problem 7.3.1. Consider the model described in Example 5. Assume

$$\lambda_i \gg \frac{N_0}{2A} \quad \text{for all} \quad A \text{ in } \chi_a.$$

1. Find an expression for $\hat{a}_0$.
2. Derive an expression for Pr $[\hat{a}_0 < 0]$.
3. Derive an expression for $\xi_{\hat{a}_0}$.
4. Compare the expression in part 3 with the Cramér-Rao bound.

Problem 7.3.2. Repeat Problem 7.3.1 for the case in which

$$\lambda_i \ll \frac{N_0}{2A} \quad \text{for all} \quad A \text{ in } \chi_a.$$

Problem 7.3.3. Consider the model in (121)–(127) and assume that the equal eigenvalue condition in (128) is valid.

1. Calculate

$$\xi_{\hat{a}_0} \triangleq E[(\hat{a}_0 - A)^2].$$

2. Compute Pr $[\hat{a}_0 < 0]$ by using (41).
3. Assume

$$\sum_{i=1}^{K} \lambda_i = \bar{E}_i.$$

Choose K to minimize ξ_{a_0}. Compare your results with those in (4.76) and (4.116).

4. Calculate $p_{\hat{a}_0}(A_0)$ exactly.
5. Evaluate Pr $[\hat{a}_0 < 0]$ using the result in part 4 and compare it with the result in part 2.

Problem 7.3.4. Consider the model in (120), but assume that

$$K_s(t, u:A) = \lambda_c(A) \sum_{i=1}^{K} \phi_i(t)\phi_i(u).$$

Assume that $\lambda_c^{-1}(A)$ exists.

1. Draw a block diagram of the optimum receiver to generate $\hat{a}_{ml}$.
2. Derive an expression for the Cramér-Rao bound.
3. What difficulty arises when you try to compute the performance exactly?

Problem 7.3.5. Consider the model in Problem 7.3.4. Let

$$\lambda_c(A) = \frac{1}{A}.$$

Assume that A is the value of a random variable whose a priori density is

$$p_a(A \mid k_1, k_2) = c(A^{k_1/2-1}) \exp\left(-\tfrac{1}{2}Ak_1k_2\right), \qquad A \geq 0, \qquad k_1, k_2 > 0,$$

where c is a normalization constant

1. Find $p_{a|r(t)}(A \mid r(t))$.
2. Find $\hat{a}_{ms}$.
3. Compute $E[(\hat{a}_{ms} - a)^2]$.

Problem 7.3.6. Consider the model in (120). Assume that

$$K_s(t, u:A) = \sum_{i=1}^{K} \lambda_i \varphi_i(t, A)\varphi_i(u, A).$$

Assume that the $\varphi_i(t, A)$ all have the same shape. The orthogonality is obtained by either time or frequency diversity (see Section 4.3 for examples).

1. Draw a block diagram of a receiver to generate $\hat{a}_{ml}$.
2. Evaluate the Cramér-Rao bound.

P.7.4 Low-energy Coherence

Problem 7.4.1. Assume that both the LEC condition and the SPLOT condition are satisfied.

1. Derive the SPLOT version of (136).
2. Derive the SPLOT version of (137).

Problem 7.4.2. Consider the model in Example 6.

1. Evaluate (142) for the case in which

$$K(t, u) = e^{-\alpha|t-u|}. \qquad \text{(P.1)}$$

2. Evaluate the SPLOT version of (142) for the covariance function in (P.1). Compare the results of parts 1 and 2.

3. Derive an expression for an upper bound on the bias. Evaluate it for the covariance function in (P.1).

Problem 7.4.3. Consider the model in Example 6. Assume that we use the LEC receiver in (140), even though the LEC condition may not be valid.

1. Prove that $\hat{a}_0$ is unbiased under all conditions.
2. Find an expression for $\xi_{\hat{a}_0}$.
3. Evaluate this expression for the covariance function in (P.1) of Problem 7.4.2. Compare your result with the result in part 1 of that problem.

Problem 7.4.4. Assume that

$$K_s(t, u:A) = Af(t)K_s(t-u)f(u)$$

and that the LEC condition is valid.

Draw a block diagram of the optimum receiver to generate $\hat{a}_{ml}$.

P.7.5 Related Topics

Problem 7.5.1. Consider the estimation model in (150)–(158). Assume that

$$m(t, \mathbf{A}) = 0,$$

$$S_s(\omega, \mathbf{A}) = \frac{A_1}{\omega^2 + A_2^2},$$

and that the SPLOT assumption is valid:

1. Draw a block diagram of the ML estimator.
2. Evaluate $\mathbf{J}(\mathbf{A})$.
3. Compute a lower bound on the variance of any unbiased estimate of A_1.
4. Compute a lower bound on the variance of any unbiased estimate of A_2.
5. Compare the result in part 4 with that in Problem 7.1.12. What effect does the unknown amplitude have on the accuracy bounds for estimating A_2?

Problem 7.5.2. Consider the estimation model in (150)–(158). Assume that

$$m(t, \mathbf{A}) = 0$$

and

$$S_s(\omega, \mathbf{A}) = \frac{2k_1A_1}{\omega^2 + k_1^2} + \frac{2k_2A_2}{\omega^2 + k_2^2}, \tag{P.1}$$

where k_1 and k_2 are known.

1. Draw a block of the ML estimator of A_1 and A_2.
2. Evaluate $\mathbf{J}(\mathbf{A})$.
3. Compute a lower bound on the variance of an unbiased estimate of A_1.
4. Compute a lower bound on the variance of an unbiased estimate of A_2.
5. Assume that A_2 is known. Compute a lower bound on the variance of any unbiased estimate of A_1. Compare this result with that in part 3.
6. Assume that the LEC condition is valid. Draw a block diagram of the optimum receiver.
7. Consider the behavior of the result in part 3 as $N_0 \to 0$.

Problem 7.5.3. Let

$$S(\omega:\mathbf{A}) = AS(\omega:\boldsymbol{\alpha}).$$

Assume that $S(\omega:\boldsymbol{\alpha})$ is bandlimited and that

$$AS(\omega:\boldsymbol{\alpha}) \gg N_0/2 \quad \text{for all } A \text{ and } \boldsymbol{\alpha}.$$

1. Assume that $\boldsymbol{\alpha}$ is fixed. Maximize $\ln \Lambda(\mathbf{A})$ over A to find $\hat{a}_{ml}(\boldsymbol{\alpha})$, the ML estimate of A.

2. Substitute this result into $\ln \Lambda(\mathbf{A})$ to find $\ln \Lambda(\hat{a}_{ml}, \boldsymbol{\alpha})$. This function must be maximized to find $\hat{\boldsymbol{\alpha}}_{ml}$.

3. Assume that $\boldsymbol{\alpha}$ is a scalar α. Find a lower bound on the variance of an unbiased estimate of α. Compare this bound with the bound for the case in which A is known. Under what conditions is the knowledge of A unimportant?

Problem 7.5.4. Repeat Problem 7.5.3 for the case in which

$$AS(\omega:\boldsymbol{\alpha}) \ll N_0/2 \quad \text{for all } A \text{ and } \boldsymbol{\alpha}$$

and $S(\omega:\boldsymbol{\alpha})$ is not necessarily bandlimited.

Problem 7.5.5. Consider the model in Problem 7.5.2. Assume that

$$S_s(\omega, \mathbf{A}) = \frac{A_1}{\omega^2} + \frac{2k_2A_2}{\omega^2 + k_2^{\,2}}.$$

Repeat Problem 7.5.2.

Problem 7.5.6. Consider the model in Problem 7.5.2. Assume that

$$S_s(\omega, \mathbf{A}) = \frac{\omega^2 + A_1^{\,2}}{\omega^4 + A_2^{\,4}}.$$

Evaluate $J(\mathbf{A})$.

Problem 7.5.7. Derive the SPLOT versions of (153) and (154).

Problem 7.5.8. Consider the estimation model in (150)–(158). Assume that

$$m(t, \mathbf{A}) = A_1 m(t),$$

$$S_s(\omega, \mathbf{A}) = A_2 S(\omega).$$

1. Draw a block diagram of the receiver to generate the ML estimates of A_1 and A_2.
2. Evaluate $\mathbf{J}(\mathbf{A})$.
3. Compute a lower bound on the variance of any unbiased estimate of A_1.
4. Compute a lower bound on the variance of any unbiased estimate of A_2.

Problem 7.5.9. Consider the binary symmetric communication system described in Section 3.4. Assume that the spectra of the received waveform on the two hypotheses are

$$S_r(\omega) = A_1 S_1(\omega) + N_0/2 : H_1,$$

$$S_r(\omega) = A_0 S_0(\omega) + N_0/2 : H_0.$$

The hypotheses are equally likely and the criterion is minimum Pr (ϵ). The parameters A_0 and A_1 are unknown nonrandom parameters. The spectra $S_1(\omega)$ and $S_0(\omega)$ are bandpass spectra that are essentially disjoint in frequency and are symmetric around their respective frequencies.

1. Derive a generalized likelihood ratio test.
2. Find an approximate expression for the Pr (ϵ) of the test.

Problem 7.5.10. Consider the following generalization of Problem 7.5.4. The covariance function of $s(t, \mathbf{A})$ is

$$K_s(t, u:\mathbf{A}) = AK_s(t, u:\boldsymbol{\alpha}).$$

Assume that the LEC condition is valid.

1. Find $\ln \Lambda(\hat{a}_{ml}, \boldsymbol{\alpha})$. Use the vector generalization of (136) as a starting point.
2. Derive $\mathbf{J}(\boldsymbol{\alpha}; A)$, the information matrix for estimating $\boldsymbol{\alpha}$.

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