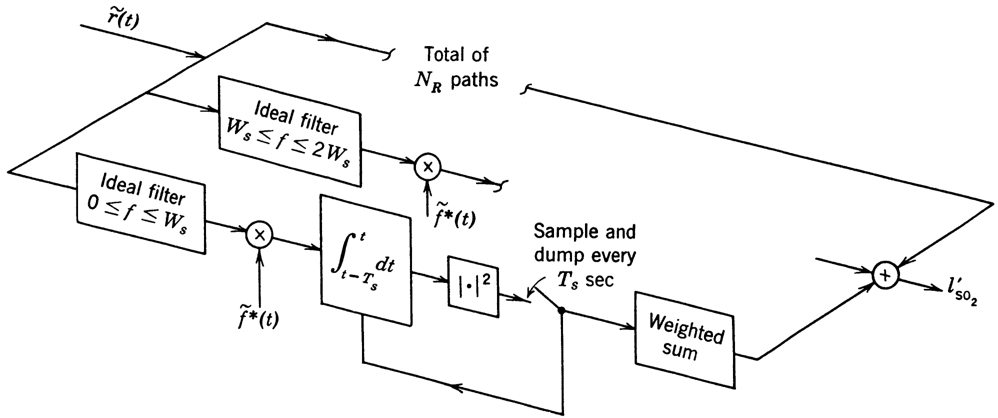


Fig. 13.31 Suboptimum receiver for detection of doubly-spread signal.

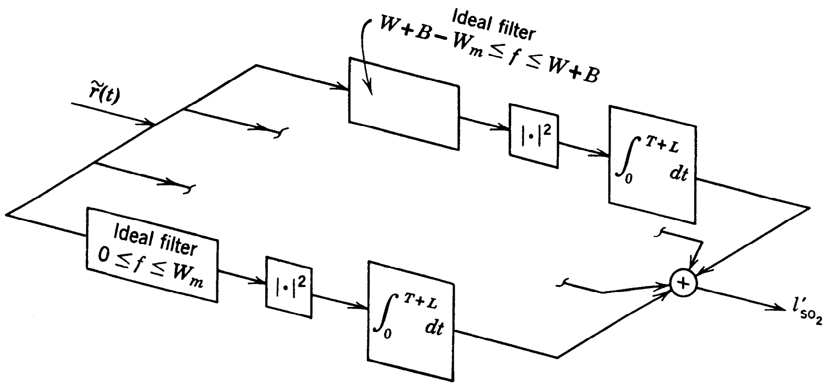


**Fig. 13.32 Approximation to suboptimum receiver No. 2 for rectangular envelope case.**

As a first approximation, we might choose

$$W_s = \frac{1}{T + L} \tag{297a}$$

$$T_s = \frac{1}{B + W}, \tag{297b}$$

so that we have a system with diversity

$$N_R N_D = (T + L)(B + W). \tag{297c}$$

In general, we leave the filter bandwidth, $W_s$ and the correlation time, $T_s$, as parameters. This receiver can be analyzed by using the techniques of Section 11.3.3. Once again, the calculations are complicated but feasible.

When $\tilde{f}(t)$ is a rectangular pulse, a good approximation to the receiver in Fig. 13.31 can be obtained as shown in Fig. 13.32. Here

$$W_m \triangleq \min\left[W_s, \frac{1}{T_s}\right]. \tag{298}$$

This receiver is essentially that suggested by Kennedy and Lebow [38].†

This completes our suboptimum receiver discussion. We now consider some other detection theory topics.

### 13.3.5 Related Topics

In this section we discuss three topics briefly. In Section 13.3.5.A we discuss equivalent channels. In Section 13.3.5.B we comment on $M$-ary

† This reference contains an excellent intuitive discussion of communication over doubly-spread channels, and most engineers will find it worthwhile reading.

communication over doubly-spread channels. In Section 13.3.5.C we re-examine the reverberation problem of Section 13.2.

***13.3.5.A. Equivalent Channels and Systems.*** The idea of an equivalent channel is due to Kennedy [37] and is a generalization of the dual-channel ideas introduced in Section 12.3. Our motivation for developing duality theory was to simplify the design of systems and their analysis. Frequently it was easier to analyze the dual of the system instead of the actual system. The motivation for our discussion of equivalent systems is identical. In many cases, it is easier to analyze an equivalent system instead of the actual system. In addition, the ability to recognize equivalent systems aids our understanding of the general problem.

The first definition is:

**Definition 1. Equivalent Processes.** Consider the two processes $\tilde{r}_1(t)$ and $\tilde{r}_2(t)$ defined over the interval $[T_i, T_f]$. If the eigenvalues of $\tilde{r}_1(t)$ equal the eigenvalues of $\tilde{r}_2(t)$, the processes are equivalent on $[T_i, T_f]$.

For simplicity we now restrict our attention to a simple binary detection problem. The complex envelopes of the received waveforms on the two hypotheses are

$$\tilde{r}(t) = \tilde{s}(t) + \tilde{w}(t), \qquad T_i \le t \le T_f : H_1, \tag{299}$$

$$\tilde{r}(t) = \tilde{w}(t) \qquad T_i \le t \le T_f : H_0. \tag{300}$$

The additive noise $\tilde{w}(t)$ is a sample function from a zero-mean complex white Gaussian noise process with spectral height $N_0$. The signal $\tilde{s}(t)$ is a sample function from a zero-mean complex Gaussian process with covariance function $\tilde{K}_{\tilde{s}}(t, u)$. From our earlier results we know that the performance of the system is completely characterized by the *eigenvalues* of $\tilde{K}_{\tilde{s}}(t, u)$. Notice that the receiver depends on both the eigenfunctions and eigenvalues, but the eigenfunctions do not affect the performance. This observation suggests the following definition.

**Definition 2. Equivalent Detection Problems.** All simple binary detection problems in which the $\tilde{s}(t)$ are equivalent processes are equivalent.

This definition is a generalization of Definition 5 on page 426.

The next idea of interest is that of *equivalent channels.* The covariance function of the signal process at the output of a doubly-spread channel is

$$\tilde{K}_{\tilde{s}_1}(t, u) = \int_{-\infty}^{\infty} \tilde{f}_1(t-\lambda)\tilde{K}_{DR_1}(t-u, \lambda)\tilde{f}_1^*(u-\lambda)\, d\lambda. \tag{301}$$

Now consider the covariance function at the output of a second channel,

$$\tilde{K}_{\tilde{s}_2}(t, u) = \int_{-\infty}^{\infty} \tilde{f}_2(t - \lambda)\tilde{K}_{DR_2}(t - u, \lambda)\tilde{f}_2^*(u - \lambda)\, d\lambda. \tag{302}$$

We can now define equivalent channels.

**Definition 3. Equivalent Channels.** Channel No. 2 is equivalent to Channel No. 1, if, for every $\tilde{f}_1(t)$ with finite energy, there exists a signal $\tilde{f}_2(t)$ with finite energy such that the eigenvalues of $\tilde{K}_{\tilde{s}_2}(t, u)$ are equal to the eigenvalues of $\tilde{K}_{\tilde{s}_1}(t, u)$.

The utility of this concept is that it is frequently easier to analyze an equivalent channel instead of the actual channel.

Some typical equivalent channels are listed in Table 13.2. In columns 1 and 3 we show the relationship between the two-channel scattering functions. Notice that $\tilde{S}_{DR}\{f, \lambda\}$ is an arbitrary scattering function. The complex envelope of the transmitted signal in system 1 is $\tilde{f}(t)$. In column 4, we show the complex envelope that must be transmitted in system 2 to generate an equivalent output signal process. We have assumed an infinite observation interval for simplicity.

We study other equivalent channels and systems in the problems. Once again, we point out that it is a logical extension of the duality theory of Section 12.3 and is useful both as a labor-saving procedure and as an aid to understanding the basic limitations on a system.

***13.3.5.B. M-ary Communications over Doubly-Spread Channels.*** In Section 11.3.4 we discussed communication over Doppler-spread channels using $M$-orthogonal signals. Many of the results were based on the eigenvalues of the output processes. All these results are also applicable to the doubly-spread channel model. In particular, the idea of an optimum eigenvalue distribution is valid. When we try to analyze the performance of a particular system, we must use the new techniques developed in this chapter. The modification of the binary results is straightforward. The reader should consult [37] for a complete discussion of the $M$-ary problem.

***13.3.5.C. Reverberation.*** In Section 13.2 we studied the problem of detecting a *point* target in the presence of doubly-spread interference. One problem of interest was the design of the optimum receiver and an analysis of its performance. The appropriate equations were (116)–(121*b*), and we indicated that we would discuss their solution in this section. We see that all of our discussion in Section 13.3.2 is directly applicable to this problem. The difference is that we want to estimate the reverberation return, $\tilde{n}_r(t)$, in one case, and the reflected signal process in the other. All of the techniques carry over directly.

| | System 1 | | System 2 | |
|---|---|---|---|---|
| | 1 | 2 | 3 | 4 |
| | $\tilde{S}_{DR_1}\{f, \lambda\}$ | $\tilde{f}_1(t)$ | $S_{DR_2}\{f, \lambda\}$ | $\tilde{f}_2(t)$ |
| 1 | $\tilde{S}_{DR}\{-\lambda, f\}$ | $\tilde{f}(t)$ | $\tilde{S}_{DR}\{f, \lambda\}$ | $\tilde{F}(t)$ |
| 2 | $\tilde{S}_{DR}\left\{\frac{f}{a}, a\lambda\right\}$ [scale change] | $\tilde{f}(t)$ | $\tilde{S}_{DR}\{f, \lambda\}$ | $\sqrt{a}\,\tilde{f}(at)$ [see (10.124)] |
| 3 | $\tilde{S}_{DR}\{\lambda \sin\alpha + f\cos\alpha, \lambda\cos\alpha - f\sin\alpha\}$ [rotation] | $\tilde{f}(t)$ | $\tilde{S}_{DR}\{f, \lambda\}$ | $\frac{1}{\sqrt{\cos\alpha}} \exp\left[j\frac{t^2 \cdot \tan\alpha}{2}\right] \int_{-\infty}^{\infty} \tilde{F}\{f\} \times \exp\left[j2\pi\left[\pi t^2 \cdot \tan\alpha + \frac{ft}{\cos\alpha}\right]\right] df$ [see (10.129)] |

**Table 13.2 Typical Equivalent Channels**

### 13.3.6 Summary of Detection of Doubly-Spread Signals

In this section we have studied the detection of doubly-spread targets and communication over doubly-spread channels. In the Section 13.3.1 we formulated the model and specified the optimum receiver and its performance in terms of integral equations and differential equations. Because the problem was one of detecting complex Gaussian processes in complex Gaussian noise, the equations from Section 11.2.1 were directly applicable. The difficulty arose when we tried solve the integral equations that specified the optimum receiver.

In Section 13.3.2 we developed several approximate models. The reason for developing these models is that they reduced the problem to a format that we had encountered previously and could analyze exactly. In particular, we developed a tapped-delay line model, a general orthogonal series model, and an approximate differential-equation model. Each model had advantages and disadvantages, and the choice of which one to use depended on the particular situation.

In Section 13.3.3 we studied a binary communication problem in detail. In addition to obtaining actual results of interest, it provided a concrete example of the techniques involved. Because of the *relative* simplicity of the binary symmetric problem, it is a useful tool for obtaining insight into more complicated problems.

In Section 13.3.4, we studied the LEC problem. In this case the optimum receiver can be completely specified and its performance evaluated. The LEC receiver also suggested suboptimum receiver structures for other problems.

In section 13.3.5, we discussed some related topics briefly. This completes our discussion of the general detection problem. In the next section we consider the parameter estimation problem.

## 13.4 PARAMETER ESTIMATION FOR DOUBLY-SPREAD TARGETS

In this section we consider the problem of estimating the parameters of a doubly-spread target. The model of interest is a straightforward extension of the model of the detection problem in Section 13.1. The complex envelope of the received waveform is

$$\tilde{r}(t) = \tilde{s}(t, \mathbf{A}) + \tilde{w}(t), \qquad T_i \le t \le T_f, \tag{303}$$

where $\tilde{s}(t, \mathbf{A})$, given $\mathbf{A}$, is a sample function of a zero-mean complex Gaussian process whose covariance function is

$$\tilde{K}_{\tilde{s}}(t, u : \mathbf{A}) = E_t \int_{-\infty}^{\infty} \tilde{f}(t - \lambda)\tilde{K}_{DR}(t - u, \lambda : \mathbf{A})\tilde{f}^*(u - \lambda)\, d\lambda. \tag{304}$$

The additive noise $\tilde{w}(t)$ is a sample function of a white Gaussian noise process. The vector parameter **A** is either a nonrandom unknown vector or a value of a random vector that we want to estimate. We consider only the nonrandom unknown parameter problem in the text. Typical parameters of interest are the amplitude of the scattering function or the mean range and mean Doppler of a doubly-spread target.

To find the maximum likelihood estimate of **A**, we construct the likelihood function and choose the value of **A** at which it is maximum. Because the expression for the likelihood function can be derived by a straightforward modification of the analysis in Chapter 6 and Section 11.4, we can just state the pertinent results.

$$l(\mathbf{A}) = l_R(\mathbf{A}) + l_B(\mathbf{A}), \tag{305}$$

where

$$l_R(\mathbf{A}) = \frac{1}{N_0} \iint_{T_i}^{T_f} \tilde{r}^*(t)\tilde{h}_o(t, u:\mathbf{A})\tilde{r}(u)\, dt\, du \tag{306}$$

and

$$l_B(\mathbf{A}) = -\frac{1}{N_0} \int_{T_i}^{T_f} \xi_P(t:\mathbf{A})\, dt. \tag{307}$$

The filter $\tilde{h}_o(t, u:\mathbf{A})$ is specified by the equation

$$N_0\tilde{h}_o(t, u:\mathbf{A}) + \int_{T_i}^{T_f} \tilde{h}_o(t, z:\mathbf{A})\tilde{K}_{\tilde{s}}(z, u:\mathbf{A})\, dz = \tilde{K}_{\tilde{s}}(t, u:\mathbf{A}),$$
$$T_i \le t, u \le T_f. \tag{308}$$

The function $\tilde{\xi}_P(t:\mathbf{A})$ is the realizable minimum mean-square error in estimating $\tilde{s}(t:\mathbf{A})$, assuming that **A** is known. Notice that $l_B(\mathbf{A})$ is usually a function of **A** and cannot be neglected.

A second realization for $l_R(\mathbf{A})$ is obtained by factoring $\tilde{h}_0(t, u:\mathbf{A})$ as

$$\tilde{h}_o(t, u:\mathbf{A}) = \int_{T_i}^{T_f} \tilde{h}^{[\frac{1}{2}]*}(z, t:\mathbf{A})\tilde{h}^{[\frac{1}{2}]}(z, u:\mathbf{A})\, dz, \qquad T_i \le t, u \le T_f. \tag{309}$$

Then

$$l_R(\mathbf{A}) = \frac{1}{N_0} \int_{T_i}^{T_f} dz \left| \int_{T_i}^{T_f} \tilde{h}^{[\frac{1}{2}]}(z, t:\mathbf{A})\tilde{r}(t)\, dt \right|^2. \tag{310}$$

This is the familiar filter-squarer-integrator realization.

A third realization is

$$l_R(\mathbf{A}) = \frac{1}{N_0} \int_{T_i}^{T_f} \left(2\,\mathrm{Re}\,[\tilde{r}^*(t)\hat{\tilde{s}}_r(t:\mathbf{A})] - |\hat{\tilde{s}}_r(t:\mathbf{A})|^2\right) dt, \tag{311}$$

where $\hat{\tilde{s}}_r(t:\mathbf{A})$ is the realizable minimum mean-square estimate of $\tilde{s}(t:\mathbf{A})$, assuming that $\mathbf{A}$ is known. This is the optimum realizable filter realization.

We see that these realizations are analogous to the realizations encountered in the detection problem. Now we must find the realization for a set of values of $\mathbf{A}$ that span the range of possible parameter values. In the general case, we must use one of the approximate target models (e.g., a tapped delay-line model or a general orthogonal series model) developed in Section 13.2 to find the receiver. The computations are much more lengthy, because we must do them for many values of $\mathbf{A}$, but there are no new concepts involved.

In the following sections we consider a special case in which a more direct solution can be obtained. This is the low-energy-coherence (LEC) case, which we have encountered previously in Section 13.3.4.

There are four sections. In Section 13.4.1, we give the results for the general parameter estimation problem under the LEC condition. In Section 13.4.2, we consider the problem of estimating the amplitude of an otherwise known scattering function. In Section 13.4.3, we consider the problem of estimating the mean range and mean Doppler of a doubly-spread target. Finally, in Section 13.4.4, we summarize our results.

### 13.4.1 Estimation under LEC Conditions

The basic derivation is identical with that in Chapter 6, and so we simply state the results. The LEC condition is

$$\tilde{\lambda}_i(\mathbf{A}) \ll N_0, \tag{312a}$$

for all $\mathbf{A}$ in the parameter space and $i$. The $\tilde{\lambda}_i(\mathbf{A})$ are the eigenvalues of $\tilde{K}_{\tilde{s}}(t, u:\mathbf{A})$. Under these restrictions,

$$\begin{aligned} l(\mathbf{A}) = {} & \left(\frac{1}{N_0}\right)^2 \iint_{T_i}^{T_f} \tilde{r}^*(t)\tilde{K}_{\tilde{s}}(t, u:\mathbf{A})\tilde{r}(u)\,dt\,du \\ & - \left(\frac{1}{N_0}\right)\int_{T_i}^{T_f} \tilde{K}_{\tilde{s}}(t, t:\mathbf{A})\,dt - \frac{1}{2}\left(\frac{1}{N_0}\right)^2 \iint_{T_i}^{T_f} |\tilde{K}_{\tilde{s}}(t, u:\mathbf{A})|^2\,dt\,du. \end{aligned} \tag{312b}$$

This result is analogous to (7.136).

For simplicity, we assume that $T_i = -\infty$ and $T_f = \infty$ in the remainder of our discussion. Observe that $\tilde{f}(t)$ has unit energy, so that $\tilde{s}(t:\mathbf{A})$ is a nonstationary process whose energy has a finite expected value. Specifically,

$$E\left[\int_{-\infty}^{\infty} |\tilde{s}(t:\mathbf{A})|^2\,dt\right] = E_t \iint_{-\infty}^{\infty} \tilde{S}_{DR}\{f, \lambda:\mathbf{A}\}\,df\,d\lambda. \tag{313a}$$

Thus, the infinite observation interval does not lead to a singular problem, as it would if $\tilde{s}(t:\mathbf{A})$ were stationary. Substituting (304) into (312*b*) gives

$$\begin{aligned}
l(\mathbf{A}) = {} & \frac{E_t}{N_0^2}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} d\lambda\, \tilde{r}^*(t)\tilde{f}(t-\lambda)\tilde{K}_{DR}(t-u,\lambda:\mathbf{A})\tilde{f}^*(u-\lambda)\tilde{r}(u) \\
& - \frac{E_t}{N_0}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\lambda\, |\tilde{f}(t-\lambda)|^2\, \tilde{K}_{DR}(0,\lambda:\mathbf{A}) \\
& - \frac{E_t^2}{2N_0^2}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} d\lambda_1 \int_{-\infty}^{\infty} d\lambda_2 \tilde{f}(t-\lambda_1)\tilde{K}_{DR}(t-u,\lambda_1:\mathbf{A}) \\
& \times \tilde{f}^*(u-\lambda_1)\tilde{f}^*(t-\lambda_2)\tilde{K}_{DR}(t-u,\lambda_2:\mathbf{A})\tilde{f}(u-\lambda_2).
\end{aligned} \tag{313b}$$

The last two terms are bias terms, which can be written in a simpler manner. We can write the second term in (313*b*) as

$$\begin{aligned}
l_B^{[1]}(\mathbf{A}) &= -\frac{E_t}{N_0}\int_{-\infty}^{\infty} d\lambda \quad {}_{DR}(0,\lambda:\mathbf{A})\int_{-\infty}^{\infty} |\tilde{f}(t-\lambda)|^2\, dt \\
&= -\frac{E_t}{N_0}\int_{-\infty}^{\infty} d\lambda\, \tilde{K}_{DR}(0,\lambda:\mathbf{A}) = -\frac{\bar{E}_r(\mathbf{A})}{2N_0}.
\end{aligned} \tag{314}$$

Here, $\bar{E}_r(\mathbf{A})$ is the average received energy written as a function of the unknown parameter $\mathbf{A}$.

To simplify the third term, we use the two-frequency correlation function $\tilde{R}_{DR}(\tau, v:\mathbf{A})$. Recall from (21) that

$$\tilde{K}_{DR}(\tau,\lambda:\mathbf{A}) = \int_{-\infty}^{\infty} \tilde{R}_{DR}\{\tau, v:\mathbf{A}\}e^{j2\pi v\lambda}\, dv. \tag{315}$$

Using (315) in the last term of (313*b*) and performing a little manipulation, we find that the third term can be written as

$$l_B^{[2]}(\mathbf{A}) = -\frac{E_t^2}{2N_0^2}\iint_{-\infty}^{\infty} dx\, dy\, \theta\{x,y\}\, |\tilde{R}_{DR}\{x,y:\mathbf{A}\}|^2, \tag{316}$$

where $\theta\{\cdot,\cdot\}$ is the ambiguity function of $\tilde{f}(t)$. We denote the sum of the last two terms in (313*b*) as $l_B(\mathbf{A})$. Thus,

$$l_B(\mathbf{A}) = -\frac{\bar{E}_r(\mathbf{A})}{2N_0} - \frac{E_t^2}{4N_0^2}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, \theta\{x,y\}\, |\tilde{R}_{DR}\{x,y:\mathbf{A}\}|^2. \tag{317}$$

The last step is to find a simpler realization of the first term in (313*b*). The procedure here is identical with that in Section 13.3.4. We factor $\tilde{K}_{DR}(t-u,\lambda:\mathbf{A})$, using the relation

$$\tilde{K}_{DR}(t-u,\lambda:\mathbf{A}) = \int_{-\infty}^{\infty} \tilde{K}_{DR}^{[1/2]*}(z-t,\lambda:\mathbf{A})\tilde{K}_{DR}^{[1/2]}(z-u,\lambda:\mathbf{A})\, dz. \tag{318}$$

Since the time interval is infinite and $\tilde{K}_{DR}(\tau, \lambda:\mathbf{A})$ is stationary, we can find a realizable (with respect to $\tau$) $\tilde{K}_{DR}^{[1/2]}(\tau, \lambda:\mathbf{A})$ by spectrum factorization. In the frequency domain

$$\tilde{S}_{DR}^{[1/2]}\{f, \lambda\} = [\tilde{S}_{DR}\{f, \lambda\}]^{+}. \tag{319}$$

Substituting (318) into (313*b*) and denoting the first term by $l_R(\mathbf{A})$, we have

$$l_R(\mathbf{A}) = \frac{E_t}{N_0^{\,2}} \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} d\lambda \left| \int_{-\infty}^{\infty} \tilde{K}_{DR}^{[1/2]}(z - u, \lambda:\mathbf{A}) \tilde{f}^*(u - \lambda) \tilde{r}(u)\, du \right|^2. \tag{320}$$

Combining (320) and (317) gives an expression for $l(\mathbf{A})$, which is

$$\boxed{\begin{aligned} l(\mathbf{A}) = {} & \frac{E_t}{N_0^{\,2}} \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} d\lambda \left| \int_{-\infty}^{\infty} \tilde{K}_{DR}^{[1/2]}(z - u, \lambda:\mathbf{A}) \tilde{f}^*(u - \lambda) \tilde{r}(u)\, du \right|^2 \\ & - \frac{\bar{E}_r(\mathbf{A})}{N_0} - \frac{E_t^{\,2}}{2N_0^{\,2}} \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} dv\, \theta\{\tau, v\}\, |\tilde{R}_{DR}\{\tau, v:\mathbf{A}\}|^2. \end{aligned}} \tag{321}$$

For *each* value of $\mathbf{A}$ we can realize $l_R(\mathbf{A})$ approximately by sampling in $\lambda$ and replacing the integral in $\lambda$ by a sum. We then add $l_B(\mathbf{A})$ to obtain an approximate likelihood function. This realization is an obvious modification of the structure in Fig. 13.30. Notice that we must carry out this calculation for a set of values of $\mathbf{A}$, so that the entire procedure is quite tedious.

We now have the receiver structure specified. The performance analysis for the general case is difficult. The Cramér-Rao bound gives a bound on the variance of any unbiased estimate. For a *single* parameter, we differentiate (312*b*) to obtain

$$\operatorname{Var}[a - A] \geq \frac{N_0^{\,2}}{\displaystyle\iint_{-\infty}^{\infty} dt\, du\, |[\partial \tilde{K}_{\tilde{s}}(t, u:A)]/\partial A|^2}. \tag{322}$$

For multiple parameters, we modify (7.155) to obtain the elements in the information matrix as

$$J_{ij}(\mathbf{A}) = \frac{1}{N_0^{\,2}} \operatorname{Re} \iint_{T_i}^{T_f} dt\, du\, \frac{\partial \tilde{K}_{\tilde{s}}(t, u:\mathbf{A})}{\partial A_i} \frac{\partial \tilde{K}_{\tilde{s}}^*(t, u:\mathbf{A})}{\partial A_j}. \tag{323}$$

Substituting (304) into (323) gives the $\tilde{J}_{ij}(\mathbf{A})$ for the doubly-spread target. Using (316), we can write $\tilde{J}_{ij}(\mathbf{A})$ compactly as

$$\boxed{\tilde{J}_{ij}(\mathbf{A}) = \frac{E_t^{\,2}}{N_0^{\,2}} \operatorname{Re} \iint\limits_{-\infty}^{\infty} \frac{\partial \tilde{R}_{DR}\{x, y:\mathbf{A}\}}{\partial A_i}\, \theta\{x, y\}\, \frac{\partial \tilde{R}_{DR}^{*}\{x, y:\mathbf{A}\}}{\partial A_j}\, dx\, dy.} \tag{324}$$

The principal results of this section are the expressions in (321) and (324). They specify the optimum receiver and the performance bound, respectively. We next consider two typical estimation problems.

### 13.4.2 Amplitude Estimation

In this section we consider the problem of estimating the amplitude of an otherwise known scattering function.† We assume that

$$\tilde{K}_{DR}(t-u, \lambda:A) = A\tilde{K}_{DR}(t-u, \lambda), \tag{325}$$

where $\tilde{K}_{DR}(t-u, \lambda)$ is normalized such that

$$\int_{-\infty}^{\infty} \tilde{K}_{DR}(0, \lambda)\, d\lambda = 1. \tag{326}$$

Thus,

$$A = \frac{\bar{E}_r}{E_t}. \tag{327}$$

The covariance function of the received signal process is

$$\tilde{K}_{\tilde{s}}(t, u:A) = E_t A \int_{-\infty}^{\infty} \tilde{f}(t-\lambda)\tilde{K}_{DR}(t-u, \lambda)\tilde{f}^*(u-\lambda)\, d\lambda \triangleq A\tilde{K}_{\tilde{s}}(t, u). \tag{328}$$

The parameter $A$ is an unknown positive number.

In this case, the likelihood function in (312*b*) has a single maximum, which is located at

$$\hat{a}_0 = \frac{\displaystyle\iint\limits_{-\infty}^{\infty} \tilde{r}^*(t)\tilde{K}_{\tilde{s}}(t, u)\tilde{r}(u)\, dt\, du - N_0\int_{-\infty}^{\infty} \tilde{K}_{\tilde{s}}(t, t)\, dt}{\displaystyle\iint\limits_{-\infty}^{\infty} |\tilde{K}_s(t, u)|^2\, dt\, du}. \tag{329}$$

† This problem was solved originally by Price [39]. Our problem is actually a degenerate case of the problem he considers.

Since $\hat{a}_0$ may be negative, the maximum likelihood estimate is

$$\hat{a}_{ml} = \max\,[0, \hat{a}_0]. \tag{330}$$

We discussed the truncation problem in detail in Section 7.1.2. For simplicity we assume that the parameters are such that the truncation effect is negligible. Using (326), (328), and (315) in (329), we obtain

$$\hat{a}_0 = \frac{\displaystyle\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} d\lambda\, \tilde{r}^*(t)\tilde{f}(t-\lambda)\tilde{K}_{DR}(t-u, \lambda)\tilde{f}^*(u-\lambda)\tilde{r}(u) - N_0}{\displaystyle\int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} dv\, \theta\{\tau, v\}\, |\tilde{R}_{DR}\{\tau, v\}|^2}. \tag{331}$$

It follows easily that $\hat{a}_0$ is unbiased. An approximate receiver realization is shown in Fig. 13.33.

If we neglect the bias on $\hat{a}_{ml}$, we can bound its normalized variance by $J_n^{-1}$, where $J_n$ is obtained from (324) as

$$J_n \triangleq \frac{J(A)}{A^2} = \frac{E_t^{\,2}}{N_0^{\,2}} \iint_{-\infty}^{\infty} \theta\{\tau, v\}\, |\tilde{R}_{DR}\{\tau, v\}|^2\, d\tau\, dv. \tag{332}$$

It is worthwhile observing that we can compute the variance of $\hat{a}_0$ exactly (see Problem 13.4.4). The result is identical with $J_n^{-1}$, except for a term that can be neglected when the LEC condition holds.

To illustrate the ideas involved, we consider an example.

**Example.** We assume that the target has the doubly Gaussian scattering function in Fig. 13.4. Then

$$\tilde{S}_{DR}\{f, \lambda\} = \frac{1}{2\pi BL} \exp\left\{-\frac{f^2}{2B^2} - \frac{\lambda^2}{2L^2}\right\}, \qquad -\infty < f, \lambda < \infty \tag{333}$$

and

$$\tilde{R}_{DR}\{\tau, v\} = \exp\left(-\frac{(2\pi B)^2\tau^2}{2} - \frac{(2\pi L)^2 v^2}{2}\right), \qquad -\infty < \tau, v < \infty. \tag{334}$$

To simplify the algebra, we assume that $\tilde{f}(t)$ is a Gaussian pulse.

$$\tilde{f}(t) = \left(\frac{1}{\pi T^2}\right)^{1/4} \exp\left(-\frac{t^2}{2T^2}\right), \qquad -\infty < t < \infty. \tag{335}$$

Then, from (10.28),

$$\theta\{\tau, f\} = \exp\left[-\frac{1}{2}\left(\frac{\tau^2}{T^2} + T^2(2\pi f)^2\right)\right], \qquad -\infty < \tau, f < \infty. \tag{336}$$

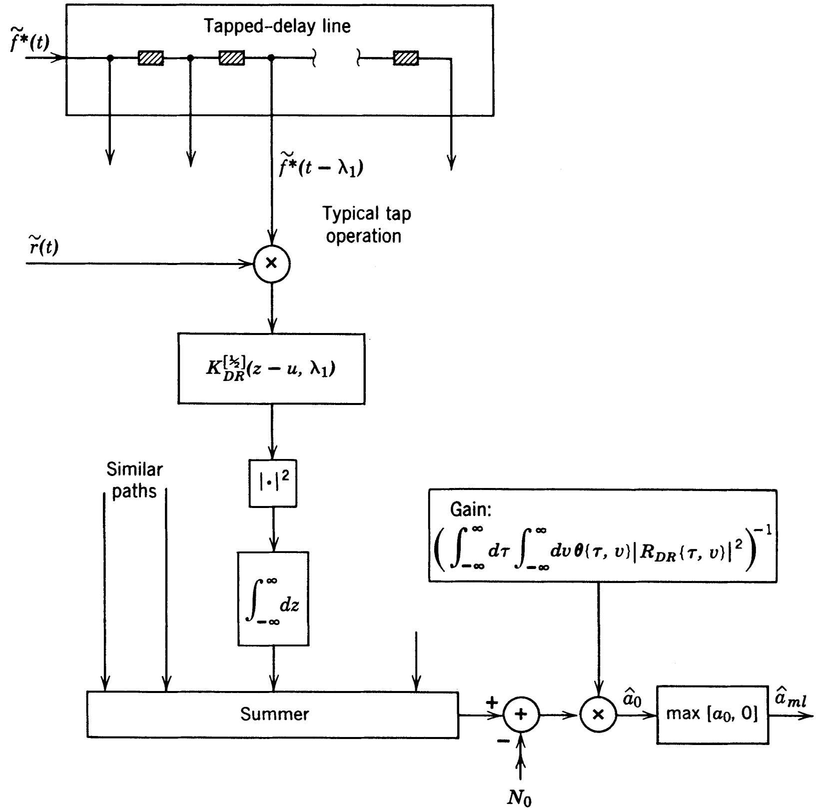


**Fig. 13.33 Maximum likelihood estimator of the amplitude of a scattering function under LEC conditions.**

Substituting in (332), we have

$$J_n = \left(\frac{E_t}{N_0}\right)^2 \iint_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\left[\tau^2\left(\frac{1}{T^2} + 2(2\pi B)^2\right) + v^2((2\pi T)^2 + 2(2\pi L)^2)\right]\right\} d\tau\, dv. \tag{337}$$

Integrating, we obtain

$$J_n = \left(\frac{E_t}{N_0}\right)^2 \left[\left(1 + 2\left(\frac{L}{T}\right)^2\right)(1 + 2(2\pi BT)^2)\right]^{-1/2}. \tag{338}$$

Looking at (338), we see that $J_n$ will be maximized [and therefore the variance bound in (322) will be minimized] by some intermediate value of $T$. Specifically, the maximum occurs at

$$T = \sqrt{\frac{L}{2\pi B}}. \tag{339}$$

Comparing (339) and (281), we see that if we let

$$k = 2\pi B, \tag{340}$$

then this value of $T$ corresponds to the point of minimum diversity in the output signal process. (Notice that $B$ and $k$ have different meanings in the two scattering functions.)

Intuitively we would expect the minimum diversity point to be optimum, because of the original LEC assumption. This is because there is an optimum "energy per eigenvalue $/N_0$" value in the general amplitude estimation problem (see Problem 13.4.8). The LEC condition in (311) means that we already have the energy distributed among too many eigenvalues. Thus we use the fewest eigenvalues possible. When the LEC condition does not hold, a curve of the form shown in Fig. 13.24*b* would be obtained. If we use the value of $T$ in (339), then

$$J_n = \left(\frac{E_t}{N_0}\right)^2 (1 + 4\pi BL)^{-1} \tag{341}$$

and

$$\text{Var}\,[\hat{a}_{ml} - A] \geq \left(\frac{N_0}{E_t}\right)^2 (1 + 4\pi BL). \tag{342}$$

We see that the variance bound increases linearly with the $BL$ product for $BL > 1$. This linear behavior with $BL$ also depends on the LEC condition and does not hold in general.

This completes our discussion of the amplitude estimation problem. We were able to obtain a closed-form solution for $\hat{a}_{ml}$ because $l(A)$ had a unique maximum. We now consider a different type of estimation problem.

### 13.4.3 Estimation of Mean Range and Doppler

In this subsection we consider the problem of estimating the mean range and mean Doppler of a doubly-spread target. A typical configuration in the $\tau, f$ plane is shown in Fig. 13.34. We denote the mean range by

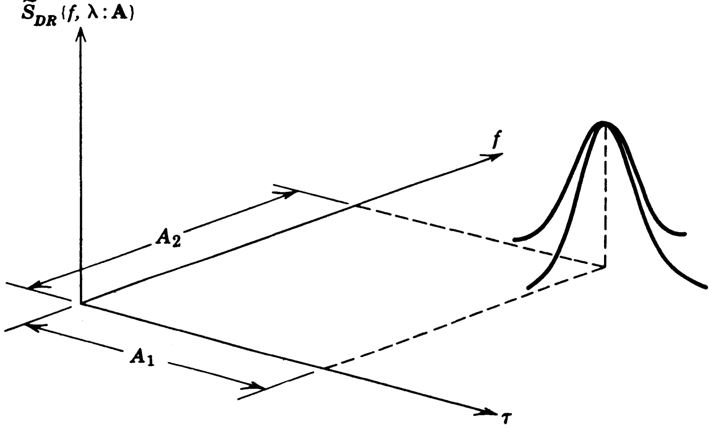


**Fig. 13.34 Target location in $\tau, f$ plane.**

$A_1$, and the mean Doppler by $A_2$. The scattering function is denoted by

$$\tilde{S}_{DR}\{f, \lambda:\mathbf{A}\} = \tilde{S}_{D_0R_0}\{f - A_2, \lambda - A_1\}, \tag{343}$$

where the scattering function on the right side of (343) is defined to have zero mean range and zero mean Doppler. Another function that will be useful is the two-frequency correlation function, which can be written as

$$\tilde{R}_{DR}\{\tau, v:\mathbf{A}\} = \tilde{R}_{D_0R_0}\{\tau, v\} \exp\left[-j2\pi v A_1 + j2\pi\tau A_2\right]. \tag{344}$$

The purpose of (343) and (344) is to express the parametric dependence explicitly.

The returned signal is given by (303). To find the maximum likelihood estimate of $\mathbf{A}$, we first divide the $\tau$, $\omega$ plane into a set of range-Doppler cells. We denote the coordinates of the center of the $i$th cell as $\mathbf{A}_i$. We next construct $l(\mathbf{A}_i)$ for each cell and choose that value of $\mathbf{A}_i$ where $l(\mathbf{A}_i)$ is maximum.

First, we consider the general case and *do not* impose the LEC condition. Then, $l(\mathbf{A}_i)$ is given by (305)–(307). As before, we let $T_i = -\infty$ and $T_f = \infty$. Looking at (307), we see that $\xi_P(t:\mathbf{A})$ does not depend on the mean range or Doppler, so that we do not need to compute $l_B(\mathbf{A})$. Thus,

$$l(\mathbf{A}_i) = l_R(\mathbf{A}_i) = \frac{1}{N_0} \iint\limits_{-\infty}^{\infty} \tilde{r}^*(t)\tilde{h}_o(t, u:\mathbf{A}_i)\tilde{r}(u)\, dt\, du, \tag{345}$$

where $\tilde{h}_o(t, u:\mathbf{A}_i)$ is specified by (308) with $\mathbf{A} = \mathbf{A}_i$. For each cell we must solve (308) [or find one of the equivalent forms given in (309)–(311)]. Actually to carry out the solution, we would normally have to use one of the orthogonal series models in Section 13.3.2.

In analyzing the performance, we must consider both global and local accuracy. To study the global accuracy problem we use the spread ambiguity function that we defined in (11.181). For doubly-spread targets the definition is

$$\begin{aligned}\theta_{\Omega_{DR}}(\mathbf{A}_a, \mathbf{A}) = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} d\lambda \tilde{f}^*(t - \lambda) \\ \times \tilde{h}_o(t, u:\mathbf{A})\tilde{K}_{DR}(t - u, \lambda:\mathbf{A}_a)\tilde{f}(u - \lambda), \end{aligned} \tag{346}$$

where $\mathbf{A}_a$ corresponds to the actual mean range and mean Doppler of the target. To analyze the local accuracy, we use the Cramér-Rao bound. There is no conceptual difficulty in carrying out these analyses, but the calculations are involved.

When the LEC condition is satisfied, the solution is appreciably simpler. From (320), the likelihood function is

$$l_R(\mathbf{A}_i) = \frac{E_t}{N_0^2}\int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} d\lambda \left| \int_{-\infty}^{\infty} \tilde{K}_{DR}^{[1/2]}(z-u, \lambda:\mathbf{A}_i)\tilde{f}^*(u-\lambda)\tilde{r}(u)\, du \right|^2. \tag{347}$$

The spread ambiguity function under LEC conditions is

$$\boxed{\begin{aligned}\theta_{\Omega_{DR},\mathrm{LEC}}(\mathbf{A}_a, \mathbf{A}) &= \frac{E_t^2}{N_0^2}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} d\lambda_1 \int_{-\infty}^{\infty} d\lambda_2 \tilde{f}(t-\lambda_1) \\ &\quad \times \tilde{K}_{DR}(t-u, \lambda_1:\mathbf{A})\tilde{f}^*(u-\lambda_1)\tilde{f}^*(t-\lambda_2) \\ &\quad \times \tilde{K}_{DR}^*(t-u, \lambda_2:\mathbf{A}_a)\tilde{f}(u-\lambda_2) \\ &= \frac{E_t^2}{N_0^2}\iint_{-\infty}^{\infty} dx\, dy \tilde{R}_{DR}\{x, y:\mathbf{A}\}\theta\{x, y\}\tilde{R}_{DR}^*\{x, y:\mathbf{A}_a\}.\end{aligned}} \tag{348}$$

Notice that

$$\tilde{J}_{ij}(\mathbf{A}) = \left.\frac{\partial^2\theta_{\Omega_{DR},\mathrm{LEC}}(\mathbf{A}_a, \mathbf{A})}{\partial A_i\, \partial A_{aj}}\right|_{\mathbf{A}=\mathbf{A}_a}, \tag{349}$$

which is identical with (324). To evaluate the Cramér-Rao bound, we use (344) in (324) to obtain

$$\tilde{J}_{11}(\mathbf{A}) = \frac{E_t^2}{N_0^2}\iint_{-\infty}^{\infty}(2\pi v)^2\theta\{\tau, v\}\, |\tilde{R}_{D_0R_0}\{\tau, v\}|^2\, d\tau\, dv, \tag{350}$$

$$\tilde{J}_{12}(\mathbf{A}) = -\frac{E_t^2}{N_0^2}\iint_{-\infty}^{\infty}(2\pi)^2\tau v\theta\{\tau, v\}\, |\tilde{R}_{D_0R_0}\{\tau, v\}|^2\, d\tau\, dv, \tag{351}$$

and

$$\tilde{J}_{22}(\mathbf{A}) = \frac{E_t^2}{N_0^2}\iint_{-\infty}^{\infty}(2\pi\tau)^2\theta\{\tau, v\}\, |\tilde{R}_{D_0R_0}\{\tau, v\}|^2\, d\tau\, dv. \tag{352}$$

As we would expect, the error performance depends on both the signal ambiguity function and the target-scattering function. Some typical situations are analyzed in the problems.

### 13.4.4 Summary

In this section we studied parameter estimation for doubly-spread targets. We first formulated the general estimation problem and developed the expressions for the likelihood ratio. The resulting receiver was closely related to those encountered earlier in the detection problem.

In the remainder of the section, we emphasized the low-energy-coherence case. In Section 13.4.1 we developed the expressions for the likelihood function and the Cramér-Rao bound under the LEC assumption. In Section 13.4.2 we found an explicit solution for the estimate of the amplitude of the scattering function. A simple example illustrated the effect of the pulse length and the $BL$ product. In Section 13.4.3, we studied the problem of estimating the mean range and Doppler of a doubly-spread target. This problem is a generalization of the range-Doppler estimation problem that we studied in Chapter 10.

Our goal in this section was to illustrate some of the important issues in the estimation problem. Because of the similarity to the detection problem, a detailed discussion was not necessary.

## 13.5 SUMMARY OF DOUBLY-SPREAD TARGETS AND CHANNELS

In this chapter we have studied targets and channels that are spread in both range and Doppler. The complex envelope of the signal returned from the target is

$$\tilde{s}(t) = \sqrt{E_t}\int_{-\infty}^{\infty} \tilde{f}(t-\lambda)\tilde{b}(t,\lambda)\,d\lambda. \tag{353}$$

The target reflection process is a sample function of zero-mean complex Gaussian random processes, which can be characterized in two ways:

1. By a scattering function $\tilde{S}_{DR}\{f,\lambda\}$ or an equivalent form such as $\tilde{K}_{DR}(\tau,\lambda)$, $\tilde{R}_{DR}\{\tau,v\}$, or $\tilde{P}_{DR}\{f,v\}$.
2. By a distributed state-variable description in which the state equations are ordinary differential equations containing the spatial variable $\lambda$ as a parameter and $\tilde{s}(t)$ is related to the state vector by a modulation functional.

After formulating the model and discussing its general characteristics, we looked at three areas in which we encounter doubly-spread targets.

In Section 13.2 we discussed the problem of resolution in a dense environment. Here, the desired signal was a nonfluctuating point target and the interference was a doubly-spread environment. We examined both conventional and optimum receivers and compared their performance. We found that when a conventional matched filter was used, the spread interference entered through a double convolution of the signal ambiguity function and the target-scattering function. As in the discrete resolution problem, examples indicated that proper signal design is frequently more important than optimum receiver design.

In Section 13.3 we discussed the problem of detecting the return from a doubly-spread target and the problem of digital communication over a doubly-spread channel. After formulating the general problem, we developed several approximate target/channel models using orthogonal series expansions. The purpose of these models was to reduce the problem to a form that we could analyze. The tapped-delay line model was the easiest to implement, but the general orthogonal series model offered some computation advantages. We next studied the binary communication problem. For underspread channels we found signals that enabled us to approach the performance bound for any system. For overspread channels we could only approach the bound for large $\bar{E}_r/N_0$ with the simple signals we considered. To verify our intuitive argument, we carried out a detailed performance analysis for a particular system. The effect of the signal parameters and the scattering function parameters on the performance of a binary communication system was studied. Finally, we indicated the extensions to several related problems.

In Section 13.4 we studied the problem of estimating the parameters of a doubly-spread target. We first formulated the general estimation problem and noted its similarity to the detection problem in Section 13.3. We then restricted our attention to the LEC case. Two particular problems, amplitude estimation and mean range and Doppler estimation, were studied in detail.

There are two important problems which we have not considered that should be mentioned. The first is the problem of measuring the instantaneous behavior of $\tilde{b}(t, \lambda)$. We encountered this issue in the estimator-correlator receiver but did not discuss it fully. The second problem is that of measuring (or estimating) the scattering function of the target or channel. We did not discuss this problem at all. An adequate discussion of these problems would take us too far afield; the interested reader should consult the references (e.g., [39]–[52] and [66]–[69]).

This completes our discussion of doubly-spread targets and channels. In the next chapter we summarize our discussion of the radar-sonar problem.

## 13.6 PROBLEMS

### P.13.1 Target Models

**Problem 13.1.1.** Read the discussion in the Appendix of [5]. Verify that the scattering function of a rough rotating sphere is as shown in Fig. 13.2.

**Problem 13.1.2.** Consider the target shown in Fig. P.13.1. The antenna pattern is constant over the target dimensions. The discs are perpendicular to the axis of propagation (assume plane wave propagation). The carrier is at $f_c$ cps. The dimensions $x_0$, $y_0$, $d_0$, and $d_1$ are in meters. The rates of rotation, $g_0$ and $g_1$, are in revolutions per second.

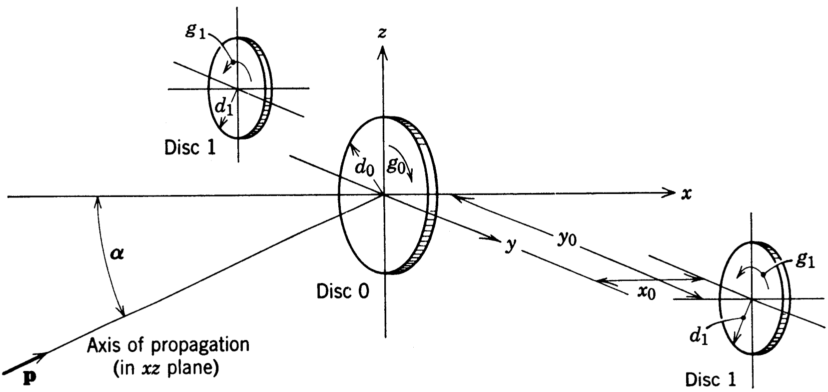


**Fig. P.13.1**

The reflectivity of disc 0 is uniform and equals $\rho_0$ per $m^2$. The reflectivities of the two disc 1's are constant, $\rho_1$ per $m^2$. These reflectivities are along the **p**-axis. Assume $y_0 \gg x_0$. The target geometry is symmetric about the $xz$ plane.

Compute the scattering function of the target as a function of $\alpha$. Sketch your result.

**Problem 13.1.3.** The quantities $\sigma_R{}^2$, $\sigma_D{}^2$, and $\rho_{DR}$ are defined in terms of $\tilde{S}_{DR}$, $\{f, \lambda\}$. Find equivalent expressions for them in terms of $\tilde{P}_{DR}\{f, v\}$, $\tilde{R}_{DR}\{\tau, v\}$, and $\tilde{K}_{DR}(\tau, \lambda)$.

**Problem 13.1.4.** Assume that

$$\tilde{S}_{DR}\{f, \lambda\} = \frac{2\sigma_b{}^2}{2\pi BL} \exp\left[-\frac{f^2}{2B^2} - \frac{\lambda^2}{2L^2}\right], \qquad -\infty < f < \infty, \quad -\infty < \lambda < \infty.$$

1. Find $\tilde{P}_{DR}\{f, v\}$, $\tilde{R}_{DR}\{\tau, v\}$, and $\tilde{K}_{DR}(\tau, \lambda)$.
2. Calculate $\sigma_R{}^2$, $\sigma_D{}^2$, and $\rho_{DR}$.

**Problem 13.1.5.** Assume that

$$\tilde{S}_{DR}\{f, \lambda\} = \frac{2\sigma_b{}^2}{2\pi BL(1-\rho^2)^{1/2}} \times \exp\left[-\frac{L^2(f-m_D)^2 - 2BL\rho(f-m_D)(\lambda-m_R) + B^2(\lambda-m_R)^2}{2B^2L^2(1-\rho^2)}\right].$$

1. Find $\tilde{P}_{DR}\{f, v\}$, $\tilde{R}_{DR}\{\tau, v\}$, and $\tilde{K}_{DR}(\tau, \lambda)$.
2. Find the five quantities defined in (14)–(19).

**Problem 13.1.6.** Assume that

$$\tilde{S}_{DR}\{f, \lambda\} = \frac{4k\sigma_b^2}{L[(2\pi f)^2 + k^2]}, \qquad -\infty < f < \infty, \qquad 0 < \lambda < L.$$

Find $\tilde{P}_{DR}\{f, v\}$, $\tilde{R}_{DR}\{\tau, v\}$, and $\tilde{K}_{DR}\{\tau, \lambda\}$.

**Problem 13.1.7.** Assume that

$$\tilde{S}_{DR}\{f, \lambda\} = \begin{cases} \dfrac{2\sigma_b^2}{BL}, & |f| \le \dfrac{B}{2}, \quad |\lambda| \le \dfrac{L}{2}, \\ 0, & \text{elsewhere}. \end{cases}$$

1. Find $\tilde{P}_{DR}\{f, v\}$, $\tilde{R}_{DR}\{\tau, v\}$, and $\tilde{K}_{DR}\{\tau, \lambda\}$.
2. Find $\sigma_R^2$ and $\sigma_D^2$.

**Problem 13.1.8.** Assume that

$$\tilde{S}_{DR}\{f, \lambda\} = \left[\frac{8\sqrt{2}\sigma_b^2 \sin^2(\pi\lambda/L)}{kL[(2\pi f/k)^4 + 1]}\right], \qquad -\infty < f < \infty, \qquad 0 \le \lambda \le L.$$

1. Find $\tilde{P}_{DR}\{f, \nu\}$, $\tilde{R}_{DR}\{\tau, v\}$, and $\tilde{K}_{DR}(\tau, \lambda)$.
2. Find $\sigma_D^2$ and $\sigma_R^2$.

**Problem 13.1.9.** Consider the target process whose scattering function is given in Problem 13.1.8.

1. Describe this process with a differential-equation model.
2. Describe the received signal process $\tilde{s}(t)$ in terms of the results in part 1.

**Problem 13.1.10.** We frequently use the doubly-Gaussian scattering function in Problem 13.1.4. Construct a differential-equation model to represent it approximately. (*Hint:* Recall Case 2 on page I-505.)

**Problem 13.1.11.** Assume that the scattering function is

$$\tilde{S}_{DR}\{f, \lambda\} = \frac{a(\lambda)}{[(j2\pi f)^2 + \tilde{k}_1^2(\lambda)][(j2\pi f)^2 + \tilde{k}_2^2(\lambda)]}, \qquad -\infty < f < \infty, \quad 0 < \lambda < L.$$

1. Sketch the scattering function for various allowable $\tilde{k}_1(\lambda)$, $\tilde{k}_2(\lambda)$, and $a(\lambda)$.
2. Write out the differential equations that characterize this target. (*Hint:* Recall Example 2 in the Appendix, page 594.)

## P.13.2 Detection in Reverberation

### Conventional Receivers

In Problem 13.2.1–13.2.9, we use the model in (69)–(72) and assume that a conventional receiver is used.

**Problem 13.2.1.** The transmitted signal in given in (10.43). The scattering function is

$$\tilde{S}_{DR}\{f, \lambda\} = \frac{1}{2\pi BL(1 - \rho^2)^{1/2}} \exp\left[-\frac{L^2 f^2 - 2BL\rho f\lambda + B^2\lambda^2}{2B^2L^2(1 - \rho^2)}\right].$$

Find $\rho_r$ [see (13.83)] as a function of $E_t$, $N_0$, $B$, $L$, $\rho$, and $T$.

**Problem 13.2.2.** Consider the reverberation model in (132). Assume that

$$\tilde{f}(t) = \sqrt{2\alpha}\, e^{-\alpha t} u_{-1}(t).$$

Calculate $\rho_r$ as a function of $E_t$, $P_c$, $f_d$, and $\alpha$.

**Problem 13.2.3.** Consider the reverberation model in (132).

1. Verify that

$$\rho_r = \frac{E_t P_c}{N_0} \int_{-\infty}^{\infty} \tilde{S}_{\tilde{f}}^{\,2}\{f\}\, df$$

for a zero-velocity target.

2. Choose $\tilde{S}_{\tilde{f}}\{f\}$ subject to the energy constant

$$\int_{-\infty}^{\infty} \tilde{S}_{\tilde{f}}\{f\}\, df = 1,$$

so that $\rho_r$ is minimized.

**Problem 13.2.4.** Consider the reverberation model in (132).

1. Verify that

$$\rho_r = \frac{E_t P_c}{N_0} \int_{-\infty}^{\infty} \tilde{S}_{\tilde{f}}\{f\} \tilde{S}_{\tilde{f}}\{f - f_d\}\, df \tag{P.1}$$

for a target with Doppler shift $f_d$.

2. What type of constraints must be placed on $\tilde{f}(t)$ in order to obtain a meaningful result when we try to minimize $\rho_r$?

3. Assume that we require

$$\int_{-\infty}^{\infty} f^2 \tilde{S}_{\tilde{f}}\{f\}\, df = \sigma_w^{\,2}. \tag{P.2}$$

Minimize $\rho_r$ subject to the constraint in (P.2) and an energy constraint.

**Problem 13.2.5.** Assume that we have the *constant-height* reverberation scattering function shown in Fig. P.13.2. The signal is the pulse train shown in Fig. 10.9.

1. Show how to choose $T_s$, $T_p$, and $n$ to minimize the effect of the reverberation.
2. Calculate $\rho_r$ (13.83) for the signal parameters that you selected.

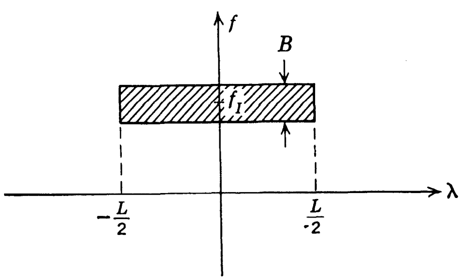


**Fig. P.13.2**

**Problem 13.2.6.** Consider the signal given in (10.145), which has $3N$ parameters to choose. Consider the scattering function in Fig. P.13.2.

1. Write an expression for $\rho_r$ (13.83) in terms of the signal parameters and $B$, $L$, and $f_I$. Assume that $L/T_s$ is an integer for simplicity.
2. Consider the special case of (10.145) in which $\omega_n = 0$, and define

$$\tilde{\mathbf{a}} = \begin{bmatrix} a_1 e^{j\theta_1} \\ \cdot \\ \cdot \\ \cdot \\ a_N e^{j\theta_N} \end{bmatrix}.$$

Express $\rho_r$ in terms of $\tilde{\mathbf{a}}$.

3. We want to minimize $\rho_r$ by choosing $\tilde{\mathbf{a}}$ properly. Formulate the optimization problems and derive the necessary equations.

**Problem 13.2.7.** Repeat parts 2 and 3 of Problem 13.2.6 for the following special cases:

1. We require

$$\omega_n = 0, \qquad n = 1, \ldots, N,$$
$$a_n = 1, \qquad n = 1, \ldots, N.$$

2. We require

$$\theta_n = 0, \qquad n = 1, \ldots, N,$$
$$a_n = 1 \quad \text{or} \quad 0, \qquad n = 1, \ldots, N.$$

**Problem 13.2.8.** We want to estimate the range and Doppler of a nonfluctuating point target in the presence of reverberation. The conventional receiver in Section 10.2 is used Derive a bound on the variance of the range and Doppler estimation errors.

**Problem 13.2.9** In Section 12.3 we developed duality theory. These ideas are also useful in reverberation problems. Assume that a conventional receiver is used.
Derive the dual of the result in (83).

## Optimum Receivers

**Problem 13.2.10.** Consider the model in (101)–(107). One procedure for solving (107) is to approximate the integrals with finite sums. Carry out the details of this procedure and obtain a matrix equation specifying $\tilde{g}(t_i)$, $i = 1, \ldots, N$. Discuss how you selected the sampling interval and the resulting computational requirements.

**Problem 13.2.11.** Consider the model in (101)–(107). The complex envelope of the transmitted signal is given by (10.25), and the scattering function is given by (13.333). Assume that $\tau_d = \omega_d = 0$.

1. Find a series solution to (107) by using Mehler's expansion (e.g., [53] or [54].)
2. Evaluate $\Delta_o$.

**Problem 13.2.12.** Generalize the result in Problem 13.2.11 to include a nonzero target range and Doppler, the transmitted signal in (10.43), and the scattering function in Problem 13.2.1.

**Problem 13.2.13.** One procedure for obtaining an approximate solution to (107) is to model $\tilde{S}_{DR}\{f, \lambda\}$ as a piecewise constant function and then replace the each piecewise constant segment by an impulse that is located at the center of the segment with the same volume as the segment. This reduces the problem to that in Section 10.5.

1. Discuss how one selects the grid dimensions.
2. Carry out the details of the procedure. Use (10.202) to write an explicit solution to the approximate problem. Identify the various matrices explicitly.
3. The performance is given by (10.203). We would like to choose $\tilde{f}(t)$ to maximize $\Delta_o$. What constraints are necessary? Carry out the optimization.

**Problem 13.2.14.** Assume that $\tilde{K}_{DR}(t-u, \lambda)$ can be factored as

$$\tilde{K}_{DR}(t-u, \lambda) = \tilde{K}_{Du}(t-u)\tilde{K}_{Ru}(\lambda).$$

1. Evaluate $\tilde{K}_{\tilde{n}_r}(t, u)$ in (104) for this case.
2. We want to approximate $\tilde{K}_{\tilde{n}_r}(t, u)$ by a separable kernel. What functions would minimize the approximation error? Discuss other choices that might be more practical. Consider, for example,

$$\tilde{K}_{Ru}(\lambda) = \sum_{i=1}^{M} a_i \psi_i(\lambda)$$

as a preliminary expansion.

**Problem 13.2.15.** In this problem we derive the optimum estimator equations in (116)–(121).

1. The first step is to derive the generalization of (I-6.55). The linear operation is

$$\hat{\tilde{\mathbf{x}}}(t, \lambda) = \int_{T_i}^{t} \tilde{\mathbf{h}}_o(t, \tau:\lambda)\tilde{r}(\tau)\, d\tau. \tag{P.1}$$

We want to minimize the realizable MMSE error. Show that the optimum impulse response must satisfy

$$E[\tilde{\mathbf{x}}(t, \lambda)\tilde{r}^*(u)] = \int_{T_i}^{t} \tilde{\mathbf{h}}_o(t, \tau:\lambda)\tilde{K}_{\tilde{r}}(\tau, u)\, d\tau, \qquad T_i < u < t. \tag{P.2}$$

2. Using (P.2) as a starting point, carry out an analysis parallel to that in Section 6.3.2 to obtain (116)–(121).

**Problem 13.2.16.** Consider the scattering function given in (53)–(63). Assume that

$$\tilde{f}(t) = \begin{cases} \dfrac{1}{\sqrt{T}}, & 0 \le t \le T, \\ 0, & \text{elsewhere.} \end{cases}$$

Write out the optimum receiver equations (116)–(121) in detail for this case.

**Problem 13.2.17.** Consider the model in (101)–(104) and assume that $\tau_d = \omega_d = 0$. We use a receiver that computes

$$\tilde{l}_m = \int_{-\infty}^{\infty} \tilde{v}^*(t)\tilde{r}(t)\, dt$$

and compares $|\tilde{l}_m|^2$ with a threshold. The function $\tilde{v}(t)$ is an arbitrary function that we want to choose. Do not confuse $\tilde{v}(t)$ and $\tilde{g}(t)$ in (106). The performance of this receiver is a monotonic function of $\Delta_m$, where

$$\Delta_m = \frac{E[|\tilde{l}_m|^2 \mid H_1] - E[|\tilde{l}_m|^2 \mid H_0]}{E[|\tilde{l}_m|^2 \mid H_0]}$$

[see (9.49)].

1. Derive an expression for $\Delta_m$.
2. Find an equation that specifies the $\tilde{v}(t)$ which maximizes $\Delta_m$. Call the solution $\hat{\tilde{v}}_1(t)$.
3. In [32], this problem is studied from a different viewpoint. Stutt and Spafford define

$$J = E[|\tilde{l}_m|^2 \mid \tilde{n}_r(t) \quad \text{only}].$$

Prove that

$$J = \iiint_{-\infty}^{\infty} \tilde{v}^*(t-\lambda)\tilde{f}(t-\lambda)\tilde{K}_{DR}(t-u,\lambda)\tilde{f}^*(u-\lambda)\tilde{v}(u-\lambda)\,dt\,du\,d\lambda$$

$$= \iint_{-\infty}^{\infty} \tilde{S}_{DR}\{f,\lambda\}\theta_{fv}\{\lambda,-f\}\,d\lambda\,df,$$

where $\theta_{fv}\{\cdot,\cdot\}$ is the cross-ambiguity function defined in (10.222).
4. We want to minimize $J$ subject to the constraints

$$\int_{-\infty}^{\infty} |\tilde{v}(t)|^2\,dt = 1$$

and

$$\int_{-\infty}^{\infty} \tilde{f}^*(t)\tilde{v}(t)\,dt = K,$$

where

$$0 \le |K| \le 1.$$

a. Explain these constraints in the context of the result in part 1.
b. Carry out the minimization using two Lagrange multipliers. Call the solution $\hat{\tilde{v}}_2(t)$.
c. Does

$$\hat{\tilde{v}}_2(t) = \hat{\tilde{v}}_1(t)$$

in general?
d. Verify that we can force

$$\hat{\tilde{v}}_2(t) = \hat{\tilde{v}}_1(t)$$

by choosing the two constraints appropriately.
e. Read [32] and discuss why one might want to use $\hat{\tilde{v}}_2(t)$ instead of $\hat{\tilde{v}}_1(t)$.

*Comments:*

1. You should have solved part 2 by inspection, since $\hat{\tilde{v}}_1(t)$ must equal $\tilde{g}(t)$ in (106).
2. The equation in part 3b is solved by a sampling approach in [32]. The same procedures can be used to solve (106).

**Problem 13.2.18.** Consider the model in Problem 13.2.17.
1. Verify that $\Delta_m$ can be written as

$$\Delta_m = \frac{\left|\int_{-\infty}^{\infty} \tilde{f}(t)\tilde{v}^*(t)\,dt\right|^2}{\iiint_{-\infty}^{\infty} \tilde{v}^*(t-\lambda)[N_0\,\delta(t-u) + \tilde{f}(t-\lambda)\tilde{K}_{DR}(t-u,\lambda)\tilde{f}^*(u-\lambda)]\tilde{v}(u-\lambda)\,d\lambda\,dt\,du}. \tag{P.1}$$

2. We require $\tilde{v}(t)$ to be of the form

$$\tilde{v}(t) = \tilde{a}_1\tilde{f}(t) + \tilde{a}_2\tilde{f}(t - T_s),$$

where $T_s$ is a fixed constant and $\tilde{a}_1$ and $\tilde{a}_2$ are complex weightings. Choose $\tilde{a}_1$ and $\tilde{a}_2$ to maximize $\Delta_m$. Call these values $\hat{\tilde{a}}_1$ and $\hat{\tilde{a}}_2$.

3. Now maximize $\Delta_m(\hat{\tilde{a}}_1, \hat{\tilde{a}}_2)$ as a function of $T_s$.

**Problem 13.2.19.** Consider the result in (P.1) in Problem 13.2.18. We would like to optimize $\tilde{f}(t)$ and $\tilde{v}^*(t)$ jointly. We studied this problem for Doppler-spread reverberation in Problem 11.2.14. Our procedure resulted in a set of nonlinear differential equations that we were unable to solve. The basic difficulty was that both the conventional and optimum receivers were related to $\tilde{f}(t)$.

We now try a new procedure. For simplicity we begin with Doppler-spread reverberation,

$$\tilde{K}_{DR}(t - u, \lambda) = \tilde{K}_D(t - u)\,\delta(\lambda).$$

We select an initial $v(t)$ with unity energy, which we denote as $\tilde{v}_1(t)$. Now conduct the following minimization:

(i) Constrain

$$\int_0^T |\tilde{f}(t)|^2\,dt = 1.$$

(ii) Constrain

$$\int_{-\infty}^{\infty} f^2\,|\tilde{F}\{f\}|^2\,df = B^2$$

and

$$\tilde{f}(0) = \tilde{f}(T) = 0.$$

(iii) Constrain

$$\int_0^T \tilde{f}(t)\tilde{v}_1^*(t)\,dt = K.$$

(iv) Minimize

$$J = N_0 + \iint_0^T \tilde{v}_1^*(t)\tilde{f}(t)\tilde{K}_D(t - u)\tilde{f}^*(u)\tilde{v}_1(u)\,dt\,du,$$

subject to these constraints.

1. Carry out the required minimization. Verify that the resulting equation is linear. Reduce the problems to a set of differential equations that specify the solution. Observe that these can be solved using Baggeroer's algorithm [55], [56]. Denote the solution as $\tilde{f}_1(t)$.

2. Assume that $\tilde{f}_1(t)$ is transmitted. Choose $\tilde{v}(t)$ to maximize $\Delta_m$. Denote the solution as $\tilde{v}_2(t)$. Is there any difficulty in carrying out this procedure?

3. Repeat part 1, using $\tilde{v}_2(t)$. What is the difficulty with this procedure?

4. Discuss the problems in extending this procedure to the doubly-spread case. Using the distributed state-variable model, derive a set of differential equations that specify the optimum signal as in part 1.

**Problem 13.2.20.** The complex envelope of the transmitted signal is $\sqrt{E_t}\tilde{f}(t)$, where

$$\tilde{f}(t) \triangleq a \sum_{n=1}^{N} \tilde{u}(t - nT_p), \qquad \textbf{(P.1)}$$

with $\tilde{u}(t)$ defined as in (10.29). The desired target is located at the origin. Instead of correlating with $\tilde{f}^*(t)$, we correlate with $\tilde{x}^*(t)$;

$$\tilde{x}(t) = \sum_{n=1}^{N} W_n T_s \tilde{u}(t - nT_p),$$

where $W_n$ is an arbitrary complex number. Both $\tilde{f}(t)$ and $\tilde{x}(t)$ are normalized to have unit energy. The receiver output is

$$l \triangleq \left| \int_{-\infty}^{\infty} \tilde{f}(t)\tilde{x}^*(t)\, dt \right|^2.$$

1. Calculate $\Delta$ when the complex input is the signal plus complex white noise with spectral height $N_0$.
2. Denote the complex weightings by the vector $\mathbf{W}$. Choose $\mathbf{W}$ to maximize $\Delta$.
3. Calculate $\Delta$ for the case in which there is clutter that has a rectangular scattering function

$$\tilde{S}_{DR}\{f, \lambda\} = \begin{cases} \dfrac{2\sigma_b^2}{BL}, & B_1 \le f \le B_2, \quad L_1 \le \lambda \le L_2, \\ 0, & \text{elsewhere.} \end{cases}$$

Write $\Delta$ in the form

$$\frac{\Delta}{\bar{E}_r/N_0} = \frac{\tilde{\mathbf{W}}^\dagger \mathbf{U} \tilde{\mathbf{W}}}{\tilde{\mathbf{W}}^\dagger [N\mathbf{I} + \lambda \mathbf{C}] \tilde{\mathbf{W}}},$$

where

$$\tilde{u}_{ij} = 1.$$

Specify the other matrices.

4. We want to choose $\tilde{\mathbf{W}}$ to maximize $\Delta$. Carry out the maximization and find the equations specifying the optimum $\tilde{\mathbf{W}}$.

*Comment:* This problem and generalizations of it are studied in detail in [22], [24], and [34].

**Problem 13.2.21.** Consider the reverberation model in (132). From (136).

$$\Delta_o = \bar{E}_r \int_{-\infty}^{\infty} \frac{\tilde{S}_{\tilde{f}}\{f\}}{N_0 + \tilde{S}_{\tilde{n}_r}\{f\}}\, df,$$

where $\tilde{S}_{\tilde{n}_r}\{f\}$ is specified by (124) and (132). We constrain the transmitted signal to be bandlimited with unit energy,

$$\tilde{S}_{\tilde{f}}\{f\} = 0, \qquad |f| \ge W.$$

Find an equation specifying the optimum $S_{\tilde{f}}\{f\}$ to maximize $\Delta_o$.

**Problem 13.2.22.** Consider the reverberation model in (132). If the target is moving, then

$$\Delta_o = \bar{E}_r \int_{-\infty}^{\infty} \frac{\tilde{S}_{\tilde{f}}\{f - f_d\}}{N_0 + \tilde{S}_{\tilde{n}_r}\{f\}}\, df.$$

Repeat Problem 13.2.21 for this case.

**Problem 13.2.23.** Consider the reverberation model in (132). Assume that

$$\tilde{f}(t) = a \sum_{i=1}^{2} \tilde{u}(t - iT_p),$$

where $\tilde{u}(\cdot)$ is defined in (10.29).The desired target has a *known* velocity corresponding to a Doppler shift of $f_d$ cps.

1. Draw a block diagram of the optimum receiver and evaluate its performance.
2. Now assume that we generate two random variables,

$$\tilde{r}_1 \triangleq \int_0^{T_s} \tilde{r}(t)\tilde{u}^*(t)\,dt,$$

$$\tilde{r}_2 \triangleq \int_{T_p}^{T_p+T_s} \tilde{r}(t)\tilde{u}^*(t-T_p)\,dt.$$

Derive a formula for the optimum operations on $\tilde{r}_1$ and $\tilde{r}_2$. Evaluate the performance of the resulting receiver.

3. Consider the receiver shown in Fig. P.13.3. The target is assumed to be at zero-range. Analyze the performance of this receiver as a function of $E_t$, $N_0$, $P_c$, and $f_d$. Compare the results in parts 1, 2, and 3.

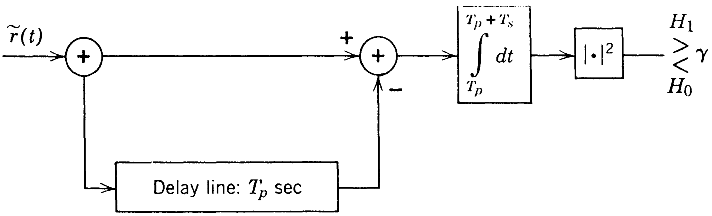


**Fig. P.13.3**

4. Read Steinberg's discussion of MTI (moving target indication) radars [57]. Compare his model and results with our model. Other interesting discussions of MTI systems are given in [58]–[60].

## P.13.3 Detection of Doubly-Spread Targets

DETECTION MODELS

**Problem 13.3.1.** Consider the binary detection problem in which the complex envelopes of the received waveforms on the two hypotheses are

$$\tilde{r}(t) = \tilde{s}_1(t) + \tilde{w}(t), \qquad -\infty < t < \infty : H_1,$$

$$\tilde{r}(t) = \tilde{s}_0(t) + \tilde{w}(t), \qquad -\infty < t < \infty : H_0,$$

where $\tilde{s}_0(t)$, $\tilde{s}_1(t)$, and $\tilde{w}(t)$ are statistically independent complex Gaussian random processes with covariance functions.

$$\tilde{K}_{\tilde{s}_0}(t, u) = E_t \int_{-\infty}^{\infty} \tilde{f}(t-\lambda)\tilde{K}_{DR,0}(t-u, \lambda)\tilde{f}^*(u-\lambda)\,d\lambda,$$

$$\tilde{K}_{\tilde{s}_1}(t, u) = E_t \int_{-\infty}^{\infty} \tilde{f}(t-\lambda)\tilde{K}_{DR\,1}(t-u, \lambda)\tilde{f}^*(u-\lambda)\,d\lambda,$$

and

$$\tilde{K}_{\tilde{w}}(t, u) = N_0 \delta(t - u).$$

Derive the equations specifying the optimum receiver.

**Problem 13.3.2.** Consider the model in Problem 13.3.1. Assume that

$$\tilde{K}_{\tilde{s}_0}(t, u) = E_t \int_{-\infty}^{\infty} \tilde{f}_0(t - \lambda)\tilde{K}_{DR}(t - u, \lambda)\tilde{f}_0^*(u - \lambda)\, d\lambda$$

and

$$\tilde{K}_{\tilde{s}_1}(t, u) = E_t \int_{-\infty}^{\infty} \tilde{f}_1(t - \lambda)\tilde{K}_{DR}(t - u, \lambda)\tilde{f}_1^*(u - \lambda)\, d\lambda.$$

Derive the equations specifying the optimum receiver.

**Problem 13.3.3.** The complex envelopes of the received waveforms on the two hypotheses are

$$\tilde{r}(t) = \sqrt{E_t}\, \tilde{b}\tilde{f}(t) + \sqrt{E_t} \int_{-\infty}^{\infty} \tilde{b}(t, \lambda)\tilde{f}(t - \lambda)\, d\lambda + \tilde{w}(t), \qquad -\infty < t < \infty : H_1,$$

$$\tilde{r}(t) = \tilde{w}(t), \qquad -\infty < t < \infty : H_0.$$

The process $\tilde{b}(t, \lambda)$ is characterized in (4). The random variable $\tilde{b}$ is a complex Gaussian variable ($E(|\tilde{b}|^2) = 2\sigma_b^2$) and is statistically independent of $\tilde{b}(t, \lambda)$.

Derive the equations specifying the optimum receiver.

**Problem 13.3.4.** Consider the statement below (176) regarding the statistical independence of the tap gain processes. Investigate the issue quantitatively.

**Problem 13.3.5.** Consider the scattering function in Problem 13.1.6. Assume that we approximate it with the tapped-delay line in Fig. 13.18.

1. Specify the spectrum of the tap gain processes.
2. Find the cross-correlation (or cross-spectrum) of tap gain processes as a function of $W_s$.
3. Assume that we use three taps and that the tap gain processes are statistically independent. Write out the state equations specifying the model.
4. Draw a block diagram of the optimum receiver for the detection model in (142)–(152). Write an expression for $\tilde{\mu}(s)$.

**Problem 13.3.6** [61]. Assume that the transmitted signal is time-limited; that is,

$$\tilde{f}(t) = 0, \qquad |t| > \frac{T}{2}.$$

Develop the dual of the tapped-delay line model.

**Problem 13.3.7.** In the Doppler-spread case the SPLOT condition enabled us to obtain answers reasonably easily. Consider the doubly-spread problem in which $\tilde{f}(t)$ is a time-limited rectangular pulse $[0, T]$ and $\tilde{S}_{DR}\{f, \lambda\}$ is range-limited $[0, L]$. The observation interval is $[-\infty, \infty]$.

1. Is the output signal process stationary?
2. Is any time segment of the output signal process stationary?
3. Consider the following procedure:

(i) Analyze the SPLOT problem for the observation interval $[L, T]$.
(ii) Analyze the SPLOT problem for the observation interval $[0, L + T]$.

a. Will the performance of the system in (i) underbound the performance of the actual system?

b. Will the performance of the system in (ii) overbound the performance of the actual system?

c. For what ranges of parameter values would this procedure be useful?

**Problem 13.3.8.** The scattering function is

$$\tilde{S}_{DR}\{f, \lambda\} = \frac{4k\sigma_b^2[1 - \cos(2\pi\lambda/L)]}{L[(2\pi f)^2 + k^2]}, \qquad -\infty < f < \infty, \qquad 0 < \lambda < L.$$

We expand the channel using the general orthogonal signal model in (196)–(217). The transmitted signal is a rectangular pulse [0, T]. The orthogonal functions are

$$\varphi_1(t) = \frac{1}{\sqrt{L}}, \qquad 0 \le \lambda \le L,$$

$$\varphi_2(t) = \sqrt{\frac{2}{L}} \cos\left(\frac{2\pi\lambda}{L}\right), \qquad 0 \le \lambda \le L,$$

$$\varphi_3(t) = \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi\lambda}{L}\right), \qquad 0 \le \lambda \le L,$$

and so forth.

Evaluate the various quantities needed to specify the model completely. Be careful about the intervals.

**Problem 13.3.9.** Prove that the tapped-delay line model is a special case of the general orthogonal signal model.

**Problem 13.3.10.** The scattering function is

$$\tilde{S}_{DR}\{f, \lambda\} = \frac{8k\sigma_b^2}{L} \frac{(1 - (2|\lambda|)/L)}{((2\pi f)^2 + k^2)}, \qquad -\infty < f < \infty, \qquad |\lambda| < L/2.$$

Repeat Problem 13.3.8.

**Problem 13.3.11.** Consider the model in (224)–(228). Show that a direct orthogonal series expansion leads back to the model in Section 13.3.2.B.

**Problem 13.3.12.** Consider the expression for $\tilde{\mathbf{p}}_{ij}(t)$ in (233).

1. Derive a set of differential equations that the $\tilde{\mathbf{p}}_{ij}(t)$ must satisfy.
2. Compare the result in part 1 with that in Problem 13.3.11. Identify the impulsive term in (229).

**Problem 13.3.13.** Consider the decomposition in (229). Verify that the results in (230) is correct. [*Hint:* Recall (47).]

## Binary Communication

**Problem 13.3.14** [38]. Assume that

$$B = 1 \text{ kcps}$$

and

$$L = 250\ \mu\text{sec}.$$

The power-to-noise ratio at the receiver is

$$\frac{P_R}{N_0} = 5 \times 10^5 \qquad (57 \text{ db}).$$

We require a probability of error of $10^{-3}$.

1. Show that the maximum achievable rate using a binary system with the above parameters in 15,000 bits/sec.
2. Design a simple system that achieves this rate.

**Problem 13.3.15.** Consider the scattering function in Fig. 13.1.6 and assume that

$$kL = 10.$$

Design signals for a binary system to communicate effectively over this channel.

**Problem 13.3.16.** Consider the model in (254)–(264) and (279).

1. Find the terms in (277) for the case when $K = 3$.
2. Repeat part 1 for $K = 5$.

LEC CONDITIONS

*Comment:* The next three problems develop some simple tests to verify the LEC condition.

**Problem 13.3.17** [5].

1. Prove that

$$\tilde{\lambda}_{\max} \leq E_t \max_t |\tilde{f}(t)|^2 \int_{-\infty}^{\infty} (\max_f \tilde{S}_{DR}\{f, \lambda\})\, d\lambda. \tag{P.1}$$

2. Consider the special case in which $\tilde{f}(t)$ is a constant. Prove that

$$\tilde{\lambda}_{\max} \leq \max_f \tilde{S}_{\tilde{s}}\{f\}. \tag{P.2}$$

**Problem 13.3.18** [5]. Derive the dual of the bound in (P.1) of Problem 13.3.17. Specifically, prove that

$$\tilde{\lambda}_{\max} \leq \tfrac{1}{2}\{\max_f |\tilde{F}\{f\}|^2\} \int_{-\infty}^{\infty} (\max_\lambda \tilde{S}_{D\dot{R}}\{f, \lambda\})\, df. \tag{P.1}$$

**Problem 13.3.19** [5]. Prove that

$$\tilde{\lambda}_{\max} \leq \bar{E}_r \max_{f,\lambda} S_{DR}\{f, \lambda\}.$$

**Problem 13.3.20.** In this problem we develop lower bounds on $\tilde{\lambda}_{\max}$.

1. Prove that

$$\tilde{\lambda}_{\max} \geq \iint_{T_i}^{T_f} \tilde{z}(t)\tilde{K}_{\tilde{s}}(t, u)\tilde{z}^*(u)\, dt\, du \tag{P.1}$$

for any $\tilde{z}(t)$ such that

$$\int_{T_i}^{T_f} |\tilde{z}(t)|^2\, dt = 1. \tag{P.2}$$

2. Assume that $T_i = -\infty$ and $T_f = \infty$. Prove

$$\tilde{\lambda}_{\max} \geq E_t \iint_{-\infty}^{\infty} \theta\{\lambda, f\}\tilde{S}_{DR}\{f, \lambda\}\, df\, d\lambda. \tag{P.3}$$

3. Give an example of an $\tilde{S}_{DR}\{f, \lambda\}$ in which (P.3) is satisfied with equality.

**Problem 13.3.21.** Consider the model in Section 13.3.4.A.

1. Derive (292).
2. Derive (293)–(295).

**Problem 13.3.22.** Consider the signal in (10.44a) and the scattering function in Problem 13.1.5 with $m_R = m_D = 0$. Evaluate $\tilde{\mu}(s)$ in (295).

**Problem 13.3.23.** Consider a binary communication system operating under LEC conditions and using a rectangular pulse. Assume that $T$ is fixed.

1. Prove that

$$\ln \Pr(\epsilon) \simeq a\left(\frac{P_r}{N_0}\right)^2 + b,$$

where $P_r/N_0$ is the received power-to-noise level ratio. Find $a$ and $b$.

2. Compare this result with (239).

**Problem 13.3.24.** Consider the suboptimum receiver in (296). Set up the equations necessary to analyze its performance.

**Problem 13.3.25.** Consider the suboptimum receiver in Fig. 13.31. Set up the equations necessary to analyze its performance.

**Problem 13.3.26.** Consider the suboptimum receiver in Fig. 13.32.

1. Set up the equations necessary to analyze its performance.
2. Discuss the utility of the SPLOT approach suggested in Problem 13.3.7 for this particular problem.

**Problem 13.3.27.** Consider the equivalent channel definition on page 523. Verify that the relations in Table 13.2 are correct.

**Problem 13.3.28.** Consider the channel whose scattering function is shown in Fig. P.13.4. The height is $2\sigma_b^2/BL$ in the shaded rectangle and zero elsewhere. Assume that

$$BL = 0.01.$$

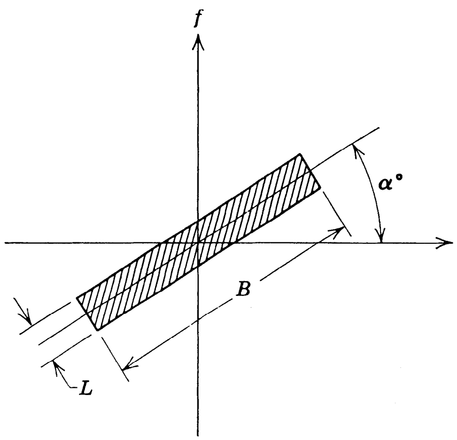


**Fig. P.13.4**

Design a binary communication system that will operate over this channel with

$$\tilde{\mu}_{BS}(\tfrac{1}{2}) \simeq -0.149.$$

Specify both the transmitted signal and the optimum receiver.

**Problem 13.3.29.** Consider the degenerate scattering function

$$S_{DR}\{f, \lambda\} = \sum_{i=1}^{N} \frac{2\sigma_b^2}{N} \delta\{f - f_i\}\, \delta\{\lambda - \lambda_i\}. \tag{P.1}$$

1. Assume that $N = 2$. Prove that all channels with this scattering function are equivalent.
2. Is this result true for $N \geq 3$?

**Problem 13.3.30** [38]. Consider the system in Problem 13.3.15. We require a bit error rate of $10^{-3}$.

1. Show that by using a system with four orthogonal signals we can achieve a rate of 25,000 bits/sec. (*Hint:* Use the results of Problem 5.1.4.)
2. Design a system to achieve this rate.

**Problem 13.3.31** [37]. Prove that all channels whose scattering functions have the form

$$\tilde{S}_{DR}\{f, \lambda : a, k, c\} = \tilde{S}_{DR}\left\{ak\lambda + \frac{1 - kc}{a} f, a\lambda - \frac{cf}{a}\right\}$$

are equivalent for any values of $c$, $k$, and $a$.

## P.13.4 Parameter Estimation

**Problem 13.4.1.**

1. Derive the expression for $l_R(A)$ in (306).
2. Derive an expression for the elements of the information matrix $\mathbf{J}$. Do not assume LEC conditions.

**Problem 13.4.2.** Derive the expression for $l_B^{[2]}(A)$ given in (317).

**Problem 13.4.3.** Assume that the LEC condition is valid. Derive the result in (324).

**Problem 13.4.4.** Consider the amplitude estimation problem in Section 13.4.2.

1. Verify that $\hat{a}_0$ [defined in (331)] is unbiased under all conditions (i.e., the LEC condition is not required).
2. Find an exact expression for

$$\xi_{\hat{a}_0} \triangleq E[(\hat{a}_0 - A)^2].$$

3. Verify that

$$\xi_{\hat{a}_0} \triangleq J^{-1}(A)$$

when the LEC condition holds.

**Problem 13.4.5.** Express the result in (332) in an alternative form that contains $\tilde{S}_{DR}\{f, \lambda\}$ instead of $\tilde{R}_{DR}\{\tau, \nu\}$.

*Comment:* Notice that the LEC assumption is *not* made in Problems 13.4.6–13.4.8.

**Problem 13.4.6.** Consider the degenerate case of the amplitude estimation problem in which $\tilde{s}(t, A)$ has a finite number of *equal* eigenvalues. Thus,

$$\tilde{K}_{\tilde{s}}(t, u : A) = A\tilde{\lambda}_c \sum_{i=1}^{N} \tilde{\varphi}_i(t) \tilde{\varphi}_i^*(u), \qquad -\infty < t, u < \infty.$$

1. Find the receiver to generate $\hat{a}_0$ and $\hat{a}_{ml}$.
2. Evaluate $\xi_{\hat{a}_0}$ and $J(A)$. Verify that $\hat{a}_0$ is an efficient unbiased estimate.
3. Constrain

$$AN\tilde{\lambda}_c = \bar{E}_r.$$

Treat $N$ as a continuous variable that is greater than or equal to 1. Find the value of $N$ that minimizes $\xi_{\hat{a}_0}$. Notice that the answer depends on $A$, the unknown parameter. How would you use this result in an actual system? (*Hint:* Is $\xi_{\hat{a}_0}$ sensitive to the exact choice of $N$?) Plot $\xi_{\hat{a}_0}$ as a function of $N$.

**Problem 13.4.7.** Consider the general amplitude estimation problem. Assume that

$$\tilde{K}_{\tilde{s}}(t, u:A) = A\tilde{K}_{\tilde{s}}(t, u), \qquad -\infty < t, u < \infty.$$

1. Express the Cramér-Rao bound of the variance of any unbiased estimate of $A$ in terms of the eigenvalues of $\tilde{K}_{\tilde{s}}(t, u)$.
2. Constrain

$$A \int_{-\infty}^{\infty} \tilde{K}_s(t, t)\, dt = \bar{E}_r.$$

Find the eigenvalue distribution that minimizes the value of the bound in part 1. Compare your result with that in part 3 of Problem 13.4.6.

3. Interpret the result in part 2 in the context of estimating the amplitude of an otherwise known scattering function. Notice that this gives a bound on the variance of an unbiased estimate of the amplitude that does not depend on the scattering function.

**Problem 13.4.8.** Assume that

$$\tilde{S}_{DR}\{f, \lambda:A\} = A\tilde{S}_{DR}\{f, \lambda\},$$

where $\tilde{S}_{DR}\{f, \lambda\}$ is known. We know that

$$\frac{A\bar{E}_r}{N_0} = \frac{AE_t}{N_0} \iint_{-\infty}^{\infty} \tilde{S}_{DR}\{f, \lambda\}\, df\, d\lambda \simeq 20,$$

and want to estimate $A$ more exactly.

1. Assume that

$$L = 10$$

and

$$BL = 0.001.$$

Design a signal $\tilde{f}(t)$ that will result in an unbiased estimate whose variance is close to that in part 3 of Problem 13.4.7. Draw a block diagram of the optimum receiver.

2. Repeat part 1 for the case in which

$$B = 10$$

and

$$BL = 0.001.$$

**Problem 13.4.9.** Consider the generalization of the example on page 531, in which the Gaussian pulse has a linear FM [see (10.44a)] and the scattering function has a skewed Gaussian shape.

$$\tilde{S}_{DR}\{f, \lambda\} = \frac{2\sigma_b^2}{2\pi\sigma_D\sigma_R(1 - \rho_{DR}^2)^{1/2}} \exp\left[-\frac{\sigma_R^2 f^2 - 2\sigma_R\sigma_D\rho_{DR} f\lambda + \sigma_D^2\lambda^2}{2\sigma_D^2\sigma_R^2(1 - \rho_{DR}^2)}\right]. \tag{P.1}$$

The LEC condition is assumed.

1. Show that this can be reduced to an equivalent problem with a nonskewed Gaussian scattering function.

2. Evaluate the bound in (332).
3. What linear sweep rate minimizes the variance bound?

**Problem 13.4.10.** Consider the problem of estimating the scale of range axis under LEC conditions,

$$\tilde{K}_{DR}(\tau, \lambda{:}A) = \tilde{K}_{D_1R_1}\left(\tau, \frac{\lambda}{A}\right),$$

where $\tilde{K}_{D_1R_1}(\cdot, \cdot)$ is known.

1. Derive a lower bound on the variance of an unbiased estimate [62].
2. Consider the special case in which $\tilde{f}(t)$ is given by (335) and $\tilde{S}_{D_1R_1}\{f, \lambda\}$ satisfies (333). Evaluate the bound in part 1.
3. Choose $T$ to minimize the bound.

**Problem 13.4.11.** Consider the problem of estimating the frequency scale under LEC conditions,

$$\tilde{S}_{DR}\{f, \lambda{:}A\} = \tilde{S}_{D_1R_1}\left\{\frac{f}{A}, \lambda\right\},$$

where $\tilde{S}_{D_1R_1}\{\cdot, \cdot\}$ is known.

1. Repeat Problem 13.4.10.
2. Solve this problem by using duality theory and the results of Problem 13.4.10.

**Problem 13.4.12.** Consider the generalization of the two previous problems in which

$$\tilde{R}_{DR}\{\tau, v{:}\mathbf{A}\} = A_1A_2\tilde{R}_{D_1R_1}\{A_1\tau, A_2v\}.$$

1. Derive an expression for the element in the bound matrix, $\mathbf{J}(\mathbf{A})$ [62].
2. Evaluate the terms for the case in part 2 of Problem 13.4.10.

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