Detection, Estimation, and Modulation Theory, Part III: Radar–Sonar Signal Processing and Gaussian Signals in Noise. Harry L. Van Trees

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Appendix: *Complex Representation of Bandpass Signals, Systems, and Processes*

In this appendix we develop complex representations for narrow-band signals, systems, and random processes. The idea of representing an actual signal as the real part of a complex signal is familiar to most electrical engineers. Specifically, the signal,

$$\cos \omega_1 t = \text{Re}\,[e^{j\omega_1 t}], \tag{A.1}$$

and the associated phasor diagram in Fig. A.1 are encountered in most introductory circuit courses. The actual signal is just the projection of the complex signal on the horizontal axis. The ideas in this appendix are generalizations of this familiar notation.

In Section A.1, we consider bandpass deterministic signals. In Section A.2, we consider bandpass linear systems. In Section A.3, we study bandpass random processes. In Section A.4, we summarize the major results. Section A.5 contains some problems to demonstrate the application of the ideas discussed.

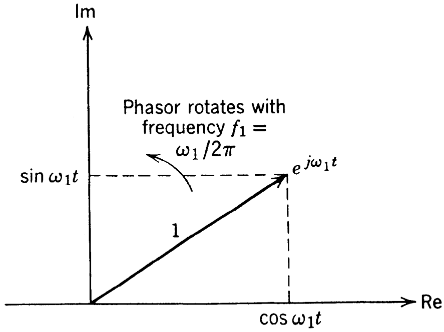


Fig. A.1 Phasor diagram.

The treatment through Section A.3.1 is reasonably standard (e.g., [1]–[8]), and readers who are familiar with complex representation can skim these sections in order to learn our notation. The material in Section A.3.2 is less well known but not new. The material in Section A.3.3 is original [9], and is probably not familiar to most readers. With the exception of Sections A.2.3 and A.3.3, the results are needed in order to understand Chapters 9–14.

A.1 DETERMINISTIC SIGNALS

In this section we consider deterministic finite-energy signals. We denote the signal by $f(t)$, and its Fourier transform by $F(j\omega)$.

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt. \qquad \text{(A.2)}\dagger$$

For simplicity, we assume that $f(t)$ has unit energy. A typical signal of interest might have the Fourier transform shown in Fig. A.2. We see that

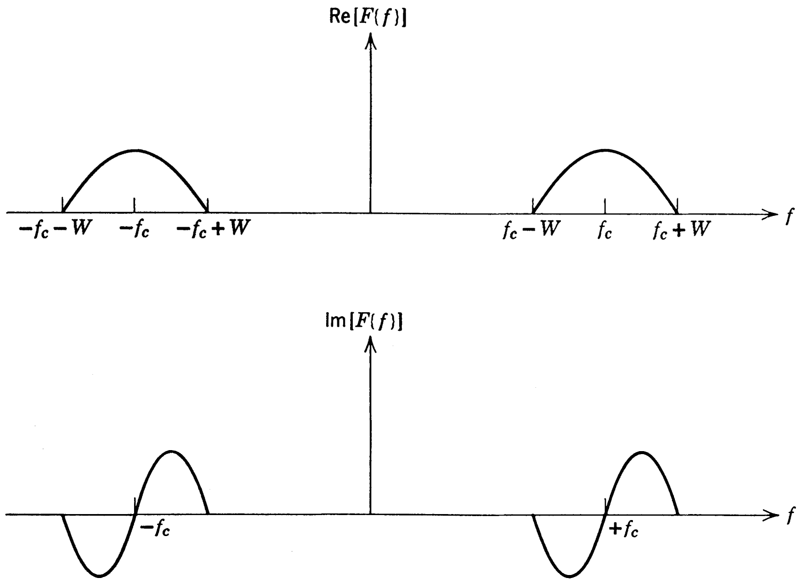


Fig. A.2 Fourier transform of a bandpass signal.

† We also use the transform $F\{f\} = \int_{-\infty}^{\infty} f(t)e^{-j2\pi ft}\,dt$. The braces $\{\cdot\}$ imply this definition.

the transform is bandlimited to frequencies within $\pm W$ cps of the carrier f_c. In practice, very few signals are strictly bandlimited. If the energy outside the band is negligible, it is convenient to neglect it. We refer to a signal that is essentially bandlimited around a carrier as a *bandpass* signal. Normally the bandwidth around the carrier is small compared to ω_c, and so we also refer to the signal as a *narrow-band* signal. A precise statement about how small the bandwidth must be in order for the signal to be considered narrow-band is not necessary for our present discussion.

It is convenient to represent the signal in terms of two low-pass quadrature components, $f_c(t)$ and $f_s(t)$,

$$f_c(t) \triangleq [(\sqrt{2}\cos\omega_c t)f(t)]_{LP}, \tag{A.3}$$

$$f_s(t) \triangleq [(\sqrt{2}\sin\omega_c t)f(t)]_{LP}. \tag{A.4}$$

The symbol $[\cdot]_{LP}$ denotes the operation of passing the argument through an ideal low-pass filter with unity gain. The low-pass waveforms $f_c(t)$ and $f_s(t)$ can be generated physically, as shown in the left of Fig. A.3. The transfer function of the ideal low-pass filters is shown in Fig. A.4. Given $f_c(t)$ and $f_s(t)$, we could reconstruct $f(t)$ by multiplying $\cos\omega_c t$ and $\sin\omega_c t$ and adding the result as shown in the right side of Fig. A.3. Thus,

$$f(t) = \sqrt{2}\,[f_c(t)\cos(\omega_c t) + f_s(t)\sin(\omega_c t)]. \tag{A.5}$$

One method of verifying that the representation in (A.5) is valid is to follow the Fourier transforms through the various operations. A much easier procedure is to verify that the entire system in Fig. A.3 is identical

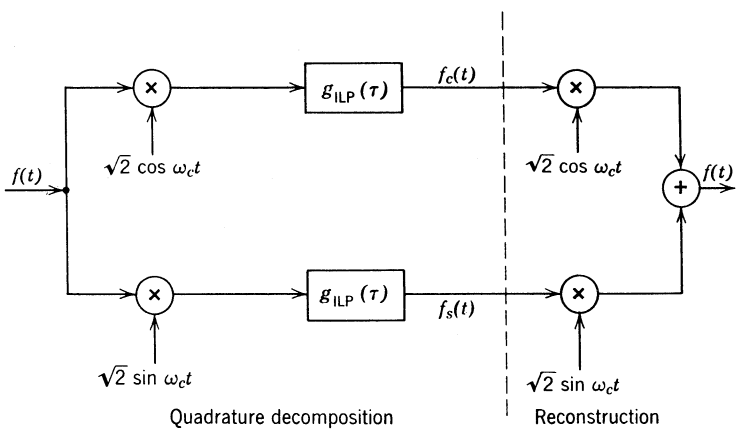


Fig. A.3 Generation of quadrature components and reconstruction of bandpass signal.

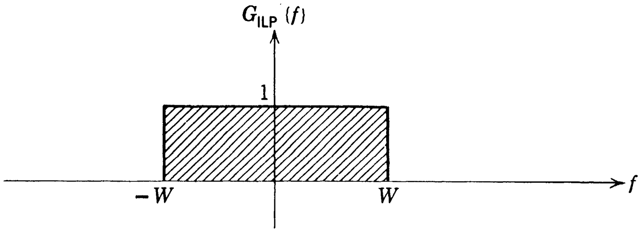


Fig. A.4 Transfer function of ideal low-pass filter.

with the ideal bandpass filter whose transfer function is shown in Fig. A.5.† To do this we put an impulse into the system and calculate its output. We denote the output due to an impulse at time $t = \tau$ as $g_\tau(t)$.

$$\begin{aligned} g_\tau(t) &= \sqrt{2}\cos(\omega_c t)\int_{-\infty}^{\infty} \delta(t-\tau)\sqrt{2}\cos(\omega_c u_1) g_{\text{ILP}}(t-u_1)\, du_1 \\ &\quad + \sqrt{2}\sin(\omega_c t)\int_{-\infty}^{\infty} \delta(t-\tau)\sqrt{2}\sin(\omega_2 u_2) g_{\text{ILP}}(t-u_2)\, du_2 \\ &= 2g_{\text{ILP}}(t-\tau)[\cos(\omega_c t)\cos(\omega_c \tau) + \sin(\omega_c t)\sin(\omega_c \tau)] \\ &= 2g_{\text{ILP}}(t-\tau)\cos[\omega_c(t-\tau)]. \end{aligned} \tag{A.6}$$

The transfer function is the Fourier transform of the impulse response,

$$\begin{aligned} &\int_{-\infty}^{\infty} 2g_{\text{ILP}}(\sigma)\cos(\omega_c\sigma)e^{-j\omega\sigma}\, d\sigma \\ &\qquad = \int_{-\infty}^{\infty} g_{\text{ILP}}(\sigma)e^{-j(\omega+\omega_c)\sigma}\, d\sigma + \int_{-\infty}^{\infty} g_{\text{ILP}}(\sigma)e^{-j(\omega-\omega_c)\sigma}\, d\sigma \\ &\qquad = G_{\text{ILP}}\{f+f_c\} + G_{\text{ILP}}\{f-f_c\}. \end{aligned} \tag{A.7}$$

The right side of (A.7) is just the transfer function shown in Fig. A.5, which is the desired result. Therefore the system in Fig. A.3 is just an ideal bandpass filter and any bandlimited input will pass through it undistorted. This verifies that our representation is valid. Notice that the

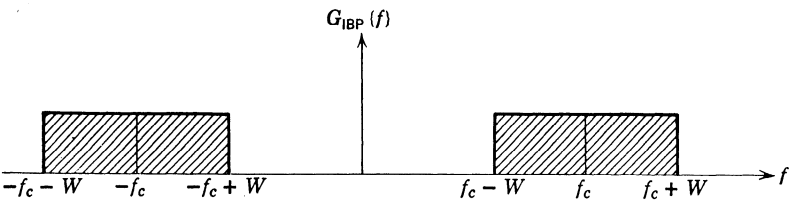


Fig. A.5 Transfer function of overall system in Fig. A.3.

† This procedure is due to [10, page 497].

assumption that $f(t)$ has unit energy implies

$$\int_{-\infty}^{\infty} (f_c^2(t) + f_s^2(t))\, dt = \int_{-\infty}^{\infty} f^2(t)\, dt = 1. \tag{A.8}$$

We can represent the low pass waveforms more compactly by defining a *complex signal*,

$$\boxed{\tilde{f}(t) \triangleq f_c(t) - jf_s(t).} \tag{A.9}$$

Equivalently,

$$\tilde{f}(t) = |\tilde{f}(t)|\, e^{j\phi_{\tilde{f}}(t)}, \tag{A.10}$$

where

$$|\tilde{f}(t)| = \sqrt{f_c^2(t) + f_s^2(t)} \tag{A.11}$$

and

$$\phi_{\tilde{f}}(t) = \tan^{-1}\left(\frac{f_s(t)}{f_c(t)}\right). \tag{A.12}$$

Notice that we can also write

$$\tilde{f}(t) = [f(t)\sqrt{2}\, e^{-j\omega_c t}]_{\text{LP}}. \tag{A.13}$$

The actual bandpass signal is

$$\begin{aligned} f(t) &= \sqrt{2E_t}\,\text{Re}\,[\tilde{f}(t)e^{j\omega_c t}] \\ &= \sqrt{2E_t}\,|\tilde{f}(t)|\, e^{j(\omega_c t + \phi_{\tilde{f}}(t))}. \end{aligned} \tag{A.14}$$

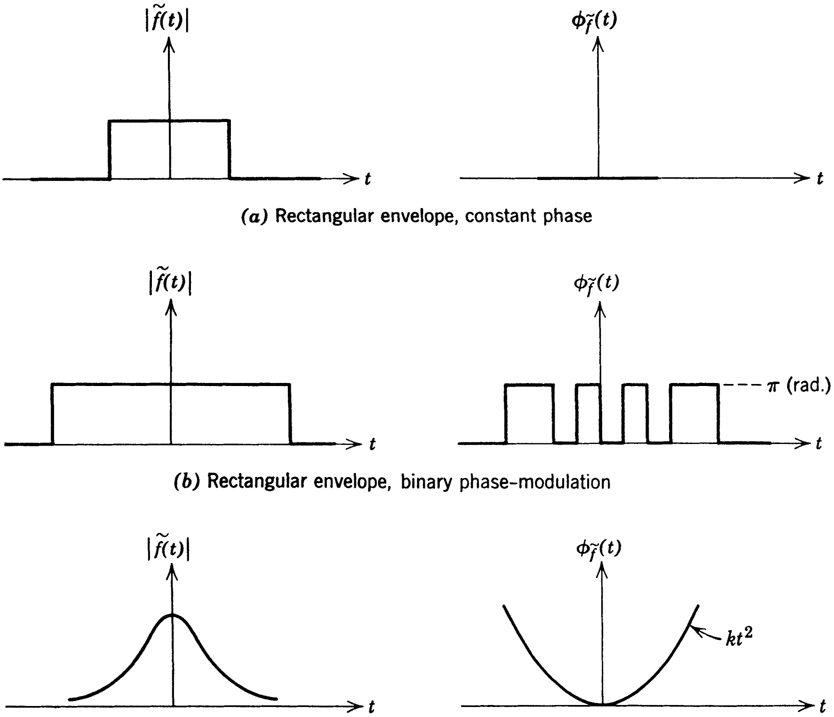
Some typical signals are shown in Fig. A.6. Notice that the signals in Fig. A.6*a–c* are not strictly bandlimited but do have negligible energy outside a certain frequency band. We see that $|\tilde{f}(t)|$ is the actual envelope of the narrow-band signal and $\phi_{\tilde{f}}(t) + \omega_c t$ is the *instantaneous phase.* The function $\tilde{f}(t)$ is commonly referred to as the *complex envelope.*

The utility of the complex representation for a bandpass signal will become more apparent as we proceed. We shall find that the results of interest can be derived and evaluated more easily in terms of the complex envelope.

There are several properties and definitions that we shall find useful in the sequel. All of the properties are straightforward to verify.

Property 1. Since the transmitted energy is unity, it follows from (A.8) that

$$\int_{-\infty}^{\infty} |\tilde{f}(t)|^2\, dt = 1. \tag{A.15}$$



(c) Gaussian envelope, linear frequency sweep

Fig. A.6 Typical signals.

Property 2. The mean frequency of the envelope is defined as the first moment of the *energy* spectrum of the complex envelope,

$$\bar{\omega} \triangleq \int_{-\infty}^{\infty} \omega \, |\tilde{F}(j\omega)|^2 \frac{d\omega}{2\pi} = \int_{-\infty}^{\infty} \omega \tilde{S}_{\tilde{f}}(\omega) \frac{d\omega}{2\pi}, \tag{A.16}$$

where $\tilde{F}(j\omega)$ is the Fourier transform of $\tilde{f}(t)$,

$$\tilde{F}(j\omega) = \int_{-\infty}^{\infty} \tilde{f}(t) e^{-j\omega t} \, dt. \tag{A.17}$$

In our model the actual signal is $f(t)$ and is fixed. The complex envelope $\tilde{f}(t)$ depends on what frequency we denote as the carrier. Since the carrier frequency is at our disposal, we may always choose it so that

$$\bar{\omega} = 0 \tag{A.18}$$

(see Problem A.1.1). Later we shall see that other considerations enter into the choice of the carrier frequency, so that (A.18) may not apply in all cases.

Property 3. The *mean time* of the envelope is defined as the first moment of the squared magnitude of the complex envelope,

$$\bar{t} \triangleq \int_{-\infty}^{\infty} t\,|\tilde{f}(t)|^2\,dt = 0. \tag{A.19}$$

Since the time origin is arbitrary, we may always choose it so that

$$\bar{t} = \int_{-\infty}^{\infty} t\,|\tilde{f}(t)|^2\,dt = 0. \tag{A.20}$$

The assumptions in (A.18) and (A.20) will lead to algebraic simplifications in some cases.

Property 4. There are several quadratic quantities that are useful in describing the signal. The first two are

$$\overline{\omega^2} \triangleq \int_{-\infty}^{\infty} \omega^2\,|\tilde{F}(j\omega)|^2\,\frac{d\omega}{2\pi} \tag{A.21a}$$

and

$$\sigma_w^{\ 2} \triangleq \overline{\omega^2} - (\bar{\omega})^2 \tag{A.21b}$$

The latter quantity is called the *mean-square bandwidth.* It is an approximate measure of the frequency spread of the signal.

Similarly, we define

$$\overline{t^2} = \int_{-\infty}^{\infty} t^2\,|\tilde{f}(t)|^2\,dt \tag{A.22a}$$

and

$$\sigma_t^{\,2} \triangleq \overline{t^2} - (\bar{t})^2. \tag{A.22b}$$

The latter quantity is called the *mean-square duration* and is an approximate measure of the time spread of the signal.

The definitions of the final quantities are

$$\overline{\omega t} = \mathrm{Im}\int_{-\infty}^{\infty} t\tilde{f}(t)\,\frac{d\tilde{f}^*(t)}{dt}\,dt \tag{A.23a}$$

and

$$\rho_{\omega t} = \frac{\overline{\omega t} - \bar{\omega}\bar{t}}{\sigma_\omega \sigma_t}. \tag{A.23b}$$

These definitions are less obvious. Later we shall see that $\rho_{\omega t}$ is a measure of the frequency modulation in $\tilde{f}(t)$.

The relations in (21)–(23) can be expressed in alternative ways using Fourier transform properties (see Problem A.1.2). Other useful interpretations are also developed in the problems.

Property 5. Consider two unit-energy bandpass signals $f_1(t)$ and $f_2(t)$. The correlation between the two signals is

$$\rho = \int_{-\infty}^{\infty} f_1(t) f_2(t)\, dt. \tag{A.24}$$

We now represent the two signals in terms of complex envelopes and the *same* carrier, ω_c.

$$\tilde{f}_i(t) = [\sqrt{2} f_i(t) e^{-j\omega_c t}]_{\text{LP}}, \qquad i = 1, 2. \tag{A.25}$$

The complex correlation $\tilde{\rho}$ is defined to be

$$\tilde{\rho} = \int_{-\infty}^{\infty} \tilde{f}_1(t) \tilde{f}_2^*(t)\, dt. \tag{A.26a}$$

Then

$$\rho = \text{Re}\, \tilde{\rho}. \tag{A.26b}$$

To verify this, we write ρ in terms of complex envelopes, perform the integration, and observe that the double frequency terms can be neglected.

This concludes our discussion of the complex envelope of a deterministic bandpass signal. We now consider bandpass systems.

A.2 BANDPASS LINEAR SYSTEMS

We now develop a complex representation for bandpass linear systems. We first consider time-invariant systems and define a bandpass system in that context.

A.2.1 Time-Invariant Systems

Consider a time-invariant linear system with impulse response $h(\sigma)$ and transfer function $H(j\omega)$.

$$H(j\omega) = \int_{-\infty}^{\infty} h(\sigma) e^{-j\omega\sigma}\, d\sigma. \tag{A.27}$$

A typical transform of interest has the magnitude shown in Fig. A.7. We see that it is bandlimited to a region about the carrier ω_c. We want to represent the bandpass impulse response in terms of two quadrature components. Because $h(\sigma)$ is deterministic, we may use the results of Section A.1 directly. We define two low-pass functions,

$$h_c(\sigma) \triangleq [h(\sigma) \cos \omega_c \sigma]_{\text{LP}} \tag{A.28}$$

and

$$h_s(\sigma) \triangleq [h(\sigma) \sin \omega_c \sigma]_{\text{LP}}. \tag{A.29}$$

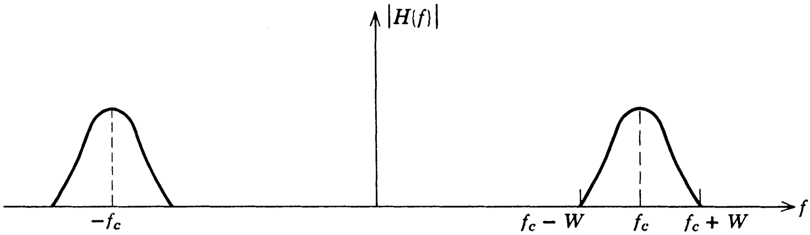


Fig. A.7 Magnitude of the transfer function of a bandpass linear system.

Then

$$h(\sigma) = 2h_c(\sigma)\cos(\omega_c\sigma) + 2h_s(\sigma)\sin(\omega_c\sigma). \tag{A.30}$$

Defining a *complex impulse response* as

$$\tilde{h}(\sigma) = h_c(\sigma) - jh_s(\sigma), \tag{A.31}$$

we have the complex representation

$$h(\sigma) \triangleq \text{Re}\,[2\tilde{h}(\sigma)e^{j\omega_c\sigma}]. \tag{A.32}$$

The introduction of the factor $\sqrt{2}$ is for convenience only.

We now derive an expression for the output of a bandpass linear system $h(t)$ when the input to the system is a bandpass waveform $f(t)$,

$$f(t) = \sqrt{2}\,\text{Re}\,[\tilde{f}(t)e^{j\omega_c t}]. \tag{A.33}$$

Notice that the carrier frequencies of the input signal and the system are identical. This common carrier frequency is implied in all our subsequent discussions. The output $y(t)$ is obtained by convolving $f(t)$ and $h(t)$.

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(t-\sigma)f(\sigma)\,d\sigma \\ &= \int_{-\infty}^{\infty} [\tilde{h}(t-\sigma)e^{j\omega_c(t-\sigma)} + \tilde{h}^*(t-\sigma)e^{-j\omega_c(t-\sigma)}] \\ &\quad \times \left[\frac{\tilde{f}(\sigma)e^{j\omega_c\sigma} + \tilde{f}^*(\sigma)e^{-j\omega_c\sigma}}{\sqrt{2}}\right] d\sigma. \end{aligned} \tag{A.34}$$

Now we define

$$\boxed{\tilde{y}(t) \triangleq \int_{-\infty}^{\infty} \tilde{h}(t-\sigma)\tilde{f}(\sigma)\,d\sigma, \qquad -\infty < t < \infty.} \tag{A.35}$$

Because $\tilde{h}(t)$ and $\tilde{f}(t)$ are low-pass, the two terms in (A.34) containing $e^{\pm 2j\omega_c\sigma}$ integrate to approximately zero and can be neglected. Using (A.35) in the other two terms, we have

$$y(t) = \sqrt{2}\,\text{Re}\,[\tilde{y}(t)e^{j\omega_c t}]. \tag{A.36}$$

This result shows that the complex envelope of the output of a bandpass system is obtained by convolving the complex envelope of the input with the complex impulse response. [The $\sqrt{2}$ was introduced in (A.21) so that (A.35) would have a familiar form.]

A.2.2 Time-Varying Systems

For time-varying bandpass systems, the complex impulse response is $\tilde{h}(t, \tau)$ and

$$h(t, u) = \text{Re}\,[2\tilde{h}(t, u)e^{j\omega_c(t-u)}]. \tag{A.37}$$

The complex envelope of the output is

$$\tilde{y}(t) = \int_{-\infty}^{\infty} \tilde{h}(t, u)\tilde{f}(u)\,du. \tag{A.38}$$

The actual bandpass output is given by (A.36).

A.2.3 State-Variable Systems

In our work in Chapters I-6, II-2, and II-3, we encountered a number of problems in which a state-variable characterization of the system led to an efficient solution procedure. This is also true in the radar-sonar area. We now develop a procedure for characterizing bandpass systems using complex state variables.† The complex input is $\tilde{f}(t)$. The complex state equation is

$$\frac{d\tilde{\mathbf{x}}(t)}{dt} = \tilde{\mathbf{F}}(t)\tilde{\mathbf{x}}(t) + \tilde{\mathbf{G}}(t)\tilde{f}(t), \qquad T_i \le t, \tag{A.39}$$

with initial condition $\tilde{\mathbf{x}}(T_i)$. The observation equation is

$$\tilde{y}(t) = \tilde{\mathbf{C}}(t)\tilde{\mathbf{x}}(t). \tag{A.40}$$

The matrices $\tilde{\mathbf{F}}(t)$, $\tilde{\mathbf{G}}(t)$, and $\tilde{\mathbf{C}}(t)$ are complex matrices. The complex state vector $\tilde{\mathbf{x}}(t)$ and complex output $\tilde{y}(t)$ are low-pass compared to ω_c. The complex block diagram is shown in Fig. A.8.

We define a complex state transition matrix $\tilde{\boldsymbol{\phi}}(t, \tau)$ such that

$$\frac{d}{dt}\tilde{\boldsymbol{\phi}}(t, \tau) = \tilde{\mathbf{F}}(t)\tilde{\boldsymbol{\phi}}(t, \tau), \tag{A.41}$$

$$\tilde{\boldsymbol{\phi}}(t, t) = \mathbf{I}. \tag{A.42}$$

† This discussion is based on our work, which originally appeared in [9]. It is worthwhile emphasizing that most of the complex state-variable results are logical extensions of the real state-variable results.

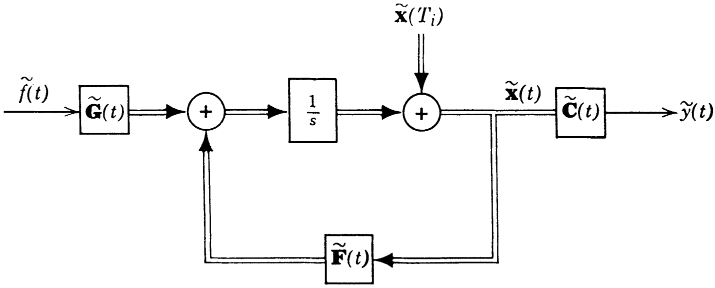


Fig. A.8 State-variable model for a complex linear system.

Then it is readily verified that

$$\tilde{y}(t) = \tilde{\mathbf{C}}(t)\left[\tilde{\boldsymbol{\phi}}(t, T_i)\tilde{\mathbf{x}}(T_i) + \int_{T_i}^{t} \tilde{\boldsymbol{\phi}}(t, \tau)\tilde{\mathbf{G}}(\tau)\tilde{f}(\tau)\, d\tau\right]. \tag{A.43}$$

The first term is the output due to the initial conditions, and the second term is the output due to $\tilde{f}(t)$. The complex impulse response is obtained by letting the initial condition $\tilde{\mathbf{x}}(T_i)$ equal zero and $T_i = -\infty$. Then

$$\tilde{y}(t) = \tilde{\mathbf{C}}(t)\int_{-\infty}^{t} \tilde{\boldsymbol{\phi}}(t, \tau)\tilde{\mathbf{G}}(\tau)\tilde{f}(\tau)\, d\tau, \qquad -\infty < t. \tag{A.44}$$

Recall from (A.38) that

$$\tilde{y}(t) = \int_{-\infty}^{\infty} \tilde{h}(t, \tau)\tilde{f}(\tau)\, d\tau, \qquad -\infty < t. \tag{A.45}$$

Thus,

$$\tilde{h}(t, \tau) = \begin{cases} \tilde{\mathbf{C}}(t)\tilde{\boldsymbol{\phi}}(t, \tau)\tilde{\mathbf{G}}(\tau), & -\infty < \tau \leq t, \\ 0, & \text{elsewhere.} \end{cases} \tag{A.46}$$

Notice that this is a realizable impulse response. Using (A.46) in (A.37) gives the actual bandpass impulse response,

$$\begin{aligned} h(t, \tau) &= \operatorname{Re}\,[2\tilde{h}(t, \tau)e^{j\omega_c(t-\tau)}] \\ &= \begin{cases} \operatorname{Re}\,[2\tilde{\mathbf{C}}(t)\tilde{\boldsymbol{\phi}}(t, \tau)\tilde{\mathbf{G}}(\tau)e^{j\omega_c(t-\tau)}], & -\infty < \tau \leq t, \\ 0, & \text{elsewhere.} \end{cases} \end{aligned} \tag{A.47}$$

There are two alternative procedures that we can use actually to implement the system shown in Fig. A.8. The first procedure is to construct a circuit that is essentially a bandpass analog computer. The second procedure is to perform the operations digitally using complex arithmetic.

We now consider the problem of representing bandpass random processes.

A.3 BANDPASS RANDOM PROCESSES

We would expect that an analogous complex representation could be obtained for bandpass random processes. In this section we discuss three classes of random processes:

1. Stationary processes.
2. Nonstationary processes.
3. Finite state processes.

Throughout our discussion we assume that the random processes have zero means. We begin our discussion with stationary processes.

A.3.1 Stationary Processes

A typical bandpass spectrum is shown in Fig. A.9. It is bandlimited to $\pm W$ cps about ω_c. We want to represent $n(t)$ as

$$n(t) = \sqrt{2}\, n_c(t) \cos(\omega_c t) + \sqrt{2}\, n_s(t) \sin(\omega_c t), \tag{A.48}$$

where $n_c(t)$ and $n_s(t)$ are low-pass functions that are generated as shown in the left side of Fig. A.10.

$$n_c(t) = [(\sqrt{2} \cos(\omega_c t))n(t)]_{\text{LP}} \tag{A.49}$$

and

$$n_s(t) = [(\sqrt{2} \sin(\omega_c t))n(t)]_{\text{LP}}. \tag{A.50}$$

The over-all system, showing both the decomposition and reconstruction, is depicted in Fig. A.10. In Section A.1 we showed that this system was just an ideal bandpass filter (see Fig. A.5). Thus, the representation is valid for all processes that are bandlimited to $\pm W$ cps around ω_c.

Alternatively, in complex notation, we define

$$\boxed{\tilde{n}(t) \triangleq n_c(t) - jn_s(t)} \tag{A.51}$$

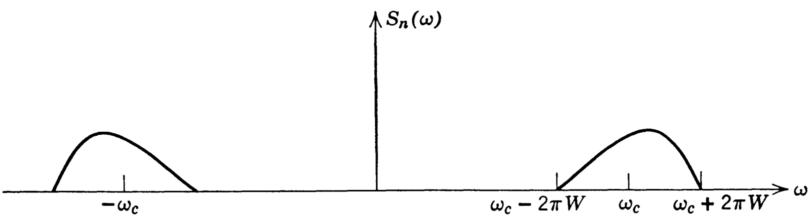


Fig. A.9 Typical spectrum for bandpass process.

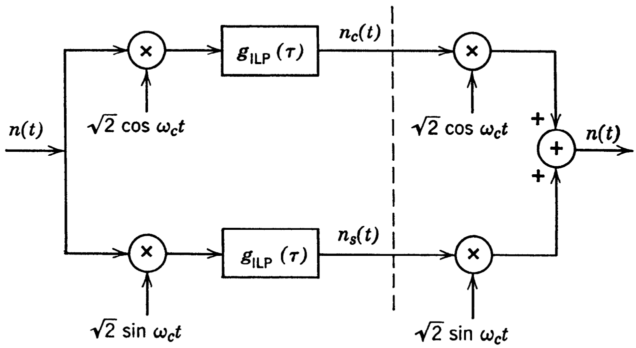


Fig. A.10 Generation of quadrature components and reconstruction of bandpass process.

or

$$\boxed{\tilde{n}(t) = [\sqrt{2}\, n(t)e^{-j\omega_c t}]_{\text{LP}}} \tag{A.52}$$

and write

$$n(t) = \sqrt{2}\,\text{Re}\,[\tilde{n}(t)e^{j\omega_c t}]. \tag{A.53}$$

We now derive the statistical properties of $\tilde{n}(t)$.

The operation indicated in (A.52) corresponds to the block diagram in Fig. A.11. We first compute the covariance function of $\tilde{z}(t)$,

$$\begin{aligned}\tilde{K}_{\tilde{z}}(t, t-\tau) \triangleq E[\tilde{z}(t)\tilde{z}^*(t-\tau)] &= E[\sqrt{2}\, n(t)e^{-j\omega_c t} \cdot \sqrt{2}\, n(t-\tau)e^{j\omega_c(t-\tau)}] \\ &= 2(E[n(t)n(t-\tau)])e^{-j\omega_c \tau} \\ &= 2K_n(\tau)e^{-j\omega_c\tau}.\end{aligned} \tag{A.54}$$

The spectrum of $\tilde{z}(t)$ is

$$\begin{aligned}\tilde{S}_{\tilde{z}}(\omega) &= \int_{-\infty}^{\infty} K_{\tilde{z}}(\tau)e^{-j\omega\tau}\, d\tau = 2\int_{-\infty}^{\infty} K_n(\tau)e^{-j(\omega+\omega_c)\tau}\, d\tau \\ &= 2S_n(\omega + \omega_c).\end{aligned} \tag{A.55}$$

Now $\tilde{n}(t)$ is related to $\tilde{z}(t)$ by an ideal low-pass filter. Thus,

$$\boxed{\tilde{S}_{\tilde{n}}(\omega) = 2[S_n(\omega + \omega_c)]_{\text{LP}}} \tag{A.56}$$

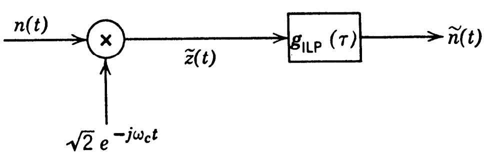


Fig. A.11 Generation of $\tilde{n}(t)$.

and

$$E[\tilde{n}(t)\tilde{n}^*(t-\tau)] \triangleq \tilde{K}_{\tilde{n}}(\tau) = 2\int_{-2\pi W}^{2\pi W} S_n(\omega+\omega_c)e^{j\omega\tau}\,\frac{d\omega}{2\pi}. \tag{A.57}$$

The next result of interest concerns the expectation without a conjugate on the second term. We shall prove that

$$\boxed{E[\tilde{n}(t_1)\tilde{n}(t_2)] = 0, \qquad \text{for all} \quad t_1, t_2.} \tag{A.58}$$

$$E[\tilde{n}(t_1)\tilde{n}(t_2)] =$$

$$2E\left\{\int_{-\infty}^{\infty} n(x_1)e^{-j\omega_c x_1} g_{ILP}(t_1-x_1)\,dx_1 \int_{-\infty}^{\infty} n(x_2)e^{-j\omega_c x_2} g_{ILP}(t_2-x_2)\,dx_2\right\}$$

$$= 2\iint_{-\infty}^{\infty} K_n(x_1-x_2)e^{-j\omega_c(x_1+x_2)} g_{ILP}(t_1-x_1)g_{ILP}(t_2-x_2)\,dx_1\,dx_2$$

$$= 2\iiint_{-\infty}^{\infty} S_n(f)e^{j2\pi f(x_1-x_2)-j2\pi f_c(x_1+x_2)} g_{ILP}(t_1-x_1)g_{ILP}(t_2-x_2)\,dx_1\,dx_2\,df$$

$$= 2\int_{-\infty}^{\infty} S_n(f)\,df \int_{-\infty}^{\infty} g_{ILP}(t_1-x_1)e^{j2\pi x_1(f-f_c)}\,dx_1$$

$$\times \int_{-\infty}^{\infty} g_{ILP}(t_2-x_2)e^{-j2\pi x_2(f+f_c)}\,dx_2$$

$$= 2\int_{-\infty}^{\infty} S_n(f)\{G_{ILP}(f-f_c)G_{ILP}^*(f+f_c)\}e^{j2\pi t_1(f-f_c)-j2\pi t_2(f+f_c)}\,df. \tag{A.59}$$

Now the term in the braces is identically zero for all f if $f_c > W$. Thus the integral is zero and (A.58) is valid. The property in (A.58) is important because it enables us to characterize the complex process in terms of a single covariance function. We can obtain the correlation function of the actual process from this single covariance function.

$$K_n(t, t-\tau) = E[n(t)n(t-\tau)]$$

$$= E\left\{\left[\frac{\sqrt{2}\,\tilde{n}(t)e^{j\omega_c t} + \sqrt{2}\,\tilde{n}^*(t)e^{-j\omega_c t}}{2}\right]\right.$$

$$\left.\times\left[\frac{\sqrt{2}\,\tilde{n}(t-\tau)e^{j\omega_c(t-\tau)} + \sqrt{2}\,\tilde{n}^*(t-\tau)e^{-j\omega_c(t-\tau)}}{2}\right]\right\}$$

$$= \text{Re}\,[\tilde{K}_{\tilde{n}}(\tau)e^{j\omega_c\tau}] + \text{Re}\,\{E[\tilde{n}(t)\tilde{n}(t-\tau)]e^{j\omega_c(2t-\tau)}\}. \tag{A.60}$$

Using (A.58) gives

$$\boxed{K_n(\tau) = \text{Re}\,[\tilde{K}_{\tilde{n}}(\tau)e^{j\omega_c\tau}].} \tag{A.61}$$

In terms of spectra,

$$S_n(\omega) = \int_{-\infty}^{\infty} \frac{\tilde{K}_{\tilde{n}}(\tau)e^{j\omega_c\tau} + \tilde{K}_{\tilde{n}}^*(\tau)e^{-j\omega_c\tau}}{2}\, e^{-j\omega\tau}\, d\tau, \tag{A.62}$$

or

$$\boxed{S_n(\omega) = \frac{\tilde{S}_{\tilde{n}}(\omega - \omega_c) + \tilde{S}_{\tilde{n}}(-\omega - \omega_c)}{2},} \tag{A.63}$$

where we have used the fact that $\tilde{S}_{\tilde{n}}(\omega)$ is a real function of ω.

The relations in (A.61) and (A.63) enable us to obtain the statistics of the bandpass process from the complex process, and vice versa. In Fig. A.12, we indicate this for some typical spectra. Notice that the spectrum of the complex process is even if and only if the bandpass process is symmetric about the carrier ω_c. In Fig. A.13, we indicate the behavior for some typical pole-zero plots. We see that the pole-zero plots are always symmetric about the $j\omega$-axis. This is because $\tilde{S}_{\tilde{n}}(\omega)$ is real. The plots are not necessarily symmetric about the σ-axis, because $\tilde{S}_{\tilde{n}}(\omega)$ is not necessarily even.

Although we shall normally work with the complex process, it is instructive to discuss the statistics of the quadrature components briefly. These follow directly from (A.57) and (A.58).

$$\begin{aligned} E[\tilde{n}(t)\tilde{n}^*(t-\tau)] &= E[(n_c(t) + jn_s(t))(n_c(t-\tau) - jn_s(t-\tau))] \\ &= K_c(\tau) + K_s(\tau) + j[K_{sc}(\tau) - K_{cs}(\tau)] \\ &= \tilde{K}_{\tilde{n}}(\tau) \end{aligned} \tag{A.64}$$

and

$$\begin{aligned} E[\tilde{n}(t)\tilde{n}(t-\tau)] &= E[(n_c(t) + jn_s(t))(n_c(t-\tau) + jn_s(t-\tau))] \\ &= K_c(\tau) - K_s(\tau) + j[K_{sc}(\tau) + K_{cs}(\tau)] \\ &= 0. \end{aligned} \tag{A.65}$$

Therefore,

$$\boxed{K_c(\tau) = K_s(\tau) = \tfrac{1}{2}\text{Re}\,[\tilde{K}_{\tilde{n}}(\tau)]} \tag{A.66}$$

and

$$\boxed{K_{sc}(\tau) = -K_{cs}(\tau) = -K_{sc}(-\tau) = \tfrac{1}{2}\,\text{Im}\,[\tilde{K}_{\tilde{n}}(\tau)].} \tag{A.67}$$

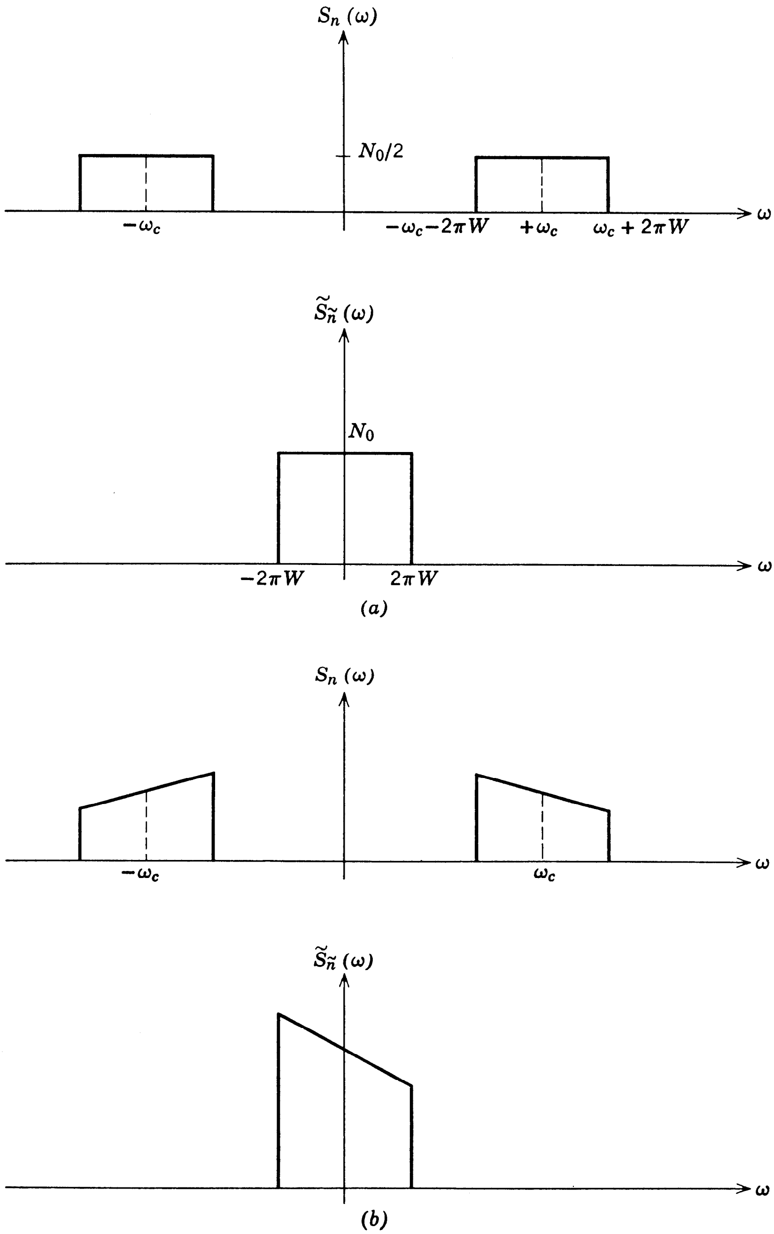


Fig. A.12 Representative spectra.

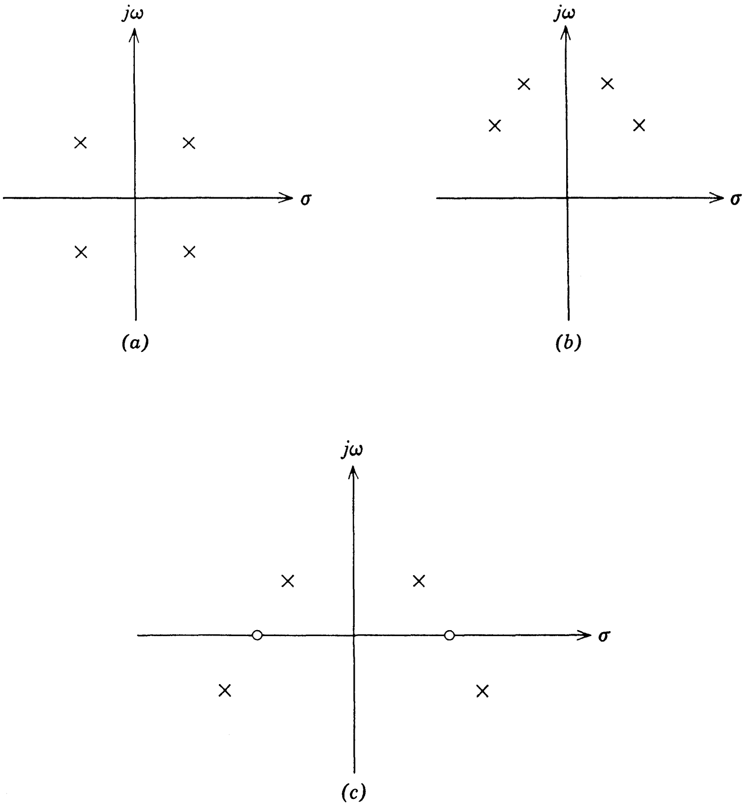


Fig. A.13 Possible pole-zero plots for spectra of complex processes.

In terms of spectra,

$$S_c(\omega) = S_s(\omega) = \frac{1}{2}\int_{-\infty}^{\infty} (\text{Re}\,[\tilde{K}_{\tilde{n}}(\tau)])e^{-j\omega\tau}\,d\tau$$
$$= \frac{1}{2}\left[\frac{\tilde{S}_{\tilde{n}}(\omega) + \tilde{S}_{\tilde{n}}(-\omega)}{2}\right], \tag{A.68}$$

or

$$S_c(\omega) = \tfrac{1}{2}[\tilde{S}_{\tilde{n}}(\omega)]_{\text{EV}}$$
$$= [[S_n(\omega + \omega_c)]_{\text{LP}}]_{\text{EV}}, \tag{A.69}$$

where $[\cdot]_{\text{EV}}$ denotes the operation of taking the even part. Similarly,

$$S_{cs}(\omega) = -\frac{1}{2}\int_{-\infty}^{\infty} \text{Im}\,[\tilde{K}_{\tilde{n}}(\tau)]e^{-j\omega\tau}\,d\tau$$
$$= \frac{j}{2}[\tilde{S}_{\tilde{n}}(\omega)]_{\text{ODD}}$$
$$= j[[S_n(\omega + \omega_c)]_{\text{LP}}]_{\text{ODD}}. \tag{A.70}$$

Notice that $S_{cs}(\omega)$ is imaginary. [This is obvious from the asymmetry in (A.67).] From (A.70) we see that the quadrature processes are correlated unless the spectrum is even around the carrier. Notice that, at any single time instant, $n_c(t_1)$ and $n_s(t_1)$ are uncorrelated. This is because (A.67) implies that

$$K_{cs}(0) = 0. \tag{A.71}$$

Complex White Processes. Before leaving our discussion of the second-moment characterization of complex processes, we define a particular process of interest. Consider the process $w(t)$ whose spectrum is shown in Fig. A.14. The complex envelope is

$$\tilde{w}(t) = w_c(t) - jw_s(t). \tag{A.72}$$

Using (A.69) and (A.70) gives

$$S_{n_c}\{f\} = S_{n_s}\{f\} = \begin{cases} \dfrac{N_0}{2}, & |f| \leq W, \\ 0, & \text{elsewhere}, \end{cases} \tag{A.73}$$

and

$$S_{n_c n_s}\{f\} = 0. \tag{A.74}$$

The covariance function of $\tilde{w}(t)$ is

$$\tilde{K}_{\tilde{w}}(t, u) \triangleq \tilde{K}_{\tilde{w}}(\tau) = 2K_{w_c}(\tau)$$

$$= N_0\left(\frac{\sin(2\pi W\tau)}{\pi\tau}\right), \qquad -\infty < \tau < \infty. \tag{A.75}$$

Now, if W is larger than the other bandwidths in the system of interest, we can approximate (A.75) with an impulse. Letting $W \to \infty$ in (A.75) gives

$$\boxed{\tilde{K}_{\tilde{w}}(\tau) = N_0\, \delta(\tau).} \tag{A.76}$$

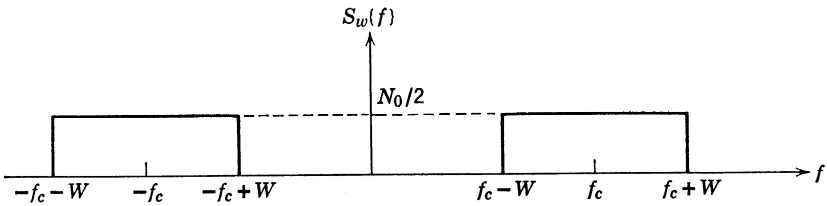


Fig. A.14 Spectrum of bandpass white noise.

We refer to $\tilde{w}(t)$ as a *complex white noise process*. We refer to the actual process $w(t)$ as a bandpass white noise process. Notice that, just as in the case of white noise, they are convenient approximations to the actual physical process.

Complex Gaussian Processes. In many cases of interest to us the processes are *Gaussian* random processes. If $n(t)$ is a stationary Gaussian process, then $n_c(t)$ and $n_s(t)$ are stationary jointly Gaussian processes, because they are obtained by linear operations on $n(t)$. The complex envelope is

$$\tilde{n}(t) = n_c(t) - jn_s(t), \tag{A.77}$$

so that we might logically refer to it as a *stationary complex Gaussian random process*. Since we shall use this idea frequently, an exact definition is worthwhile.

Definition. Let $n_c(t)$ and $n_s(t)$ be two zero-mean stationary jointly Gaussian random processes with identical covariance functions. The process $\tilde{n}(t)$ is defined by (A.77). The relation

$$E[\tilde{n}(t)\tilde{n}(t - \tau)] = 0, \qquad \text{for all} \quad t \text{ and } \tau, \tag{A.78}$$

is satisfied. The process $\tilde{n}(t)$ is a *zero-mean stationary complex Gaussian random process*.

The modification to include a mean value function is straightforward. Notice that a complex process whose real and imaginary parts are both Gaussian processes is not necessarily Gaussian. The condition in (A.78) must be satisfied. This implies that the real part and imaginary part are Gaussian processes with identical characteristics that are related by the covariance function in (A.67). They are statistically independent *if and only if* the original spectrum is symmetric around the carrier. We also note that a real Gaussian process is *not* a special case of a complex Gaussian process.

If we sample the complex envelope at time t_1, we get a complex random variable $\tilde{n}(t_1)$. To specify the density of a complex random variable, we need the joint density of the real part, $n_c(t_1)$, and the imaginary part, $n_s(t_1)$. Since $n_c(t_1)$ and $n_s(t_1)$ are samples of a jointly Gaussian process, they are jointly Gaussian random variables. Since (A.71) implies that they are uncorrelated, we know that they are statistically independent. Therefore,

$$p_{n_{c_{t_1}} n_{s_{t_1}}}(N_c, N_s) = \frac{1}{2\pi\sigma_n^2} \exp\left\{-\frac{N_c^2 + N_s^2}{2\sigma_n^2}\right\}, \qquad -\infty < N_c, N_s < \infty, \tag{A.79}$$

where

$$\sigma_{\tilde{n}}^2 = K_c(0) = K_s(0) = \tfrac{1}{2}K_{\tilde{n}}(0). \tag{A.80}$$

Equivalently,

$$\boxed{p_{\tilde{n}_{t_1}}(\tilde{N}) = \frac{1}{2\pi\sigma_n^2} \exp\left\{-\frac{|\tilde{N}|^2}{2\sigma_n^2}\right\}, \qquad -\infty < \mathrm{Re}\,[\tilde{N}], \mathrm{Im}\,[\tilde{N}] < \infty.} \tag{A.81}$$

We define a *complex Gaussian random variable* as a random variable whose probability density has the form in (A.81). Notice that

$$E(|\tilde{n}_{t_1}|^2) = 2\sigma_n^2. \tag{A.82}$$

Properties analogous to those for real Gaussian random variables and real Gaussian random processes follow easily (see Problems A.3.1–A.3.7). One property that we need corresponds to the definitions on page I-183. Define

$$\tilde{y} = \int_{T_\alpha}^{T_\beta} \tilde{g}(u)\tilde{x}(u)\,du, \tag{A.83}$$

where $\tilde{g}(u)$ is a function such that $E[|\tilde{y}|^2] < \infty$. If $\tilde{x}(u)$ is a complex Gaussian process, then $\tilde{y}$ is a complex Gaussian random variable. This result follows immediately from the above definitions.

A particular complex Gaussian process that we shall use frequently is the complex Gaussian white noise process $\tilde{w}(t)$. It is a complex Gaussian process whose covariance function is given by (A.76).

Two other probability densities are of interest. We can write $\tilde{n}(t)$ in terms of a magnitude and phase angle.

$$\tilde{n}(t) = |\tilde{n}(t)|\, e^{j\phi_{\tilde{n}}(t)}. \tag{A.84}$$

The magnitude corresponds to the envelope of the actual random process. It is easy to demonstrate that it is a Rayleigh random variable at any given time. The phase angle corresponds to the instantaneous phase of the actual random process minus $\omega_c t$, and is a uniform random variable that is independent of the envelope variable. Notice that the envelope and phase processes are not independent processes.

We now turn our attention to nonstationary processes.

A.3.2 Nonstationary Processes

A physical situation in which we encounter nonstationary processes is the reflection of a deterministic signal from a fluctuating point target. We shall see that an appropriate model for the complex envelope of the return is

$$\tilde{s}(t) = \tilde{f}(t)\tilde{b}(t), \tag{A.85}$$

where $\tilde{f}(t)$ is a complex deterministic signal and $\tilde{b}(t)$ is a zero-mean stationary complex Gaussian process. We see that $\tilde{s}(t)$ is a zero-mean nonstationary process whose second-moment characteristics are

$$E[\tilde{s}(t)\tilde{s}^*(u)] = \tilde{f}(t)\tilde{K}_{\tilde{b}}(t-u)\tilde{f}^*(u) \tag{A.86}$$

and

$$E[\tilde{s}(t)\tilde{s}(u)] = \tilde{f}(t)E[(\tilde{b}(t)\tilde{b}(u))]\tilde{f}^*(u) = 0. \tag{A.87}$$

The condition in (A.87) corresponds to the result for stationary processes in (A.58) and enables us to characterize the complex process in terms of a single covariance function. Without this condition, the complex notation is less useful, and so we include it as a condition on the nonstationary processes that we study. Specifically, we consider processes that can be represented as

$$n(t) \triangleq \sqrt{2}\,\text{Re}\,[\tilde{n}(t)e^{j\omega_c t}], \tag{A.88}$$

where $n(t)$ is a complex low-pass process such that

$$E[\tilde{n}(t)\tilde{n}^*(\tau)] = \tilde{K}_{\tilde{n}}(t, \tau) \tag{A.89}$$

and

$$E[\tilde{n}(t)\tilde{n}^*(\tau)] = 0, \qquad \text{for all} \quad t \text{ and } \tau. \tag{A.90†}$$

For a nonstationary process to be low-pass, all of its eigenfunctions with non-negligible eigenvalues must be low-pass compared to ω_c. This requirement is analogous to the spectrum requirement for stationary processes.

The covariance of the actual bandpass process is

$$\begin{aligned} K_n(t, u) &= E[n(t)n(u)] \\ &= E\left\{\left[\frac{\tilde{n}(t)e^{j\omega_c t} + \tilde{n}^*(t)e^{-j\omega_c t}}{\sqrt{2}}\right]\left[\frac{\tilde{n}(u)e^{j\omega_c u} + \tilde{n}^*(u)e^{-j\omega_c u}}{\sqrt{2}}\right]\right\} \\ &= \text{Re}\,\{\tilde{K}_{\tilde{n}}(t, u)e^{j\omega_c(t-u)}\} + \text{Re}\,\{E[\tilde{n}(t)\tilde{n}(u)]e^{j\omega_c(t+u)}\}. \end{aligned} \tag{A.91}$$

The second term on the right-hand side is zero because of the assumption in (A.90). Thus, we have the desired one-to-one correspondence between the second-moment characteristics of the two processes $n(t)$ and $\tilde{n}(t)$. The assumption in (A.90) is not particularly restrictive, because most of the processes that we encounter in practice satisfy it.

As before, the eigenvalues and eigenfunctions of a random process play an important role in many of our discussions. All of our discussion in

† It is worthwhile emphasizing that (A.90) has to be true for the complex envelope of a *stationary* bandpass process. For nonstationary processes it is an additional assumption. Examples of nonstationary processes that do not satisfy (A.90) are given in [11] and [12].

Chapter I-3 (page I-166) carries over to complex processes. The equation specifying the eigenvalues and eigenfunctions is

$$\tilde{\lambda}_i \tilde{\phi}_i(t) = \int_{T_i}^{T_f} \tilde{K}_{\tilde{n}}(t, u) \tilde{\phi}_i(u)\, du, \qquad T_i \le t \le T_f. \tag{A.92}$$

We assume that the kernel is Hermitian,

$$\tilde{K}_{\tilde{n}}(t, u) = \tilde{K}_{\tilde{n}}^*(u, t). \tag{A.93}$$

This is analogous to the symmetry requirement in the real case and is satisfied by all complex covariance functions. The eigenvalues of a Hermitian kernel are all real. We would expect this because the spectrum is real in the stationary case. We now look at the complex envelope process and the actual bandpass process and show how their eigenfunctions and eigenvalues are related. We first write the pertinent equations for the two processes and then show their relationship.

For the bandpass random process, we have from Chapter I-3 that

$$n(t) = \underset{K\to\infty}{\text{l.i.m.}} \sum_{i=1}^{K} n_i \phi_i(t), \qquad T_i \le t \le T_f, \tag{A.94}$$

where the $\phi_i(t)$ satisfy

$$\lambda_i \phi_i(t) = \int_{T_i}^{T_f} K_n(t, u) \phi_i(u)\, du, \qquad T_i \le t \le T_f \tag{A.95}$$

and the coefficients are

$$n_i = \int_{T_i}^{T_f} n(t) \phi_i(t)\, dt. \tag{A.96}$$

This implies that

$$E[n_i n_j] = \lambda_i \delta_{ij} \tag{A.97}$$

and

$$K_n(t, u) = \sum_{i=1}^{\infty} \lambda_i \phi_i(t) \phi_i(u), \qquad T_i \le t, u \le T_f. \tag{A.98}$$

Similarly, for the complex envelope random process,

$$\tilde{n}(t) = \underset{K\to\infty}{\text{l.i.m.}} \sum_{i=1}^{K} \tilde{n}_i \tilde{\phi}_i(t), \qquad T_i \le t \le T_f, \tag{A.99}$$

where the $\tilde{\phi}_i(t)$ satisfy the equation

$$\tilde{\lambda}_i \tilde{\phi}_i(t) = \int_{T_i}^{T_f} \tilde{K}_{\tilde{n}}(t, u) \tilde{\phi}_i(u)\, du, \qquad T_i \le t \le T_f. \tag{A.100}$$

The complex eigenfunctions are orthonormal,

$$\int_{T_i}^{T_f} \tilde{\phi}_i(t)\tilde{\phi}_j^*(t)\,dt = \delta_{ij}. \tag{A.101}$$

The coefficients are

$$\tilde{n}_i = \int_{T_i}^{T_f} \tilde{n}(t)\tilde{\phi}_i^*(t)\,dt. \tag{A.102}$$

We can then show that

$$E[\tilde{n}_i\tilde{n}_j^*] = \tilde{\lambda}_i\delta_{ij}, \tag{A.103}$$

$$E[n_i n_j] = 0, \qquad \text{for all} \quad i \text{ and } j, \tag{A.104}$$

and

$$\tilde{K}_n(t, u) = \sum_{i=1}^{\infty} \tilde{\lambda}_i\tilde{\phi}_i(t)\tilde{\phi}_i^*(u), \qquad T_i \le t, u \le T_f. \tag{A.105}$$

The processes are related by (A.88) and (A.91),

$$n(t) = \sqrt{2}\,\mathrm{Re}\,[\tilde{n}(t)e^{j\omega_c t}], \tag{A.106}$$

$$K_n(t, u) = \mathrm{Re}\,[\tilde{K}_n(t, u)e^{j\omega_c(t-u)}]. \tag{A.107}$$

To find how the eigenfunctions are related, we substitute

$$\phi_i(t) = \sqrt{2}\,\mathrm{Re}\,[\tilde{\phi}_i(t)e^{j(\omega_c t+\theta)}], \qquad T_i \le t \le T_f \tag{A.108}$$

into (A.95) and use (A.107). The result is

$$\lambda_i[\tilde{\phi}_i(t)e^{j(\omega_c t+\theta)} + \tilde{\phi}_i^*(t)e^{-j(\omega_c t+\theta)}]$$
$$= \frac{1}{2}\left\{e^{j(\omega_c t+\theta)}\int_{T_i}^{T_f} \tilde{K}_{\tilde{n}}(t, u)\tilde{\phi}_i(u)\,du + e^{-j(\omega_c t+\theta)}\int_{T_i}^{T_f} \tilde{K}_{\tilde{n}}^*(t, u)\tilde{\phi}_i^*(u)\,du\right\}. \tag{A.109}$$

Equivalently,

$$\mathrm{Re}\left\{\left(\lambda_i\tilde{\phi}_i(t) - \frac{1}{2}\int_{T_i}^{T_f} \tilde{K}_{\tilde{n}}(t, u)\tilde{\phi}_i(u)\,du\right)e^{j\theta+j\omega_c t}\right\} = 0. \tag{A.110}$$

If we require that

$$\lambda_i = \frac{\tilde{\lambda}_i}{2}, \tag{A.111}$$

then (A.109) will be satisfied for any θ. Because (A.109) is valid for any θ, each eigenvalue and eigenfunction of the complex process corresponds to an eigenvalue and a family of eigenfunctions of the bandpass process. Clearly, not more than two of these can be algebraically linearly independent. These can be chosen to be orthogonal by using $\theta = 0$ and $\theta = -\pi/2$. (Any two values of θ that differ by 90° are also satisfactory.) Thus,

Table A.1

Complex process	Actual bandpass process	
$\tilde{\lambda}_1\tilde{\phi}_1(t)$	$\lambda_1 = \dfrac{\tilde{\lambda}_1}{2}$	$\phi_1(t) = \sqrt{2}\,\text{Re}\,[\tilde{\phi}_1(t)e^{j\omega_c t}]$
	$\lambda_2 = \dfrac{\tilde{\lambda}_1}{2}$	$\phi_2(t) = \sqrt{2}\,\text{Re}\,[\tilde{\phi}_1(t)e^{j(\omega_c t-\pi/2)}] = \sqrt{2}\,\text{Im}\,[\tilde{\phi}_1(t)e^{j\omega_c t}]$
$\tilde{\lambda}_2\tilde{\phi}_2(t)$	$\lambda_3 = \dfrac{\tilde{\lambda}_2}{2}$	$\phi_3(t) = \sqrt{2}\,\text{Re}\,[\tilde{\phi}_2(t)e^{j\omega_c t}]$
	$\lambda_4 = \dfrac{\tilde{\lambda}_2}{2}$	$\phi_4(t) = \sqrt{2}\,\text{Im}\,[\tilde{\phi}_2(t)e^{j\omega_c t}]$

we can index the eigenvalues and eigenfunctions as shown in Table A.1. The result that the eigenvalues of the actual process occur in pairs is important in our succeeding work. It leads to a significant simplification in our analyses.

The relationship between the coefficients in the Karhunen-Loève expansion follows by direct substitution:

$$\begin{aligned} n_1 &= \text{Re}\,[\tilde{n}_1], \\ n_2 &= \text{Im}\,[\tilde{n}_1], \\ n_3 &= \text{Re}\,[\tilde{n}_2], \\ n_4 &= \text{Im}\,[\tilde{n}_2], \end{aligned} \tag{A.112}$$

and so forth.

From (A.89) and (A.90) we know that $n_c(t)$ and $n_s(t)$ have identical covariance functions. When they are uncorrelated processes, the eigenvalues of $\tilde{n}(t)$ are just twice the eigenvalues of $n_c(t)$. In the general case, there is no simple relationship between the eigenvalues of the complex envelope process and the eigenvalues of the quadrature process.

Up to this point we have considered only second-moment characteristics. We frequently are interested in Gaussian processes. If $n(t)$ is a nonstationary Gaussian process and (A.88)–(A.90) are true, we could define $\tilde{n}(t)$ to be a complex Gaussian random process. It is easier to define a complex Gaussian process directly.

Definition. Let $\tilde{n}(t)$ be a random process defined over some interval $[T_\alpha, T_\beta]$ with a mean value $\tilde{m}_{\tilde{n}}(t)$ and covariance function

$$E[(\tilde{n}(t) - \tilde{m}_{\tilde{n}}(t))(\tilde{n}^*(u) - \tilde{m}_{\tilde{n}}^*(u))] = \tilde{K}_{\tilde{n}}(t, u), \tag{A.113}$$

which has the property that

$$E[(\tilde{n}(t) - \tilde{m}_{\tilde{n}}(t))(\tilde{n}(u) - \tilde{m}_{\tilde{n}}(u))] = 0, \qquad \text{for all} \quad t \text{ and } u. \qquad \text{(A.114)}$$

If every complex linear functional of $\tilde{n}(t)$ is a complex Gaussian random variable, $\tilde{n}(t)$ is a complex Gaussian random process. In other words, assume that

$$\tilde{y} = \int_{T_\alpha}^{T_\beta} \tilde{g}(u)\tilde{x}(u)\, du, \qquad \text{(A.115)}$$

where $\tilde{g}(u)$ is any function such that $E[|\tilde{y}|^2] < \infty$. Then, in order for $\tilde{x}(u)$ to be a complex Gaussian random process, $\tilde{y}$ must be a complex Gaussian random variable for every $\tilde{g}(u)$ in the above class.

Notice that this definition is exactly parallel to the definition of a real Gaussian process on page I-183. Various properties of nonstationary Gaussian processes are derived in the problems. Since stationary complex Gaussian processes are a special case, they must satisfy the above definition. It is straightforward to show that the definition on page 583 is equivalent to the above definition when the processes are stationary.

Returning to the Karhunen-Loève expansion, we observe that if $\tilde{n}(t)$ is a complex Gaussian random process, $\tilde{n}_i$ is a complex Gaussian random variable whose density is given by (A.81), with $\sigma_n^2 = \tilde{\lambda}_i/2$,

$$\boxed{p_{\tilde{n}_i}(\tilde{N}_i) = \frac{1}{\pi\tilde{\lambda}_i} \exp\left(-\frac{|\tilde{N}_i|^2}{\tilde{\lambda}_i}\right), \qquad -\infty < \text{Re}\,[\tilde{N}_i],\ \text{Im}\,[\tilde{N}_i] < \infty.} \qquad \text{(A.116)}$$

The complex Gaussian white noise process has the property that a series expansion using any set of orthonormal functions has statistically independent coefficients. Denoting the ith coefficient as $\tilde{w}_i$, we have

$$p_{\tilde{w}_i}(\tilde{W}_i) = \frac{1}{\pi N_0} \exp\left(-\frac{|\tilde{W}_i|^2}{N_0}\right), \qquad -\infty < \text{Re}\,[W_i],\ \text{Im}\,[W_i] < \infty. \qquad \text{(A.117)}$$

This completes our general discussion of nonstationary processes. We now consider complex processes with a finite state representation.

A.3.3 Complex Finite-State Processes†

In our previous work we found that an important class of random processes consists of those which can be generated by exciting a finite-dimensional linear dynamic system with a white noise process. Instead of

† This section is based on [9].

working with the bandpass process, we shall work with the complex envelope process. We want to develop a class of complex processes that we can generate by exciting a finite-dimensional complex system with complex white noise. We define it in such a manner that its properties will be consistent with the properties of stationary processes when appropriate. The complex state equation of interest is

$$\dot{\tilde{\mathbf{x}}}(t) = \tilde{\mathbf{F}}(t)\tilde{\mathbf{x}}(t) + \tilde{\mathbf{G}}(t)\tilde{\mathbf{u}}(t). \tag{A.118}$$

This is just a generalization of (A.39) to include a vector-driving function $\tilde{\mathbf{u}}(t)$. The observation equation is

$$\tilde{\mathbf{y}}(t) = \tilde{\mathbf{C}}(t)\tilde{\mathbf{x}}(t). \tag{A.119}$$

A block diagram of the system is shown in Fig. A.15. We assume that $\tilde{\mathbf{u}}(t)$ is a complex vector white noise process with zero mean and covariance matrix

$$E[\tilde{\mathbf{u}}(t)\tilde{\mathbf{u}}^{\dagger}(\sigma)] = \tilde{\mathbf{K}}_{\tilde{\mathbf{u}}}(t, \sigma) = \tilde{\mathbf{Q}}\,\delta(t - \sigma), \tag{A.120}$$

where

$$\tilde{\mathbf{u}}^{\dagger}(t) \triangleq [\tilde{\mathbf{u}}(t)^*]^T. \tag{A.121}$$

We further assume that

$$E[\tilde{\mathbf{u}}(t)\tilde{\mathbf{u}}^T(\sigma)] = \mathbf{0}, \qquad \text{for all} \quad t \text{ and } \sigma. \tag{A.122}$$

This is just the vector analog to the assumption in (A.90). In terms of the quadrature components,

$$\begin{aligned}\tilde{\mathbf{K}}_{\tilde{\mathbf{u}}}(t, \sigma) &= E[(\mathbf{u}_c(t) - j\mathbf{u}_s(t))(\mathbf{u}_c^T(\sigma) + j\mathbf{u}_s^T(\sigma))]\\ &= \mathbf{K}_{\mathbf{u}_c}(t, \sigma) + \mathbf{K}_{\mathbf{u}_s}(t, \sigma) + j\mathbf{K}_{\mathbf{u}_c\mathbf{u}_s}(t, \sigma) - j\mathbf{K}_{\mathbf{u}_s\mathbf{u}_c}(t, \sigma)\\ &= \tilde{\mathbf{Q}}\,\delta(t - \sigma).\end{aligned} \tag{A.123}$$

The requirement in (A.122) implies that

$$\mathbf{K}_{\mathbf{u}_c}(t, \sigma) = \mathbf{K}_{\mathbf{u}_s}(t, \sigma) = \tfrac{1}{2}\,\mathrm{Re}\,[\tilde{\mathbf{Q}}]\,\delta(t - \sigma), \tag{A.124}$$

$$\mathbf{K}_{\mathbf{u}_c\mathbf{u}_s}(t, \sigma) = -\mathbf{K}_{\mathbf{u}_s\mathbf{u}_c}(t, \sigma) = \tfrac{1}{2}\,\mathrm{Im}\,[\tilde{\mathbf{Q}}]\,\delta(t - \sigma). \tag{A.125}$$

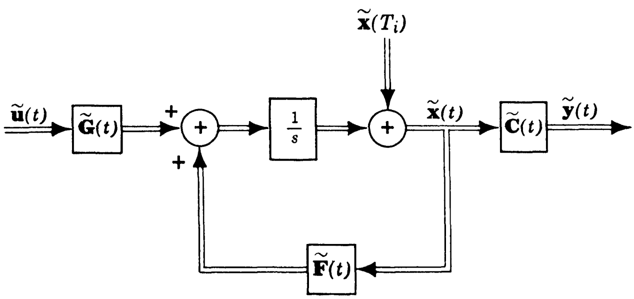


Fig. A.15 Generation of a finite-state complex process.

The covariance matrices for the two quadrature components are identical non-negative-definite matrices, and the cross-covariance matrix is a skew symmetric matrix (i.e., $a_{ij} = -a_{ji}$). This implies that $\tilde{\mathbf{Q}}$ is a Hermitian matrix with a non-negative-definite real part.

Usually we do not need a correlation between the components of $\tilde{\mathbf{u}}(t)$ (i.e., we can let $E[\mathbf{u}_c(t)\mathbf{u}_s^T(t)] = \mathbf{0}$), since any correlation between the components of the state vector may be represented in the coefficient matrices $\tilde{\mathbf{F}}(t)$ and $\tilde{\mathbf{G}}(t)$. In this case $\tilde{\mathbf{Q}}$ is a real non-negative-definite symmetric matrix.

The next issue that we want to consider is the initial conditions. In order that we be consistent with the concept of state, whatever symmetry assumptions we make regarding the state vector at the initial time T_i should be satisfied at an arbitrary time t $(t \geq T_i)$.

First, we shall assume that $\tilde{\mathbf{x}}(T_i)$ is a complex random vector (we assume zero mean for simplicity). The complex covariance matrix for this random vector is

$$\begin{aligned}\tilde{\mathbf{P}}_i \triangleq \tilde{\mathbf{K}}_{\tilde{\mathbf{x}}}(T_i, T_i) &= E[\tilde{\mathbf{x}}(T_i)\tilde{\mathbf{x}}^{\dagger}(T_i)] \\ &= \mathbf{K}_{\mathbf{x}_c}(T_i, T_i) + \mathbf{K}_{\mathbf{x}_s}(T_i, T_i) + j\mathbf{K}_{\mathbf{x}_c\mathbf{x}_s}(T_i, T_i) - j\mathbf{K}_{\mathbf{x}_s\mathbf{x}_c}(T_i, T_i). \end{aligned} \quad \text{(A.126)}$$

We assume that

$$E[\tilde{\mathbf{x}}(T_i)\tilde{\mathbf{x}}^T(T_i)] = \mathbf{0}. \quad \text{(A.127)}$$

Notice that (A.126) and (A.127) are consistent with our earlier ideas. They imply that

$$\mathbf{K}_{\mathbf{x}_c}(T_i, T_i) = \mathbf{K}_{\mathbf{x}_s}(T_i, T_i) = \tfrac{1}{2}\,\mathrm{Re}\,[\tilde{\mathbf{P}}_i], \quad \text{(A.128)}$$

$$\mathbf{K}_{\mathbf{x}_c\mathbf{x}_s}(T_i, T_i) = -\mathbf{K}_{\mathbf{x}_s\mathbf{x}_c}(T_i, T_i) = \tfrac{1}{2}\,\mathrm{Im}\,(\tilde{\mathbf{P}}_i). \quad \text{(A.129)}$$

Consequently, the complex covariance matrix of the initial condition is a Hermitian matrix with a non-negative-definite real part.

Let us now consider what these assumptions imply about the covariance of the state vector $\tilde{\mathbf{x}}(t)$ and the observed signal $\tilde{\mathbf{y}}(t)$. Since we can relate the covariance of $\tilde{\mathbf{y}}(t)$ directly to that of the state vector, we shall consider $\tilde{\mathbf{K}}_{\tilde{\mathbf{x}}}(t, u)$ first.

For real state-variable random processes, we can determine $\mathbf{K}_{\mathbf{x}}(t, \sigma)$ in terms of the state equation matrices, the matrix $\mathbf{Q}$ associated with the covariance of the excitation noise $\mathbf{u}(t)$, and the covariance $\mathbf{K}_{\mathbf{x}}(T_i, T_i)$ of the initial state vector, $\mathbf{x}(T_i)$. The results for complex state variables are parallel. The only change is that the transpose operation is replaced by a conjugate transpose operation. Because of the similarity of the derivations, we shall only state the results (see Problem A.3.19).

The matrix $\tilde{\mathbf{K}}_{\tilde{\mathbf{x}}}(t, t)$ is a Hermitian matrix that satisfies the linear matrix differential equation

$$\frac{d\tilde{\mathbf{K}}_{\tilde{\mathbf{x}}}(t, t)}{dt} = \tilde{\mathbf{F}}(t)\tilde{\mathbf{K}}_{\tilde{\mathbf{x}}}(t, t) + \tilde{\mathbf{K}}_{\tilde{\mathbf{x}}}(t, t)\tilde{\mathbf{F}}^{\dagger}(t) + \tilde{\mathbf{G}}(t)\tilde{\mathbf{Q}}\tilde{\mathbf{G}}^{\dagger}(t), \quad \text{(A.130)}$$

where the initial condition $\tilde{\mathbf{K}}_{\tilde{\mathbf{x}}}(T_i, T_i)$ is given as part of the system description. [This result is analogous to (I-6.279).] $\tilde{\mathbf{K}}_{\tilde{\mathbf{x}}}(t, \sigma)$ is given by

$$\tilde{\mathbf{K}}_{\tilde{\mathbf{x}}}(t, \sigma) = \begin{cases} \tilde{\boldsymbol{\phi}}(t, \sigma)\tilde{\mathbf{K}}_{\tilde{\mathbf{x}}}(\sigma, \sigma), & t > u, \\ \tilde{\mathbf{K}}_{\tilde{\mathbf{x}}}(t, t)\tilde{\boldsymbol{\phi}}^{\dagger}(\sigma, t), & u > t, \end{cases} \quad \text{(A.131)}$$

where $\tilde{\boldsymbol{\phi}}(t, \sigma)$ is the complex transition matrix associated with $\tilde{\mathbf{F}}(t)$. (This result is analogous to that in Problem I-6.3.16.) In addition,

$$\tilde{\mathbf{K}}_{\tilde{\mathbf{x}}}(t, \sigma) = \tilde{\mathbf{K}}_{\tilde{\mathbf{x}}}^{\dagger}(\sigma, t) \quad \text{(A.132)}$$

and

$$E[\tilde{\mathbf{x}}(t)\tilde{\mathbf{x}}^{T}(\sigma)] = \mathbf{0}, \qquad \text{for all} \quad t \text{ and } \sigma. \quad \text{(A.133)}$$

Therefore the assumptions that we have made on the covariance of the initial state vector $\tilde{\mathbf{x}}(T_i)$ are satisfied by the covariance of the state vector $\tilde{\mathbf{x}}(t)$ for all $t \geq T_i$.

Usually we are not concerned directly with the state vector of a system. The vector of interest is the observed signal, $\tilde{\mathbf{y}}(t)$, which is related to the state vector by (A.119). We can simply indicate the properties of the covariance $\tilde{\mathbf{K}}_{\tilde{\mathbf{y}}}(t, \sigma)$, since it is related directly to the covariance of the state vector by

$$\tilde{\mathbf{K}}_{\tilde{\mathbf{y}}}(t, \sigma) = \tilde{\mathbf{C}}(t)\tilde{\mathbf{K}}_{\tilde{\mathbf{x}}}(t, \sigma)\tilde{\mathbf{C}}^{\dagger}(\sigma). \quad \text{(A.134)}$$

Consequently, it is clear that $\tilde{\mathbf{K}}_{\tilde{\mathbf{y}}}(t, t)$ is Hermitian. Similarly, from (A.133) we have the result that $E[\tilde{\mathbf{y}}(t)\tilde{\mathbf{y}}^{T}(\sigma)]$ is zero.

The properties of the quadrature components follow easily:

$$E[\mathbf{y}_c(t)\mathbf{y}_c^{T}(\sigma)] = E[\mathbf{y}_s(t)\mathbf{y}_s^{T}(\sigma)] = \tfrac{1}{2}\,\mathrm{Re}\,[\tilde{\mathbf{K}}_{\tilde{\mathbf{y}}}(t, \sigma)] \quad \text{(A.135)}$$

and

$$E[\mathbf{y}_c(t)\mathbf{y}_s^{T}(\sigma)] = \tfrac{1}{2}\,\mathrm{Im}\,[\tilde{\mathbf{K}}_{\tilde{\mathbf{y}}}(t, \sigma)]. \quad \text{(A.136)}$$

In this section we have introduced the idea of generating a complex random process by exciting a linear system having a complex state variable description with a complex white noise. We then showed how we could describe the second-order statistics of this process in terms of a complex covariance function, and we discussed how we could determine this function from the state-variable description of the system. The only assumptions that we made were on the second-order statistics of $\tilde{\mathbf{u}}(t)$ and $\tilde{\mathbf{x}}(T_i)$. Our results were independent of the form of the coefficient matrices $\tilde{\mathbf{F}}(t)$, $\mathbf{G}(t)$, and $\tilde{\mathbf{C}}(t)$. Our methods were exactly parallel to those for real state variables. It is easy to verify that all of the results are consistent with those derived in Sections A.1 and A.2 for stationary and nonstationary random processes.

We now consider a simple example to illustrate some of the manipulations involved.

Example 1. In this example we consider a first-order (scalar) state equation. We shall find the covariance function for the nonstationary case and then look at the special case in which the process is stationary, and find the spectrum. The equations that describe this system are

$$\frac{d\tilde{x}(t)}{dt} = -\tilde{k}\tilde{x}(t) + \tilde{u}(t), \qquad T_i \le t \tag{A.137}$$

and

$$\tilde{y}(t) = \tilde{x}(t). \tag{A.138}$$

The assumptions on $\tilde{u}(t)$ and $\tilde{x}(T_i)$ are

$$E[\tilde{u}(t)\tilde{u}^*(\sigma)] = 2\,\mathrm{Re}[\tilde{k}]P\,\delta(t-\sigma) \tag{A.139}$$

and

$$E[|\tilde{x}(T_i)|^2] = P_i. \tag{A.140}$$

Because we have a scalar process, both P and P_i must be real. In addition, we have again assumed zero means.

First, we shall find $K_x(t, t)$. The differential equation (A.130) that it satisfies is

$$\begin{aligned}\frac{d\tilde{K}_{\tilde{x}}(t,t)}{dt} &= -\tilde{k}\tilde{K}_{\tilde{x}}(t,t) - \tilde{k}^*\tilde{K}_{\tilde{x}}(t,t) + 2\,\mathrm{Re}\,[\tilde{k}]P \\ &= -2\,\mathrm{Re}\,[\tilde{k}]\tilde{K}_{\tilde{x}}(t,t) + 2\,\mathrm{Re}\,[\tilde{k}]P, \qquad t \ge T_i.\end{aligned} \tag{A.141}$$

The solution to (A.141) is

$$\tilde{K}_{\tilde{x}}(t,t) = P - (P - P_i)e^{-2\,\mathrm{Re}\,[\tilde{k}](t-T_i)}, \qquad t \ge T_i. \tag{A.142}$$

In order to find $\tilde{K}_{\tilde{x}}(t, \sigma)$ by using (A.131), we need to find $\tilde{\phi}(t, \sigma)$, the transition matrix for this system. This is

$$\tilde{\phi}(t,\sigma) = e^{-\tilde{k}(t-\sigma)}, \qquad t > \sigma. \tag{A.143}$$

By substituting (A.142) and (A.143) into (A.131), we can find $\tilde{K}_{\tilde{x}}(t, \sigma)$, which is also $\tilde{K}_{\tilde{y}}(t, \sigma)$ for this particular example.

Let us now consider the stationary problem in more detail. This case arises when we observe a segment of a stationary process. To make $\tilde{x}(t)$ stationary, we let

$$P_i = P. \tag{A.144}$$

If we perform the indicated substitutions and define $\tau = t - u$, we obtain

$$\tilde{K}_{\tilde{x}}(\tau) = \begin{cases} Pe^{-\tilde{k}\tau}, & \tau \ge 0, \\ Pe^{\tilde{k}^*\tau}, & \tau \le 0. \end{cases} \tag{A.145}$$

This may be written as

$$\tilde{K}_{\tilde{x}}(\tau) = Pe^{-\mathrm{Re}\,[\tilde{k}]|\tau|}e^{-j\,\mathrm{Im}\,[\tilde{k}]\tau}. \tag{A.146}$$

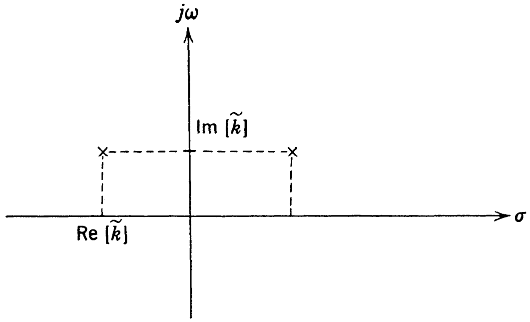


Fig. A.16 Pole location for stationary process generated by first-order system.

The spectrum of the complex process is

$$\tilde{S}_{\tilde{y}}(\omega) = \tilde{S}_{\tilde{x}}(\omega) = \frac{2 \operatorname{Re}[\tilde{k}]P}{(\omega + \operatorname{Im}[\tilde{k}])^2 + (\operatorname{Re}[\tilde{k}])^2}. \tag{A.147}$$

From (A.147), we see that in the stationary case, the net effect of the complex pole $\tilde{k}$ is that the complex spectrum has a frequency shift equal to the imaginary part of $\tilde{k}$. In the actual bandpass process, this corresponds to shifting the carrier frequency. This is obvious if we look at the pole-zero plot of $\tilde{S}_{\tilde{x}}(\omega)$ as shown in Fig. A.16.

Example 2. Consider the pole-zero plot shown in Fig. A.17. The spectrum is

$$\tilde{S}_{\tilde{y}}(\omega) = \frac{C}{(j\omega - \tilde{k}_1)(j\omega - \tilde{k}_2)(-j\omega - \tilde{k}_1^*)(-j\omega - \tilde{k}_2^*)}. \tag{A.148}$$

We can generate this spectrum by driving a two-state system with complex white noise. The eigenvalues of the $\tilde{\mathbf{F}}$ matrix must equal $-\tilde{k}_1$ and $-\tilde{k}_2$. If we use the state representation

$$\tilde{x}_1(t) = \tilde{y}(t) \tag{A.149}$$

$$\tilde{x}_2(t) = \dot{\tilde{x}}_1(t), \tag{A.150}$$

then the equations are

$$\frac{d}{dt}\begin{bmatrix}\tilde{x}_1(t)\\ \tilde{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\tilde{k}_1\tilde{k}_2 & -(\tilde{k}_1 + \tilde{k}_2)\end{bmatrix}\begin{bmatrix}\tilde{x}_1(t)\\ \tilde{x}_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u(t), \tag{A.151}$$

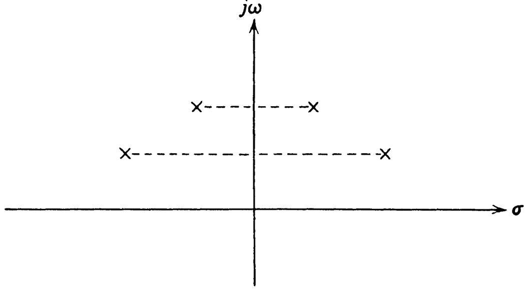


Fig. A.17 Pole location for a particular second-order spectrum.

and

$$\tilde{y}(t) = [1 \quad 0]\begin{bmatrix}\tilde{x}_1(t)\\ \tilde{x}_2(t)\end{bmatrix}, \tag{A.152}$$

where we assume that

$$\text{Re}\,[\tilde{k}_1] > 0, \tag{A.153a}$$

$$\text{Re}\,[\tilde{k}_2] > 0, \tag{A.153b}$$

and

$$\tilde{k}_1 \neq \tilde{k}_2. \tag{A.153c}$$

We can carry out the same type of analysis as in Example 1 (see Problem A.3.14).

Our two examples emphasized stationary processes. The use of complex state variables is even more important when we must deal with nonstationary processes. Just as with real state variables, they enable us to obtain complete solutions to a large number of important problems in the areas of detection, estimation, and filtering theory. Many of these applications arise logically in Chapters 9–13. There is one application that is easy to formulate, and so we include it here.

Optimal Linear Filtering Theory. In many communication problems, we want to estimate the complex envelope of a narrow-band process. The efficiency of real state-variable techniques in finding estimator structures suggests that we can use our complex state variables to find estimates of the complex envelopes of narrow-band processes. In this section we shall indicate the structure of the realizable complex filter for estimating the complex envelope of a narrow-band process. We shall only quote the results of our derivation, since the methods used are exactly parallel to those for real state variables. The major difference is that the transpose operations are replaced by conjugate transpose operations.

We consider complex random processes that have a finite-dimensional state representation. In the state-variable formulation of an optimal linear filtering problem, we want to estimate the state vector $\tilde{\mathbf{x}}(t)$ of a linear system when we observe its output $\tilde{\mathbf{y}}(t)$ corrupted by additive white noise, $\tilde{\mathbf{w}}(t)$. Therefore, our received signal $\tilde{\mathbf{r}}(t)$ is given by

$$\begin{aligned}\tilde{\mathbf{r}}(t) &= \tilde{\mathbf{y}}(t) + \tilde{\mathbf{w}}(t)\\ &= \tilde{\mathbf{C}}(t)\tilde{\mathbf{x}}(t) + \tilde{\mathbf{w}}(t), \qquad T_i \leq t \leq T_f,\end{aligned} \tag{A.154}$$

where

$$E[\tilde{\mathbf{w}}(t)\tilde{\mathbf{w}}^{\dagger}(\tau)] = \tilde{\mathbf{R}}(t)\,\delta(t-\tau). \tag{A.155}$$

We assume that $\tilde{\mathbf{R}}(t)$ is a positive-definite Hermitian matrix.

In the realizable filtering problem, we estimate the state vector at the endpoint time of the observation interval, i.e., at T_f. This endpoint time, however, is usually a variable that increases as the data are received. Consequently, we want to have our estimate $\tilde{\mathbf{x}}(t)$ evolve as a function of the endpoint time of the observation interval $[T_i, t]$. We choose the estimate $\hat{\tilde{\mathbf{x}}}(t)$ to minimize the mean-square error,

$$\tilde{\boldsymbol{\xi}}_P(t) \triangleq E\{[\tilde{\mathbf{x}}(t) - \hat{\tilde{\mathbf{x}}}(t)][\tilde{\mathbf{x}}(t) - (\hat{\tilde{\mathbf{x}}}(t)]^\dagger\}. \tag{A.156}$$

We assume that $\hat{\tilde{\mathbf{x}}}(t)$ is obtained by a linear filter. For complex Gaussian processes, this gives the best MMSE estimate without a linearity assumption.

We can characterize the optimum realizable filter in terms of its impulse response $\tilde{\mathbf{h}}_o(t, \tau)$, so that the optimal estimate is given by

$$\hat{\tilde{\mathbf{x}}}(t) = \int_{T_i}^{t} \tilde{\mathbf{h}}_o(t, \tau)\tilde{\mathbf{r}}(\tau)\, d\tau, \qquad t > T_i. \tag{A.157}$$

It is easy to show that this impulse response $\tilde{\mathbf{h}}_o(t, \tau)$ is the solution of the complex Wiener-Hopf integral equation,

$$\tilde{\mathbf{K}}_{\tilde{\mathbf{x}}}(t, \tau)\tilde{\mathbf{C}}^\dagger(\tau) = \int_{T_i}^{t} \tilde{\mathbf{h}}_o(t, \sigma)\tilde{\mathbf{K}}_{\tilde{\mathbf{r}}}(\sigma, \tau)\, d\sigma, \qquad T_i \leq \tau < t \tag{A.158}$$

(see Problem A.3.15). In the state-variable formulation we find $\hat{\tilde{\mathbf{x}}}(t)$ directly without finding the optimum impulse response explicitly. By paralleling the development for real state variables, we can implicitly specify $\hat{\tilde{\mathbf{x}}}(t)$ as the solution of the differential equation

$$\frac{d\hat{\tilde{\mathbf{x}}}(t)}{dt} = \tilde{\mathbf{F}}(t)\hat{\tilde{\mathbf{x}}}(t) + \tilde{\mathbf{z}}(t)[\tilde{\mathbf{r}}(t) - \tilde{\mathbf{C}}(t)\hat{\tilde{\mathbf{x}}}(t)], \qquad T_i \leq t, \tag{A.159}$$

where

$$\tilde{\mathbf{z}}(t) = \tilde{\mathbf{h}}_o(t, t) = \tilde{\boldsymbol{\xi}}_P(t)\tilde{\mathbf{C}}^\dagger(t)\tilde{\mathbf{R}}^{-1}(t). \tag{A.160}$$

The covariance matrix $\tilde{\boldsymbol{\xi}}_P(t)$ is given by the nonlinear equation

$$\frac{d\tilde{\boldsymbol{\xi}}_P(t)}{dt} = \tilde{\mathbf{F}}(t)\tilde{\boldsymbol{\xi}}_P(t) + \tilde{\boldsymbol{\xi}}_P(t)\tilde{\mathbf{F}}^\dagger(t) - \tilde{\mathbf{z}}(t)\tilde{\mathbf{R}}(t)\tilde{\mathbf{z}}^\dagger(t) + \tilde{\mathbf{G}}(t)\tilde{\mathbf{Q}}\tilde{\mathbf{G}}^\dagger(t), \qquad T_i \leq t, \tag{A.161}$$

which can also be written as

$$\frac{d\tilde{\boldsymbol{\xi}}_P(t)}{dt} = \tilde{\mathbf{F}}(t)\tilde{\boldsymbol{\xi}}_P(t) + \tilde{\boldsymbol{\xi}}_P(t)\tilde{\mathbf{F}}^\dagger(t) - \tilde{\boldsymbol{\xi}}_P(t)\tilde{\mathbf{C}}^\dagger(t)\tilde{\mathbf{R}}^{-1}(t)\tilde{\mathbf{C}}(t)\tilde{\boldsymbol{\xi}}_P(t) + \tilde{\mathbf{G}}(t)\tilde{\mathbf{Q}}\tilde{\mathbf{G}}(t), \qquad T_i \leq t. \tag{A.162}$$

The initial conditions reflect our a-priori information about the initial state of the system.

$$\hat{\tilde{\mathbf{x}}}(T_i) = E[\tilde{\mathbf{x}}(T_i)], \tag{A.163}$$

$$\tilde{\boldsymbol{\xi}}_P(T_i) = \tilde{\mathbf{P}}_i. \tag{A.164}$$

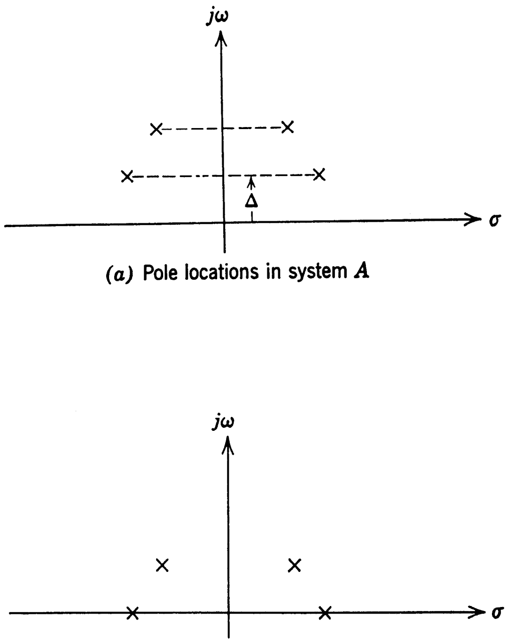
$\hat{\tilde{\mathbf{x}}}(T_i)$ is an a-priori estimate of the initial state. (Often it is assumed to be zero for zero-mean processes.) $\tilde{\mathbf{P}}_i$ is the covariance of this a-priori estimate.

As in the case of real variables, the variance equation may be computed independently of the estimator equation. In order to obtain solutions, it may be integrated numerically or the solution may be computed in terms of the transition matrix of an associated set of linear equations. Several interesting examples are discussed in the problems.

A particular case of interest corresponds to a scalar received waveform. Then we can write

$$\tilde{\mathbf{R}}(t) = N_0. \tag{A.165}$$

We also observe that $\tilde{\mathbf{C}}(t)$ is a $1 \times n$ matrix. In Fig. A.18 we show two pole-zero plots for the spectrum of $\tilde{\mathbf{y}}(t)$. We denote the modulations matrix of the two systems as $\tilde{\mathbf{C}}_a(t)$ and $\tilde{\mathbf{C}}_b(t)$, respectively. Clearly, we can



(a) Pole locations in system *A*

(b) Pole locations in system *B*

Fig. A.18 Effect of carrier frequency shift on pole location.

use the same state equations for the two systems and let

$$\tilde{\mathbf{C}}_b(t) = e^{-j\Delta t}\tilde{\mathbf{C}}_a(t). \tag{A.166}$$

Using (A.165) and (A.166) in (A.162), we see that $\tilde{\xi}_P(t)$ is not a function of Δ. Since Δ corresponds to a carrier frequency shift in the actual bandpass problem, this result is just what we would expect. Notice that $\tilde{\xi}_P(t)$ is also invariant to an arbitrary phase modulation on $\tilde{\mathbf{C}}_a(t)$,

$$\tilde{\mathbf{C}}_n(t) = e^{-j\phi(t)}\tilde{\mathbf{C}}_a(t). \tag{A.167}$$

This result is less obvious intuitively, but follows easily from (A.162).

The results in (A.157)–(A.164) are valid for nonstationary processes and arbitrary observation intervals. For stationary processes and semi-infinite observation intervals, the problem is equivalent to the complex version of the Wiener filtering problem. All of the techniques carry over with obvious modifications (see Problem A.3.15).

Our discussion has considered the MMSE estimate of a complex random process. As we would expect from our work in Chapters 2 and 3, we shall encounter the problem of estimating a complex Gaussian random process in the detection problem.

A.4 SUMMARY

In this appendix we have developed a complex representation for bandpass signals, systems, and processes. Several important ideas should be re-emphasized at this point. The first idea is that of a *complex envelope*, $\tilde{f}(t)$. It is a low-pass function whose magnitude is the actual envelope and whose phase is the phase modulation of the carrier. We shall find that the complex envelope plays the same role as the signal itself did in our earlier discussions. The second idea is that of a *complex Gaussian random process*. It plays the same role in the bandpass problem that the real Gaussian random process played previously. The third idea is that of *complex state variables*. They play the same role as real state variables did earlier.

We have spent a fair amount of time developing the complex notation. As we proceed through Chapters 9–14, we shall find that it was time well spent, because of the efficiency and insight it adds to the development.

A.5 PROBLEMS

P.A.1 Complex Signals

Problem A.1.1. The mean frequency $\bar{\omega}$ is defined in (A.16). Prove that we can always choose the carrier frequency so that

$$\bar{\omega} = 0.$$

Problem A.1.2. Derive the following expressions:

$$\overline{\omega^2} = \int_{-\infty}^{\infty} \left| \frac{d\tilde{f}(t)}{dt} \right|^2 dt,$$

$$\bar{\omega} = -j\int_{-\infty}^{\infty} \frac{d\tilde{f}(t)}{dt} \tilde{f}^*(t)\, dt,$$

$$\overline{t^2} = \int_{-\infty}^{\infty} \left| \frac{d\tilde{F}(j\omega)}{d\omega} \right|^2 \frac{d\omega}{2\pi},$$

$$\bar{t} = j \int_{-\infty}^{\infty} \frac{d\tilde{F}(j\omega)}{d\omega} \tilde{F}^*(j\omega) \frac{d\omega}{2\pi},$$

$$\overline{\omega t} = \operatorname{Im} \int_{-\infty}^{\infty} \omega \frac{d\tilde{F}(j\omega)}{d\omega} \tilde{F}^*(j\omega) \frac{d\omega}{2\pi}.$$

Problem A.1.3 [18]. Write

$$\tilde{f}(t) \triangleq A(t)e^{j\varphi(t)},$$

where

$$A(t) \triangleq |\tilde{f}(t)|$$

is the signal envelope. Assume that

$$\bar{\omega} = \bar{t} = 0.$$

1. Prove that

$$\overline{t^2} = \int_{-\infty}^{\infty} t^2 A^2(t)\, dt, \tag{P.1}$$

$$\overline{\omega^2} = \int_{-\infty}^{\infty} \left(\frac{dA(t)}{dt}\right)^2 dt + \int_{-\infty}^{\infty} \left(\frac{d\varphi(t)}{dt}\right)^2 A^2(t)\, dt. \tag{P.2}$$

Notice that the first term in (P.2) is the frequency spread due to amplitude modulation and the second term is the frequency spread due to frequency modulation.

2. Derive an expression for $\overline{\omega t}$ in terms of $A(t)$ and $\varphi(t)$. Interpret the result.

Problem A.1.4 [2].

1. Prove that

$$\operatorname{Re}\left[\int_{-\infty}^{\infty} t\tilde{f}(t) \frac{d\tilde{f}^*(t)}{dt}\, dt\right] = \frac{1}{2}, \tag{P.1}$$

and therefore

$$\int_{-\infty}^{\infty} t\tilde{f}(t) \frac{d\tilde{f}^*(t)}{dt}\, dt = \frac{1}{2} + j\overline{\omega t}. \tag{P.2}$$

2. Use the Schwarz inequality on (P.2) to prove

$$\overline{\omega^2}\,\overline{t^2} - (\overline{\omega t})^2 \geq \tfrac{1}{4}, \tag{P.3}$$

assuming

$$\bar{\omega} = \bar{t} = 0.$$

This can also be written as

$$\sigma_\omega \sigma_t [1 - \rho_{\omega t}^2]^{1/2} \geq \tfrac{1}{2}. \tag{P.4}$$

3. Prove that

$$\sigma_\omega \sigma_t \geq \tfrac{1}{2}. \tag{P.5}$$

An alternative way of stating (P.5) is to define

$$\sigma_f \triangleq \frac{\sigma_\omega}{2\pi}.$$

Then

$$\sigma_f \sigma_t \geq \pi. \tag{P.6}$$

The relation in (P.5) and (P.6) is called the *uncertainty relation.*

Problem A.1.5. Assume that

$$\tilde{f}(t) = \left(\frac{1}{\pi T^2}\right)^{1/4} \exp\left[-\left(\frac{1}{2T^2} - jb\right)t^2\right].$$

Find $\overline{\omega^2}$, $\overline{t^2}$, and $\overline{\omega t}$.

Problem A.1.6. In Chapter 10 we define a function

$$\phi(\tau, \omega) \triangleq \int_{-\infty}^{\infty} \tilde{f}\left(t - \frac{\tau}{2}\right) \tilde{f}^*\left(t + \frac{\tau}{2}\right) e^{-j\omega t}\, dt.$$

Evaluate σ_ω^2, σ_t^2, and $\overline{\omega t} - \bar{\omega}\bar{t}$ in terms of derivatives of $\phi(\tau, \omega)$ evaluated at $\tau = \omega = 0$.

P.A.3 Complex Processes

Problem A.3.1. Consider a complex Gaussian random variable whose density is given by (A.81). The characteristic function of a complex random variable is defined as

$$\tilde{M}_{\tilde{y}}(j\tilde{v}) = E[e^{j\,\mathrm{Re}\,[\tilde{v}^*\tilde{y}]}].$$

1. Find $\tilde{M}_{\tilde{y}}(j\tilde{v})$ for a complex Gaussian random variable.
2. How are the moments of $\tilde{y}$ related to $\tilde{M}_{\tilde{y}}(j\tilde{v})$ in general (i.e., $\tilde{y}$ is not necessarily complex Gaussian)?

Problem A.3.2. Consider the N-dimensional complex random vector $\tilde{\mathbf{x}}$, where

$$E[\tilde{\mathbf{x}}] = \mathbf{0},$$

$$E[\tilde{\mathbf{x}}\tilde{\mathbf{x}}^\dagger] \triangleq 2\tilde{\mathbf{\Lambda}}_{\tilde{\mathbf{x}}},$$

and

$$E[\tilde{\mathbf{x}}\tilde{\mathbf{x}}^T] = \mathbf{0}.$$

We define $\tilde{\mathbf{x}}$ to be a complex Gaussian vector if

$$p_{\tilde{\mathbf{x}}}(\tilde{\mathbf{X}}) = \frac{1}{(2\pi)^N |\tilde{\mathbf{\Lambda}}_{\tilde{\mathbf{x}}}|} \exp\left(-\tfrac{1}{2}\tilde{\mathbf{X}}^\dagger \tilde{\mathbf{\Lambda}}_{\tilde{\mathbf{x}}}^{-1}\tilde{\mathbf{X}}\right),$$

$$-\infty < \mathrm{Re}\,[\tilde{\mathbf{X}}] < \infty, \quad -\infty < \mathrm{Im}\,[\tilde{\mathbf{X}}] < \infty.$$

We refer to the components of $\tilde{\mathbf{x}}$ as *joint complex Gaussian random variables.*

The characteristic function of a complex random vector is defined as

$$\tilde{M}_{\tilde{\mathbf{x}}}(\tilde{\mathbf{v}}) \triangleq E[e^{j\,\mathrm{Re}[\tilde{\mathbf{v}}^\dagger\tilde{\mathbf{x}}]}].$$

Prove that

$$\tilde{M}_{\tilde{\mathbf{x}}}(\tilde{\mathbf{v}}) = \exp\left(-\tfrac{1}{2}\tilde{\mathbf{v}}^{\dagger}\tilde{\boldsymbol{\Lambda}}_{\tilde{\mathbf{x}}}\tilde{\mathbf{v}}\right)$$

for a complex Gaussian random vector.

Problem A.3.3. A complex Gaussian random variable is defined in (A.81). Define

$$\tilde{y} = \tilde{\mathbf{g}}^{\dagger}\tilde{\mathbf{x}}.$$

If $\tilde{y}$ is a complex Gaussian random variable for every finite $\mathbf{g}$, we say that $\tilde{\mathbf{x}}$ is a complex Gaussian random vector. Prove that this definition is equivalent to the one in Problem A.3.2.

Problem A.3.4. Assume that $\tilde{y}$ is a complex Gaussian random variable. Prove that

$$E[|\tilde{y}|^{2n}] = n!\,(E(|\tilde{y}|^2))^n.$$

Problem A.3.5. Assume that $\tilde{y}_1$ and $\tilde{y}_2$ are joint complex Gaussian random variables. Prove that

$$E[(\tilde{y}_1\tilde{y}_2^*)^n] = n!\,[E(\tilde{y}_1\tilde{y}_2^*)]^n.$$

Problem A.3.6. Assume that $\tilde{y}_1$, $\tilde{y}_2$, $\tilde{y}_3$, and $\tilde{y}_4$ are joint complex Gaussian random variables. Prove that

$$E[\tilde{y}_1^*\tilde{y}_2^*\tilde{y}_3\tilde{y}_4] = E[\tilde{y}_1^*\tilde{y}_3]E[\tilde{y}_2^*\tilde{y}_4] + E[\tilde{y}_2^*\tilde{y}_3]E[\tilde{y}_1^*\tilde{y}_4].$$

(This result is given in [16].)

Problem A.3.7 [8]. Derive the "factoring-of-moments" property for complex Gaussian random processes. (Recall Problem I-3.3.12.)

Problem A.3.8. Consider the problem outlined on pages 161–165 of [15]. Reformulate this problem using complex notation and solve it. Compare the efficiency of the two procedures.

Problem A.3.9. In Problem I-6.2.1, we developed the properties of power density spectra of real random processes.

Let $n(t)$ be a stationary narrow-band process with a rational spectra $S_n(\omega)$. Denote the complex envelope process by $\tilde{n}(t)$, and its spectrum by $\tilde{S}_{\tilde{n}}(\omega)$.

1. Derive properties similar to those in Problem I-6.2.1.
2. Sketch the pole-zero plots of some typical complex spectra.

Problem A.3.10. The definition of a complex Gaussian process is given on page 588. Derive the complex versions of Properties 1 through 4 on pages I-183–I-185.

Problem A.3.11. Prove that the eigenvalues of a complex envelope process are invariant to the choice of the carrier frequency.

Problem A.3.12. Consider the results in (A.99)–(A.105).

1. Verify that one gets identical results by working with a real vector process,

$$\mathbf{n}(t) = \begin{bmatrix} n_c(t) \\ n_s(t) \end{bmatrix}.$$

[Review Section 3.7 and observe that $\mathbf{K}_{\mathbf{n}}(t, u)$ has certain properties because of the assumption in (A.90).]

2. What is the advantage of working with the complex process instead of $\mathbf{n}(t)$?

Problem A.3.13 [14]. Consider the process described by (A.118)–(A.122) with

$$\tilde{\mathbf{G}}(t) = 1,$$

$$\tilde{\mathbf{F}}(t) = a - \frac{jbt^2}{2},$$

$$\tilde{\mathbf{C}}(t) = 1.$$

1. Find the covariance function of $\tilde{y}(t)$.
2. Demonstrate that $\tilde{y}(t)$ has the same covariance as the output of a Doppler-spread channel with a one-pole fading spectrum and input signal given by (10.52).

Problem A.3.14. Consider the process described by (A.148)–(A.153).

1. Find the covariance function of $\tilde{y}(t)$.
2. Calculate $E[|\tilde{y}|^2]$.

Problem A.3.15. Consider the linear filtering model in (A.154)–(A.156).

1. Derive the Wiener-Hopf equation in (A.158).
2. Derive the complex Kalman-Bucy equations in (A.159)–(A.164).
3. Assume that $T_i = -\infty$ and $\tilde{y}(t)$ is stationary. Give an explicit solution to (A.158) by using spectrum factorization.
4. Prove that, with a complex Gaussian assumption, a linear filter is the optimum MMSE processor.

Problem A.3.16. The complex envelope of the received waveform is

$$\tilde{r}(u) = \tilde{s}(u) + \tilde{w}(u), \qquad -\infty < u \le t,$$

where $\tilde{s}(u)$ and $\tilde{w}(u)$ are statistically independent complex Gaussian processes with spectra

$$\tilde{S}_{\tilde{s}}(\omega) = \frac{\text{Re}\,[\tilde{k}_1]P}{(\omega + \text{Im}\,[\tilde{k}_1])^2 + \text{Re}\,[\tilde{k}_1])^2} + \frac{k_2 P}{\omega^2 + k_2{}^2},$$

and

$$\tilde{S}_{\tilde{w}}(\omega) = N_0.$$

1. Find the minimum mean-square realizable estimate of $\tilde{s}(t)$.
2. Evaluate the minimum mean-square error.

Problem A.3.17. The complex envelope of the received waveform is

$$\tilde{r}(u) = \tilde{s}(u) + \tilde{n}_c(u) + \tilde{w}(u), \qquad -\infty < u \le t,$$

where $\tilde{s}(u)$, $\tilde{n}_c(u)$, and $\tilde{w}(u)$ are statistically independent Gaussian processes with spectra

$$\tilde{S}_{\tilde{s}}(\omega) = \frac{2\,\text{Re}\,[\tilde{k}_1]P_s}{(\omega + \text{Im}\,[\tilde{k}_1])^2 + (\text{Re}\,[\tilde{k}_1]^2)},$$

$$\tilde{S}_{\tilde{n}_c}(\omega) = \frac{2\,\text{Re}\,[\tilde{k}_1]P_c}{\omega^2 + (\text{Re}\,[\tilde{k}_1])^2},$$

and

$$\tilde{S}_{\tilde{w}}(\omega) = N_0,$$

respectively.

1. Find the minimum mean-square realizable estimate of $\tilde{s}(t)$.
2. Evaluate the minimum mean-square error. Do your results behave correctly as $\text{Im}\,[\tilde{k}_1] \to \infty$?

Problem A.3.18. Consider the system shown in Fig. P.A.1. The input $u(t)$ is a sample function of a real white Gaussian process.

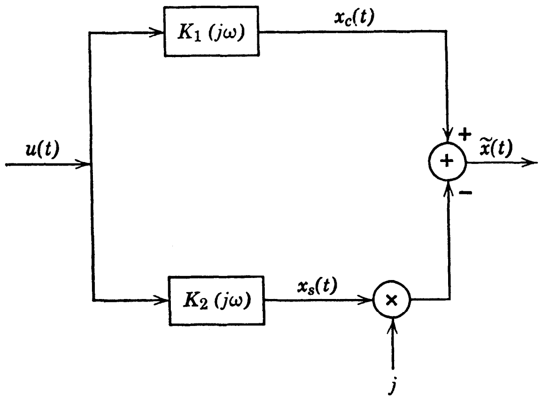


Fig. P.A.1

1. Compute $S_{x_c}(\omega)$, $S_{x_s}(\omega)$, and $S_{x_cx_s}(\omega)$.
2. Under what conditions is $\tilde{x}(t)$ a complex Gaussian process (according to our definition)?
3. Let $K_1(j\omega)$ and $K_2(j\omega)$ be arbitrary transfer functions. We observe that

$$\tilde{r}(t) = \tilde{x}(t) + \tilde{w}(t),$$

where $\tilde{w}(t)$ is a sample function of a complex white Gaussian process with spectral height N_0. Find the minimum mean-square *unrealizable* estimate of $\tilde{x}(t)$.

(This type of problem is discussed in [12] and [17].)

Problem A.3.19. Verify the results in (A.130)–(A.133).

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