Chapter

1

Mathematical Review

Electronic signals are complicated phenomena, and their exact behavior is impossible to describe completely. However, simple mathematical models can describe the signals well enough to yield some very useful results that can be applied in a variety of practical situations. Furthermore, linear systems and digital filters are inherently mathematical beasts. This chapter is devoted to a concise review of the mathematical techniques that are used throughout the rest of the book.

1.1 Exponentials and Logarithms

Exponentials

There is an irrational number, usually denoted as e, that is of great importance in virtually all fields of science and engineering. This number is defined by

$$e \triangleq \lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^{x} \simeq 2.71828 \cdots \tag{1.1}$$

Unfortunately, this constant remains unnamed, and writers are forced to settle for calling it "the number e" or perhaps "the base of natural logarithms." The letter e was first used to denote the irrational in (1.1) by Leonhard Euler (1707–1783), so it would seem reasonable to refer to the number under discussion as "Euler's constant." Such is not the case, however, as the term *Euler's constant* is attached to the constant γ defined by

$$\gamma = \lim_{N \to \infty} \left(\sum_{n=1}^{N} \frac{1}{n} - \log_e N \right) \simeq 0.577215664 \cdots \tag{1.2}$$

The number e is most often encountered in situations where it raised to some real or complex power. The notation $\exp(x)$ is often used in place of e^{x}, since the former can be written more clearly and typeset more easily than the latter—especially in cases where the exponent is a complicated expression rather just a single variable. The value for e raised to a complex power z can be expanded in an infinite series as

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \tag{1.3}$$

The series in (1.3) converges for all complex z having finite magnitude.

Logarithms

The *common logarithm*, or *base-10 logarithm*, of a number x is equal to the power to which 10 must be raised in order to equal x:

$$y = \log_{10} x \iff x = 10^y \tag{1.4}$$

The *natural logarithm*, or *base-e logarithm*, of a number x is equal to the power to which e must be raised in order to equal x:

$$y = \log_e x \iff x = \exp(y) \equiv e^y \tag{1.5}$$

Natural logarithms are also called *napierian logarithms* in honor of John Napier (1550–1617), a Scottish amateur mathematician who in 1614 published the first account of logarithms in *Mirifici logarithmorum canonis descripto* ("A Description of the Marvelous Rule of Logarithms") (see Boyer 1968). The concept of logarithms can be extended to any positive base b, with the base-b logarithm of a number x equaling the power to which the base must be raised in order to equal x:

$$y = \log_b x \iff x = b^y \tag{1.6}$$

The notation log without a base explicitly indicated usually denotes a common logarithm, although sometimes this notation is used to denote natural logarithms (especially in some of the older literature). More often, the notation ln is used to denote a natural logarithm. Logarithms exhibit a number of properties that are listed in Table 1.1. Entry 1 is sometimes offered as the definition of natural logarithms. The multiplication property in entry 3 is the theoretical basis upon which the design of the slide rule is based.

Decibels

Consider a system that has an output power of P_{out} and an output voltage of V_{out} given an input power of P_{in} and an input voltage of V_{in}. The gain G, in decibels (dB), of the system is given by

$$G_{dB} = 10 \log_{10}\left(\frac{P_{out}}{P_{in}}\right) = 10 \log_{10}\left(\frac{V_{out}^2/Z_{out}}{V_{in}^2/Z_{in}}\right) \tag{1.7}$$

TABLE 1.1 Properties of Logarithms

1.	$\ln x = \int_1^x \frac{1}{y}\,dy \qquad x > 0$
2.	$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad x > 0$
3.	$\log_b(xy) = \log_b x + \log_b y$
4.	$\log_b\left(\frac{1}{x}\right) = -\log_b x$
5.	$\log_b(y^x) = x \log_b y$
6.	$\log_c x = (\log_b x)(\log_c b) = \frac{\log_b x}{\log_b c}$
7.	$\ln(1+z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n} \qquad \lvert z \rvert < 1$

If the input and output impedances are equal, (1.7) reduces to

$$G_{\mathrm{dB}} = 10 \log_{10}\left(\frac{V_{\mathrm{out}}^2}{V_{\mathrm{in}}^2}\right) = 20 \log_{10}\left(\frac{V_{\mathrm{out}}}{V_{\mathrm{in}}}\right) \tag{1.8}$$

Example 1.1 An amplifier has a gain of 17.0 dB. For a 3-mW input, what will the output power be? Substituting the given data into (1.7) yields

$$17.0\ \mathrm{dB} = 10 \log_{10}\left(\frac{P_{\mathrm{out}}}{3 \times 10^{-3}}\right)$$

Solving for P_{out} then produces

$$P_{\mathrm{out}} = (3 \times 10^{-3})10^{(17/10)} = 1.5 \times 10^{-1} = 150\ \mathrm{mW}$$

Example 1.2 What is the range in decibels of the values that can be represented by an 8-bit unsigned integer?

solution The smallest value is 1, and the largest value is $2^8 - 1 = 255$. Thus

$$20 \log_{10}\left(\frac{255}{1}\right) = 48.13\ \mathrm{dB}$$

The abbreviation dBm is used to designate power levels relative to 1 milliwatt (mW). For example:

$$30\ \mathrm{dBm} = 10 \log_{10}\left(\frac{P}{10^{-3}}\right)$$

$$P = (10^{-3})(10^3) = 10^0 = 1.0\ \mathrm{W}$$

1.2 Complex Numbers

A complex number z has the form $a + bj$, where a and b are real and $j = \sqrt{-1}$. The *real part* of z is a, and the *imaginary part* of z is b. Mathematicians use i to denote $\sqrt{-1}$, but electrical engineers use j to avoid confusion with the traditional use of i for denoting current. For convenience, $a + bj$ is sometimes represented by the ordered pair (a, b). The *modulus*, or *absolute value*, of z is denoted as $|z|$ and is defined by

$$|z| = |a + bj| = \sqrt{a^2 + b^2} \tag{1.9}$$

The *complex conjugate* of z is denoted as z^* and is defined by

$$(z = a + bj) \Leftrightarrow (z^* = a - bj) \tag{1.10}$$

Conjugation distributes over addition, multiplication, and division:

$$(z_1 + z_2)^* = z_1^* + z_2^* \tag{1.11}$$

$$(z_1 z_2)^* = z_1^* z_2^* \tag{1.12}$$

$$\left(\frac{z_1}{z_2}\right)^* = \frac{z_1^*}{z_2^*} \tag{1.13}$$

Operations on complex numbers in rectangular form

Consider two complex numbers:

$$z_1 = a + bj \qquad z_2 = c + dj$$

The four basic arithmetic operations are then defined as

$$z_1 + z_2 = (a + c) + j(b + d) \tag{1.14}$$

$$z_1 - z_2 = (a - c) + j(b - d) \tag{1.15}$$

$$z_1 z_2 = (ac - bd) + j(ad + bc) \tag{1.16}$$

$$\frac{z_1}{z_2} = \frac{ac + bd}{c^2 + d^2} + j\,\frac{bc - ad}{c^2 + d^2} \tag{1.17}$$

Polar form of complex numbers

A complex number of the form $a + bj$ can be represented by a point in a coordinate plane as shown in Fig. 1.1. Such a representation is called an *Argand diagram* (Spiegel 1965) in honor of Jean Robert Argand (1768–1822), who published a description of this graphical representation of complex num-

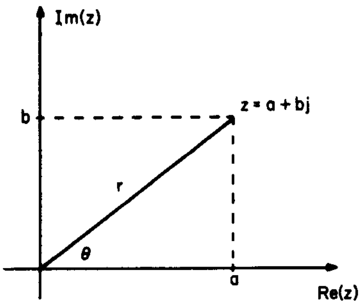

Figure 1.1 Argand diagram representation of a complex number.

bers in 1806 (Boyer 1968). The point representing $a + bj$ can also be located using an angle θ and radius r as shown. From the definitions of sine and cosine given in (1.25) and (1.26) of Sec. 1.3, it follows that

$$a = r \cos \theta \qquad b = r \sin \theta$$

Therefore,

$$z = r \cos \theta + jr \sin \theta = r(\cos \theta + j \sin \theta) \tag{1.18}$$

The quantity $(\cos \theta + j \sin \theta)$ is sometimes denoted as cis θ. Making use of (1.58) from Sec. 1.3, we can rewrite (1.18) as

$$z = r \text{ cis } \theta = r \exp(j\theta) \tag{1.19}$$

The form in (1.19) is called the *polar form* of the complex number z.

Operations on complex numbers in polar form

Consider three complex numbers:

$$z = r(\cos \theta + j \sin \theta) = r \exp(j\theta)$$

$$z_1 = r_1(\cos \theta_1 + j \sin \theta_1) = r_1 \exp(j\theta_1)$$

$$z_2 = r_2(\cos \theta_2 + j \sin \theta_2) = r_2 \exp(j\theta_2)$$

Several operations can be conveniently performed directly upon complex numbers that are in polar form, as follows.

Multiplication

$$\begin{aligned} z_1 z_2 &= r_1 r_2 [\cos(\theta_1 + \theta_2) + j \sin(\theta_1 + \theta_2)] \\ &= r_1 r_2 \exp[j(\theta_1 + \theta_2)] \end{aligned} \tag{1.20}$$

Division

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + j\sin(\theta_1 - \theta_2)]$$
$$= \frac{r_1}{r_2}\exp[j(\theta_1 - \theta_2)] \tag{1.21}$$

Powers

$$z^n = r^n[\cos(n\theta) + j\sin(n\theta)]$$
$$= r^n \exp(jn\theta) \tag{1.22}$$

Roots

$$\sqrt[n]{z} = z^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + j\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$
$$= r^{1/n}\exp\left[\frac{j(\theta + 2k\pi)}{n}\right] \qquad k = 0, 1, 2, \ldots \tag{1.23}$$

Equation (1.22) is known as *De Moivre's theorem.* In 1730, an equation similar to (1.23) was published by Abraham De Moivre (1667–1754) in his *Miscellanea analytica* (Boyer 1968). In Eq. (1.23), for a fixed n as k increases, the sinusoidal functions will take on only n distinct values. Thus there are n different nth roots of any complex number.

Logarithms of complex numbers

For the complex number $z = r\exp(j\theta)$, the natural logarithm of z is given by

$$\ln z = \ln[r\exp(j\theta)]$$
$$= \ln\{r\exp[j(\theta + 2k\pi)]\}$$
$$= (\ln r) + j(\theta + 2k\pi) \qquad k = 0, 1, 2, \ldots \tag{1.24}$$

The *principal value* is obtained when $k = 0$.

1.3 Trigonometry

For x, y, r, and θ as shown in Fig. 1.2, the six trigonometric functions of the angle θ are defined as

$$\text{Sine:} \qquad \sin\theta = \frac{y}{r} \tag{1.25}$$

$$\text{Cosine:} \qquad \cos\theta = \frac{x}{r} \tag{1.26}$$

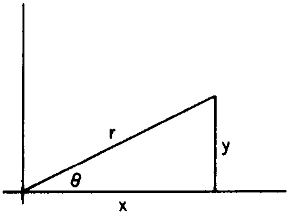

Figure 1.2 An angle in the cartesian plane.

$$\text{Tangent:} \qquad \tan\theta = \frac{y}{x} \tag{1.27}$$

$$\text{Cosecant:} \qquad \csc\theta = \frac{r}{y} \tag{1.28}$$

$$\text{Secant:} \qquad \sec\theta = \frac{r}{x} \tag{1.29}$$

$$\text{Cotangent:} \qquad \cot\theta = \frac{x}{y} \tag{1.30}$$

Phase shifting of sinusoids

A number of useful equivalences can be obtained by adding particular phase angles to the arguments of sine and cosine functions:

$$\cos(\omega t) = \sin\left(\omega t + \frac{\pi}{2}\right) \tag{1.31}$$

$$\cos(\omega t) = \cos(\omega t + 2n\pi) \qquad n = \text{any integer} \tag{1.32}$$

$$\sin(\omega t) = \sin(\omega t + 2n\pi) \qquad n = \text{any integer} \tag{1.33}$$

$$\sin(\omega t) = \cos\left(\omega t - \frac{\pi}{2}\right) \tag{1.34}$$

$$\cos(\omega t) = \cos[\omega t + (2n+1)\pi] \qquad n = \text{any integer} \tag{1.35}$$

$$\sin(\omega t) = -\sin[\omega t + (2n+1)\pi] \qquad n = \text{any integer} \tag{1.36}$$

Trigonometric identities

The following trigonometric identities often prove useful in the design and analysis of signal processing systems.

$$\tan x = \frac{\sin x}{\cos x} \tag{1.37}$$

$$\sin(-x) = -\sin x \tag{1.38}$$

$$\cos(-x) = \cos x \tag{1.39}$$

$$\tan(-x) = -\tan x \tag{1.40}$$

$$\cos^2 x + \sin^2 x = 1 \tag{1.41}$$

$$\cos^2 x = \tfrac{1}{2}[1 + \cos(2x)] \tag{1.42}$$

$$\sin(x \pm y) = \sin x)(\cos y) \pm (\cos y)(\sin y) \tag{1.43}$$

$$\cos(x \pm y) = (\cos x)(\cos y) \mp (\sin x)(\sin y) \tag{1.44}$$

$$\tan(x + y) = \frac{(\tan x) + (\tan y)}{1 - (\tan x)(\tan y)} \tag{1.45}$$

$$\sin(2x) = 2(\sin x)(\cos x) \tag{1.46}$$

$$\cos(2x) = \cos^2 x - \sin^2 x \tag{1.47}$$

$$\tan(2x) = \frac{2(\tan x)}{1 - \tan^2 x} \tag{1.48}$$

$$(\sin x)(\sin y) = \tfrac{1}{2}[-\cos(x + y) + \cos(x - y)] \tag{1.49}$$

$$(\cos x)(\cos y) = \tfrac{1}{2}[\cos(x + y) + \cos(x - y)] \tag{1.50}$$

$$(\sin x)(\cos y) = \tfrac{1}{2}[\sin(x + y) + \sin(x - y)] \tag{1.51}$$

$$(\sin x) + (\sin y) = 2 \sin \frac{x + y}{2} \cos \frac{x - y}{2} \tag{1.52}$$

$$(\sin x) - (\sin y) = 2 \sin \frac{x - y}{2} \cos \frac{x + y}{2} \tag{1.53}$$

$$(\cos x) + (\cos y) = 2 \cos \frac{x + y}{2} \cos \frac{x - y}{2} \tag{1.54}$$

$$(\cos x) - (\cos y) = -2 \sin \frac{x + y}{2} \sin \frac{x - y}{2} \tag{1.55}$$

$$A \cos(\omega t + \psi) + B \cos(\omega t + \phi) = C \cos(\omega t + \theta) \tag{1.56}$$

where $C = [A^2 + B^2 - 2AB \cos(\phi - \psi)]^{1/2}$

$$\theta = \tan^{-1}\left(\frac{A \sin \psi + B \sin \phi}{A \cos \psi + B \cos \phi}\right)$$

$$A \cos(\omega t + \psi) + B \sin(\omega t + \phi) = C \cos(\omega t + \theta) \tag{1.57}$$

where $C = [A^2 + B^2 - 2AB \sin(\phi - \psi)]^{1/2}$

$$\theta = \tan^{-1}\left(\frac{A \sin \psi - B \cos \phi}{A \cos \psi + B \sin \phi}\right)$$

Euler's identities

The following four equations, called *Euler's identities*, relate sinusoids and complex exponentials.

$$e^{jx} = \cos x + j \sin x \tag{1.58}$$

$$e^{-jx} = \cos x - j \sin x \tag{1.59}$$

$$\cos x = \frac{e^{jx} + e^{-jx}}{2} \tag{1.60}$$

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j} \tag{1.61}$$

Series and product expansions

Listed below are infinite series expansions for the various trigonometric functions (Abramowitz and Stegun 1966).

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \tag{1.62}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \tag{1.63}$$

$$\tan x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n}(2^{2n}-1) B_{2n} x^{2n-1}}{(2n)!} \qquad |x| < \frac{\pi}{2} \tag{1.64}$$

$$\cot x = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} B_{2n} x^{2n-1}}{(2n)!} \qquad |x| < \pi \tag{1.65}$$

$$\sec x = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n} x^{2n}}{(2n)!} \qquad |x| < \frac{\pi}{2} \tag{1.66}$$

$$\csc x = \sum_{n=0}^{\infty} \frac{(-1)^{n-1} 2(2^{2n-1}-1) B_{2n} x^{2n-1}}{(2n)!} \qquad |x| < \pi \tag{1.67}$$

Values for the Bernoulli number B_n and Euler number E_n are listed in Tables 1.2 and 1.3, respectively. In some instances, the infinite product expansions for sine and cosine may be more convenient than the series expansions.

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2 \pi^2}\right) \tag{1.68}$$

$$\cos x = \prod_{n=1}^{\infty} \left[1 - \frac{4x^2}{(2n-1)^2 \pi^2}\right] \tag{1.69}$$

TABLE 1.2 Bernoulli Numbers

$B_n = N/D \quad B_n = 0 \quad \text{for } n = 3, 5, 7, \ldots$

n	N	D
0	1	1
1	−1	2
2	1	6
4	−1	30
6	1	42
8	−1	30
10	5	66
12	−691	2730
14	7	6
16	−3617	510
18	43867	798
20	−174611	330

TABLE 1.3 Euler Numbers

$E_n = 0 \quad \text{for } n = 1, 3, 5, 7, \ldots$

n	E_n
0	1
2	−1
4	5
6	−61
8	1385
10	−50521
12	2,702,765
14	−199,360,981
16	19,391,512,145
18	−2,404,879,675,441
20	370,371,188,237,525

Orthonormality of sine and cosine

Two functions $\phi_1(t)$ and $\phi_2(t)$ are said to form an *orthogonal* set over the interval $[0, T]$ if

$$\int_0^T \phi_1(t)\, \phi_2(t)\, dt = 0 \tag{1.70}$$

The functions $\phi_1(t)$ and $\phi_2(t)$ are said to form an *orthonormal* set over the interval $[0, T]$ if in addition to satisfying (1.70) each function has unit energy over the interval

$$\int_0^T [\phi_1(t)]^2\, dt = \int_0^T [\phi_2(t)]^2\, dt = 1 \tag{1.71}$$

Consider the two signals given by

$$\phi_1(t) = A \sin(\omega_0 t) \tag{1.72}$$

$$\phi_2(t) = A \cos(\omega_0 t) \tag{1.73}$$

The signals ϕ_1 and ϕ_2 will form an orthogonal set over the interval $[0, T]$ if $\omega_0 T$ is an integer multiple of π. The set will be orthonormal as well as orthogonal if $A^2 = 2/T$. The signals ϕ_1 and ϕ_2 will form an approximately orthonormal set over the interval $[0, T]$ if $\omega_0 T \gg 1$ and $A^2 = 2/T$. The orthonormality of sine and cosine can be derived as follows.

Substitution of (1.72) and (1.73) into (1.70) yields

$$\begin{aligned}\int_0^T \phi_1(t)\,\phi_2(t)\,dt &= A^2\int_0^T \sin\omega_0 t\,\cos\omega_0 t\,dt \\ &= \frac{A^2}{2}\int_0^T [\sin(\omega_0 t+\omega_0 t)+\sin(\omega_0 t-\omega_0 t)]\,dt \\ &= \frac{A^2}{2}\int_0^T \sin 2\omega_0 t\,dt = \frac{A^2}{2}\left(\frac{\cos 2\omega_0 t}{2\omega_0}\right)\Bigg|_{t=0}^{T} \\ &= \frac{A^2}{4\omega_0 T}(1-\cos 2\omega_0 T) \end{aligned} \tag{1.74}$$

Thus if $\omega_0 T$ is an integer multiple of π, then $\cos(2\omega_0 T)=1$ and ϕ_1 and ϕ_2 will be orthogonal. If $\omega_0 T \gg 1$, then (1.74) will be very small and reasonably approximated by zero; thus ϕ_1 and ϕ_2 can be considered as approximately orthogonal. The energy of $\phi_1(t)$ on the interval $[0, T]$ is given by

$$\begin{aligned}E_1 &= \int_0^T [\phi_1(t)]^2\,dt = A^2\int_0^T \sin^2\omega_0 t\,dt \\ &= A^2\left(\frac{t}{2}-\frac{\sin 2\omega_0 t}{4\omega_0}\right)\Bigg|_{t=0}^{T} \\ &= A^2\left(\frac{T}{2}-\frac{\sin 2\omega_0 T}{4\omega_0}\right)\end{aligned} \tag{1.75}$$

For ϕ_1 to have unit energy, A^2 must satisfy

$$A^2 = \left(\frac{T}{2}-\frac{\sin 2\omega_0 T}{4\omega_0}\right)^{-1} \tag{1.76}$$

When $\omega_0 T = n\pi$, then $\sin 2\omega_0 T = 0$. Thus (1.76) reduces to

$$A = \sqrt{\frac{2}{T}} \tag{1.77}$$

Substituting (1.77) into (1.75) yields

$$E_1 = 1-\frac{\sin 2\omega_0 T}{2\omega_0 T} \tag{1.78}$$

When $\omega_0 T \gg 1$, the second term of (1.78) will be very small and reasonably approximated by zero, thus indicating that ϕ_1 and ϕ_2 are approximately orthonormal. In a similar manner, the energy of $\phi_2(t)$ can be found to be

$$\begin{aligned}E_2 &= A^2\int_0^T \cos^2\omega_0 t\,dt \\ &= A^2\left(\frac{T}{2}+\frac{\sin 2\omega_0 T}{4\omega_0}\right)\end{aligned} \tag{1.79}$$

Thus
$$E_2 = 1 \qquad \text{if } A = \sqrt{\frac{2}{T}} \quad \text{and} \quad \omega_0 T = n\pi$$
$$E_2 \doteq 1 \qquad \text{if } A = \sqrt{\frac{2}{T}} \quad \text{and} \quad \omega_0 T \gg 1$$

1.4 Derivatives

Listed below are some derivative forms that often prove useful in theoretical analysis of communication systems.

$$\frac{d}{dx}\sin u = \cos u \frac{du}{dx} \tag{1.80}$$

$$\frac{d}{dx}\cos u = -\sin u \frac{du}{dx} \tag{1.81}$$

$$\frac{d}{dx}\tan u = \sec^2 u \frac{du}{dx} = \frac{1}{\cos^2 u}\frac{du}{dx} \tag{1.82}$$

$$\frac{d}{dx}\cot u = \csc^2 u \frac{du}{dx} = \frac{1}{\sin^2 u}\frac{du}{dx} \tag{1.83}$$

$$\frac{d}{dx}\sec u = \sec u \tan u \frac{du}{dx} = \frac{\sin u}{\cos^2 u}\frac{du}{dx} \tag{1.84}$$

$$\frac{d}{dx}\csc u = -\csc u \cot u \frac{du}{dx} = \frac{-\cos u}{\sin^2 u}\frac{du}{dx} \tag{1.85}$$

$$\frac{d}{dx}e^u = e^u \frac{du}{dx} \tag{1.86}$$

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx} \tag{1.87}$$

$$\frac{d}{dx}\log u = \frac{\log e}{u}\frac{du}{dx} \tag{1.88}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{1}{v^2}\left(v\frac{du}{dx} - u\frac{dv}{dx}\right) \tag{1.89}$$

Derivatives of polynomial ratios

Consider a ratio of polynomials given by

$$C(s) = \frac{A(s)}{B(s)} \qquad B(s) \neq 0 \tag{1.90}$$

The derivative of $C(s)$ can be obtained using Eq. (1.89) to obtain

$$\frac{d}{ds}C(s) = [B(s)]^{-1}\frac{d}{ds}A(s) - A(s)[B(s)]^{-2}\frac{d}{ds}B(s) \tag{1.91}$$

Equation (1.91) will be very useful in the application of the Heaviside expansion, which is discussed in Sec. 2.6.

1.5 Integration

Large integral tables fill entire volumes and contain thousands of entries. However, a relatively small number of integral forms appear over and over again in the study of communications, and these are listed below.

$$\int \frac{1}{x}\,dx = \ln x \tag{1.92}$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} \tag{1.93}$$

$$\int xe^{ax}\,dx = \frac{ax-1}{a^2}e^{ax} \tag{1.94}$$

$$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) \tag{1.95}$$

$$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) \tag{1.96}$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) \tag{1.97}$$

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) \tag{1.98}$$

$$\int x\sin(ax)\,dx = -\frac{x}{a}\cos(ax) + \frac{1}{a^2}\sin(ax) \tag{1.99}$$

$$\int x\cos(ax)\,dx = \frac{x}{a}\sin(ax) + \frac{1}{a^2}\cos(ax) \tag{1.100}$$

$$\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a} \tag{1.101}$$

$$\int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a} \tag{1.102}$$

$$\int x^2\sin ax\,dx = \frac{1}{a^3}(2ax\sin ax + 2\cos ax - a^2x^2\cos ax) \tag{1.103}$$

$$\int x^2 \cos ax\, dx = \frac{1}{a^3}(2ax \cos ax - 2 \sin ax + a^2x^2 \sin ax) \tag{1.104}$$

$$\int \sin^3 x\, dx = -\tfrac{1}{3} \cos x(\sin^2 x + 2) \tag{1.105}$$

$$\int \cos^3 x\, dx = \tfrac{1}{3} \sin x(\cos^2 x + 2) \tag{1.106}$$

$$\int \sin x \cos x\, dx = \tfrac{1}{2} \sin^2 x \tag{1.107}$$

$$\int \sin(mx) \cos(nx)\, dx = \frac{-\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)} \qquad (m^2 \neq n^2) \tag{1.108}$$

$$\int \sin^2 x \cos^2 x\, dx = \tfrac{1}{8}[x - \tfrac{1}{4} \sin(4x)] \tag{1.109}$$

$$\int \sin x \cos^m x\, dx = \frac{-\cos^{m+1} x}{m+1} \tag{1.110}$$

$$\int \sin^m x \cos x\, dx = \frac{\sin^{m+1} x}{m+1} \tag{1.111}$$

$$\int \cos^m x \sin^n x\, dx = \frac{\cos^{m-1} x \sin^{n+1} x}{m+n} + \frac{m-1}{m+n} \int \cos^{m-2} x \sin^n x\, dx \qquad (m \neq -n) \tag{1.112}$$

$$\int \cos^m x \sin^n x\, dx = \frac{-\cos^{m+1} x \sin^{n-1} x}{m+n} + \frac{n-1}{m+n} \int \cos^m x \sin^{n-2} x\, dx \qquad (m \neq -n) \tag{1.113}$$

$$\int u\, dv = uv - \int v\, du \tag{1.114}$$

1.6 Dirac Delta Function

In all of electrical engineering, there is perhaps nothing that is responsible for more hand-waving than is the so-called *delta function*, or *impulse function*, which is denoted $\delta(t)$ and which is usually depicted as a vertical arrow at the origin as shown in Fig. 1.3. This function is often called the *Dirac delta function* in honor of Paul Dirac (1902–1984), an English physicist who used delta functions extensively in his work on quantum mechanics. A number of nonrigorous approaches for defining the impulse function can be found throughout the literature. A *unit impulse* is often loosely described as having a zero width and an infinite amplitude at the origin such that the total area

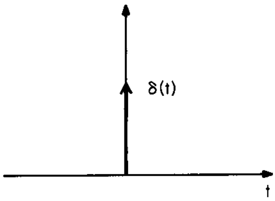

Figure 1.3 Graphical representation of the Dirac delta function.

under the impulse is equal to unity. How is it possible to claim that zero times infinity equals 1? The trick involves defining a sequence of functions $f_n(t)$ such that

$$\int_{-\infty}^{\infty} f_n(t)\,dt = 1 \tag{1.115}$$

and

$$\lim_{n \to \infty} f_n(t) = 0 \qquad \text{for } t \neq 0 \tag{1.116}$$

The delta function is then defined as

$$\delta(t) = \lim_{n \to \infty} f_n(t) \tag{1.117}$$

Example 1.3 Let a sequence of pulse functions $f_n(t)$ be defined as

$$f_n(t) = \begin{cases} \dfrac{n}{2} & |t| \leqslant \dfrac{1}{n} \\ 0 & \text{otherwise} \end{cases} \tag{1.118}$$

Equation (1.115) is satisfied since the area of pulse is equal to $(2n) \cdot (n/2) = 1$ for all n. The pulse width decreases and the pulse amplitude increases as n approaches infinity. Therefore, we intuitively sense that this sequence must also satisfy (1.116). Thus the impulse function can be defined as the limit of (1.118) as n approaches infinity. Using similar arguments, it can be shown that the impulse can also be defined as the limit of a sequence of sinc functions or gaussian pulse functions.

A second approach entails simply defining $\delta(t)$ to be that function which satisfies

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1 \qquad \text{and} \qquad \delta(t) = 0 \qquad \text{for } t \neq 0 \tag{1.119}$$

In a third approach, $\delta(t)$ is defined as that function which exhibits the property

$$\int_{-\infty}^{\infty} \delta(t)\,f(t)\,dt = f(0) \tag{1.120}$$

While any of these three approaches is adequate to introduce the delta function into an engineer's repertoire of analytical tools, none of the three is sufficiently rigorous to satisfy mathematicians or discerning theoreticians. In particular, notice that none of the approaches presented deals with the thorny issue of just what the value of $\delta(t)$ is for $t = 0$. The rigorous definition of $\delta(t)$ introduced in 1950 by Laurent Schwartz (Schwartz (1950) rejects the notion that the impulse is an ordinary function and instead defines it as a *distribution.*

Distributions

Let S be the set of functions $f(x)$ for which the nth derivative $f^{[n]}(x)$ exists for any n and all x. Furthermore, each $f(x)$ decreases sufficiently fast at infinity such that

$$\lim_{x \to \infty} x^n f(x) = 0 \qquad \text{for all } n \tag{1.121}$$

A *distribution,* often denoted $\phi(x)$, is defined as a continuous linear mapping from the set S to the set of complex numbers. Notationally, this mapping is represented as an inner product

$$\int_{-\infty}^{\infty} \phi(x)\, f(x)\, dx = z \tag{1.122}$$

or alternatively

$$\langle \phi(x), f(x) \rangle = z \tag{1.123}$$

Notice that no claim is made that ϕ is a function capable of mapping values of x into corresponding values $\phi(x)$. In some texts (such as Papoulis 1962), $\phi(x)$ is referred to as a *functional* or as a *generalized function.* The distribution ϕ is defined only through the impact that it has upon other functions. The impulse function is a distribution defined by the following:

$$\int_{-\infty}^{\infty} \delta(t)\, f(t)\, dt = f(0) \tag{1.124}$$

The equation (1.124) looks exactly like (1.120), but defining $\delta(t)$ as a distribution eliminates the need to tap dance around the issue of assigning a value to $\delta(0)$. Furthermore, the impulse function is elevated to a more substantial foundation from which several useful properties may be rigorously derived. For a more in-depth discussion of distributions other than $\delta(t)$, the interested reader is referred to Chap. 4 of Weaver (1989).

Properties of the delta distribution

It has been shown (Weaver 1989; Brigham 1974; Papoulis 1962; Schwartz and Friedland 1965) that the delta distribution exhibits the following properties:

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{1.125}$$

$$\frac{d}{dt}\delta(t) = \lim_{\tau \to 0} \frac{\delta(t) - \delta(t - \tau)}{\tau} \tag{1.126}$$

$$\int_{-\infty}^{\infty} \delta(t - t_0)\, f(t)\, dt = f(t_0) \tag{1.127}$$

$$\delta(at) = \frac{1}{|a|}\,\delta(t) \tag{1.128}$$

$$\delta(t_0) f(t) = f(t_0)\delta(t_0) \tag{1.129}$$

$$\delta_1(t - t_1) * \delta_2(t - t_2) = \delta[t - (t_1 + t_2)] \tag{1.130}$$

In Eq. (1.129), $f(t)$ is an ordinary function that is continuous at $t = t_0$, and in Eq. (1.130) the asterisk denotes convolution.

1.7 Mathematical Modeling of Signals

The distinction between a signal and its mathematical representation is not always rigidly observed in the signal processing literature. Mathematical functions that only *model* signals are commonly referred to as "signals," and properties of these models are often taken as properties of the signals themselves.

Mathematical models of signals are generally categorized as either *steady-state* or *transient models*. The typical voltage output from an oscillator is sketched in Fig. 1.4. This signal exhibits three different parts—a *turn-on transient* at the beginning, an interval of *steady-state operation* in the middle, and a *turn-off transient* at the end. It is possible to formulate a single mathematical expression that describes all three parts, but for most uses, such an expression would be unnecessarily complicated. In cases where the primary concern is steady-state behavior, simplified mathematical expres-

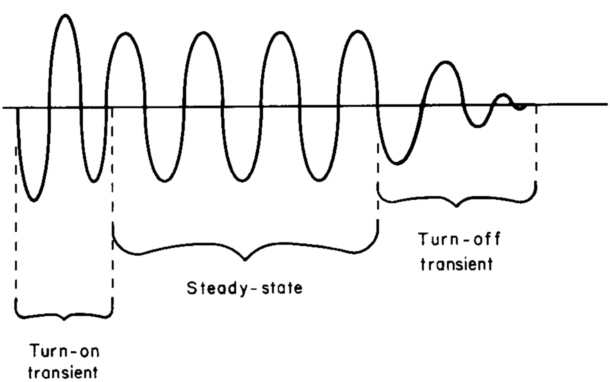

Figure 1.4 Typical output of an audio oscillator.

sions that ignore the transients will often be adequate. The steady-state portion of the oscillator output can be modeled as a sinusoid that theoretically exists for all time. This seems to be a contradiction to the obvious fact that the oscillator output exists for some limited time interval between turn-on and turn-off. However, this is not really a problem; over the interval of steady-state operation that we are interested in, the mathematical sine function accurately describes the behavior of the oscillator's output voltage. Allowing the mathematical model to assume that the steady-state signal exists over all time greatly simplifies matters since the transients' behavior can be excluded from the model. In situations where the transients are important, they can be modeled as exponentially saturating and decaying sinusoids as shown in Figs. 1.5 and 1.6. In Fig. 1.5, the saturating exponential envelope continues to increase, but it never quite reaches the steady-state value. Likewise the decaying exponential envelope of Fig. 1.6 continues to decrease, but it never quite reaches zero. In this context, the steady-state value is sometimes called an *assymptote*, or the envelope can be said to *assymptotically* approach the steady-state value. Steady-state and transient models of signal behavior inherently contradict each other, and neither constitutes a "true" description of a particular signal. The formulation of the appropriate model requires an understanding of the signal to be modeled and of the implications that a particular choice of model will have for the intended application.

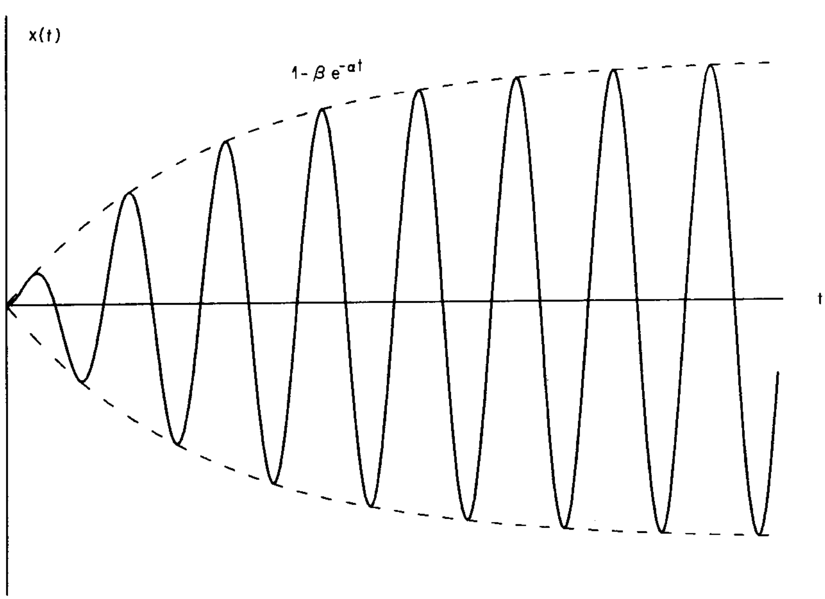

Figure 1.5 Exponentially saturating sinusoid.

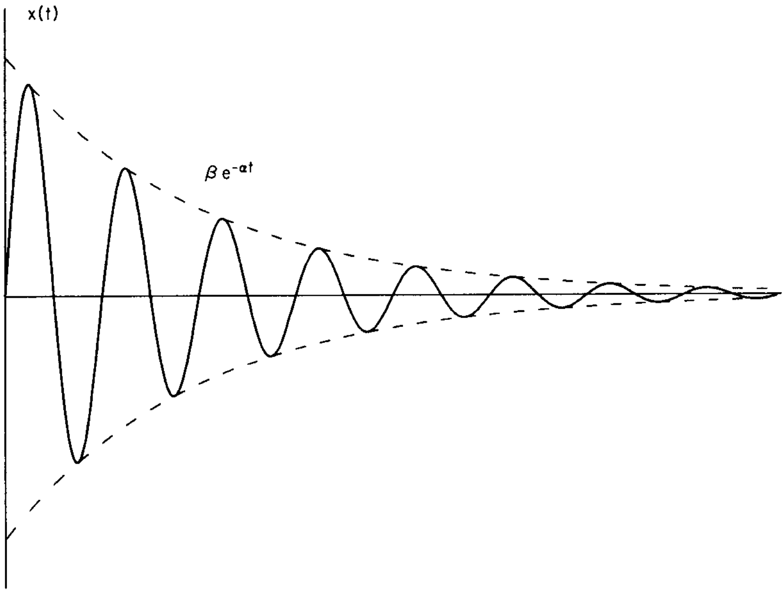

Figure 1.6 Exponentially decaying sinusoid.

Steady-state signal models

Generally, steady-state signals are limited to just sinusoids or sums of sinusoids. This will include virtually any periodic signals of practical interest since such signals can be resolved into sums of weighted and shifted sinusoids using the Fourier analysis techniques presented in Sec. 1.8.

Periodicity. Sines, cosines, and square waves are all periodic functions. The characteristic that makes them periodic is the way in which each of the complete waveforms can be formed by repeating a particular cycle of the waveform over and over at a regular interval as shown in Fig. 1.7.

> **Definition.** A function $x(t)$ is periodic with a period of T if and only if $x(t + nT) = x(t)$ for all integer values of n.

Functions that are not periodic are called *aperiodic*, and functions that are "almost" periodic are called *quasi-periodic*.

Symmetry. A function can exhibit a certain symmetry regarding its position relative to the origin.

> **Definition.** A function $x(t)$ is said to be *even*, or to exhibit *even symmetry*, if for all t, $x(t) = x(-t)$.

> **Definition.** A function $x(t)$ is said to be *odd*, or to exhibit *odd symmetry*, if for all t, $x(t) = -x(-t)$.

An even function is shown in Fig. 1.8, and an odd function is shown in Fig. 1.9.

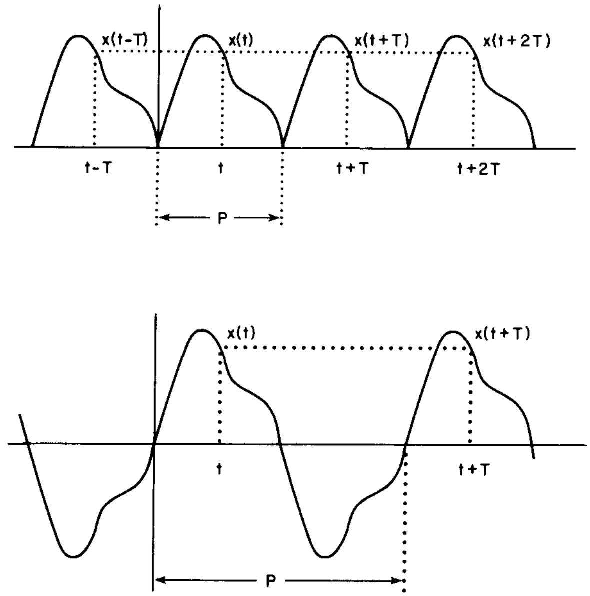

Figure 1.7 Periodic functions.

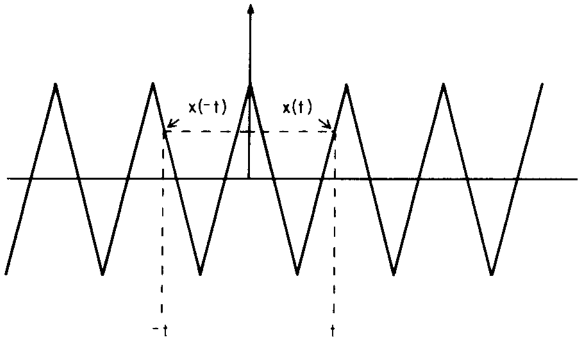

Figure 1.8 Even-symmetric function.

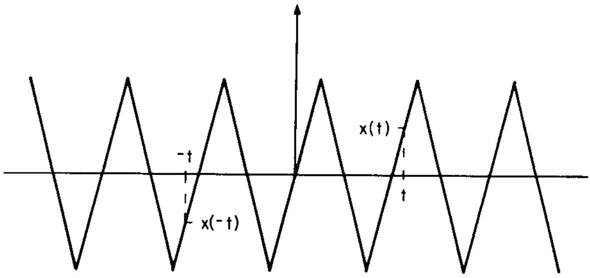

Figure 1.9 Odd-symmetric function.

Symmetry may appear at first to be something that is only "nice to know" and not particularly useful in practical applications where the definition of time zero is often somewhat arbitrary. This is far from the case, however, because symmetry considerations play an important role in Fourier analysis—especially the discrete Fourier analysis that will be discussed in Chap. 7. Some functions are neither odd nor even, but any *periodic* function can be resolved into a sum of an even function and an odd function as given by

$$x(t) = x_{\text{even}}(t) + x_{\text{odd}}(t)$$

where $x_{\text{even}}(t) = \frac{1}{2}[x(t) + x(-t)]$

$$x_{\text{odd}}(t) = \tfrac{1}{2}[x(t) - x(-t)]$$

Addition and multiplication of symmetric functions will obey the following rules:

$$\text{Even} + \text{even} = \text{even}$$

$$\text{Odd} + \text{odd} = \text{odd}$$

$$\text{Odd} \times \text{odd} = \text{even}$$

$$\text{Even} \times \text{even} = \text{even}$$

$$\text{Odd} \times \text{even} = \text{odd}$$

Energy signals versus power signals

It is a common practice to deal with mathematical functions representing abstract signals as though they are either voltages across a 1-Ω resistor or currents through a 1-Ω resistor. Since, in either case, the resistance has an assumed value of unity, the voltage and current for any particular signal will be numerically equal—thus obviating the need to select one viewpoint over the other. Thus for a signal $x(t)$, the instantaneous power $p(t)$ dissipated in the 1-Ω resistor is simply the squared amplitude of the signal

$$p(t) = |x(t)|^2 \tag{1.131}$$

regardless of whether $x(t)$ represents a voltage or a current. To emphasize the fact that the power given by (1.131) is based upon unity resistance, it is often referred to as the *normalized power*. The total energy of the signal $x(t)$ is then obtained by integrating the right-hand side of (1.131) over all time:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt \tag{1.132}$$

and the average power is given by

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt \tag{1.133}$$

A few texts (for example, Haykin 1983) equivalently define the average power as

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt \tag{1.134}$$

If the total energy is finite and nonzero, $x(t)$ is referred to as an *energy signal*. If the average power is finite and nonzero, $x(t)$ is referred to as a *power signal*. Note that a power signal has infinite energy, and an energy signal has zero average power; thus the two categories are mutually exclusive. Periodic signals and most random signals are power signals, while most deterministic aperiodic signals are energy signals.

1.8 Fourier Series

Trigonometric forms

Periodic signals can be resolved into linear combinations of phase-shifted sinusoids using the *Fourier series*, which is given by

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)] \tag{1.135}$$

where $$a_0 = \frac{2}{T} \int_{-T/2}^{T/2} x(t) \, dt \tag{1.136}$$

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} x(t) \cos(n\omega_0 t) \, dt \tag{1.137}$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} x(t) \sin(n\omega_0 t) \, dt \tag{1.138}$$

T = period of $x(t)$

$$\omega_0 = \frac{2\pi}{T} = 2\pi f_0 = \text{fundamental radian frequency of } x(t)$$

Upon application of the appropriate trigonometric identities, Eq. (1.135) can be put into the following alternative form:

$$x(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t - \theta_n) \tag{1.139}$$

where the c_n and θ_n are obtained from a_n and b_n using

$$c_0 = \frac{a_0}{2} \tag{1.140}$$

$$c_n = \sqrt{a_n^2 + b_n^2} \tag{1.141}$$

$$\theta_n = \tan^{-1}\left(\frac{b_n}{a_n}\right) \tag{1.142}$$

Examination of (1.135) and (1.136) reveals that a periodic signal contains only a dc component plus sinusoids whose frequencies are integer multiples of the original signal's *fundamental frequency*. (For a fundamental frequency of f_0, $2f_0$ is the *second harmonic*, $3f_0$ is the *third harmonic*, and so on.) Theoretically, periodic signals will generally contain an infinite number of harmonically related sinusoidal components. In the real world, however, periodic signals will contain only a finite number of measurable harmonics. Consequently, pure mathematical functions are only approximately equal to the practical signals which they model.

Exponential form

The trigonometric form of the Fourier series given by (1.135) makes it easy to visualize periodic signals as summations of sine and cosine waves, but mathematical manipulations are often more convenient when the series is in the exponential form given by

$$x(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{j2\pi n f_0 t} \tag{1.143}$$

where $c_n = \dfrac{1}{T}\displaystyle\int_T x(t)\, e^{-j2\pi n f_0 t}\, dt$ (1.144)

The integral notation used in (1.144) indicates that the integral is to be evaluated over one period of $x(t)$. In general, the values of c_n are complex; and they are often presented in the form of a magnitude spectrum and phase spectrum as shown in Fig. 1.10. The magnitude and phase values plotted in such spectra are obtained from c_n using

$$|c_n| = \sqrt{[\mathrm{Re}(c_n)]^2 + [\mathrm{Im}(c_n)]^2} \tag{1.145}$$

$$\theta_n = \tan^{-1}\left[\frac{\mathrm{Im}(c_n)}{\mathrm{Re}(c_n)}\right] \tag{1.146}$$

The complex c_n of (1.144) can be obtained from the a_n and b_n of (1.137) and

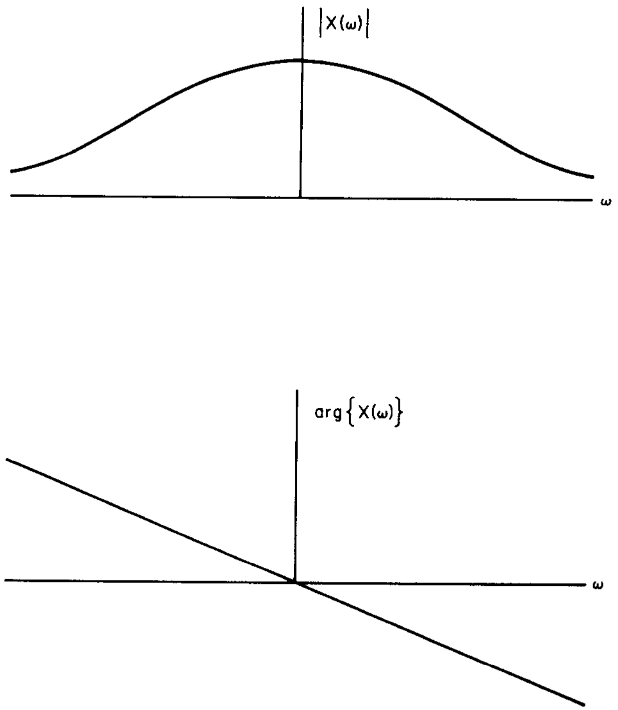

Figure 1.10 Magnitude and phase spectra.

(1.138) using

$$c_n = \begin{cases} \dfrac{a_n + jb_n}{2} & n < 0 \\ a_0 & n = 0 \\ \dfrac{a_n - jb_n}{2} & n > 0 \end{cases} \tag{1.147}$$

Conditions of applicability

The Fourier series can be applied to almost all periodic signals of *practical* interest. However, there are some functions for which the series will not converge. The Fourier series coefficients are guaranteed to exist and the series will converge uniformly if $x(t)$ satisfies the following conditions:

1. The function $x(t)$ is a single-valued function.
2. The function $x(t)$ has at most a finite number of discontinuities within each period.
3. The function $x(t)$ has at most a finite number of extrema (that is, maxima and minima) within each period.

4. The function $x(t)$ is absolutely integrable over a period:

$$\int_T |x(t)|\, dt < \infty \tag{1.148}$$

These conditions are often called the *Dirichlet conditions* in honor of Peter Gustav Lejeune Dirichlet (1805–1859) who first published them in the 1828 issue of *Journal für die reine und angewandte Mathematik* (commonly known as *Crelle's Journal*). In applications where it is sufficient for the Fourier series coefficients to be convergent in the mean, rather than uniformly convergent, it suffices for $x(t)$ to be integrable square over a period:

$$\int_T |x(t)|^2\, dt < \infty \tag{1.149}$$

For most engineering purposes, the Fourier series is usually assumed to be identical to $x(t)$ if conditions 1 through 3 plus either (1.148) or (1.149) are satisfied.

Properties of the Fourier series

A number of useful Fourier series properties are listed in Table 1.4. For ease of notation, the coefficients c_n corresponding to $x(t)$ are denoted as $X(n)$, and the c_n corresponding to $y(t)$ are denoted as $Y(n)$. In other words, the Fourier series representations of $x(t)$ and $y(t)$ are given by

$$x(t) = \sum_{n=-\infty}^{\infty} X(n) \exp\left(\frac{j2\pi nt}{T}\right) \tag{1.150}$$

$$y(t) = \sum_{n=-\infty}^{\infty} Y(n) \exp\left(\frac{j2\pi nt}{T}\right) \tag{1.151}$$

TABLE 1.4 Properties of the Fourier Series
[*Note*: $x(t)$, $y(t)$, $X(n)$, and $Y(n)$ are as given in Eqs. (1.150) and (1.151).]

Property	Time function	Transform
1. Homogeneity	$ax(t)$	$aX(n)$
2. Additivity	$x(t) + y(t)$	$X(n) + Y(n)$
3. Linearity	$ax(t) + by(t)$	$aX(n) + bY(n)$
4. Multiplication	$x(t)y(t)$	$\sum_{m=-\infty}^{\infty} X(n-m)Y(m)$
5. Convolution	$\frac{1}{T}\int_0^T x(t-\tau)y(\tau)\, d\tau$	$X(n)Y(n)$
6. Time shifting	$x(t-\tau)$	$\exp\left(\frac{-j2\pi n\tau}{T}\right)X(n)$
7. Frequency shifting	$\exp\left(\frac{j2\pi mt}{T}\right)x(t)$	$X(n-m)$

where T is the period of both $x(t)$ and $y(t)$. In addition to the properties listed in Table 1.4, the Fourier series coefficients exhibit certain symmetries. If (and only if) $x(t)$ is real, the corresponding FS coefficients will exhibit even symmetry in their real part and odd symmetry in their imaginary part:

$$\mathrm{Im}[x(t)] = 0 \Leftrightarrow \mathrm{Re}[X(-n)] = \mathrm{Re}[X(n)]$$
$$\mathrm{Im}[X(-n)] = -\mathrm{Im}[X(n)] \tag{1.152}$$

Equation (1.152) can be rewritten in a more compact form as

$$\mathrm{Im}[x(t)] = 0 \Leftrightarrow X(-n) = X^*(n) \tag{1.153}$$

where the superscript asterisk indicates complex conjugation. Likewise for purely imaginary $x(t)$, the corresponding FS coefficients will exhibit odd symmetry in their real part and even symmetry in their imaginary part:

$$\mathrm{Re}[x(t)] = 0 \Leftrightarrow X(-n) = -[X^*(n)] \tag{1.154}$$

If and only if $x(t)$ is (in general) complex with even symmetry in the real part and odd symmetry in the imaginary part, then the corresponding FS coefficients will be purely real:

$$x(-t) = x^*(t) \Leftrightarrow \mathrm{Im}[X(n)] = 0 \tag{1.155}$$

If and only if $x(t)$ is (in general) complex with odd symmetry in the real part and even symmetry in the imaginary part, then the corresponding FS coefficients will be purely imaginary:

$$x(-t) = -[x^*(t) \Leftrightarrow \mathrm{Re}[X(n)] = 0 \tag{1.156}$$

In terms of the amplitude and phase spectra, Eq. (1.153) means that for real signals, the amplitude spectrum will have even symmetry and the phase spectrum will have odd symmetry. If $x(t)$ is both real and even, then both (1.153) and (1.155) apply. In this special case, the FS coefficients will be both real and even symmetric. At first glance, it may appear that real even-symmetric coefficients are in contradiction to the expected odd-symmetric phase spectrum; but in fact there is no contradiction. For all the positive real coefficients, the corresponding phase is of course zero. For each of the negative real coefficients, we can choose a phase value of either plus or minus 180°. By appropriate selection of positive and negative values, odd symmetry in the phase spectrum can be maintained.

Fourier series of a square wave

Consider the square wave shown in Fig. 1.11. The Fourier series representation of this signal is given by

$$x(t) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{j2\pi nt}{T}\right) \tag{1.157}$$

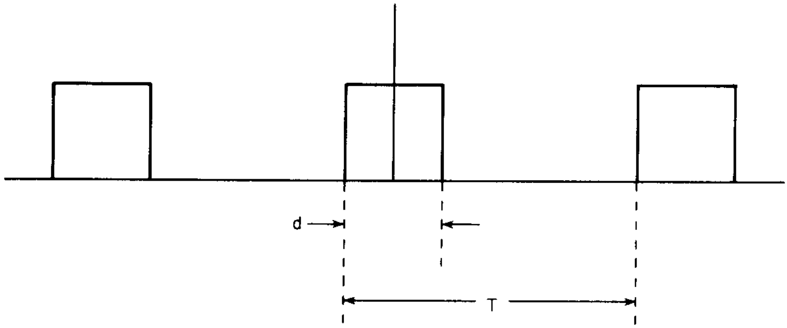

Figure 1.11 Square wave.

$$\text{where } c_n = \frac{\tau A}{T} \operatorname{sinc}\left(\frac{n\tau}{T}\right) \tag{1.158}$$

Since the signal is both real and even symmetric, the FS coefficients are real and even symmetric as shown in Fig. 1.12. The corresponding magnitude spectrum will be even, as shown in Fig. 1.13a. Appropriate selection of $\pm 180°$ values for the phase of negative coefficients will allow an odd-symmetric phase spectrum to be plotted as in Fig. 1.13b.

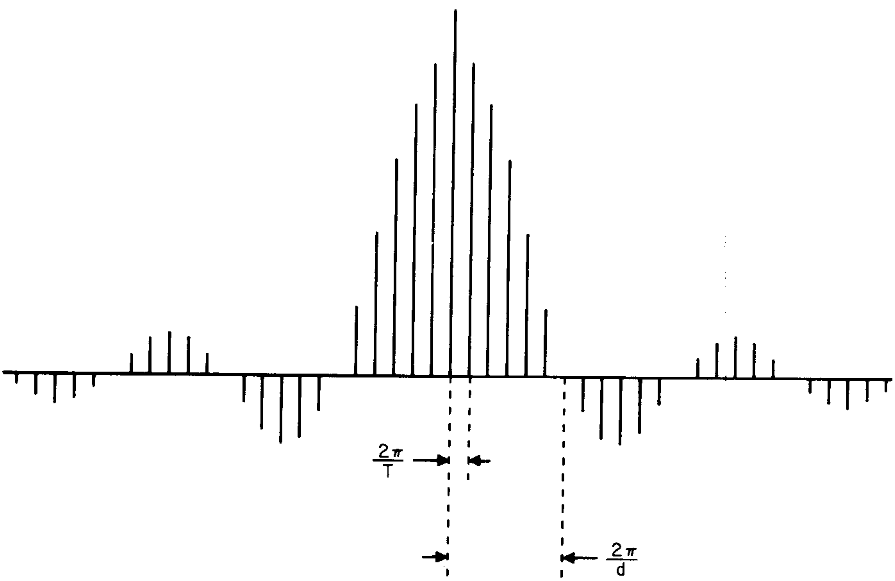

Figure 1.12 Fourier series for a square wave.

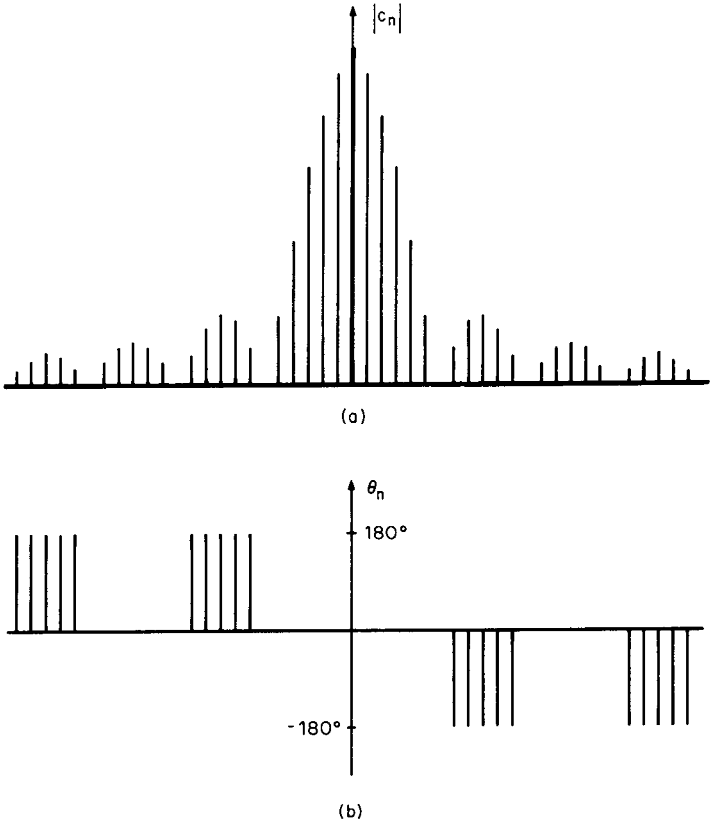

Figure 1.13 Fourier series (*a*) amplitude and (*b*) phase spectra for a square wave.

Parseval's theorem

The average power (normalized for 1 Ω) of a real-valued periodic function of time can be obtained directly from the Fourier series coefficients by using Parseval's theorem:

$$P = \frac{1}{T} \int_T |x(t)|^2 \, dt$$

$$= \sum_{n=-\infty}^{\infty} |c_n|^2 = c_0^2 + \sum_{n=1}^{\infty} \tfrac{1}{2}|2c_n|^2 \tag{1.159}$$

1.9 Fourier Transform

The *Fourier transform* is defined as

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft} \, dt \tag{1.160}$$

or in terms of the radian frequency $\omega = 2\pi f$:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \qquad (1.161)$$

The *inverse transform* is defined as

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft}\, df \qquad (1.162a)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega \qquad (1.162b)$$

There are a number of different shorthand notations for indicating that $x(t)$ and $X(f)$ are related via the Fourier transform. Some of the more common notations include:

$$X(f) = \mathscr{F}[x(t)] \qquad (1.163)$$

$$x(t) = \mathscr{F}^{-1}[X(f)] \qquad (1.164)$$

$$x(t) \overset{\text{FT}}{\longleftrightarrow} X(f) \qquad (1.165)$$

$$x(t) \underset{\text{IFT}}{\overset{\text{FT}}{\rightleftarrows}} X(f) \qquad (1.166)$$

$$x(t) \Leftrightarrow X(f) \qquad (1.167)$$

The notation used in (1.163) and (1.164) is easiest to typeset, while the notation of (1.167) is probably the most difficult. However, the notation of (1.167) is used in the classic work on fast Fourier transforms described by Brigham (1974). The notations of (1.165) and (1.166), while more difficult to typeset, offer the flexibility of changing the letters FT to FS, DFT, or DTFT to indicate, respectively, "Fourier series," "discrete Fourier transform," or "discrete-time Fourier transform" as is done in Roberts and Mullis (1987). (The latter two transforms will be discussed in Chap. 6.) The form used in (1.166) is perhaps best saved for tutorial situations (such as Rorabaugh 1986) where the distinction between the transform and inverse transform needs to be emphasized. Strictly speaking, the equality shown in (1.164) is incorrect, since the inverse transform of $X(f)$ is only guaranteed to approach $x(t)$ in the sense of convergence in the mean. Nevertheless, the notation of Eq. (1.164) appears often throughout the engineering literature. Often the frequency domain function is written as $X(j\omega)$ rather than $X(\omega)$ in order to facilitate comparison with the Laplace transform. We can write

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \qquad (1.168)$$

and realize that this is identical to the two-sided Laplace transform defined by Eq. (2.21) with $j\omega$ substituted for s. A number of useful Fourier transform properties are listed in Table 1.5.

TABLE 1.5 Properties of the Fourier Transform

Property	Time function $x(t)$	Transform $X(f)$
1. Homogeneity	$ax(t)$	$aX(f)$
2. Additivity	$x(t) + y(t)$	$X(f) + Y(f)$
3. Linearity	$ax(t) + by(t)$	$aX(f) + bY(f)$
4. Differentiation	$\frac{d^n}{dt^n} x(t)$	$(j2\pi f)^n X(f)$
5. Integration	$\int_{-\infty}^{t} x(\tau)\, d\tau$	$\frac{X(f)}{j2\pi f} + \frac{1}{2} X(0)\, \delta(f)$
6. Frequency shifting	$e^{-j2\pi f_0 t} x(t)$	$X(f + f_0)$
7. Sine modulation	$x(t) \sin(2\pi f_0 t)$	$\frac{1}{2}[X(f - f_0) + X(f + f_0)]$
8. Cosine modulation	$x(t) \cos(2\pi f_0 t)$	$\frac{1}{2}[X(f - f_0) - X(f + f_0)]$
9. Time shifting	$x(t - \tau)$	$e^{-j\omega\tau} X(f)$
10. Time convolution	$\int_{-\infty}^{\infty} h(t - \tau)\, x(\tau)\, d\tau$	$H(f)X(f)$
11. Multiplication	$x(t)y(t)$	$\int_{-\infty}^{\infty} X(\lambda) Y(f - \lambda)\, d\lambda$
12. Time and frequency scaling	$x\left(\frac{t}{a}\right) \quad a > 0$	$aX(af)$
13. Duality	$X(t)$	$x(-f)$
14. Conjugation	$x^*(t)$	$X^*(-f)$
15. Real part	$\mathrm{Re}[x(t)]$	$\frac{1}{2}[X(f) + X^*(-f)]$
16. Imaginary part	$\mathrm{Im}[x(t)]$	$\frac{1}{2j}[X(f) - X^*(-f)]$

Fourier transforms of periodic signals

Often there is a requirement to analyze systems that include both periodic power signals and aperiodic energy signals. The mixing of Fourier transform results and Fourier series results implied by such an analysis may be quite cumbersome. For the sake of convenience, the spectra of most periodic signals can be obtained as Fourier transforms that involve the Dirac delta function. When the spectrum of a periodic signal is determined via the Fourier series, the spectrum will consist of lines located at the fundamental frequency and its harmonics. When the spectrum of this same signal is obtained as a Fourier transform, the spectrum will consist of Dirac delta functions located at the fundamental frequency and its harmonics. Obviously, these two different mathematical representations must be equivalent in their physical significance. Specifically, consider a periodic signal $x_p(t)$ having a period of T. The Fourier series representation of $x_p(t)$ is obtained from Eq. (1.143) as

$$x_p(t) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{j2\pi nt}{T}\right) \tag{1.169}$$

We can then define a *generating function* $x(t)$ that is equal to a single period of $x_p(t)$:

$$x(t) = \begin{cases} x_p(t) & |t| \le \dfrac{T}{2} \\ 0 & \text{elsewhere} \end{cases} \tag{1.170}$$

The periodic signal $x_p(t)$ can be expressed as an infinite summation of time-shifted copies of $x(t)$:

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(t - nT) \tag{1.171}$$

The Fourier series coefficients c_n appearing in (1.169) can be obtained as

$$c_n = \frac{1}{T} X\left(\frac{n}{T}\right) \tag{1.172}$$

where $X(f)$ is the Fourier transform of $x(t)$. Thus, the Fourier transform of $x_p(t)$ can be obtained as

$$\mathscr{F}[x_p(t)] = \frac{1}{T} \sum_{n=-\infty}^{\infty} X\left(\frac{n}{T}\right) \delta\left(f - \frac{n}{T}\right) \tag{1.173}$$

Common Fourier transform pairs

A number of frequently encountered Fourier transform pairs are listed in Table 1.6. Several of these pairs are actually obtained as Fourier transforms-in-the-limit.

1.10 Spectral Density

Energy spectral density

The *energy spectral density* of an energy signal is defined as the squared magnitude of the signal's Fourier transform:

$$S_e(f) = |X(f)|^2 \tag{1.174}$$

Analogous to the way in which Parseval's theorem relates the Fourier series coefficients to the average power of a power signal, *Rayleigh's energy theorem*

TABLE 1.6 Some Common Fourier Transform Pairs

Pair No.	$x(t)$	$X(\omega)$	$X(f)$
1	1	$2\pi\,\delta(\omega)$	$\delta(f)$
2	$u_1(t)$	$\dfrac{1}{j\omega}+\pi\,\delta(\omega)$	$\dfrac{1}{2\pi f}+\dfrac{1}{2}\delta(f)$
3	$\delta(t)$	1	1
4	t^n	$2\pi j^n\,\delta^{(n)}(\omega)$	$\left(\dfrac{j}{2\pi}\right)^n \delta^{(n)}(f)$
5	$\sin\omega_0 t$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$	$\dfrac{j}{2}[\delta(f+f_0)-\delta(f-f_0)]$
6	$\cos\omega_0 t$	$\pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$	$\tfrac{1}{2}[\delta(f+f_0)+\delta(f-f_0)]$
7	$e^{-at}u_1(t)$	$\dfrac{1}{j\omega+a}$	$\dfrac{1}{j2\pi f+a}$
8	$u_1(t)\,e^{-at}\sin\omega_0 t$	$\dfrac{\omega_0}{(a+j\omega)^2+\omega_0^2}$	$\dfrac{2\pi f_0}{(a+j2\pi f)^2+(2\pi f_0)^2}$
9	$u_1(t)\,e^{-at}\cos\omega_0 t$	$\dfrac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$	$\dfrac{a+j2\pi f}{(a+j2\pi f)^2+(2\pi f_0)^2}$
10	$\begin{cases}1 & \lvert t\rvert\le \tfrac{1}{2}\\ 0 & \text{elsewhere}\end{cases}$	$\operatorname{sinc}\left(\dfrac{\omega}{2\pi}\right)$	$\operatorname{sinc} f$
11	$\operatorname{sinc} t \triangleq \dfrac{\sin\pi t}{\pi t}$	$\begin{cases}1 & \lvert\omega\rvert\le\pi\\ 0 & \text{elsewhere}\end{cases}$	$\begin{cases}1 & \lvert f\rvert\le \tfrac{1}{2}\\ 0 & \text{elsewhere}\end{cases}$
12	$\begin{cases}at\exp(-at) & t>0\\ 0 & \text{elsewhere}\end{cases}$	$\dfrac{a}{(a+j\omega)^2}$	$\dfrac{a}{(a+j2\pi f)^2}$
13	$\exp(-a\lvert t\rvert)$	$\dfrac{2a}{a^2+\omega^2}$	$\dfrac{2a}{a^2+4\pi^2 f^2}$
14	$\operatorname{signum} t \triangleq \begin{cases}1 & t>0\\ 0 & t=0\\ -1 & t<0\end{cases}$	$\dfrac{2}{j\omega}$	$\dfrac{1}{j\pi f}$

relates the Fourier transform to the total energy of an energy signal as follows:

$$E=\int_{-\infty}^{\infty}\lvert x(t)\rvert^2\,dt=\int_{-\infty}^{\infty}S_e(f)\,df=\int_{-\infty}^{\infty}\lvert X(f)\rvert^2\,df \tag{1.175}$$

In many texts where $x(t)$ is assumed to be real valued, the absolute-value signs are omitted from the first integrand in (1.175). In some texts (such as Kanefsky 1985), Eq. (1.175) is loosely referred to as "Parseval's theorem."

Power spectral density of a periodic signal

The *power spectral density* (PSD) of a periodic signal is defined as the squared magnitude of the signal's line spectrum obtained via either a Fourier series or a Fourier transform with impulses. Using the Dirac delta notational conventions of the latter, the PSD is defined as

$$S_p(f) = \frac{1}{T^2} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right) \left| X\left(\frac{n}{T}\right) \right|^2 \tag{1.176}$$

where T is the period of the signal $x(t)$. Parseval's theorem as given by Eq. (1.159) of Sec. 1.8 can be restated in the notation of Fourier transform spectra as

$$P = \frac{1}{T^2} \sum_{n=-\infty}^{\infty} \left| X\left(\frac{n}{T}\right) \right|^2 \tag{1.177}$$