Nonlinear Signal Processing
ELEG 833

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8 Myriad Smoothers

8.1 FLOM Smoothers

Under the framework of Gaussian processes, the sample mean ($\bar{\beta}$) minimizes the second moment of the shifted variable $X - \beta$ over all possible shifts.

$$\mathrm{E}(X) = \bar{\beta} = \arg\min_{\beta} \mathrm{E}(X - \beta)^2. \tag{1}$$

Second-order moments do not exist with stable processes, but fractional-order moments do. The second moment in (1) can be replaced by fractional lower-order moments (FLOMs) to obtain the following measure of location

$$\beta_p = \arg\min_{\beta} \mathrm{E}(|X - \beta|^p), \quad p < 2. \tag{2}$$

FLOM smoothers follow from (2) where FLOM estimates are computed in the running window $\mathbf{X}(n) = [X_1(n), X_2(n), \ldots, X_N(n)]^T$ as

$$Y(n) = \arg\min_{\beta} \sum_{i=1}^{N} |X_i(n) - \beta|^p \tag{3}$$

with $p < 2$.

The behavior of FLOM smoothers is markedly dependant on the choice of p. As $p \rightarrow 2$, FLOM smoothers resemble the running mean. As p is reduced in value, FLOM smoothers become more robust and its output can tract discontinuities more effectively.

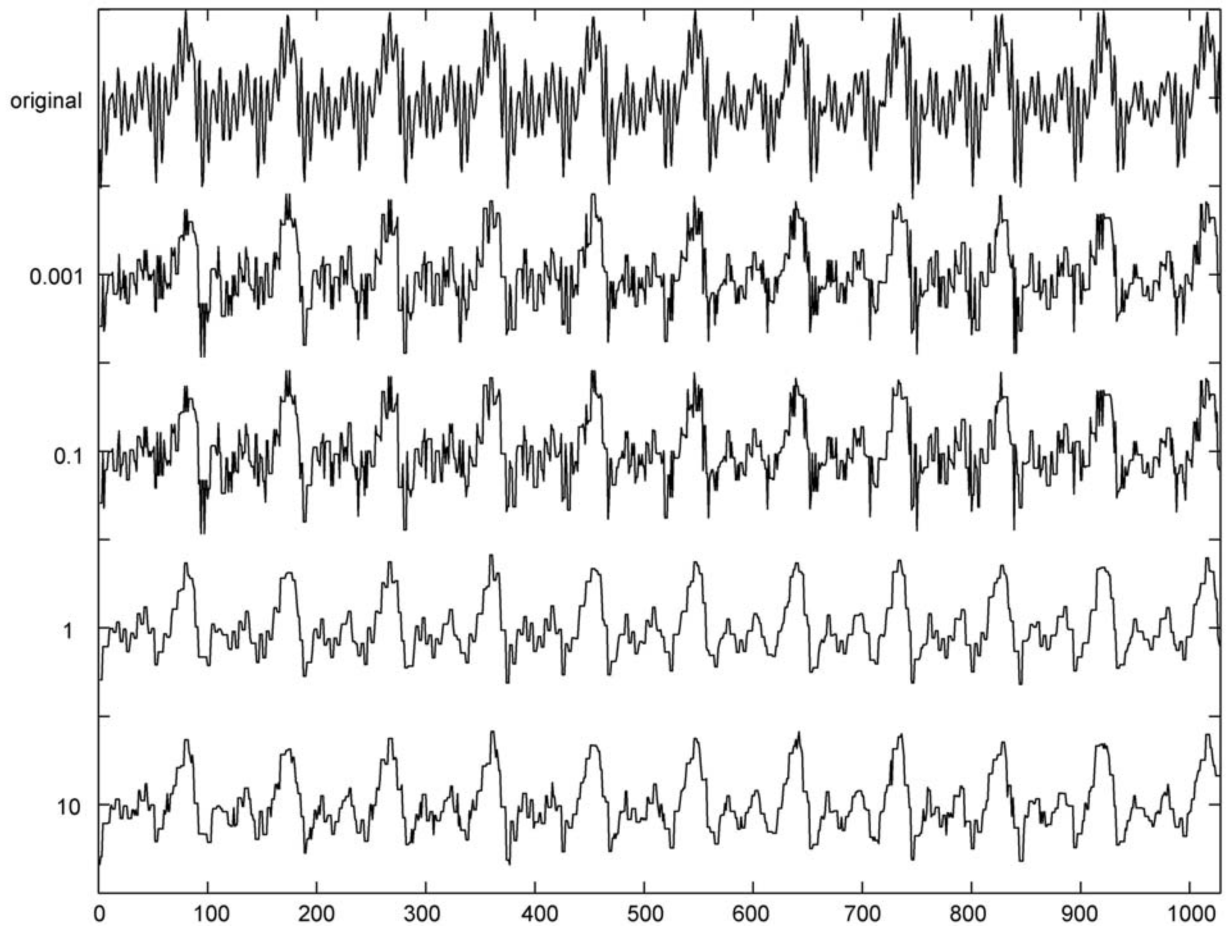

Figure 1: FLOM smoothing of a speech signal for different values of *p* and window size 5.

- FLOM smoothers arise from the location estimation problem under the generalized Gaussian distribution.
- For $p < 1$, FLOM smoothers are selection type.
- For $p > 1$, the cost function is convex and the output is not necessarily equal in value to one of the input samples.
- FLOM smoother computation is in general nontrivial.
- A method to overcome this limitation is to force the output of the smoother to be identical in value to one of the input samples.
- Selection type FLOM smoothers are suboptimal and are referred to as gamma filters.

EXAMPLE 8.1 (IMAGE DENOISING)

Figure 2: FLOM smoothing of an image for different values of p. (*a*) Image contaminated with salt-and-pepper noise (PSNR=17.75dB) and output of the FLOM smoother for: (*b*) $p = 0.01$ (PSNR=26.12dB).

Figure 3: FLOM smoothing of an image for different values of p. Outputs of the FLOM smoother for: (*c*) $p = 0.1$ (PSNR=31.86dB), (*d*) $p = 1$ (median smoother, PSNR=37.49dB).

Figure 4: FLOM smoothing of an image for different values of p (continued). (*a*) $p = 2$ (mean smoother, PSNR=33.53dB), (*b*) $p = 10$ (PSNR=31.15).

Figure 5: Gamma smoothing of an image for different values of p and different window sizes. (*a*) Original image and output of the 3×3 gamma smoother for (*b*) $p = 2$ (sample closest to the mean, PSNR=32.84dB).

Figure 6: Gamma smoothing of an image for different values of p and different window sizes. Output of the 3×3 gamma smoother for (*c*) $p = 10$ (PSNR=32.32dB), and the 5×5 gamma smoother for (*d*) $p = 0.1$ (PSNR=28.84dB).

Figure 7: Gamma smoothing of an image for different values of p and different window sizes (continued). (*a*) $p = 1$ (PSNR=29.91dB), (*b*) $p = 10$ (PSNR=28.13dB).

8.2 Running Myriad Smoothers

Given an observation vector $\mathbf{X}(n) = [X_1(n), X_2(n), \ldots, X_N(n)]$ and a fixed positive (tunable) value of K, the running myriad smoother output at time n is computed as

$$\begin{aligned} Y_K(n) &= \mathrm{MYRIAD}[K; X_1(n), X_2(n), \ldots, X_N(n)] \\ &= \arg\min_{\beta} \prod_{i=1}^{N} \left[K^2 + (X_i(n) - \beta)^2\right]. \quad (4) \\ &= \arg\min_{\beta} \sum_{i=1}^{N} \log\left[K^2 + (X_i(n) - \beta)^2\right]. \quad (5) \end{aligned}$$

The myriad $Y_K(n)$ is thus the value of β that minimizes the above cost function.

The definition of the sample myriad involves the free-tunable parameter K. This parameter will be shown to play a critical role in characterizing the behavior of the myriad.

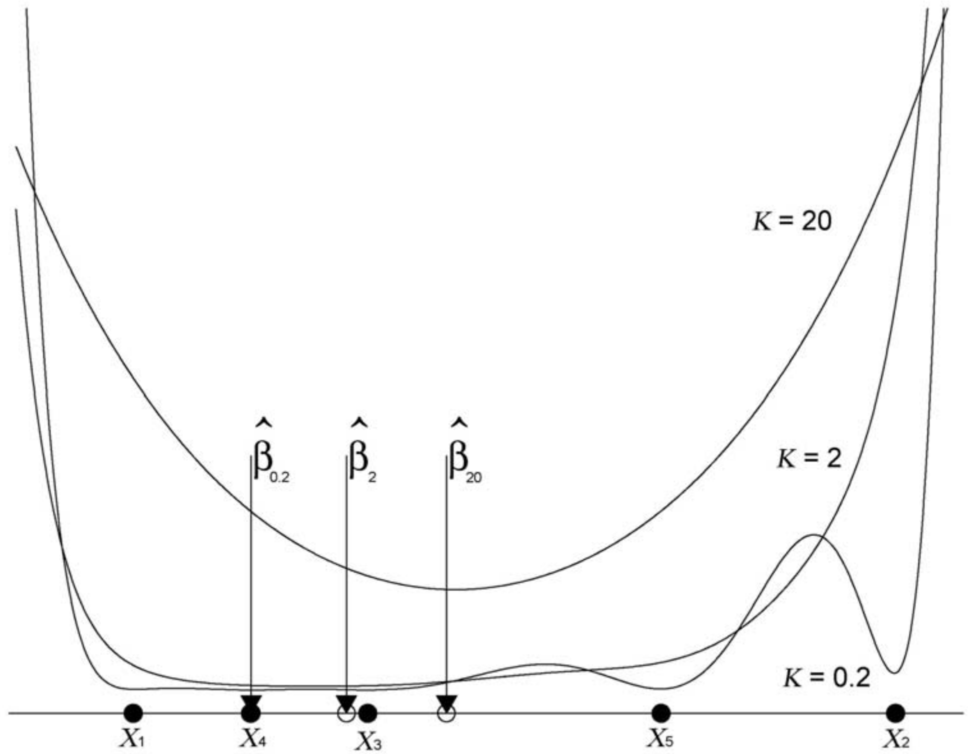

Figure 8: Myriad cost functions for different values of *k*

Property 8.1 (Linear Property) *Given a set of samples,* $X_1, X_2, \ldots, X_N$*, the sample myriad* $\hat{\beta}_K$ *converges to the sample average as* $K \to \infty$*. This is,*

$$\begin{aligned}\lim_{K\to\infty} \hat{\beta}_K &= \lim_{K\to\infty} \mathrm{MYRIAD}(K; X_1, \ldots, X_N) \\ &= \frac{1}{N}\sum_{i=1}^{N} X_i. \end{aligned} \tag{6}$$

Note that $\hat{\beta}_K \leq X_{(N)}$ by checking that for any i, and for $\beta > X_{(N)}, K^2 + (X_i - \beta)^2 > K^2 + (X_i - X_{(N)})^2$. In the same way, $\hat{\beta}_K \geq X_{(1)}$. Hence,

$$\hat{\beta}_K = \arg\min_{X_{(1)} \leq \beta \leq X_{(N)}} \prod_{i=1}^{N} [K^2 + (X_i - \beta)^2] \tag{7}$$

$$= \arg\min_{X_{(1)} \leq \beta \leq X_{(N)}} \left\{ K^{2N} + K^{2N-2} \sum_{i=1}^{N} (X_i - \beta)^2 + f(K) \right\} \tag{8}$$

where $f(K) = O(K^{2N-4})$ and O denotes the *asymptotic order* as $K \to \infty$.

Since adding or multiplying by constants does not affect the $\arg\min$ operator, Equation (8) can be rewritten as

$$\hat{\beta}_K = \arg\min_{X_{(1)} \leq \beta \leq X_{(N)}} \left\{ \sum_{i=1}^{N} (X_i - \beta)^2 + \frac{O(K^{2N-4})}{K^{2N-2}} \right\}. \qquad (9)$$

Letting $K \to \infty$, the term $O(K^{2N-4})/K^{2N-2}$ becomes negligible, and

$$\hat{\beta}_K \to \arg\min_{X_{(1)} \leq \beta \leq X_{(N)}} \left\{ \sum_{i=1}^{N} (X_i - \beta)^2 \right\} = \frac{1}{N} \sum_{i=1}^{N} X_i.$$

■

Definition 8.1 (Sample mode-myriad) *Given a set of samples* $X_1,\ X_2, \ldots,$ X_N*, the mode-myriad estimator,* $\hat{\beta}_0$*, is defined as*

$$\hat{\beta}_0 = \lim_{K \to 0} \hat{\beta}_K, \tag{10}$$

where $\hat{\beta}_K = \text{MYRIAD}(K; X_1, X_2, \ldots, X_N)$.

Property 8.2 (Mode Property) *The mode-myriad* $\hat{\beta}_0$ *is always equal to one of the most repeated values in the sample. Furthermore,*

$$\hat{\beta}_0 = \arg\min_{X_j \in \mathcal{M}} \prod_{i=1, X_i \neq X_j}^{N} |X_i - X_j|, \tag{11}$$

where $\mathcal{M}$ *is the set of most repeated values.*

EXAMPLE 8.2 (BEHAVIOR OF THE MODE MYRIAD)

$\vec{X} = [1,\ 4,\ 2.3,\ S,\ 2.5,\ 2,\ 5,\ 4.25,\ 6]$. S varies from 0 to 7.

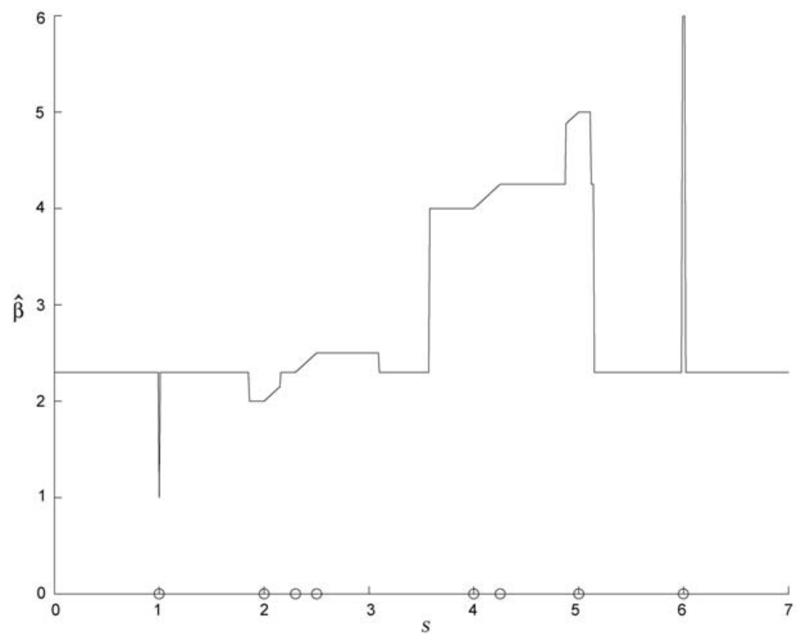

Figure 9: Mode myriad of a sample set with one variable sample. The constant samples are indicated with "◯"

EXAMPLE 8.3 (MODE-MYRIAD PERFORMANCE IN α-STABLE NOISE)

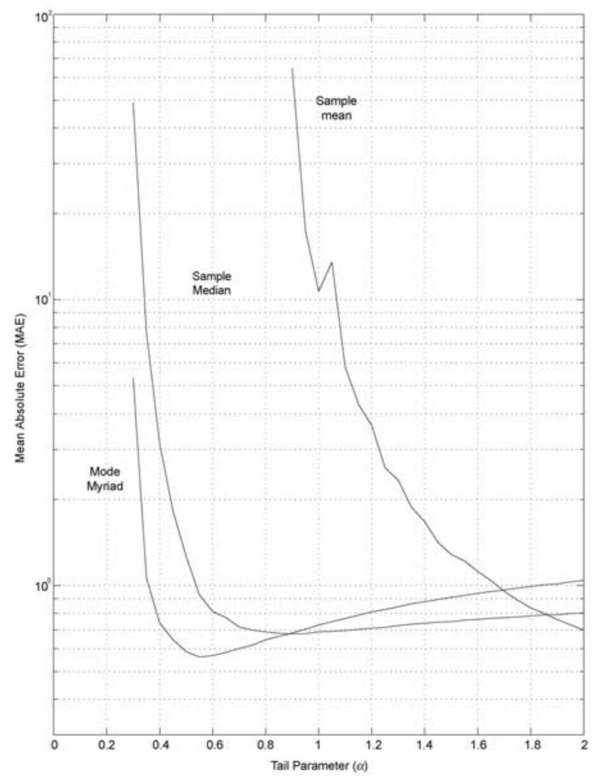

Figure 10: Estimated Mean Absolute Error of the sample mean, sample median and mode-myriad location estimator in α-stable noise ($N = 5$).

EXAMPLE 8.4 (DENOISING OF A VERY IMPULSIVE SIGNAL.)

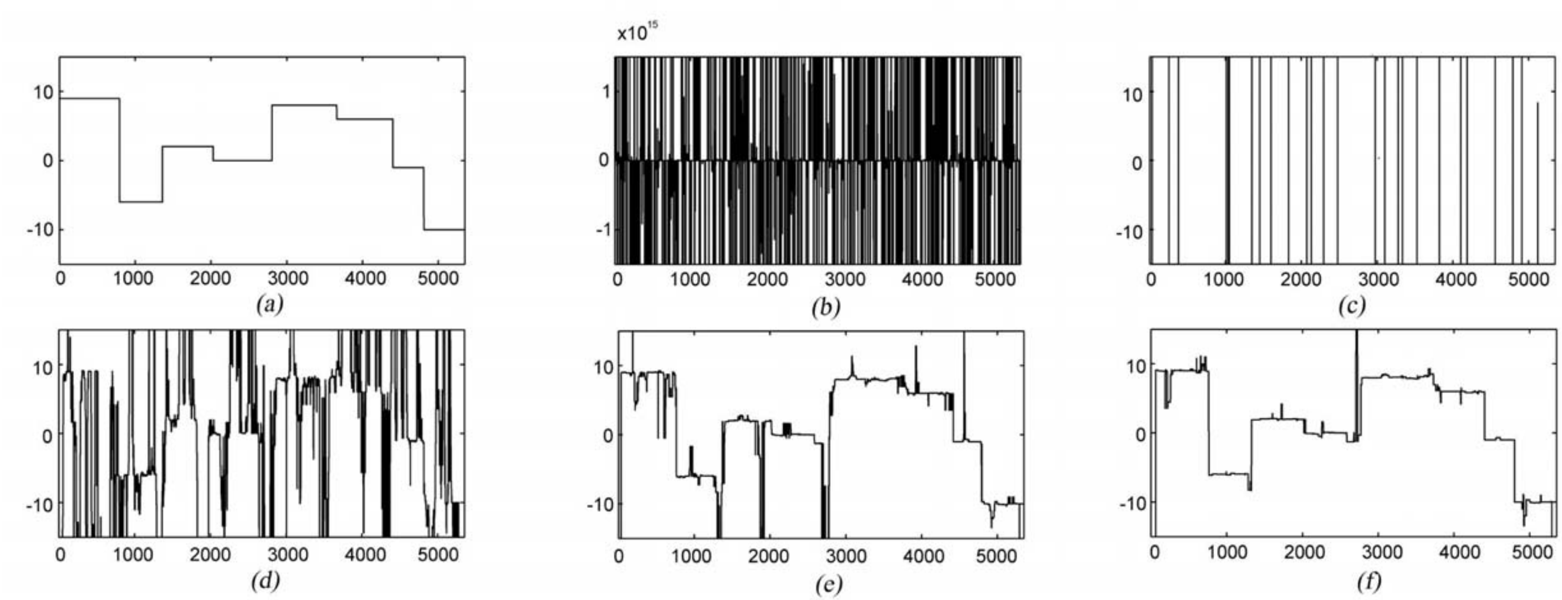

Figure 11: Running smoothers in stable noise ($\alpha = 0.2$). All smoothers of size 121; (*a*) original blocks signal, (*b*) corrupted signal with stable noise, (*c*) the output of the running mean, (*d*) the running median, (*e*) the running FLOM smoother, and (*f*) the running mode-myriad smoother.

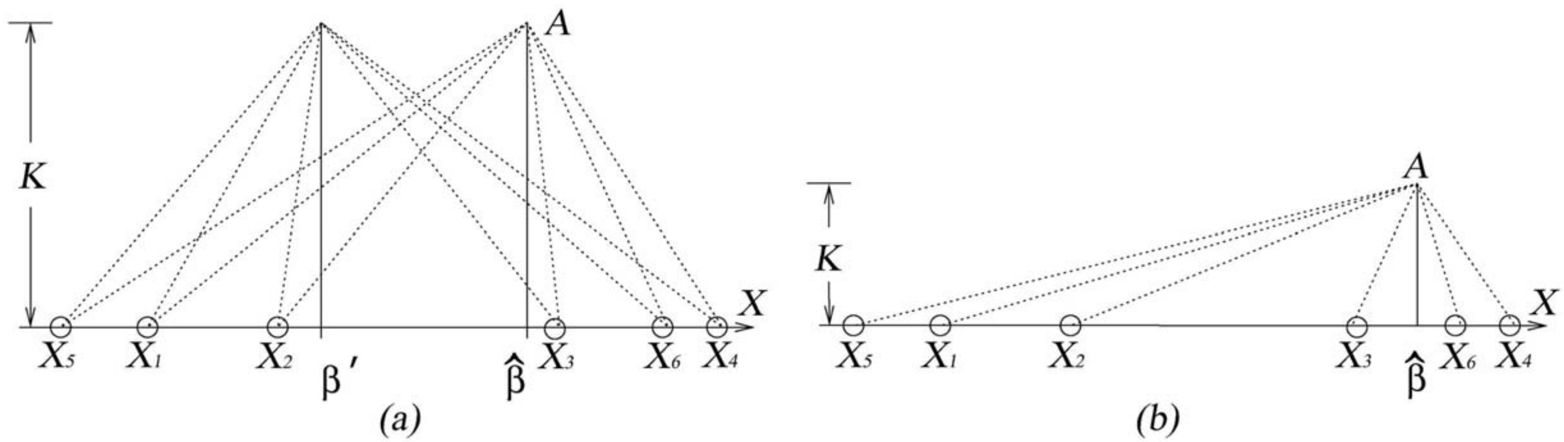

Figure 12: (*a*) The sample myriad, $\hat{\beta}$, minimizes the product of distances from point A to all samples. Any other value, such as $x = \beta'$, produces a higher product of distances; (*b*) the myriad as K is reduced.

EXAMPLE 8.5

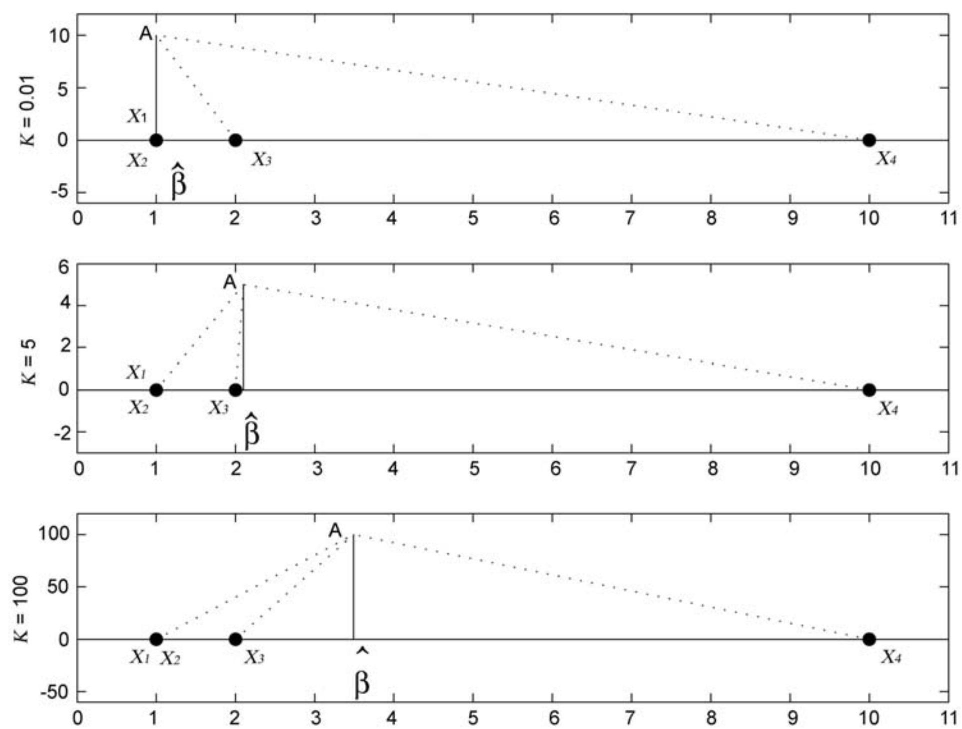

Figure 13: Sample myriad of the sample set $\{1,\ 1,\ 2,\ 10\}$ for (*a*) $K = 0.01$, (*b*) $K = 5$, (*c*) $K = 100$.

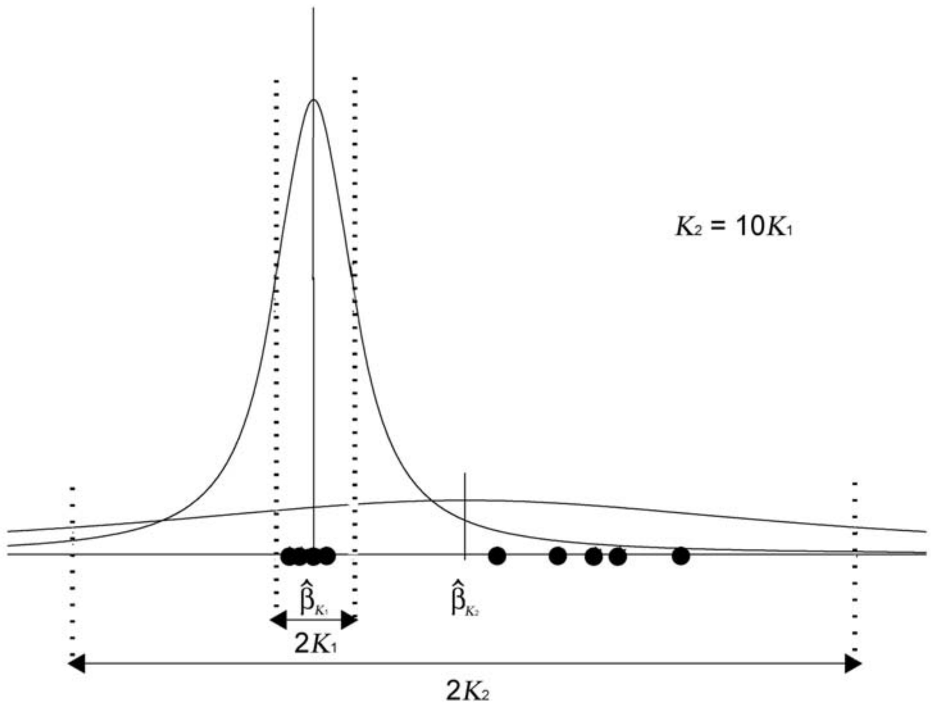

Figure 14: The role of the linearity parameter when the myriad is looked as a maximum likelihood estimator. When K is large, the generating density function is spread and the data are visualized as well-behaved (the optimal estimator is the sample average). For small values of K, the generating density becomes highly localized, and the data are visualized as very impulsive (the optimal estimator is a cluster locator).

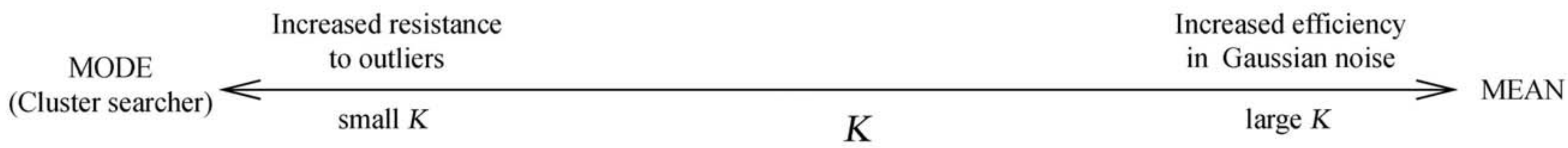

Figure 15: Functionality of the myriad as K is varied. Tuning the linearity parameter K adapts the behavior of the myriad from impulse-resistant mode-type estimators (small K) to the Gaussian-efficient sample mean (large K).

Empirical selection of K:

- Linear type $K \approx X_{(N)} - X_{(1)}$
- Mode type $K \approx \min_{i,j} |X_i - X_j|$

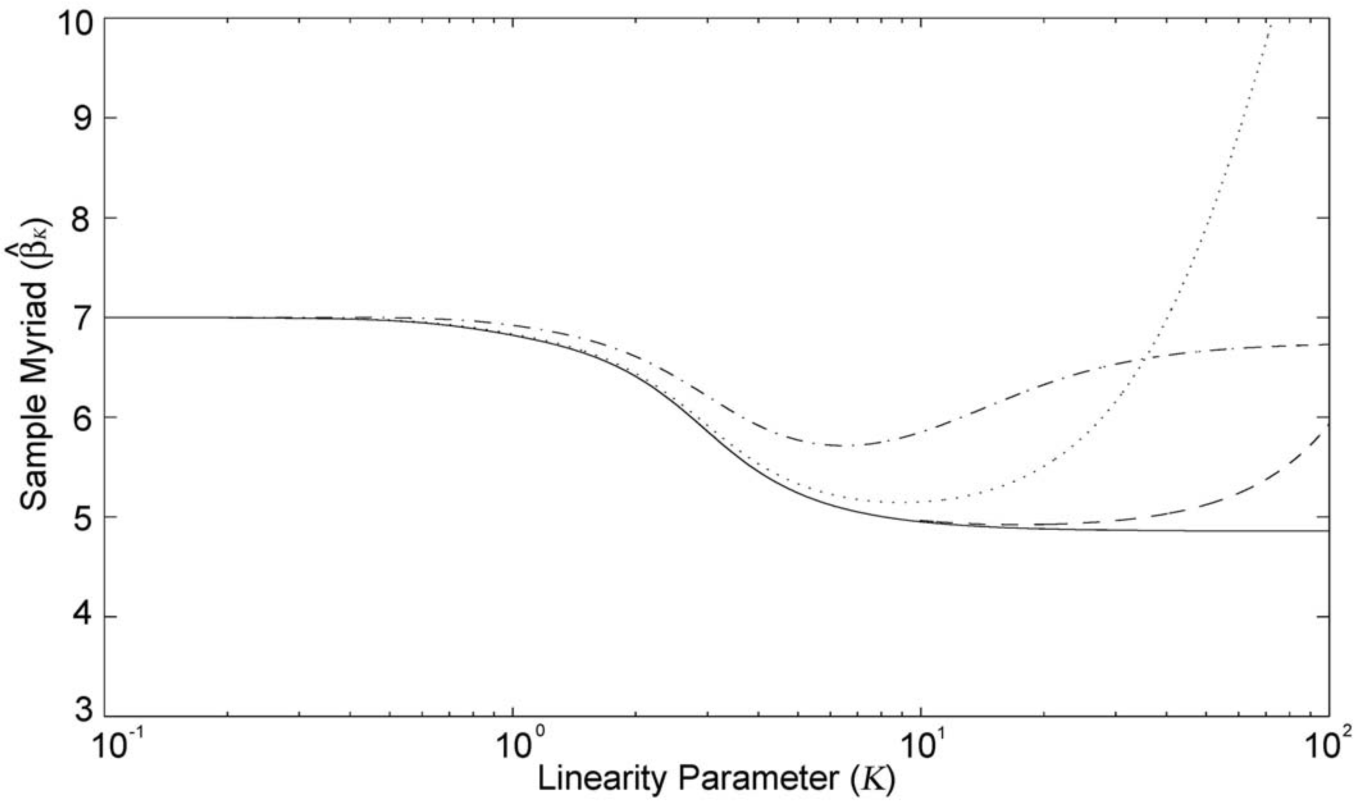

Figure 16: Values of the myriad as a function of K for the following data sets: (solid) original data set $= 0, 1, 3, 6, 7, 8, 9$; (dash-dot) original set plus an additional observation at 20; (dotted) additional observation at 100; (dashed) additional observations at 800, -500, and 700.

Property 8.3 (Scale Invariance) *Let $\hat{\beta}_K(\mathbf{X})$ denote the myriad of order K of the data in the vector $\mathbf{X}$. Then, for $c > 0$,*

$$\hat{\beta}_K(c\mathbf{X}) = c\hat{\beta}_{K/c}(\mathbf{X}). \tag{12}$$

Proof : Let $X_1, X_2, \ldots, X_N$ denote the data in $\mathbf{X}$. Then,

$$\begin{aligned} \hat{\beta}_K(c\mathbf{X}) &= \arg\min_{\beta} \prod_{i=1}^{N} \left[K^2 + (cX_i - \beta)^2\right] \\ &= \arg\min_{\beta} \prod_{i=1}^{N} \left[\left(\frac{K}{c}\right)^2 + \left(X_i - \frac{\beta}{c}\right)^2\right] \\ &= c\left(\arg\min_{\beta} \prod_{i=1}^{N} \left[\left(\frac{K}{c}\right)^2 + (X_i - \beta)^2\right]\right). \end{aligned} \tag{13}$$

■

8.3 Optimality of the Sample Myriad

Optimality In The α-Stable Model

Proposition 8.1 *Let $T_{\alpha,\gamma}(X_1, X_2, \ldots, X_N)$ denote the maximum likelihood location estimator derived from a symmetric α-stable distribution with characteristic exponent α and dispersion γ. Then,*

$$\lim_{\alpha \to 0} T_{\alpha,\gamma}(X_1, X_2, \ldots, X_N) = \text{MYRIAD}\left\{0; X_1, X_2, \ldots, X_N\right\}. \tag{14}$$

The α-stable triplet of optimality points satisfied by the myriad:

- $\alpha = 2 \leftrightarrow K = \infty$
- $\alpha = 1 \leftrightarrow K = \gamma$
- $\alpha = 0 \leftrightarrow K = 0$

Proposition 8.2 *Let* α *and* γ *denote the characteristic exponent and dispersion parameter of a symmetric* α*-stable distribution. Let* $K_o(\alpha, \gamma)$ *denote the optimal tuning value of* K *in the sense that* $\hat{\beta}_{K_o}$ *minimizes a given performance criterion (usually the variance) among the class of sample myriads with non negative linearity parameter. Then,*

$$K_o(\alpha, \gamma) = K_o(\alpha, 1)\gamma. \tag{15}$$

A simple empirical formula is

$$K(\alpha) = \sqrt{\frac{\alpha}{2 - \alpha}}, \tag{16}$$

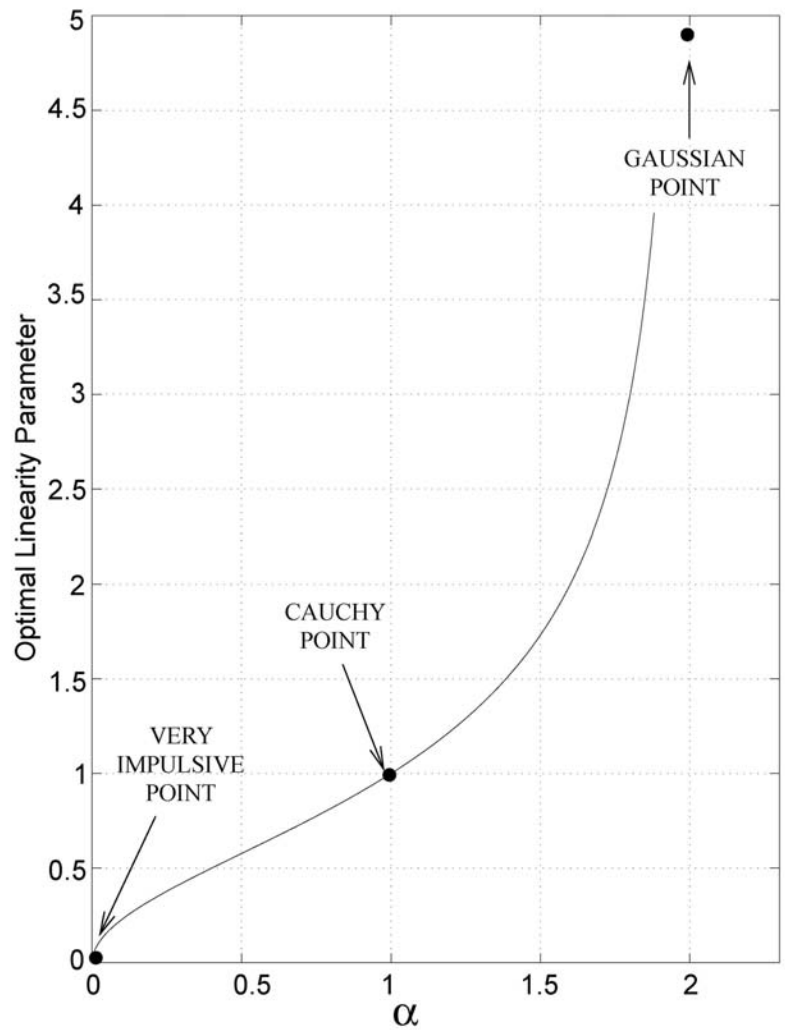

Figure 17: Empirical α-K curve for α-stable distributions. The curve values at $\alpha = 0, 1,$ and 2 constitute the optimality points of the α-stable triplet.

8.4 Weighted Myriad Smoothers

Given N observations $\{X_i\}_{i=1}^N$ and nonnegative weights $\{W_i \geq 0\}_{i=1}^N$, let the input and weight vectors be defined as $\mathbf{X} \stackrel{\triangle}{=} [X_1, X_2, \ldots, X_N]^T$ and $\mathbf{W} \stackrel{\triangle}{=} [W_1, W_2, \ldots, W_N]^T$, respectively. For a given *nominal* scale factor K, the underlying random variables are assumed to be independent and Cauchy distributed with a common location parameter β, but varying scale factors $\{S_i\}_{i=1}^N$: $X_i \sim$ Cauchy(β, S_i):

$$f_{X_i}(X_i; \beta, S_i) = \frac{1}{\pi} \frac{S_i}{S_i^2 + (X_i - \beta)^2}, \quad -\infty < X_i < \infty, \tag{17}$$

and where

$$S_i \stackrel{\triangle}{=} \frac{K}{\sqrt{W_i}} > 0, \; i = 1, 2, \ldots, N. \tag{18}$$

The weighted myriad smoother output $\hat{\beta}_K(\mathbf{W}, \mathbf{X})$ is defined as follows

$$\begin{aligned} \hat{\beta}_K(\mathbf{W}, \mathbf{X}) &= \arg\max_{\beta} \prod_{i=1}^{N} \frac{S_i}{S_i^2 + (X_i - \beta)^2}, \\ &= \arg\min_{\beta} \prod_{i=1}^{N} \left[K^2 + W_i (X_i - \beta)^2\right] \quad (19) \\ &\triangleq \arg\min_{\beta} P(\beta); \quad (20) \end{aligned}$$

Alternatively, we can write $\hat{\beta}_K(\mathbf{W}, \mathbf{X}) \triangleq \hat{\beta}_K$ as

$$\hat{\beta}_K = \arg\min_{\beta} Q(\beta) \triangleq \arg\min_{\beta} \sum_{i=1}^{N} \log\left[K^2 + W_i (X_i - \beta)^2\right]; \quad (21)$$

$\hat{\beta}_K$ is the global minimizer of $P(\beta)$ and $Q(\beta) \triangleq \log(P(\beta))$.

Definition 8.2 (Weighted myriad) *Let* $\mathbf{W} = [W_1, W_2, \ldots, W_N]$ *be a vector of nonnegative weights. Given* $K > 0$*, the weighted myriad of order* K *for the data* $X_1, X_2, \ldots, X_N$ *is defined as*

$$\hat{\beta}_K = \text{MYRIAD}\left\{K; W_1 \circ X_1, \ldots, W_N \circ X_N\right\}$$

$$= \arg\min_{\beta} \sum_{i=1}^{N} \log\left[K^2 + W_i(X_i - \beta)^2\right], \tag{22}$$

where $W_i \circ X_i$ *represents the weighting operation in (22). In some situations, the following equivalent expression can be computationally more convenient*

$$\hat{\beta}_K = \arg\min_{\beta} \prod_{i=1}^{N} \left[K^2 + W_i(X_i - \beta)^2\right]. \tag{23}$$

Note that the weighted myriad has only N independent parameters

$$\hat{\beta}_K(\mathbf{W}, \mathbf{X}) = \hat{\beta}_1\left(\frac{\mathbf{W}}{K^2}, \mathbf{X}\right) \tag{24}$$

Equivalently:

$$\hat{\beta}_{K_1}(\mathbf{W}_1, \mathbf{X}) = \hat{\beta}_{K_2}(\mathbf{W}_2, \mathbf{X}) \quad \text{iff} \quad \frac{\mathbf{W}_1}{K_1^2} = \frac{\mathbf{W}_2}{K_2^2}. \tag{25}$$

Hence, the output depends only on $\frac{\mathbf{W}}{K^2}$.

The objective function $P(\beta)$ is a polynomial in β of degree $2N$, with well-defined derivatives of all orders. Therefore, it can have at most $(2N-1)$ local extremes, one of which is the output:

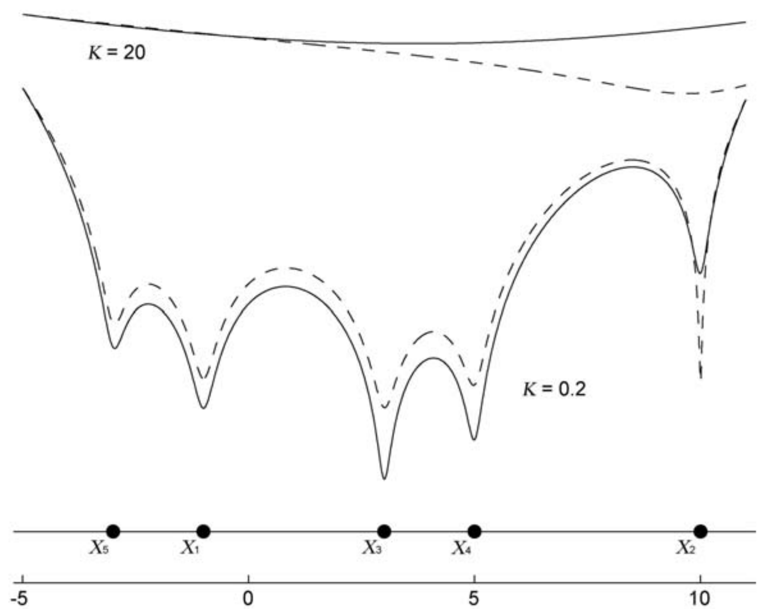

Figure 18: Sketch of a typical weighted myriad objective function $Q(\beta)$ for the weights $[1,\ 2,\ 3,\ 2,\ 1]$ (solid line), and $[1,\ 100,\ 3,\ 2,\ 1]$ (dashed line), and the sample set $[-1,\ 10,\ 3,\ 5,\ -3]$.

Property 8.4 (Linear Property) *In the limit as $K \to \infty$, the weighted myriad reduces to the normalized linear estimate*

$$\lim_{K \to \infty} \hat{\beta}_K = \frac{\sum_{i=1}^{N} W_i X_i}{\sum_{i=1}^{N} W_i}. \tag{26}$$

Property 8.5 (No undershoot/overshoot) *The output of a weighted myriad smoother is always bracketed by*

$$X_{(1)} \leq \hat{\beta}_K(\mathbf{W}; X_1, X_2, \ldots, X_N) \leq X_{(N)}, \tag{27}$$

where $X_{(1)}$ and $X_{(N)}$ denote the minimum and maximum samples in the input window.

Property 8.6 (Mode Property) *Given a vector of positive weights,* $\mathbf{W} = [W_1, \ldots, W_N]$*, the weighted mode-myriad* $\hat{\beta}_0$ *is always equal to one of the most repeated values in the sample. Furthermore,*

$$\hat{\beta}_0 = \arg\min_{X_j \in \mathcal{M}} \left(\frac{1}{W_j}\right)^{\frac{r}{2}} \prod_{i=1, X_i \neq X_j}^{N} |X_i - X_j|, \tag{28}$$

where $\mathcal{M}$ *is the set of most repeated values, and* r *is the number of times a member of* $\mathcal{M}$ *is repeated in the sample set.*

Property 8.7 (Outlier Rejection Property) *Let* $K < \infty$*, and let* $\mathbf{W}$ *denote a vector of positive and finite weights. The outlier rejection property states that:*

$$\lim_{X_N \to \pm\infty} \hat{\beta}_K(\mathbf{W}; X_1, X_2, \ldots, X_N) = \hat{\beta}_K(\mathbf{W}; X_1, X_2, \ldots, X_{N-1}). \tag{29}$$

Property 8.8 (Unbiasedness) *Let* $X_1, X_2, \ldots, X_N$ *be all independent and symmetrically distributed around the point of symmetry* c*. Then,* $\hat{\beta}_K$ *is also symmetrically distributed around* c*. In particular, if* $E\hat{\beta}_K$ *exists, then* $E\hat{\beta}_K = c$*.*

Geometrical Interpretation

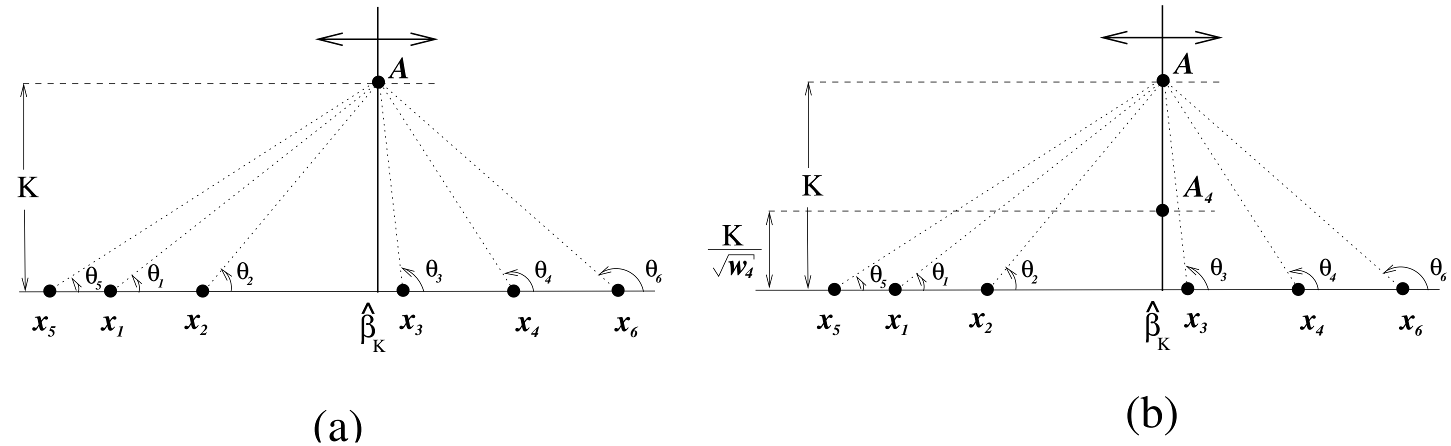

Figure 19: (*a*) The sample myriad, $\hat{\beta}_K$, indicates the position of a moving bar such that the product of distances from point A to the sample points $X_1, X_2, \ldots, X_N$ is minimum. (*b*) If the weight $W_4 > 1$ is introduced, the product of distances is more sensitive to the variations of the segment $\overline{X_4 A_4}$, very likely resulting in a weighted myriad $\hat{\beta}_K$ closer to X_4.

8.5 Fast Weighted Myriad Computation

- No explicit formulation is available to compute the weighted myriad.
- But, some characteristics of the objective function can be exploited.
- Turns out certain numerical iterative technique can be applied.

Recall that the weighted myriad is given by

$$\hat{\beta}_K = \arg\min_{\beta} \log(P(\beta)) \stackrel{\triangle}{=} \arg\min_{\beta} Q(\beta)$$
$$= \arg\min_{\beta} \sum_{i=1}^{N} \log\left[1 + \left(\frac{x_i - \beta}{S_i}\right)^2\right], \quad (30)$$

it's easi to show that

$$Q'(\beta) = 2 \sum_{i=1}^{N} \frac{W_i\,(\beta - X_i)}{K^2 + W_i\,(X_i - \beta)^2}. \quad (31)$$

$$Q'(\beta) \;=\; 2\sum_{i=1}^{N} \frac{\left(\dfrac{\beta - X_i}{S_i^2}\right)}{1 + \left(\dfrac{X_i - \beta}{S_i}\right)^2}. \tag{32}$$

Defining

$$\psi(v) \;\triangleq\; \frac{2v}{1+v^2}, \tag{33}$$

and referring to (32) the following equation is obtained for the local extremes of $Q(\beta)$:

$$Q'(\beta) \;=\; -\sum_{i=1}^{N} \frac{1}{S_i} \cdot \psi\left(\frac{X_i - \beta}{S_i}\right) \;=\; 0. \tag{34}$$

By introducing the *positive* functions

$$h_i(\beta) \;\triangleq\; \frac{1}{S_i^2} \cdot \varphi\left(\frac{X_i - \beta}{S_i}\right) \;>\; 0, \tag{35}$$

for $i = 1, 2, \ldots, N$, where

$$\varphi(v) \;\triangleq\; \frac{\psi(v)}{v} \;=\; \frac{2}{1 + v^2}, \tag{36}$$

the local extremes of $Q(\beta)$ in (34) can be formulated as

$$Q'(\beta) \;=\; -\sum_{i=1}^{N} h_i(\beta) \cdot (X_i - \beta) \;=\; 0. \tag{37}$$

Fixed Point Formulation

Equation (37) can be written as

$$\beta = \frac{\sum_{i=1}^{N} h_i(\beta) \cdot X_i}{\sum_{i=1}^{N} h_i(\beta)} \tag{38}$$

By defining the mapping

$$T(\beta) \triangleq \frac{\sum_{i=1}^{N} h_i(\beta) \cdot X_i}{\sum_{i=1}^{N} h_i(\beta)}, \tag{39}$$

the local extremes of $Q(\beta)$, or the roots of $Q'(\beta)$, are seen to be the *fixed points* of $T(\cdot)$:

$$\beta^* = T(\beta^*). \tag{40}$$

The following *fixed point iteration* results in an efficient algorithm to compute these fixed points:

$$\beta_{m+1} \triangleq T(\beta_m) = \frac{\displaystyle\sum_{i=1}^{N} h_i(\beta_m) \cdot X_i}{\displaystyle\sum_{i=1}^{N} h_i(\beta_m)}. \tag{41}$$

In the classical literature, this is also called the *method of successive approximation* for the solution of the equation $\beta = T(\beta)$.

It has been proven that the iterative method of (41) converges to a fixed point of $T(\cdot)$; thus,

$$\lim_{m\to\infty} \beta_m = \beta^* = T(\beta^*). \tag{42}$$

Fixed Point Weighted Myriad Search

Step 1: Select the initial point $\hat{\beta}_0$ among the values of the input samples:

$$\hat{\beta}_0 = \arg\min_{X_i} P(X_i).$$

Step 2: Using $\hat{\beta}_0$ as the initial value, perform L iterations of the fixed point recursion $\beta_{m+1} = T(\beta_m)$ of (41). The final value of these iterations is then chosen as the weighted myriad: $\hat{\beta}_{\mathsf{FP}} = T^{(L)}(\hat{\beta}_0)$.

This algorithm can be compactly written as

$$\hat{\beta}_{\mathsf{FP}} = T^{(L)}\left(\arg\min_{X_i} P(X_i)\right). \tag{43}$$

8.6 Weighted Myriad Smoother Design

8.6.1 Center-Weighted Myriads for Image Denoising

The notion of center weighting can be applied to the myriad structure, leading to:

$$Y = \mathrm{MYRIAD}\left\{K; X_1, \ldots, W_c \circ X_c, \ldots, X_N\right\}. \tag{44}$$

The cost function in (21) is now modified to

$$Q(\beta) = \log\left[K^2 \;+\; W_c(X_c - \beta)^2\right] + \sum_{X_i \neq X_c} \log\left[K^2 \;+\; (X_i - \beta)^2\right]. \tag{45}$$

- In addition to the center weight W_c, the CWMy has the free parameter (K) that controls the impulsiveness rejection.
- The center weight in the CWMy smoother is data dependent.
- For different applications, the center weight should be adjusted based on their data ranges.
- For grayscale image denoising (values normalized between 0 and 1), the parameters are:

  (1) Choose $K = (X_{(U)} + X_{(L)})/2$, where $1 \leq L < U \leq N$, with $X_{(U)}$ being the Uth smallest sample in the window and $X_{(L)}$ the Lth smallest sample.

  (2) Set $W_c = 10,000$.

- When there is “salt” noise in the window (outliers having large values), the myriad structure assures that they are deemphasized because of the outlier rejection property of K.
- For a single “pepper” outlier sample, the cost function (45) evaluated at $\beta = K$ will always be smaller than that at $\beta = 0$. Thus, “pepper” noise will never go through the smoother.

A 2-pass CWMy smoother can be defined as follows:

$$\mathbf{Y} = 1 - \mathrm{CWMy}(1 - \mathrm{CWMy}(\mathbf{X})). \qquad (46)$$

Figure 20: (*a*) Original image, (*b*) Image with 5% salt-and-pepper noise (PSNR=17.75dB)

Figure 21: ((*c*) smoothed with 5×5 center weighted median with $W_c = 15$(PSNR=37.48dB), (*d*) smoothed with 5×5 center weighted myriad with $W_c = 10,000$ and $K = (X_{(21)} + X_{(5)})/2$ (PSNR=39.98dB)

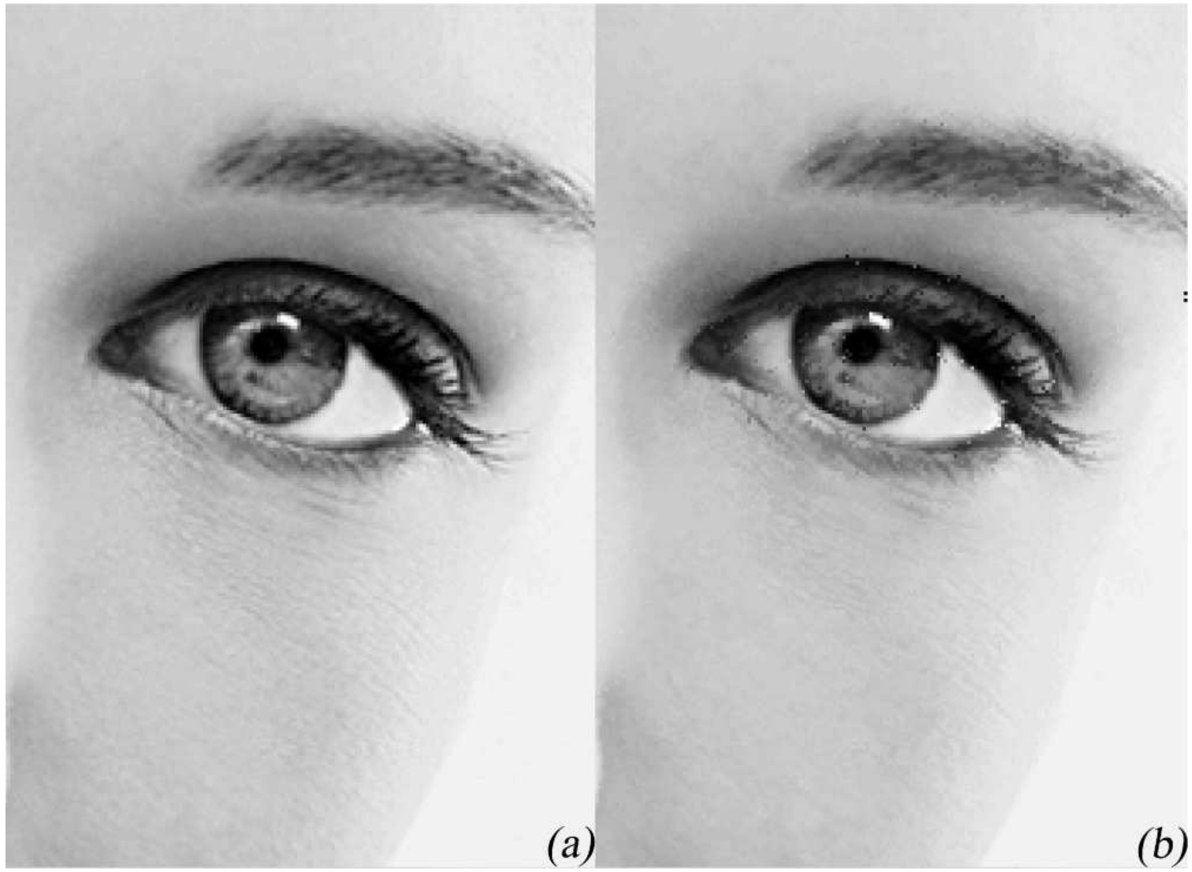

Figure 22: Comparison of different filtering schemes (Enlarged). (*a*) Original Image, (*b*) Image smoothed with a center weighted median (PSNR=37.48dB)

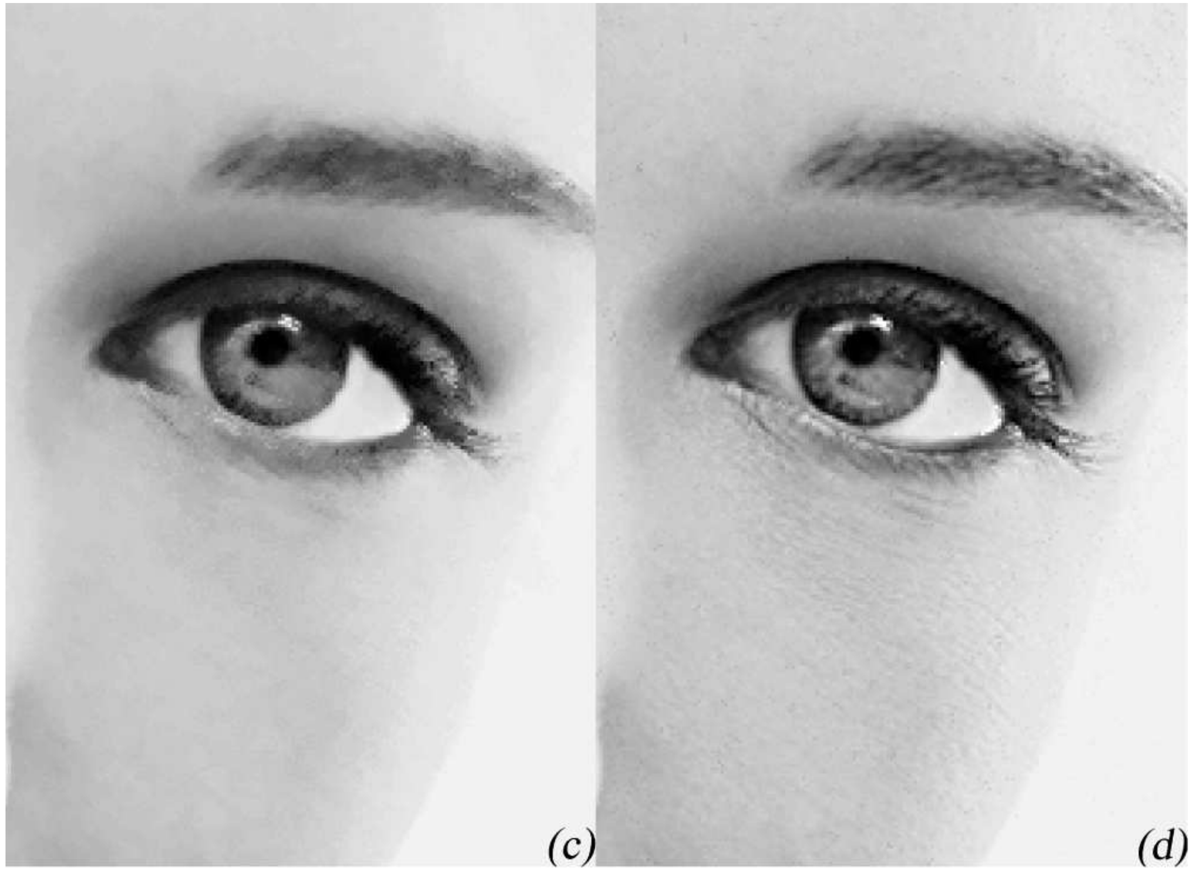

Figure 23: Comparison of different filtering schemes (Enlarged). (*c*) Image smoothed with a 5×5 permutation weighted median (PSNR=35.55dB), (*d*) Image smoothed with the center weighted myriad (PSNR=39.98dB).

Figure 24: Output of the Center weighted myriad smoother for different values of the center weight W_c (*a*) Original image, (*b*) 100 (PSNR=36.74dB)

Figure 25: Output of the Center weighted myriad smoother for different values of the center weight W_c (*c*) 10,000 (PSNR=39.98dB), (*d*) 1,000,000 (PSNR=38.15dB).

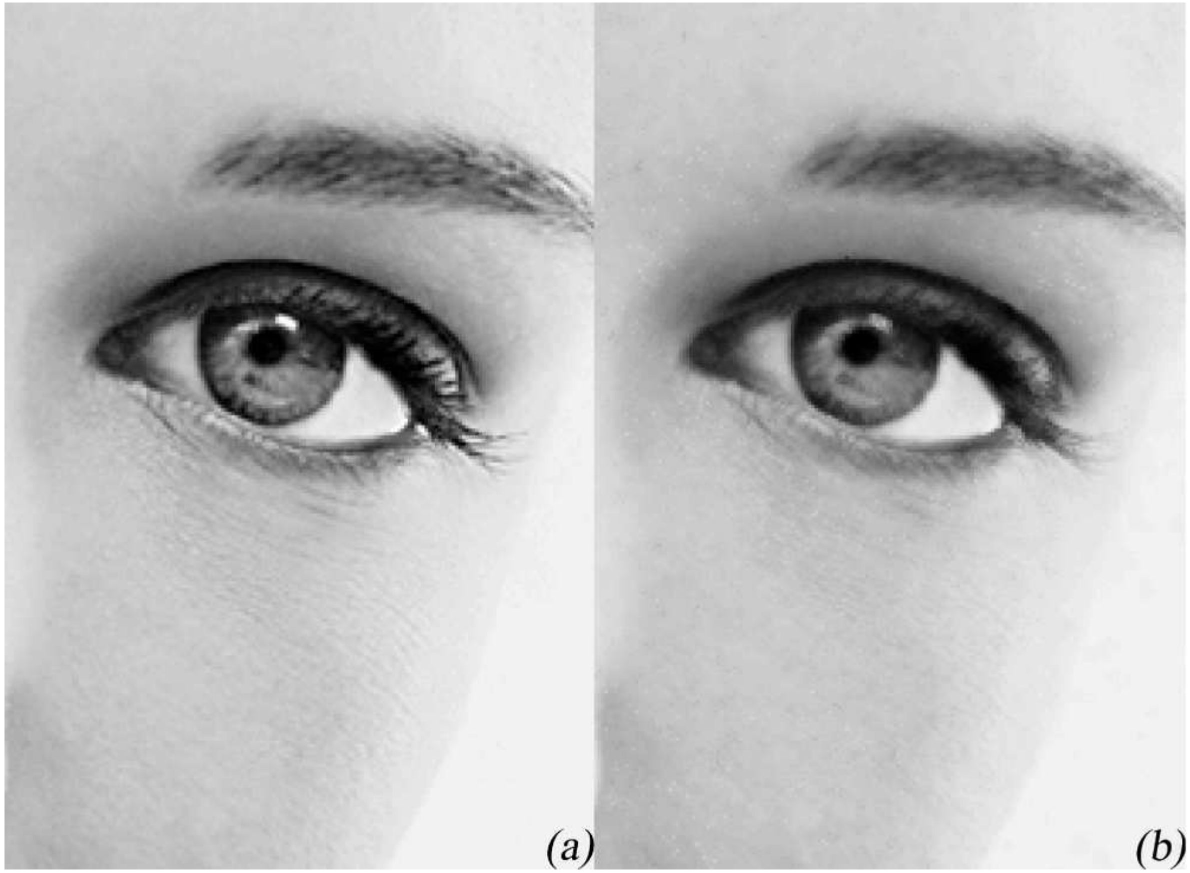

Figure 26: Output of the center-weighted myriad smoother for different values of the center weight W_c (enlarged) (*a*) Original image, (*b*) 100, (*c*) 10,000, (*d*) 1,000,000.

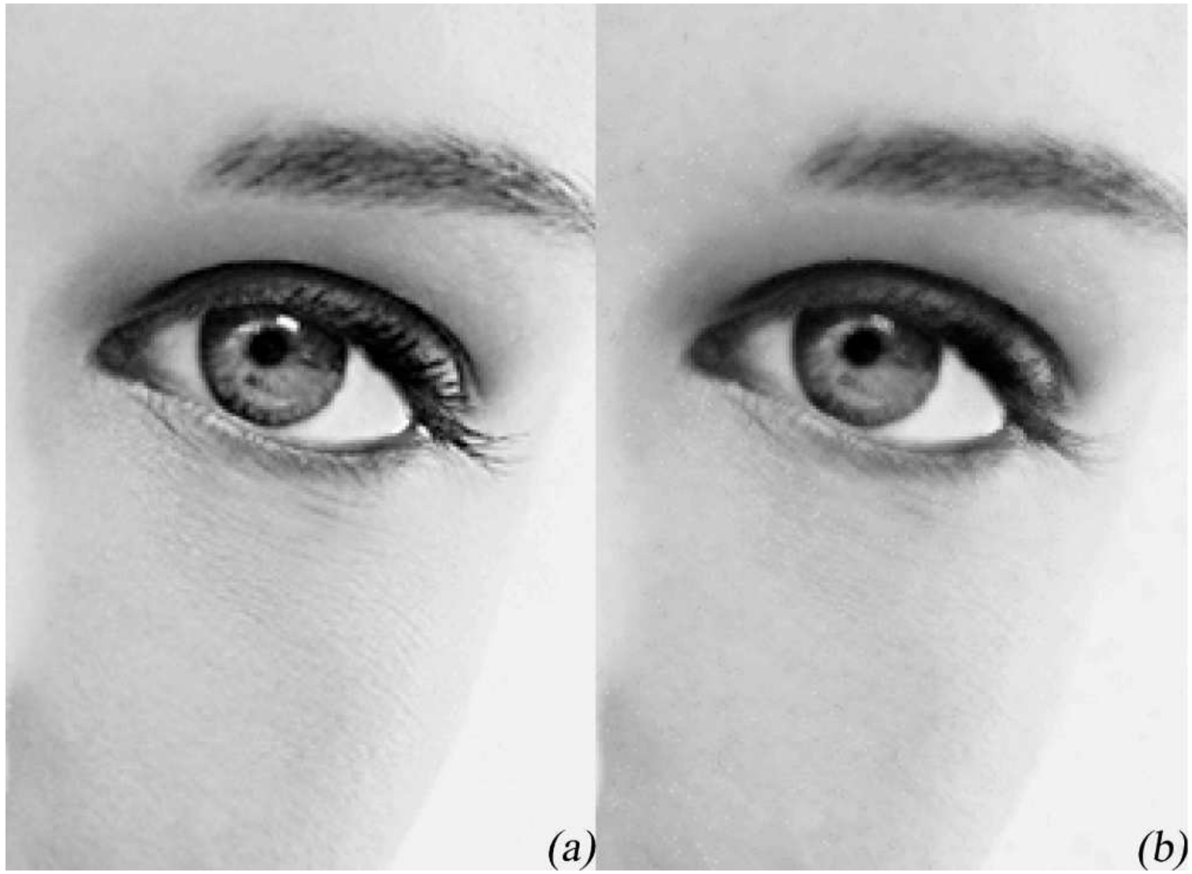

Figure 27: Output of the center-weighted myriad smoother for different values of the center weight W_c (enlarged) (*c*) 10,000, (*d*) 1,000,000.

8.6.2 Myriadization

- First, design a constrained linear smoother for Gaussian or noiseless environments using FIR filter (smoother) design techniques.
- Then, plug in these smoother coefficients into weighted myriad structure.
- Choose the suitable K according to the impulsiveness of the environment.
- Note that the smoother coefficients W_i must be non-negative and satisfy the normalization condition $\sum_{i=1}^{N} W_i = 1$

EXAMPLE 8.6 (ROBUST LOW PASS FILTER DESIGN)

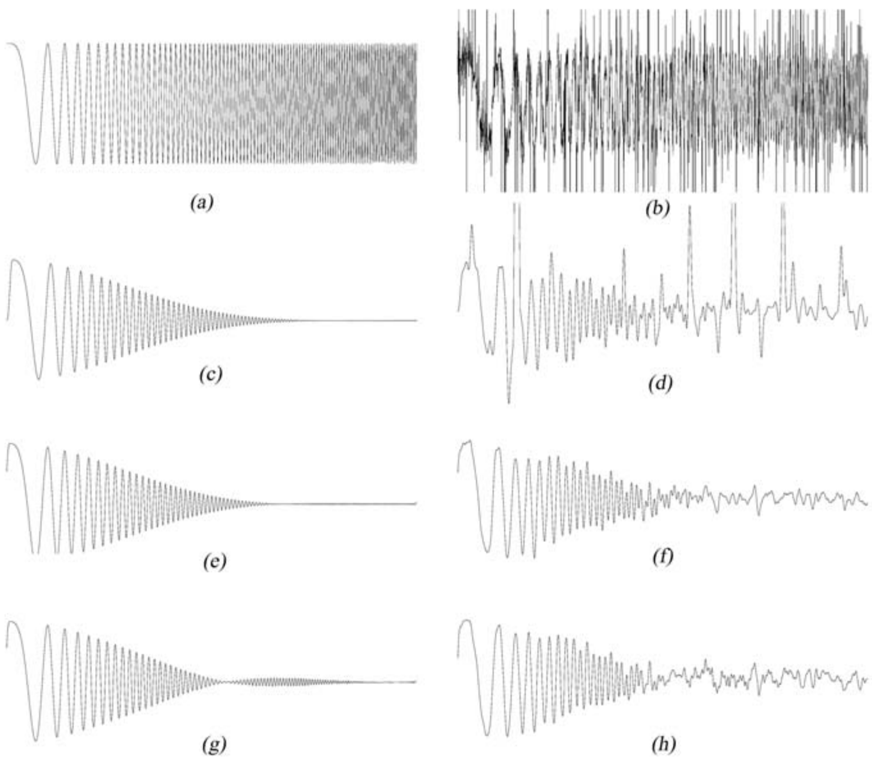

Figure 28: Myriadizing a linear low-pass smoother in an impulsive environment: (*a*) chirp signal, (*b*) chirp in additive impulsive noise, (*c*) ideal (no noise) myriad smoother output with $K = \infty$, (*e*) $K = 0.5$, and (*g*) $K = 0.2$; Myriad smoother output in the presence of noise with (*d*) $K = \infty$, (*f*) $K = 0.5$, and (*h*) $K = 0.2$.

EXAMPLE 8.7 (MYRIADIZATION OF PHASE LOCK LOOP FILTERS)

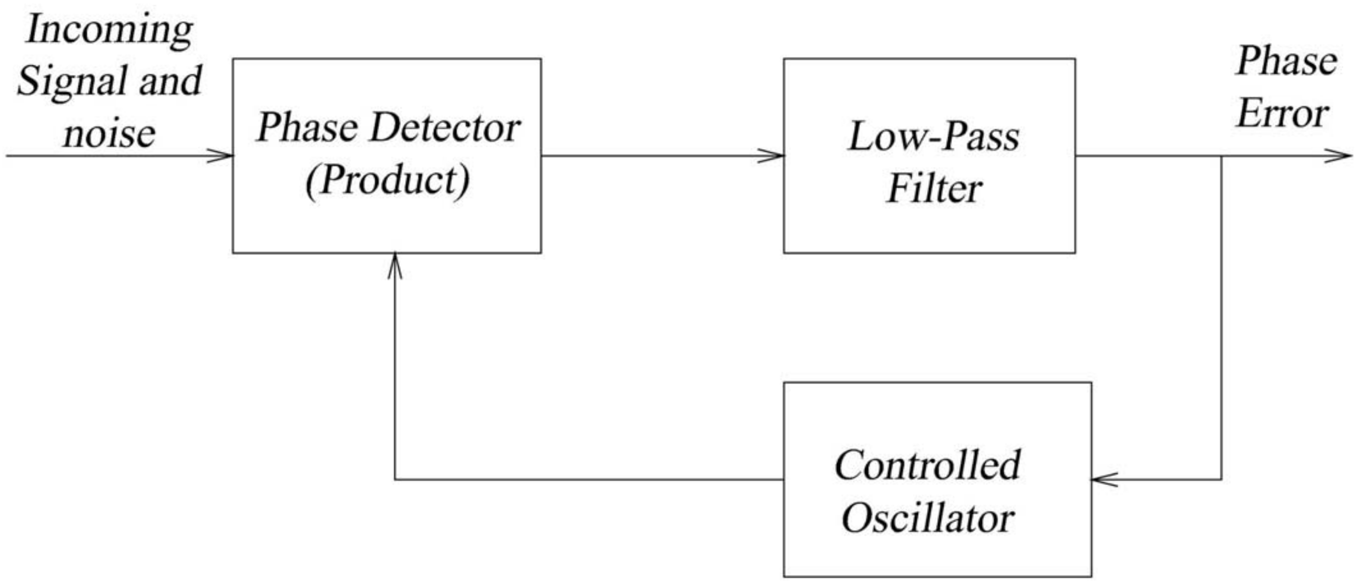

Figure 29: Block diagram of the Phase-Locked Loop system.

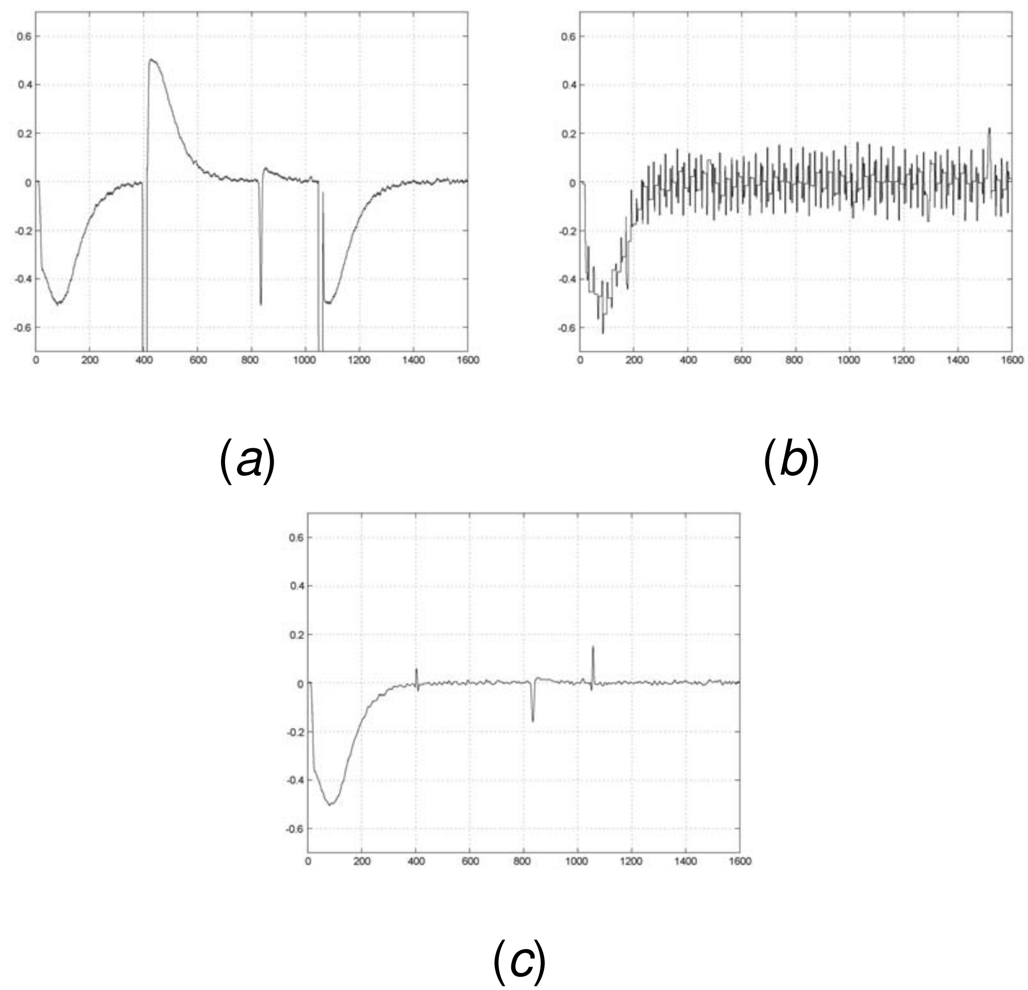

Figure 30: Phase error plot for the PLL with (*a*) a linear FIR filter; (*b*) an optimal weighted median filter; and (*c*) a *myriadized* version of the linear filter.