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Signals and Systems

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1.1 Introduction to Signals

A knowledge of a broad range of signals is of practical importance in describing human experience. In engineering systems, signals may carry information or energy. The signals with which we are concerned may be the cause of an event or the consequence of an action.

The characteristics of a signal may be of a broad range of shapes, amplitudes, time duration, and perhaps other physical properties. In many cases, the signal will be expressed in analytic form; in other cases, the signal may be given only in graphical form.

It is the purpose of this chapter to introduce the mathematical representation of signals, their properties, and some of their applications. These representations are in different formats depending on whether the signals are periodic or truncated, or whether they are deduced from graphical representations.

Signals may be classified as follows:

1. Phenomenological classification is based on the evolution type of signal, that is, a perfectly predictable evolution defines a deterministic signal and a signal with unpredictable behavior is called a **random signal**.
2. Energy classification separates signals into **energy signals**, those having finite energy, and **power signals**, those with a finite average power and infinite energy.
3. Morphological classification is based on whether signals are continuous, quantitized, sampled, or digital signals.
4. Dimensional classification is based on the number of independent variables.
5. Spectral classification is based on the shape of the frequency distribution of the signal spectrum.

1.1.1 Functions (Signals), Variables, and Point Sets

The **rule of correspondence** from a set S_x of real or complex number x to a real or complex number

$$y = f(x) \tag{1.1.1}$$

is called a function of the argument x. Equation (1.1.1) specifies a value (or values) y of the variable y (set of values in Y) corresponding to each suitable value of x in X. In (1.1.1) x is the **independent** variable and y is the **dependent** variable

A function of n variables $x_1, x_2, \ldots, x_n$ associates values

$$y = f(x_1, x_2, \ldots, x_n) \tag{1.1.2}$$

of a dependent variable y with ordered sets of values of the independent variables $x_1, x_2, \ldots, x_n$.

The set S_x of the values of x (or sets of values of $x_1, x_2, \ldots, x_n$) for which the relationships (1.1.1) and (1.1.2) are defined constitutes the **domain** of the function. The corresponding set of S_y of values of y is the S_x **range** of the function.

A **single-valued** function produces a single value of the dependent variable for each value of the argument. A **multiple-valued** function attains two or more values for each value of the argument.

The function $y(x)$ has an **inverse** function $x(y)$ if $y = y(x)$ implies $x = x(y)$.

A function $y = f(x)$ is **algebraic** of x if and only if x and y satisfy a relation of the form $F(x, y) = 0$, where $F(x, y)$ is a polynomial in x and y. The function $y = f(x)$ is **rational** if $f(x)$ is a polynomial or is a quotient of two polynomials.

A real or complex function $y = f(x)$ is bounded on a set S_x if and only if the corresponding set S_y of values y is bounded. Furthermore, a real function $y = f(x)$ has an **upper bound**, **least upper bound**, **lower bound**, **greatest lower bound**, **maximum**, or **minimum** on S_x if this is also true for the corresponding set S_y.

Neighborhood

Given any finite real number a, an open neighborhood of the point a is the set of all points $\{x\}$ such that $|x - a| < \delta$ for any positive real number δ.

An open neighborhood of the point $(a_1, a_2, \ldots, a_n)$, where all a_i are finite, is the set of all points $(x_1, x_2, \ldots, x_n)$ such that $|x_1 - a_1| < \delta$, $|x_2 - a_2| < \delta$, … and $|x_n - a_n| < \delta$ for some positive real number δ.

Open and Closed Sets

A point P is a **limit** point (accumulation point) of the point set S if and only if every neighborhood of P has a neighborhood contained entirely in S, other than P itself.

A limit point P is an interior point of S if and only if P has a neighborhood contained entirely in S. Otherwise P is a **boundary** point.

A point P is an isolated point of S if an only if P has a neighborhood in which P is the only point belonging to S.

A point set is **open** if and only if it contains only interior points.

A point set is **closed** if and only if it contains all its limit points; a finite set is closed.

1.1.2 Limits and Continuous Functions

1. A single-value function $f(x)$ has a **limit**

$$\lim_{x \to a} f(x) = L, \quad L = \text{finite}$$

as $x \to a$ $\{f(x) \to L$ as $x \to a\}$ if and only if for each positive real number ε there exists a real number δ such that $0 < |x - a| < \delta$ implies that $f(x)$ is defined and $|f(x) - L| < \varepsilon$.

2. A single-valued function $f(x)$ has a limit

$$\lim_{x\to\infty} f(x) = L, \quad L = \text{finite}$$

as $x \to \infty$ if and only if for each positive real number ε there exists a real number N such that $x > N$ implies that $f(x)$ is defined and $|f(x) - L| < \varepsilon$.

Operations with Limits

If limits exist, Table 1.2.1 gives the limit operations.

TABLE 1.2.1 Operations with Limits

$\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
$\lim_{x\to a}[b\,f(x)] = b\lim_{x\to a} f(x)$
$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x)\lim_{x\to a} g(x)$
$\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \left(\lim_{x\to a} g(x) \neq 0\right)$

a = may be finite or infinite

Asymptotic Relations Between Two Functions

Given two real or complex functions $f(x)$, $g(x)$ of a real or complex variable x, we write

1. $f(x) = O[g(x)]$; $f(x)$ is **of the order** $g(x)$ as $x \to a$ if and only if there is a neighborhood of $x = a$ such that $|f(x)/g(x)|$ is bounded.
2. $f(x) \sim g(x)$; $f(x)$ is **asymptotically proportional** to $g(x)$ as $x \to a$ if and only if $\lim_{x\to a}[f(x)/g(x)]$ exists and it is not zero.
3. $f(x) \cong g(x)$; $f(x)$ is **asymptotically equal** to $g(x)$ as $x \to a$ if and only if

$$\lim_{x\to a}\left[f(x)/g(x)\right] = 1.$$

4. $f(x) = o[g(x)]$; $f(x)$ becomes negligible compared with $g(x)$ if and only if

$$\lim_{x\to a}\left[f(x)/g(x)\right] = 0.$$

5. $f(x) = \varphi(x) + O[g(x)]$ if $f(x) - \varphi(x) = O[g(x)]$
$f(x) = \varphi(x) + o[g(x)]$ if $f(x) - \varphi(x) = o[g(x)]$

Uniform Convergence

1. A single-valued function $f(x_1, x_2)$ **converges uniformly** on a set S of values of x_2, $\lim_{x_1\to a} f(x_1, x_2) = L(x_2)$ if and only if for each positive real number ε there exists a real number δ such that $0 < |x_1 - a| < \delta$ implies that $f(x_1, x_2)$ is defined and $|f(x_1, x_2) - L(x_2)| < \varepsilon$ for all x_2 in S (δ is independent of x_2).
2. A single-valued function $f(x_1, x_2)$ **converges uniformly** on a set S of values of x_2, $\lim_{x_1\to\infty} f(x_1, x_2) = L(x_2)$ if and only if for each positive real number ε there exists a real number N such that for $x_1 > N$ implies that $f(x_1, x_2)$ is defined and $|f(x_1, x_2) - L(x_2)| < \varepsilon$ for all x_2 in S.

3. A **sequence** of functions $f_1(x)$, $f_2(x)$, … **converges uniformly** on a set S of values of x to a finite and unique function

$$\lim_{n\to\infty} f_n(x) = f(x)$$

if and only if for each positive real number ε there exists a real integer N such that for $n > N$ implies that $|f_n(x) - f(x| < \varepsilon$ for all n in S.

Continuous Functions

1. A single-valued function $f(x)$ defined in the neighborhood of $x = a$ is **continuous** at $x = a$ if and only if for every positive real number ε there exists a real number δ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$.
2. A function is **continuous on a series of points** (interval or region) if and only if it is continuous at each point of the set.
3. A real function continuous on a bounded closed interval $[a, b]$ is bounded on $[a, b]$ and assumes every value between and including its g.l.b. (greatest lower bound) and its l.u.b. (least upper bound) at least once on $[a, b]$.
4. A function $f(x)$ is **uniformly continuous** on a set S and only if for each positive real number ε there exists a real number δ such that $|x - X| < \delta$ implies $|f(x) - f(X)| < \varepsilon$ for all X in S.

If a function is continuous in a bounded closed interval $[a, b]$, it is uniformly continuous on $[a, b]$. If $f(x)$ and $g(x)$ are continuous at a point, so are the functions $f(x) + g(x)$ and $f(x)\,f(x)$.

Limits

1. A function $f(x)$ of a real variable x has the **right-hand** limit $\lim_{x\to a+} f(x) = f(a+) = L_+$ at $x = a$ if and only if for each positive real number ε there exists a real number δ such that $0 < x - a < \delta$ implies that $f(x)$ is defined and $|f(x) - L_+| < \varepsilon$.
2. A function $f(x)$ of a real variable x has the **left-hand** limit $\lim_{x\to a-} f(x) = f(a-) = L_-$ at $x = a$ if and only if for each positive real number ε there exists a real number δ such that $0 < a < \delta$ implies that $f(x)$ is defined and $|f(x) - L_-| < \varepsilon$.
3. If $\lim_{x\to a} f(x)$ exists, then $\lim_{x\to a+} f(x) = \lim_{x\to a-} f(x) = \lim_{x\to a} f(x)$. Consequently, $\lim_{x\to a-} f(x) = \lim_{x\to a+} f(x)$ implies the existence of $\lim_{x\to a} f(x)$.
4. The function $f(x)$ is **right continuous** at $x = a$ if $f(a+) = f(a)$.
5. The function $f(x)$ is **left continuous** at $x = a$ if $f(a-) = f(a)$.
6. A real function $f(x)$ has a **discontinuity of the first kind** at point $x = a$ if $f(a+)$ and $f(a-)$ exist. The greatest difference between two of these numbers $f(a)$, $f(a+)$, $f(a-)$ is the **saltus** of $f(x)$ at the discontinuity. The discontinuities of the first kind of $f(x)$ constitute a discrete and countable set.
7. A real function $f(x)$ is **piecewise continuous** in an interval I if and only if $f(x)$ is continuous throughout I except for a finite number of discontinuities of the first kind.

Monotonicity

1. A real function $f(x)$ of a real variable x is a **strongly monotonic** in the open interval (a, b) if $f(x)$ increases as x increases in (a, b) or if $f(x)$ decreases as x decreases in (a, b).
2. A function $f(x)$ is **weakly monotonic** in (a, b) if $f(x)$ does not decrease, or if $f(x)$ does not increase in (a, b). Analogous definitions apply to monotonic sequences.
3. A real function of a real variable x is of **bounded variation** in the interval (a, b) if and only if there exists a real number of M such that

$$\sum_{i=1}^{m} \left| f(x_i) - f(x_{i-1}) \right| < M \quad \text{for all partitions}$$

$$a = x_0 < x_1 < x_2 < \cdots < x_m = b$$

of the interval (a, b). If $f(x)$ and $g(x)$ are of bounded variation in (a, b), then $f(x) + g(x)$ and $f(x)g(x)$ are of bounded variation also. The function $f(x)$ is of bounded variation in every finite open interval where $f(x)$ is bounded and has a finite number of relative maxima and minima and discontinuities (Dirichlet conditions).

A function of bounded variation in (a, b) is bounded in (a, b) and its discontinuities are only of the first kind.

Table 1.2.2 presents some useful mathematical functions.

1.1.3 Energy and Power Signals

Energy Signals

If we consider any signal $f(t)$ as denoting a voltage that exists across a 1-ohm resistor, then

$$\frac{f^2(t)}{1} = f(t)\frac{f(t)}{1} = f(t)i(t) = \text{power } VA$$

Therefore, the integral

$$E = \int_a^b f^2(t)\,dt \quad \text{joule} \tag{1.3.1}$$

representing the energy dissipated in the resistor during the time interval (a, b). A signal is called **energy signal** if

$$\int_{-\infty}^{\infty} f^2(t)\,dt < \infty \tag{1.3.2}$$

Power Signals

Power signals are defined by the relation

$$0 \le \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f^2(t)\,dt < \infty \tag{1.3.3}$$

For complex-valued signals, we must introduce $|f(t)|^2$ instead of $f^2(t)$.

We may represent the energy in a finite interval in terms of the coefficients of the basis function φ_i; that is, we write the energy integral in the form

$$E = \int_a^b f^2(t)\,dt = \int_a^b f(t)\sum_{n=0}^{\infty} c_n\varphi_n(t)\,dt = \sum_{n=0}^{\infty} c_n \int_a^b f(t)\,\varphi_n(t)\,dt = \sum_{n=0}^{\infty} c_n^2 \left\|\varphi_n(t)\right\|^2 \tag{1.3.4}$$

where

$$\int_a^b f(t)\,\varphi_n(t)\,dt = c_n\int_a^b \varphi_n^2(t)\,dt = c_n\left\|\varphi_n(t)\right\|^2$$

Because the square of the norm $||\varphi_n(t)||^2$ is the energy associated with the nth orthogonal function, (1.3.4) shows that the energy of the signal is the sum of the energies of its individual orthogonal components

TABLE 1.2.2 Some Useful Mathematical Functions

1. Signum Function

$$\operatorname{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 1 \end{cases}$$

2. Step Function

$$u(t) = \frac{1}{2} + \frac{1}{2}\operatorname{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

3. Ramp Function

$$r(t) = \int_{-\infty}^{t} u(\tau)\,d\tau = tu(t)$$

4. Pulse Function

$$p_a(t) = u(t+a) - u(t-a) = \begin{cases} 1 & |t| < a \\ 0 & |t| > a \end{cases}$$

5. Triangular Pulse

$$\Lambda_a(t) = \begin{cases} 1 - \frac{|t|}{a} & |t| < a \\ 0 & |t| > a \end{cases}$$

6. Sinc Function

$$\sin c_a(t) = \frac{\sin at}{t}, \qquad -\infty < t < \infty$$

7. Gaussian Function

$$g_a(t) = e^{-at^2}, \qquad -\infty < t < \infty$$

8. Error Function

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-\tau^2}\,d\tau = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{n!(2n+1)}$$

Properties:

$$\operatorname{erf}(\infty) = 1, \operatorname{erf}(0) = 0,\ \operatorname{erf}(-t) = -\operatorname{erf}(t)$$

$$\operatorname{erfc}(t) = \text{complementary error function} = 1 - \operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_t^{\infty} e^{-\tau^2}\,d\tau$$

9. Exponential Function

$$f(t) = e^{-at}u(t), \qquad t \geq 0$$

10. Double Exponential

$$f(t) = e^{-a|t|}, \qquad -\infty < t < \infty$$

11. Lognormal Function

$$f(t) = \frac{1}{t} e^{-\ell n^2 t/2}, \qquad 0 < t < \infty$$

12. Rayleigh Function

$$f(t) = te^{-t^2/2}, \qquad 0 < t < \infty$$

weighted by c_n. Note that this is the Parseval theorem. This equation shows that the set $\{\varphi_n(t)\}$ forms an orthogonal (complete) set, and the signal energy can be calculated from this representation.

Example

(a) $\int_0^{\infty} u^2(t)dt = \int_0^{\infty} dt = \infty$; $\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} u^2(t)dt = \lim_{T\to\infty} \frac{1}{2T}\int_0^{T} dt = \lim_{T\to\infty} \frac{1}{2T}\left(t\Big|_0^T\right) = \frac{1}{2} < \infty.$
This implies that $u\ (t)$ is a power signal.

(b) The signal $e^{-at}\,u(t)$, $a > 0$ is an energy signal.

1.2 Distributions, Delta Function

1.2.1 Introduction

The **delta** function $\delta(t)$ often called the **impulse** or **Dirac delta** function, occupies a central place in signal analysis. Many physical phenomena such as point sources, point charges, concentrated loads on structures, and voltage or current sources, acting for very short times, can be modeled as delta functions.

Strictly speaking, delta functions are not functions in the accepted mathematical sense, and they cannot be treated with rigor within the framework of classical analysis. However, if distributions are introduced, then the concept of a delta function and operations on delta functions can be given a precise meaning.

1.2.2 Testing Functions

A **distribution** is a generalization of a function. Within the framework of distributions, any function encountered in applications, such as unit-step functions and pulses, may be differentiated as many times as we desire, and any convergent series of functions may be differentiated term by term.

A **testing function** $\varphi(t)$ is a real-valued function of the real variable that can be differentiated an arbitrary number of times, and which is identical to zero outside a finite interval.

Example
Testing function

$$\varphi(t, a) = \begin{cases} e^{-\frac{a^2}{a^2-t^2}} & |t| < a \\ 0 & |t| \geq a \end{cases} \tag{2.2.1}$$

Properties

1. If $f(t)$ can be differentiated arbitrarily often

$$\psi(t) = f(t)\,\varphi(t) = \text{testing function}$$

2. If $f(t)$ is zero outside a finite interval

$$\psi(t) = \int_{-\infty}^{\infty} f(\tau)\,\varphi(t-\tau)\,d\tau, \quad -\infty < t < \infty = \text{testing function}$$

3. A sequence of testing functions, $\{\varphi_n\}$ $1 \leq n < \infty$, converges to zero if all φ_n are identically zero outside some interval independent of n and each φ_n, as well as all of its derivatives, tends uniformly to zero.

Example:

$$\varphi_n(t) = \varphi\left(t + \frac{1}{n}\right) - \varphi(t)$$

4. Testing functions belong to a set D, where D is a linear vector space, and if $\varphi_1 \in D$ and $\varphi_2 \in D$, then $\varphi_1 + \varphi_2 \in D$ and $a\varphi_1 \in D$ for any number a.

1.2.3 Definition of Distributions

A **distribution** (or **generalized** function) $g(t)$ is a process of assigning to an arbitrary test function $\varphi(t)$ a **number** $N_g[\varphi(t)]$. A distribution is also a functional.

Example
An ordinary function $f(t)$ is a distribution if

$$\int_{-\infty}^{\infty} f(t)\,\varphi(t)dt = N_f\left[\varphi(t)\right] \tag{2.3.1}$$

exists for every test function $\varphi(t)$ in the set. For example, if $f(t) = u(t)$ then

$$\int_{-\infty}^{\infty} u(t)\,\varphi(t)dt = \int_0^{\infty}\varphi(t)\,dt \tag{2.3.2}$$

The function $u(t)$ is a distribution that assigns to $\varphi(t)$ a number equal to its area from zero to infinity.

Properties of Distributions

1. Linearity–Homogeneity

$$\int_{-\infty}^{\infty} g(t)\left[a_1\varphi_1(t) + a_2\varphi_2(t)\right]dt = a_1\int_{-\infty}^{\infty} g(t)\,\varphi_1(t)dt + a_2\int_{-\infty}^{\infty} g(t)\,\varphi_2(t)dt \tag{2.3.3}$$

   for all test functions and all numbers a_i.

2. Summation

$$\int_{-\infty}^{\infty}\left[g_1(t) + g_2(t)\right]\varphi(t)dt = \int_{-\infty}^{\infty} g_1(t)\,\varphi(t)dt + \int_{-\infty}^{\infty} g_2(t)\,\varphi(t)dt \tag{2.3.4}$$

3. Shifting

$$\int_{-\infty}^{\infty} g(t - t_0)\,\varphi(t)dt = \int_{-\infty}^{\infty} g(t)\varphi(t + t_0)dt \tag{2.3.5}$$

4. Scaling

$$\int_{-\infty}^{\infty} g(at)\varphi(t)dt = \frac{1}{|a|}\int_{-\infty}^{\infty} g(t)\varphi\left(\frac{t}{a}\right)dt \tag{2.3.6}$$

5. Even Distribution

$$\int_{-\infty}^{\infty} g(t)\varphi(t)dt = 0, \quad \varphi(t) = \text{odd} \tag{2.3.7}$$

6. Odd Distribution

$$\int_{-\infty}^{\infty} g(t)\varphi(t)dt = 0, \quad \varphi(t) = \text{even} \tag{2.3.8}$$

7. Derivative

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{dg(t)}{dt}\varphi(t)dt &= g(t)\varphi(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} g(t)\frac{d\varphi(t)}{dt}dt \\ &= -\int_{-\infty}^{\infty} g(t)\frac{d\varphi(t)}{dt}dt\end{aligned} \tag{2.3.9}$$

   where the integrated term is equal to zero in view of the properties of testing functions.

8. The nth Derivative

$$\int_{-\infty}^{\infty} \frac{d^n g(t)}{dt^n} \varphi(t)dt = (-1)^n \int_{-\infty}^{\infty} g(t) \frac{d^n \varphi(t)}{dt^n} dt \tag{2.3.10}$$

9. Product with Ordinary Function

$$\int_{-\infty}^{\infty} [g(t)f(t)]\varphi(t)dt = \int_{-\infty}^{\infty} g(t)[f(t)\varphi(t)]dt \tag{2.3.11}$$

provided that $f(t)\varphi(t)$ belongs to the set of test functions.

10. Convolution

$$\begin{aligned}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} g_1(\tau)g_2(t-\tau)d\tau\right]\varphi(t)dt \\ = \int_{-\infty}^{\infty} g_1(\tau)\left[\int_{-\infty}^{\infty} g_2(t-\tau)\varphi(t)dt\right]d\tau\end{aligned} \tag{2.3.12}$$

by formal change of the order of integration.

Definition

A sequence of distributions $\{g_n(t)\}_1^{\infty}$ is said to converge to the distribution $g(t)$ if

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(t)\varphi(t)dt = \int_{-\infty}^{\infty} g(t)\varphi(t)dt \tag{2.3.13}$$

for all φ belonging to the set of test functions.

11. Every distribution is the limit, in the sense of distributions, of a sequence of infinitely differentiable functions.
12. If $g_n(t) \to g(t)$ and $r_n(t) \to r(t)$ (r is a distribution), and the numbers $a_n \to a$, then

$$\frac{d}{dt} g_n(t) \to \frac{dg(t)}{dt}, \quad g_n(t) + r_n(t) \to g(t) + r(t), \quad a_n g_n(t) \to ag(t) \tag{2.3.14}$$

13. Any distribution $g(t)$ may be differentiated as many times as desired. That is, the derivative of any distribution always exists and it is a distribution.

1.2.4 The Delta Function

Properties

Based on the distribution properties, the properties of the delta function are given below.

1. The delta function is a distribution assigning to the function $\varphi(t)$ the number $\varphi(0)$; thus

$$\int_{-\infty}^{\infty} \delta(t)\varphi(t)dt = \varphi(0) \tag{2.4.1}$$

2. Shifted

$$\int_{-\infty}^{\infty} \delta(t-t_0)\varphi(t)dt = \varphi(t_0) \tag{2.4.2}$$

3. Scaled

$$\int_{-\infty}^{\infty} \delta(at)\varphi(t)dt = \frac{1}{|a|}\int_{-\infty}^{\infty} \delta(t)\,\varphi\left(\frac{t}{a}\right)dt = \frac{1}{|a|}\varphi(0)$$

From (2.4.1) we have the identity

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

and hence ($a = -1$)

$$\delta(-t) = \delta(t) = \text{even} \tag{2.4.3}$$

4. Multiplication by Continuous Function

$$\int_{-\infty}^{\infty} [\delta(t)f(t)]\varphi(t)dt = \int_{-\infty}^{\infty} \delta(t)[f(t)\varphi(t)]dt = f(0)\varphi(0)$$

If $f(t)$ is continuous at 0, then

$$f(t)\delta(t) = f(0)\delta(t) \tag{2.4.4}$$

and

$$t\delta(t) = 0 \tag{2.4.5}$$

5. Derivatives

$$\int_{-\infty}^{\infty} \frac{d\delta(t)}{dt}\varphi(t)dt = -\frac{d\varphi(0)}{dt}$$

$$\int_{-\infty}^{\infty} \frac{d\delta(t-t_0)}{dt}\varphi(t) = -\frac{d\varphi(t_0)}{dt} \tag{2.4.6}$$

$$\int_{-\infty}^{\infty} \frac{d^n\delta(t)}{dt^n}\varphi(t)dt = (-1)^n\frac{d^n\varphi(0)}{dt^n} \tag{2.4.7}$$

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{d\delta(t)}{dt} f(t)\varphi(t)dt &= -\int_{-\infty}^{\infty} \delta(t)\frac{d[f(t)\varphi(t)]}{dt}dt \\ &= -f(0)\frac{d\varphi(0)}{dt} - \frac{df(0)}{dt}\varphi(0)\end{aligned} \tag{2.4.8}$$

$$f(t)\frac{d\delta(t)}{dt} = -\frac{df(0)}{dt}\delta(t) + f(0)\frac{d\delta(t)}{dt} \tag{2.4.9}$$

$$t\frac{d\delta(t)}{dt} = -\delta(t) \tag{2.4.10}$$

Set $f(t) = \varphi(t) = 1$ in (2.4.8) to find the relation

$$\int_{-\infty}^{\infty} \frac{d\delta(t)}{dt} dt = 0 \quad \left[\frac{d\delta(t)}{dt} \text{ is an odd function} \right] \tag{2.4.11}$$

$$f(t)\frac{d^n\delta(t)}{dt^n} = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{d^k f(0)}{dt^k} \frac{d^{n-k}\delta(t)}{dt^{n-k}} \tag{2.4.12}$$

From

$$\int_{-\infty}^{\infty} \frac{du(t)}{dt}\varphi(t)dt = u(t)\varphi(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u(t)\frac{d\varphi(t)}{dt}dt$$

$$= -\int_{0}^{\infty} \frac{d\varphi(t)}{dt} dt = -\varphi(t)\Big|_{0}^{\infty} = \varphi(0)$$

and comparing with (2.4.1) we find that

$$\delta(t) = \frac{du(t)}{dt} \tag{2.4.13}$$

Therefore, the generalized derivatives of discontinuous function contain impulses. A_n is the jump at the discontinuity point $t = t_n$ of the expression $A_n\, \varphi(t - t_n)$. Also

$$\frac{d\delta(t)}{dt} = \frac{d^2u(t)}{dt^2} \quad \text{or} \quad u(t) + u(-t) = 1$$

Hence

$$\frac{du(-t)}{dt} = -\delta(t) \tag{2.4.14}$$

$$\delta(t - t_0) = \frac{du(t - t_0)}{dt} \tag{2.4.15}$$

If $r(t)$ has a finite or countably infinite number of zeros at t_n on the entire t axis and these points $r(t)$ have a continuous derivative $dr(t_n)/dt \neq 0$, then

$$\delta[r(t)] = \sum_n \frac{\delta(t - t_n)}{\left| \frac{dr(t_n)}{dt} \right|} \tag{2.4.16}$$

Hence, we obtain

$$\delta(t^2 - 1) = \frac{1}{2}\delta(t - 1) + \frac{1}{2}\delta(t + 1) \tag{2.4.17}$$

$$\delta(\sin t) = \sum_{n=-\infty}^{\infty} \delta(t - n\pi) \tag{2.4.18}$$

In addition, the following relation is also true:

$$\frac{d\delta[r(t)]}{dt} = \sum_{n} \frac{\dfrac{d\delta(t - t_n)}{dt}}{\dfrac{dr(t)}{dt}\left|\dfrac{dr(t_n)}{dt}\right|} \tag{2.4.19}$$

6. Integrals

$$\int_{-\infty}^{\infty} A\delta(t - t_0)dt = A \tag{2.4.20}$$

for all t_0

$$\begin{aligned} \delta(t - t_1) * \delta(t - t_2) &= \text{convolution} \\ &= \int_{-\infty}^{\infty} \delta(\tau - t_1)\delta(t - \tau - t_2)d\tau = \delta[t - (t_1 + t_2)] \end{aligned} \tag{2.4.21}$$

$$f(t) * \delta(t) = \int_{-\infty}^{\infty} f(t - \tau)\delta(\tau)d\tau = f(t - 0) = f(t) \tag{2.4.22}$$

Distributions as Generalized Limits

We can define a distribution as a generalized limit of a sequence $f_n(t)$ of ordinary function. If there exists a sequence $f_n(t)$ such that the limit

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(t)\varphi(t)dt \tag{2.4.23}$$

exists for every test function in the set, then the result is a number depending on $\varphi(t)$. Hence, we may define a distribution $g(t)$ as

$$g(t) = \lim f_n(t) \tag{2.4.24}$$

and, therefore, equivalently

$$\delta(t) = \lim f_n(t) \tag{2.4.25}$$

Consider the two sequences shown in Figures 2.4.1a and 2.4.1b. The rectangular pulse sequence is given by

$$p_\varepsilon(t) = \frac{u(t) - u(t - \varepsilon)}{\varepsilon}$$

and has area unity whatever the value of ε. Because $\varphi(t)$ is continuous, it follows that

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} p_\varepsilon(t)\varphi(t)dt = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^{\varepsilon}\varphi(t)dt = \lim_{\varepsilon\to 0}\varphi(0)\frac{1}{\varepsilon}\int_0^{\varepsilon} dt = \varphi(0)$$

and therefore

$$\delta(t) = \lim_{\varepsilon\to 0} p_\varepsilon(t) \tag{2.4.26}$$

Similarly, from

$$\lim_{\varepsilon\to 0}\frac{1}{\sqrt{\varepsilon\pi}}\int_{-\infty}^{\infty} e^{-t^2/\varepsilon}\varphi(t)dt \cong \frac{\varphi(0)}{\sqrt{\varepsilon\pi}}\int_{-\infty}^{\infty} e^{-t^2/\varepsilon}dt = \varphi(0)$$

it follows that

$$\delta(t) = \lim_{\varepsilon\to 0}\frac{e^{-t^2/\varepsilon}}{\sqrt{\varepsilon\pi}} \tag{2.4.27}$$

If we use the sequence

$$\delta(t) = \lim_{\omega\to\infty}\frac{\sin\omega t}{\pi t}$$

we find that

$$\delta(t) = \lim_{a\to\infty}\frac{1}{2\pi}\int_{-a}^{a} e^{+j\omega t}d\omega = \lim_{a\to\infty}\frac{\sin at}{\pi t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{+j\omega t}d\omega \tag{2.4.28}$$

Also

$$\delta(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega(t-t_0)}d\omega \tag{2.4.29}$$

Further

$$\begin{aligned}\int_{-\infty}^{\infty}\cos\omega t\, d\omega &= \lim_{\Omega\to\infty}\int_{-\Omega}^{\Omega}\cos\omega t\, d\omega \\ &= \lim_{\Omega\to\infty}\frac{2\sin\Omega t}{t} \\ &= \lim_{\Omega\to\infty} 2\pi\frac{\sin\Omega t}{\pi t} = 2\pi\delta(t)\end{aligned} \tag{2.4.30}$$

Figure 2.4.1c shows the derivatives of the sequence (2.4.27). The following examples will elucidate some of the delta properties and the use of the delta function in Table 2.4.1.

Example
Equivalence of expressions involving the delta functions:

(a) $(\cos t + \sin t)\delta(t) = \delta(t)$

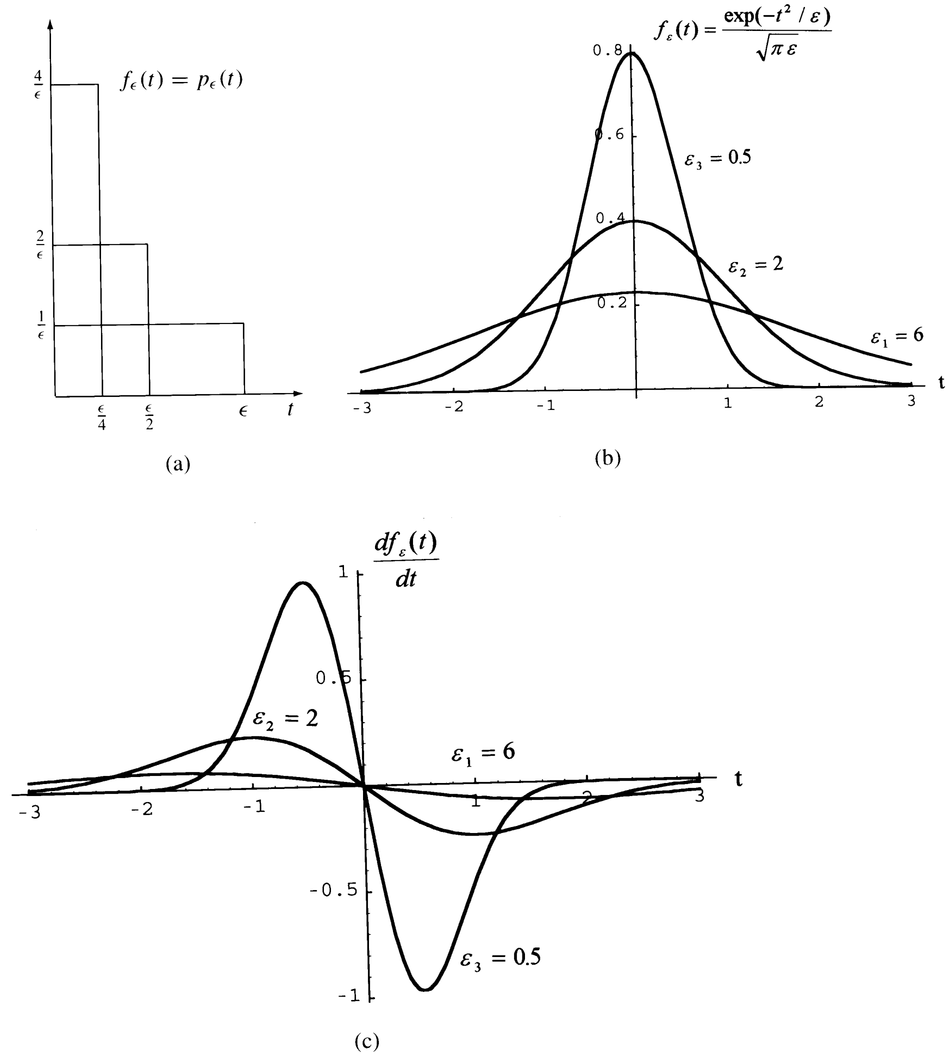


FIGURE 2.4.1

(b) $\cos 2t + \sin t\delta(t) = \cos 2t$
(c) $1 + 2e^{-t}\delta(t-1) = 1 + 2e^{-1}\delta(t-1)$

Example
The values of the following integrals are

$$\int_{-\infty}^{\infty}\left(t^2+4\,t+5\right)\delta\left(t\right)dt = 0^2+4\cdot 0+5 = 5, \quad \int_{-\infty}^{\infty}\frac{\left(1+\cos t\right)\delta\left(t\right)}{1+2e^t}dt = \frac{2}{1+2}$$

$$\int_{-\infty}^{\infty} t^2\sum_{k=1}^{n}\delta\left(t-k\right)dt = \sum_{k=1}^{n}k^2 = \frac{1}{6}\left[n\left(n+1\right)\left(2n+1\right)\right]$$

Example
The first derivative of the functions is

$$\frac{d}{dt}\left(2u\left(t+1\right)+u\left(1-t\right)\right) = \frac{d}{dt}\left(2u\left(t+1\right)+u\left[-\left(t-1\right)\right]\right) = 2\delta\left(t+1\right)-\delta\left(t-1\right)$$

TABLE 2.4.1 Delta Functional Properties

1. $\delta(at) = \dfrac{1}{|a|}\delta(t)$

2. $\delta\left(\dfrac{t-t_0}{a}\right) = |a|\delta(t-t_0)$

3. $\delta(at-t_0) = \dfrac{1}{|a|}\delta\left(t-\dfrac{t_0}{a}\right)$

4. $\delta(-t+t_0) = \delta(t-t_0)$

5. $\delta(-t) = \delta(t); \qquad \delta(t) = \text{even function}$

6. $\displaystyle\int_{-\infty}^{\infty} \delta(t) f(t)\, dt = f(0)$

7. $\displaystyle\int_{-\infty}^{\infty} \delta(t-t_0) f(t) = f(t_0)$

8. $f(t)\delta(t) = f(0)\delta(t)$

9. $f(t)\delta(t-t_0) = f(t_0)\delta(t-t_0)$

10. $t\delta(t) = 0$

11. $\displaystyle\int_{-\infty}^{\infty} A\delta(t)\, dt = \int_{-\infty}^{\infty} A\delta(t-t_0)\, dt = A$

12. $\displaystyle f(t) * \delta(t) = \text{convolution} = \int_{-\infty}^{\infty} f(t-\tau)\delta(\tau)\, d\tau = f(t)$

13. $\displaystyle\delta(t-t_1) * \delta(t-t_2) = \int_{-\infty}^{\infty} \delta(\tau-t_1)\delta(t-\tau-t_2)\, d\tau = \delta[t-(t_1+t_2)]$

14. $\displaystyle\sum_{n=-N}^{N} \delta(t-nT) * \sum_{n=-N}^{N} \delta(t-nT) = \sum_{n=-2N}^{2N} (2N+1-|n|)\delta(t-nT)$

15. $\displaystyle\int_{-\infty}^{\infty} \frac{d\delta(t)}{dt} f(t)\, dt = -\frac{df(0)}{dt}$

16. $\displaystyle\int_{-\infty}^{\infty} \frac{d\delta(t-t_0)}{dt} f(t)\, dt = -\frac{df(t_0)}{dt}$

17. $\displaystyle\int_{-\infty}^{\infty} \frac{d^n\delta(t)}{dt^n} f(t)\, dt = (-1)^n \frac{d^n f(0)}{dt^n}$

18. $\displaystyle f(t)\frac{d\delta(t)}{dt} = -\frac{df(0)}{dt}\delta(t) + f(0)\frac{d\delta(t)}{dt}$

19. $\displaystyle t\frac{d\delta(t)}{dt} = -\delta(t)$

20. $$t^n \frac{d^m\delta(t)}{dt^m} = \begin{cases} (-1)^n n!\delta(t), & m=n \\ (-1)^n \frac{m!}{m-n!}\frac{d^{m-n}\delta(t)}{dt^{m-n}}, & m>n \\ 0, & m<n \end{cases}$$

TABLE 2.4.1 Delta Functional Properties (Continued)

21. $\int_{-\infty}^{\infty} \frac{d\delta(t)}{dt} = 0, \qquad \frac{d\delta(t)}{dt} = \text{odd function}$

22. $f(t) * \frac{d\delta(t)}{dt} = \frac{df(t)}{dt}$

23. $f(t)\frac{d^n\delta(t)}{dt^n} = \sum_{k=0}^{n}(-1)^k \frac{n!}{k!(n-k)!}\frac{d^k f(0)}{dt^k}\frac{d^{n-k}\delta(t)}{dt^{n-k}}$

24. $\frac{\partial\delta(yt)}{\partial y} = -\frac{1}{y^2}\delta(t)$

25. $\delta(t) = \frac{du(t)}{dt}$

26. $\frac{d^n\delta(-t)}{dt^n} = (-1)^n\frac{d^n\delta(t)}{dt^n}, \left\{\frac{d^n\delta(t)}{dt^n} \text{ is even if } n \text{ is even, and odd if } n \text{ is odd.}\right\}$

27. $(\sin at)\frac{d\delta(t)}{dt} = -a\delta(t)$

28. $\frac{d\delta(t)}{dt} = \frac{d^2u(t)}{dt^2}$

29. $-\delta(t) = \frac{du(-t)}{dt}$

30. $\delta(t - t_0) = \frac{du(t - t_0)}{dt}$

31. $\frac{d\,\mathrm{sgn}(t)}{dt} = 2\delta(t)$

32. $\delta[r(t)] = \sum_n \frac{\delta(t - t_n)}{\left|\frac{dr(t_n)}{dt}\right|}, \qquad t_n = \text{zeros of } r(t), \quad \frac{dr(t_n)}{dt} \neq 0$

33. $\frac{d\delta[r(t)]}{dt} = \sum_n \frac{\frac{d\delta(t-t_n)}{dt}}{\frac{dr(t)}{dt}\left|\frac{dr(t_n)}{dt}\right|}, \qquad t_n = \text{zeros of } r(t), \quad \frac{dr(t_n)}{dt} \neq 0, \quad \frac{dr(t)}{dt} \neq 0$

34. $\delta(\sin t) = \sum_{n=-\infty}^{\infty} \delta(t - n\pi)$

35. $\delta(t^2 - 1) = \frac{1}{2}\delta(t - 1) + \frac{1}{2}\delta(t + 1)$

36. $\delta(t^2 - a^2) = \frac{1}{2a}[\delta(t + a) + \delta(t - a)]$

37. $\delta(t) = \lim_{\varepsilon\to 0} \frac{e^{-t^2/\varepsilon}}{\sqrt{\varepsilon\pi}}$

38. $\delta(t) = \lim_{\omega\to\infty} \frac{\sin \omega t}{\pi t}$

39. $\delta(t) = \lim_{\varepsilon\to 0} \frac{1}{\pi}\frac{\varepsilon}{t^2 + \varepsilon^2}$

40. $\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos \omega t \, d\omega$

TABLE 2.4.1 Delta Functional Properties (Continued)

41.	$\frac{df(t)}{dt} = \frac{d}{dt}[tu(t) - (t-1)u(t-1) - u(t-1)]$ $= t\delta(t) + u(t) - (t-1)\delta(t-1) - u(t-1) - \delta(t-1)$
42.	$\text{comb}_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT), \qquad f(t)\,\text{comb}_T(t) = \sum_{n=-\infty}^{\infty} f(nT)\delta(t - nT)$ $\text{COMB}_{\omega_0}(\omega) = \mathcal{F}\{\text{comb}_T(t)\} = \omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0), \qquad \omega_0 = \frac{2\pi}{T}$

$$\frac{d}{dt}\big([2-u(t)]\cos t\big) = \frac{d}{dt}\big(2\cos t - u(t)\cos t\big)$$
$$= -2\sin t - \delta(t)\cos t + u(t)\sin t$$
$$= \big(u(t)-2\big)\sin t - \delta(t)$$

$$\frac{d}{dt}\left(\left[u\left(t-\frac{\pi}{2}\right) - u(t-\pi)\right]\sin t\right) = \left[\delta\left(t-\frac{\pi}{2}\right) - \delta(t-\pi)\right]\sin t$$
$$+\left[u\left(t-\frac{\pi}{2}\right) - u(t-\pi)\right]\cos t$$
$$= \delta\left(t-\frac{\pi}{2}\right) + \left[u\left(t-\frac{\pi}{2}\right) - u(t-\pi)\right]\cos t$$

Example

The values of the following integrals are

$$\int_{-\infty}^{\infty} e^{2t}\sin 4t \frac{d^2\delta(t)}{dt^2}dt = (-1)^2 \frac{d^2}{dt^2}\left[e^{2t}\sin 4t\right]\Big|_{t=0} = 2\times 2\times 4 = 16$$

$$\int_{-\infty}^{\infty}(t^3+2t+3)\left(\frac{d\delta(t-1)}{dt} + 2\frac{d^2\delta(t-2)}{dt^2}\right)dt = \int_{-\infty}^{\infty}(t^3+2t+3)\frac{d\delta(t-1)}{dt}dt$$
$$+2\int_{-\infty}^{\infty}(t^3+2t+3)\frac{d^2\delta(t-2)}{dt^2}dt$$
$$= (-1)(3t^2+2)\Big|_{t=1} + (-1)^2 2(6t)\Big|_{t=2}$$
$$= -5 + 24 = 19$$

Example

The values of the following integrals are

$$\int_0^4 e^{4t}\delta(2t-3)dt = \int_0^4 e^{4t}\delta\left[2\left(t-\frac{3}{2}\right)\right]dt = \frac{1}{2}\int_0^4 e^{4t}\delta\left(t-\frac{3}{2}\right)dt = \frac{1}{2}e^{4\frac{3}{2}} = \frac{1}{2}e^6$$

$$\int_0^4 e^{4t}\delta(3-2t)dt = \int_0^4 e^{4t}\delta[-(2t-3)]dt = \int_0^4 e^{4t}\delta(2t-3)dt = \frac{1}{2}e^6$$

$$\int_{-\infty}^{\infty} e^{at}\delta(\sin t)dt = \int_{-\infty}^{\infty} e^{at}\sum_{n=-\infty}^{\infty}\frac{\delta(t-n\pi)}{(-1)^n}dt$$

$$= \sum_{n=-\infty}^{\infty}\frac{1}{(-1)^n}\int_{-\infty}^{\infty} e^{at}\delta(t-n\pi)dt$$

$$= \sum_{n=-\infty}^{\infty}\frac{1}{(-1)^n}e^{an\pi}$$

Example
The values of the following integrals are

$$\int_{-2\pi}^{2\pi} e^{at}\delta(t^2-\pi^2)dt = \int_{-2\pi}^{2\pi} e^{at}\frac{1}{2\pi}[\delta(t-\pi)+\delta(t+\pi)]dt$$

$$= \frac{1}{2\pi}\left[e^{a\pi}+e^{-a\pi}\right]$$

$$= \frac{\cosh a\pi}{\pi}$$

$$\int_{-\pi}^{\pi}\cosh\theta\delta(\cos\theta)d\theta = \int_{-\pi}^{\pi}\cosh\theta\left[\frac{\delta\left(\theta+\frac{\pi}{2}\right)}{\left|\sin\left(-\frac{\pi}{2}\right)\right|}+\frac{\delta\left(\theta-\frac{\pi}{2}\right)}{\left|\sin\frac{\pi}{2}\right|}\right]d\theta$$

$$= \cosh\left(-\frac{\pi}{2}\right)+\cosh\frac{\pi}{2}$$

$$= 2\cosh\frac{\pi}{2}$$

1.2.5 The Gamma and Beta Functions

The gamma function is defined by the formula

$$\Gamma(z) = \int_0^{\infty} e^{-t}t^{z-1}dt, \quad \mathrm{Re}\{z\} > 0 \tag{2.5.1}$$

We shall mainly concentrate on the positive values of z and we shall take the following relationship as the basic definition of the **gamma function**:

$$\Gamma(x) = \int_0^{\infty} e^{-t}t^{x-1}dt, \quad x > 0 \tag{2.5.2}$$

The gamma function converges for all positive values of x are shown in Figure 2.5.1.

The **incomplete gamma function** is given by

$$\gamma(x,\tau)=\int_0^{\tau} t^{x-1}e^{-t}dt, \quad x>0, \tau>0 \tag{2.5.3}$$

The **beta function** is a function of two arguments and is given by

$$B(x,y)=\int_0^{1} t^{x-1}(1-t)^{y-t}dt, \quad x>0, y>0 \tag{2.5.4}$$

The beta function is related to the gamma function as follows:

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{2.5.5}$$

Integral Expressions of $\Gamma(x)$

If we set $u = e^{-t}$ in (2.5.3), then $1/u = e^t$, $\log_e(1/u) = t$, $-(1/u)du = dt$, and $[\log_e(1/u)]^{x-1} = t^{x-1}$, for the limits $t = 0$ $u = 1$, and $t = \infty$ $u = 0$. Hence

$$\Gamma(x)=\int_0^{\infty} t^{x-1}e^{-t}dt=-\int_1^0\left[\log_e\left(\frac{1}{u}\right)\right]^{x-1}u\frac{1}{u}du=\int_0^1\left[\log_e\left(\frac{1}{u}\right)\right]^{x-1}du \tag{2.5.6}$$

Starting from the definitions and setting $t = m^2$ ($dt = 2m\ dm$) we obtain (limits are the same)

$$\Gamma(x)=\int_0^{\infty} t^{x-1}e^{-t}dt=\int_0^{\infty} m^{2(x-1)}e^{-m^2}2m\,dm=2\int_0^{\infty} m^{2x-1}e^{-m^2}dm \tag{2.5.7}$$

Properties and Specific Evaluations of $\Gamma(x)$

Setting $x + 1$ in place of x we obtain

$$\begin{aligned}\Gamma(x+1)&=\int_0^{\infty} t^{x+1-1}e^{-t}dt=\int_0^{\infty} t^{x}e^{-t}dt\\&=-\int_0^{\infty} t^{x}d\left(e^{-t}\right)=-t^{x}e^{-t}\Big|_0^{\infty}+\int_0^{\infty} xt^{x-1}e^{-t}dt\\&=x\Gamma(x)\end{aligned} \tag{2.5.8}$$

From the above relation we also obtain

$$\Gamma(x)=\frac{\Gamma(x+1)}{x} \tag{2.5.9}$$

$$\Gamma(x) = (x-1)\Gamma(x-1) \tag{2.5.10}$$

$$\Gamma(-x)=\frac{\Gamma(x-1)}{-x}, \quad x\neq 0,1,2,\ldots \tag{2.5.11}$$

From (2.5.2) with $x = 1$, we find that $\Gamma(1) = 1$. Using (2.5.8) we obtain

$$\begin{aligned}\Gamma(2) &= \Gamma(1+1) = 1\Gamma(1) = 1\cdot 1 = 1,\\ \Gamma(3) &= \Gamma(2+1) = 2\Gamma(2) = 2\cdot 1,\\ \Gamma(4) &= \Gamma(3+1) = 3\Gamma(3) = 3\cdot 2\cdot 1.\end{aligned}$$

Hence we obtain

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)! = n!, \qquad n = 0, 1, 2, \ldots \tag{2.5.12}$$

$$\Gamma(n) = (n-1)!, \qquad n = 1, 2, \ldots \tag{2.5.13}$$

To find $\Gamma\left(\frac{1}{2}\right)$ we first set $t = u^2$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-1/2} e^{-t}\,dt = \int_0^\infty 2e^{-u^2}\,du, \quad \left(t = u^2\right)$$

Hence its square value is

$$\begin{aligned}\Gamma^2\left(\frac{1}{2}\right) &= \left[\int_0^\infty 2e^{-x^2}\,dx\right]\left[\int_0^\infty 2e^{-y^2}\,dy\right]\\ &= 4\int_0^\infty \left[\int_0^\infty e^{-y^2}\,dy\right] e^{-x^2}\,dx = 4\int_0^{\pi/2}\left[\int_0^\infty e^{-r^2} r\,dr\right] d\theta\\ &= 4\frac{\pi}{2}\cdot\frac{1}{2} = \pi\end{aligned}$$

and thus

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{2.5.14}$$

Next let us find the expression for $\Gamma\left(n+\frac{1}{2}\right)$ for integer positive value of n. From (2.5.10) we obtain

$$\begin{aligned}\Gamma\left(n+\frac{1}{2}\right) &= \Gamma\left(\frac{2n+1}{2}\right) = \left(\frac{2n+1}{2}-1\right)\Gamma\left(\frac{2n+1}{2}-1\right) = \frac{2n-1}{2}\Gamma\left(\frac{2n-1}{2}\right)\\ &= \left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\Gamma\left(\frac{2n-3}{2}\right)\end{aligned}$$

If we proceed to apply (2.5.10), we finally obtain

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)(2n-3)(2n-5)\cdots(3)(1)\sqrt{\pi}}{2^n} \tag{2.5.15}$$

Similarly we obtain

$$\Gamma\left(n+\frac{3}{2}\right) = \frac{(2n+1)(2n-1)(2n-3)\cdots(3)(1)\sqrt{\pi}}{2^{n+1}} \tag{2.5.16}$$

$$\Gamma\left(n-\frac{1}{2}\right)=\frac{(2n-3)(2n-5)\cdots(3)(1)\sqrt{\pi}}{2^{n-1}} \tag{2.5.17}$$

Example

To find the ratio $\Gamma(x+n)/\Gamma(x-n)$ where n is a positive integer and $x-n \neq 0, -1, -2, \ldots$, we proceed as follows (see [2.5.10]):

$$\begin{aligned}\frac{\Gamma(x+n)}{\Gamma(x-n)}&=\frac{(x+n-1)\Gamma(x+n-1)}{\Gamma(x-n)}=\frac{(x+n-1)(x+n-2)\Gamma(x+n-2)}{\Gamma(x-n)}=\cdots\\&=\frac{(x+n-1)(x+n-2)(x+n-3)\cdots(x+n-2n)\Gamma(x+n-2n)}{\Gamma(x-n)}\\&=(x+n-1)(x+n-2)\cdots(x-n)\end{aligned} \tag{2.5.18}$$

Example

Applying (2.5.10) we find

$$\begin{aligned}2^n\Gamma(n+1)&=2^n n\Gamma(n)=2^n n(n-1)\Gamma(n-1)=\cdots=2^n n(n-1)(n-2)\cdots 2\cdot 1\\&=2^n n!=(2\cdot 1)(2\cdot 2)(2\cdot 3)\cdots(2\cdot n)=2\cdot 4\cdot 6\cdots 2n\end{aligned} \tag{2.5.19}$$

If $n-1$ is substituted in place of n, we obtain

$$2\cdot 4\cdot 6\cdots(2n-2)=2^{n-1}\,\Gamma(n) \tag{2.5.20}$$

Example

Based on the Legendre duplication formula

$$\frac{\Gamma(2n)}{\Gamma(n)}=\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,2^{1-2n}} \tag{2.5.21}$$

we can find the ratio $\Gamma\left(n+\frac{1}{2}\right)\;\left(\sqrt{\pi}\,\Gamma(n+1)\right)$ as follows:

$$\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(n+1)}=\frac{\Gamma(2n)2^{1-2n}}{\Gamma(n)\Gamma(n+1)}=\frac{\Gamma(2n)2^{1-2n}2^n}{\Gamma(n)2^n\Gamma(n+1)}=\frac{\Gamma(2n)2^{1-n}}{\Gamma(n)2\cdot 4\cdot 6\cdots 2n}$$

(see previous example). But

$$1\cdot 3\cdot 5\cdots(2n-1)=\frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots(2n-2)(2n-1)}{2\cdot 4\cdots(2n-2)}=\frac{\Gamma(2n)}{2^{n-1}\Gamma(n)} \tag{2.5.22}$$

and hence

$$\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(n+1)}=\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n} \tag{2.5.23}$$

Remarks on Gamma Function

1. The gamma function is continuous at every x except 0 and the negative integers.
2. The second derivative is positive for every $x > 0$, and this indicates that the curve $y = \Gamma(x)$ is concave upward for all $x > 0$.
3. $\Gamma(x) \to +\infty$ as $x \to 0+$ through positive values and as $x \to +\infty$.
4. $\Gamma(x)$ becomes, alternatively, negatively infinite and positively infinite at negative integers.
5. $\Gamma(x)$ attains a single minimum for $0 < x < \infty$ and is located between $x = 1$ and $x = 2$.

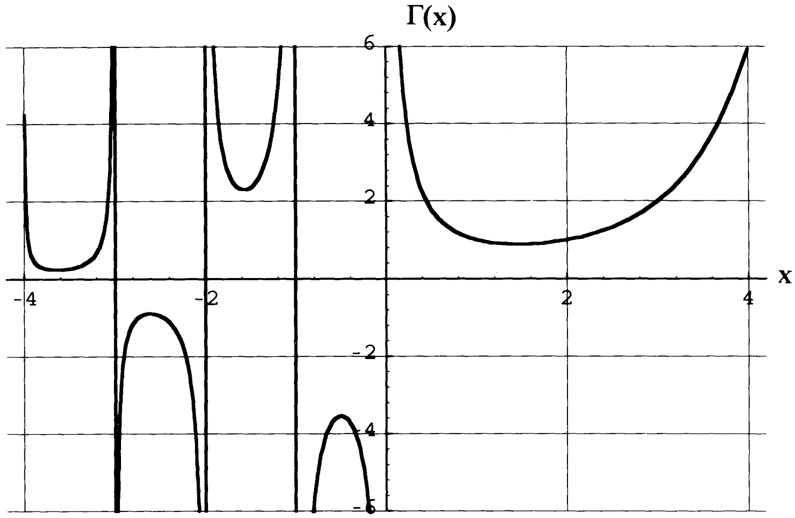


FIGURE 2.5.1 The gamma function.

The **beta function** is defined by

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt, \quad x > 0,\ y > 0 \tag{2.5.24}$$

From the above definition we write

$$\begin{aligned} B(y, x) &= \int_0^1 t^{y-1}(1-t)^{x-1}\,dt = -\int_1^0 (1-s)^{y-1} s^{x-1}\,ds = \int_0^1 s^{x-1}(1-s)^{y-1}\,ds \\ &= B(x, y) \end{aligned} \tag{2.5.25}$$

where we set $1 - t = s$.

If we set $t = \sin^2 \theta$, $dt = 2 \sin \theta \cos \theta d\theta$ and the limits of θ are 0 and $\pi/2$, then

$$B(x, y) = \int_0^{\pi/2} 2\sin^{2x-1}\theta \cos^{2y-1}\theta\, d\theta \tag{2.5.26}$$

The integral representation of the beta function is given by

$$B(x, y) = \int_0^{\infty} \frac{u^{x-1}du}{(u+1)^{x+y}}, \quad x > 0,\ y > 0 \tag{2.5.27}$$

Set $t = pt$ in (2.5.1) and find the relation

$$\int_0^\infty e^{-pt} t^{z-1} dt = \frac{\Gamma(z)}{p^z}, \quad \text{Re}\{p\} > 0 \tag{2.5.28}$$

Next set $p = 1 + u$ and $z = x + y$ in the above equation to find that

$$\frac{1}{(1+u)^{x+y}} = \frac{1}{\Gamma(x+y)} \int_0^\infty e^{-(1+u)t} t^{x+y-1} dt \tag{2.5.29}$$

Substituting (2.5.29) in (2.5.27), we obtain

$$\begin{aligned} B(x, y) &= \frac{1}{\Gamma(x+y)} \int_0^\infty e^{-t} t^{x+y-1} dt \int_0^\infty e^{-ut} u^{x-1} du \\ &= \frac{\Gamma(x)}{\Gamma(x+y)} \int_0^\infty e^{-t} t^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \end{aligned} \tag{2.5.30}$$

It can be shown that

$$B(p, 1-p) = \frac{\pi}{\sin p\pi}, \quad 0 < p < 1 \tag{2.5.31}$$

From the identities $\Gamma(x + 1) = x\Gamma(x)$, $\Gamma(-x) = \Gamma(1 - x)/(-x)$, $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$ together with (2.5.31), we obtain

$$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin p\pi}, \quad p \text{ is nonintegral} \tag{2.5.32}$$

Example

To show that

$$\int_0^\infty t^{n-1} e^{-(a+1)t} dt = \frac{\Gamma(n)}{(a+1)^n}, \quad n > 0, a > -1$$

we set $t = (a + 1)^{-1} y$. Hence

$$\begin{aligned} \int_0^\infty t^{n-1} e^{-(a+1)t} dt &= \int_0^\infty \left(\frac{y}{a+1}\right)^{n-1} e^{-y} \frac{dy}{a+1} = (a+1)^{-n} \int_0^\infty y^{n-1} e^{-y} dy \\ &= \frac{\Gamma(n)}{(a+1)^n} \end{aligned}$$

Example

To evaluate the integral $\int_0^\infty e^{-x^2} dx$, we write it in the form

$$\int_0^\infty x^0 e^{-x^2} dx$$

which, if compared with the integral in Table 2.5.1, we have the correspondence $a = 0$, $b = 1$, $c = 2$. Hence we obtain

$$\int_0^\infty e^{-x^2} dx = \frac{\Gamma\left(\frac{a+1}{c}\right)}{cb^{(a+1)/c}} = \frac{\Gamma\left(\frac{0+1}{2}\right)}{2 \cdot 1^{1/2}} = \frac{\sqrt{\pi}}{2}$$

1.3 Convolution and Correlation

1.3.1 Convolution

Convolution of functions, although a mathematical relation, is extremely important to engineers. If the impulse response of a system is known, that is, the response of the system to a delta function input, the output of the system is the convolution of the input and its impulse response. The convolution of two functions is given by

$$g(t) \doteq f(t) * h(t) = \int_{-\infty}^{\infty} f(\tau) h(t-\tau) d\tau \qquad (3.1.1)$$

Proof

Let $f(t)$ be written as a sum of elementary functions $f_i(t)$. The output $g(t)$ is also given by the sum of the outputs $g_i(t)$ due to each elementary function $f_i(t)$. Hence

$$f(t) = \sum_i f_i(t), \quad g(t) = \sum_i g_i(t) \qquad (3.1.2)$$

If $\Delta\tau$ is sufficiently small, the area of $f_i(t)$ equals $f(\tau_i)\,\Delta\tau$ (see Figure 3.1.1). Hence, the output is approximately $f(\tau_i)\,\Delta\tau\, h(t - \tau_i)$ because $f_i(t)$ is concentrated near the point τ_i. As $\Delta\tau \to 0$, we thus conclude that

$$\sum_i g_i(t) \cong \sum_i f(\tau_i) h(t-\tau_i) \Delta\tau \to \int_{-\infty}^{\infty} f(\tau) h(t-\tau) d\tau$$

For casual systems, the impulse response is

$$h(t) = 0, \qquad t < 0 \qquad (3.13)$$

and, therefore, the output of the system becomes

$$g(t) = \int_{-\infty}^{t} f(\tau) h(t-\tau) d\tau = \int_0^\infty f(t-\tau) h(\tau) d\tau \qquad (3.1.4)$$

If, also, $f(t) = 0$ for $t < 0$, then $g(t) = 0$ for $t < 0$; for $t > 0$ we obtain

$$g(t) = \int_0^t f(\tau) h(t-\tau) d\tau = \int_0^t f(t-\tau) h(\tau) d\tau \qquad (3.1.5)$$

TABLE 2.5.1 Gamma and Beta Function Relations

$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt$	$x > 0$
$\Gamma(x) = \int_0^\infty 2u^{2x-1} e^{-u^2}\, du$	$x > 0$
$\Gamma(x) = \int_0^1 \left[\log\left(\frac{1}{r}\right)\right]^{x-1} dr$	$x > 0$
$\Gamma(x) = \frac{\Gamma(x+1)}{x}$	$x \neq 0, -1, -2, \ldots$
$\Gamma(x) = (x-1)\Gamma(x-1)$	$x \neq 0, -1, -2, \ldots$
$\Gamma(-x) = \frac{\Gamma(1-x)}{-x}$	$x \neq 0, 1, 2, \ldots$
$\Gamma(n) = (n-1)!$	$n = 1, 2, 3, \ldots, \quad 0! = 1$
$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$	
$\Gamma\left(n + \frac{1}{2}\right) = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)\sqrt{\pi}}{2^n}$	$n = 1, 2, \ldots$
$\Gamma\left(n + \frac{3}{2}\right) = \frac{(2n+1)(2n-1)(2n-3)\cdots(3)(1)\sqrt{\pi}}{2^{n+1}}$	$n = 1, 2, \ldots$
$\Gamma\left(n - \frac{1}{2}\right) = \frac{(2n-3)(2n-5)\cdots(3)(1)\sqrt{\pi}}{2^{n-1}}$	$n = 1, 2, \ldots$
$\Gamma(n+1) = \frac{2 \cdot 4 \cdot 6 \cdots 2n}{2^n}$	$n = 1, 2, \ldots$
$\Gamma(2n) = 1 \cdot 3 \cdot 5 \cdots (2n-1)\Gamma(n) 2^{1-n}$	$n = 1, 2, \ldots$
$\frac{\Gamma(2n)}{\Gamma(n)} = \frac{\Gamma\left(n + \frac{1}{2}\right)}{\sqrt{\pi}\, 2^{1-2n}}$	$n = 1, 2, \ldots$
$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin x\pi}$	$x \neq 0, \pm 1, \pm 2, \ldots$
$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n} + h$	$n = 1, 2, \ldots, \quad 0 < \frac{h}{n!} < \frac{1}{12n}$
$\int_0^\infty t^a e^{-bt^c}\, dt = \frac{\Gamma\left(\frac{a+1}{c}\right)}{c b^{(a+1)/c}}$	$a > -1,\ b > 0,\ c > 0$
$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt$	$x > 0,\ y > 0$
$B(x, y) = \int_0^{\pi/2} 2 \sin^{2x-1}\theta \cos^{2y-1}\theta\, d\theta$	$x > 0,\ y > 0$
$B(x, y) = \int_0^\infty \frac{u^{x-1}}{(u+1)^{x+y}}\, du$	$x > 0,\ y > 0$
$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$	

The convolution does not exist for all functions. The sufficient conditions are

TABLE 2.5.1 Gamma and Beta Function Relations (Continued)

$B(x, y) = B(y, x)$	
$B(x, 1-x) = \dfrac{\pi}{\sin x\pi}$	$0 < x < 1$
$B(x, y) = B(x+1, y) + B(x, y+1)$	$x > 0,\ y > 0$
$B(x, n+1) = \dfrac{1 \cdot 2 \cdots n}{x(x+1)\cdots(x+n)}$	$x > 0$

TABLE 2.5.2 $\Gamma(x)$, $1 \le x \le 1.99$

x	0	1	2	3	4	5	6	7	8	9
1.0	1.0000	.9943	.9888	.9835	.9784	.9735	.9687	.9642	.9597	.9555
.1	.9514	.9474	.9436	.9399	.9364	.9330	.9298	.9267	.9237	.9209
.2	.9182	.9156	.9131	.9108	.9085	.9064	.9044	.9025	.9007	.8990
.3	.8975	.8960	.8946	.8934	.8922	.8912	.8902	.8893	.8885	.8879
.4	.8873	.8868	.8864	.8860	.8858	.8857	.8856	.8856	.8857	.8859
.5	.8862	.8866	.8870	.8876	.8882	.8889	.8896	.8905	.8914	.8924
.6	.8935	.8947	.8959	.8972	.8986	.9001	.9017	.9033	.9050	.9068
.7	.9086	.9106	.9126	.9147	.9168	.9191	.9214	.9238	.9262	.9288
.8	.9314	.9341	.9368	.9397	.9426	.9456	.9487	.9518	.9551	.9584
.9	.9618	.9652	.9688	.9724	.9761	.9799	.9837	.9877	.9917	.9958

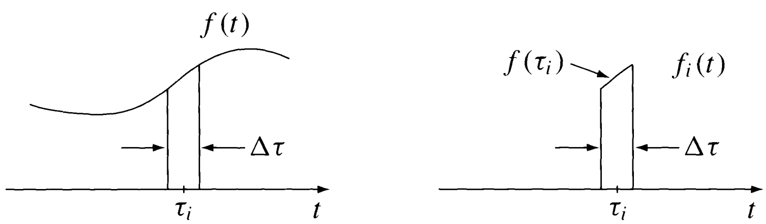


FIGURE 3.1.1

1. Both $f(t)$ and $h(t)$ must be absolutely integrable in the interval $(-\infty, 0]$.
2. Both $f(t)$ and $h(t)$ must be absolutely integrable in the interval $[0, \infty)$.
3. Either $f(t)$ or $h(t)$ (or both) must be absolutely integrable in the interval $(-\infty, \infty)$.

For example, the convolution $\cos \omega_0 t * \cos \omega_0 t$ does not exist.

Example
If the functions to be convoluted are

$$f(t) = 1, \quad 0 < t < 1, \qquad h(t) = e^{-t}u(t)$$

then the output is given by

$$g(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$$

The ranges are

1. $-\infty < t < 0$. No overlap of $f(t)$ and $h(t)$ takes place. Hence, $g(t) = 0$.
2. $0 < t < 1$. Overlap occurs from 0 to t. Hence

$$g(t) = \int_0^t 1 \cdot e^{-(t-\tau)} d\tau = e^{-t} \int_0^t e^{\tau} d\tau = 1 - e^{-t}$$

3. $1 < t < \infty$. Overlap occurs from 0 to 1. Hence

$$g(t) = \int_0^1 e^{-(t-\tau)} d\tau = e^{-t}(e-1)$$

Definition: Convolution Systems

The convolution of any continuous and discrete system is given respectively by

$$y(t) = \int_{-\infty}^{\infty} h(t, \tau) x(\tau) d\tau \qquad (3.1.6)$$

$$y(n) = \sum_{m=-\infty}^{\infty} h(n, m) x(m) \qquad (3.1.7)$$

If the systems are time invariant, the kernels $h(\cdot)$ are functions of the difference of their argument. Hence

$$h(n, m) = h(n - m), \qquad h(t, \tau) = h(t - \tau)$$

and therefore

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau \qquad (3.1.8)$$

$$y(n) = \sum_{m=-\infty}^{\infty} x(m) h(n - m) \qquad (3.1.9)$$

Definition: Impulse Response

The impulse response $h(t)$ of a system is the result of a delta function input to the system. Its value at t is the response to a delta function at $t = 0$.

Example

The voltage $\upsilon_c(t)$ across the capacitor of an RC circuit in series with an input voltage source $\upsilon(t)$ is given by

$$\frac{d\upsilon_c(t)}{dt} + \frac{1}{RC}\upsilon_c(t) = \frac{1}{RC}\upsilon(t)$$

For a given initial condition $\upsilon_c(t_0)$ at time $t = t_0$ the solution is

$$\upsilon_c(t) = e^{-(t-t_0)/RC}\upsilon_c(t_0) + \frac{1}{RC}\int_{t_0}^{t} e^{-(t-\tau)/RC}\upsilon(\tau) d\tau, \quad t \ge t_0$$

For a finite initial condition and $t_0 \to -\infty$, the above equation is written in the form

$$\upsilon_c(t) = \frac{1}{RC}\int_{-\infty}^{\infty} e^{-(t-\tau)/RC} u(t-\tau)\upsilon(\tau)d\tau = \left(\frac{1}{RC}e^{-t/RC}\right) * \upsilon(t)$$

Therefore, the impulse response of this system is

$$h(t) = \frac{1}{RC}e^{-t/RC}u(t)$$

Example
A discrete system that smooths the input signal $x(n)$ is described by the difference equation

$$y(n) = ay(n-1) + (1-a)x(n), \qquad n = 0, 1, 2, \ldots$$

By repeated substitution and assuming zero initial condition $y(-1) = 0$, the output of the system is given by

$$y(n) = (1-a)\sum_{m=0}^{n} a^{n-m}x(m), \quad n = 0, 1, 2, \ldots \tag{3.1.10}$$

If we define the impulse response of the system by

$$h(n) = (1-a)a^n, \quad n = 0, 1, 2, \ldots$$

the system has an input–output relation

$$y(n) = \sum_{m=-\infty}^{\infty} h(n-m)x(m)$$

which indicates that the system is a convolution one.

Example
A **pure delay** system is defined by

$$y(t) = \int_{-\infty}^{\infty} \delta(t - t_0 - \tau)x(\tau)d\tau = x(t - t_0) \tag{3.1.11}$$

which shows that its impulse response is $h(t) = \delta(t - t_0)$.

Definition: Nonanticipative Convolution System

A system, discrete or continuous, is nonanticipative if and only if its impulse response is

$$h(t) = 0, \quad t < 0$$

with t ranging over the range in which the system is defined.

If the delay t_0 of a pure delay system is positive, then the system is nonanticipative; and if it is negative, the system is anticipative.

1.3.2 Convolution Properties

Commutative

$$y(t)=\int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau=\int_{-\infty}^{\infty} f(t-\tau)h(\tau)d\tau$$

Set $t-\tau=\tau'$ in the first integral, and then rename the dummy variable τ' to τ.

Distributive

$$g(t) = f(t) * [h_1(t) + h_2(t)] = f(t) * h_1(t) + f(t) * h_2(t)$$

This property follows directly as a result of the linear property of integration.

Associative

$$[f(t) * h_1(t)] * h_2(t)] = f(t) * [h_1(t) * h_2(t)]$$

Shift Invariance

If $g(t) = f(t) * h(t)$, then

$$g(t-t_0)=f(t-t_0)*h(t)=\int_{-\infty}^{\infty} f(\tau-t_0)h(t-\tau)d\tau$$

Write $g(t)$ in its integral form, substitute $t-t_0$ for t, set $\tau + t_0 = \tau'$, and then rename the dummy variable.

Area Property

$$A_f=\int_{-\infty}^{\infty} f(t)dt= \text{ area}$$

$$m_f=\int_{-\infty}^{\infty} t\,f(t)dt= \text{ first moment}$$

$$K_f=\frac{m_f}{A_f}= \text{ center of gravity}$$

The convolution $g(t) = f(t) * h(t)$ leads to

$$A_g = A_f A_h$$
$$K_g = K_f + K_h$$

Proof

$$m_g=\int_{-\infty}^{\infty} tg(t)dt=\int_{-\infty}^{\infty} t\left[\int_{-\infty}^{\infty} f(\tau)h(t-\tau)dt\right]d\tau$$

$$=\int_{-\infty}^{\infty} f(\tau)\left[\int_{-\infty}^{\infty} th(t-\tau)dt\right]d\tau$$

$$=\int_{-\infty}^{\infty} f(\tau)\left[\int_{-\infty}^{\infty} (\lambda+\tau)h(\lambda)d\lambda\right]d\tau, \quad t-\tau=\lambda$$

$$=\int_{-\infty}^{\infty} f(\tau)d\tau\int_{-\infty}^{\infty} \lambda h(\lambda)d\lambda+\int_{-\infty}^{\infty} \tau f(\tau)d\tau\int_{-\infty}^{\infty} h(\lambda)d\lambda=A_f m_h+m_f A_h$$

$$\frac{m_g}{A_f A_h} = \frac{m_g}{A_g} \doteq K_g = \frac{A_f m_h + m_f A_h}{A_f A_h} = K_h + K_f$$

Scaline Property

If $g(t) = f(t) * h(t)$, then $f\left(\frac{t}{a}\right) * h\left(\frac{t}{a}\right) = |a| g\left(\frac{t}{a}\right)$.

Proof

$$\int_{-\infty}^{\infty} f\left(\frac{\tau}{a}\right) h\left(\frac{t-\tau}{a}\right) d\tau = \int_{-\infty}^{\infty} f\left(\frac{\tau}{a}\right) h\left(\frac{t}{a} - \frac{\tau}{a}\right) d\tau$$

$$= |a| \int_{-\infty}^{\infty} f(r)\, h\left(\frac{t}{a} - r\right) dr = |a| g\left(\frac{t}{a}\right)$$

Complex-Valued Functions

$$g(t) = f(t) * h(t) = [f_r(t) + jf_i(t)] * [h_r(t) + jfh_i(t)]$$
$$= [f_r(t) * h_r(t) - f_i(t) * h_i(t)] + j[f_r(t) * h_i(t) + f_i(t) * h_r(t)]$$

Derivative of Delta Function

$$g(t) = f(t) * \frac{d\delta(t)}{dt} = \int_{-\infty}^{\infty} f(\tau) \frac{d}{dt} \delta(t-\tau) d\tau = \frac{d}{dt} \int_{-\infty}^{\infty} f(\tau) \delta(t-\tau) d\tau = \frac{df(t)}{dt}$$

Moment Expansion

Expand $f(t - \tau)$ in Taylor series about the point $t = 0$

$$f(t-\tau) = f(t) - \tau f^{(1)}(t) + \frac{\tau^2}{2!} f^{(2)}(t) + \cdots + \frac{(-\tau)^{n-1}}{(n-1)!} f^{(n-1)}(t) + e_n$$

Insert into convolution integral

$$g(t) = f(t) \int_{-\infty}^{\infty} h(\tau) d\tau - f^{(1)} \int_{-\infty}^{\infty} \tau h(\tau) d\tau + \frac{f^{(2)}(t)}{2!} \int_{-\infty}^{\infty} \tau^2 h(\tau) d\tau$$

$$+ \cdots + \frac{f^{(n-1)}(t)}{(n-1)!} (-1)^{n-1} \int_{-\infty}^{\infty} \tau^{n-1} h(\tau) d\tau + E_n$$

$$= m_{h0} f(t) - m_{h1} f^{(1)}(t) + \frac{m_{h2}}{2!} f^{(2)}(t) + \cdots + \frac{(-1)^{n-1}}{(n-1)!} m_{h(n-1)} f^{(n-1)}(t) + E_n$$

where bracketed numbers in exponents indicate order of differentiation.

Truncation Error

Because

$$e_n = \frac{(-\tau)^n}{n!} f^{(n)}(t-\tau_1), \quad 0 \le \tau_1 \le \tau$$

$$E_n = \frac{1}{n!}\int_{-\infty}^{\infty} (-\tau)^n f^{(n)}(t-\tau_1) h(\tau) d\tau$$

Because τ_1 depends on τ, the function $f^{(n)}(t-\tau_1)$ cannot be taken outside the integral. However, if $f^{(n)}(t)$ is continuous and $t^n h(t) \ge 0$, then

$$E_n = \frac{1}{n!} f^{(n)}(t-\tau_0)\int_{-\infty}^{\infty} (-\tau)^n h(\tau) d\tau = \frac{(-1)^n m_{hn}}{n!} f^{(n)}(t-\tau_0)$$

where τ_0 is some constant in the interval of integration.

Fourier Transform

$$\mathcal{F}\{f(t) * h(t)\} = F(\omega)H(\omega)$$

Proof

$$\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau\right]e^{-j\omega t}dt = \int_{-\infty}^{\infty} f(\tau)\int_{-\infty}^{\infty} h(t-\tau)e^{-j\omega t}dt\,d\tau$$

$$= \int_{-\infty}^{\infty} f(\tau)e^{-j\omega\tau}d\tau\int_{-\infty}^{\infty} h(r)e^{-j\omega r}dr, \quad t-\tau = r$$

Inverse Fourier Transform

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)H(\omega)e^{-j\omega t}d\omega = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$$

Band-Limited Function

If $f(t)$ is σ-band limited, then the output of a system is

$$g(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau = \sum_{n=-\infty}^{\infty} Tf(nT)h_\sigma(t-nT)$$

where

$$h_\sigma(t) = \frac{1}{2\pi}\int_{-\sigma}^{\sigma} H(\omega)e^{j\omega t}d\omega$$

Proof

$$H_\sigma(\omega) = p_\sigma(\omega)H(\omega),$$

hence

$$\begin{aligned} G(\omega) &= F(\omega)H(\omega) \\ &= \bar{F}(\omega)p_\sigma(\omega)H(\omega) \\ &= \bar{F}(\omega)H_\sigma(\omega), \quad \bar{F}(\omega) = F(\omega) \quad \text{for } -\sigma < \omega < \sigma \end{aligned}$$

$$g(t) = \bar{f}(t) * h_\sigma(t) = \left[\sum_{n=-\infty}^{\infty} Tf(nT)\delta(t-nT)\right] * h_\sigma(t) = \sum_{n=-\infty}^{\infty} Tf(nT)h_\sigma(t-nT)$$

The convolution properties are given in Table 3.2.1.

Stability of Convolution Systems

Definition: Bounded Input Bounded Output (BIBO) Stability

A discrete or continuous convolution system with impulse response h is BIBO stable if and only if the impulse satisfies the inequality, $\Sigma_n|h| < \infty$ or $\int_R |h(t)|dt < \infty$. If the system is BIBO stable, then

$$\sup|y(n)| \le \sum_n |h(n)| \sup|x(n)|, \quad \sup|y(t)| \le \int_R |h(t)| dt \sup|x(t)|, \quad t \in R$$

for every finite amplitude input $x(t)$ (y is the input of the system).

Example

If the impulse response of a discrete system is $h(n) = ab^n$, $n = 0, 1, 2, \ldots$, then

$$\sum_{n=0}^{\infty} |h(n)| = \sum_{n=0}^{\infty} |a||b|^n = \begin{cases} |a|\dfrac{1}{1-|b|} & |b| < 1 \\ \infty & |b| \ge 1 \end{cases}$$

The above indicates that for $|b| < 1$ the system is BIBO and for $|b| \ge 1$ the system is unstable.

Example

If $h(t) = u(t)$ then $|h(t)| = \int_0^\infty |u(t)|dt = \infty$, which indicates the system is not BIBO stable.

Harmonic Inputs

If the input function is of complex exponential order $e^{j\omega t}$, then its output is

$$y(t) = \int_{-\infty}^{\infty} h(\tau)e^{j\omega(t-\tau)}d\tau = e^{j\omega t}\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau = H(\omega)e^{j\omega t}$$

The above equation indicates that the output is the same as the input $e^{j\omega t}$ with its amplitude modified by $|H(\omega)|$ and its phase by $\tan^{-1}(H_i(\omega)/H_r(\omega))$ where $H_r(\omega) = \text{Re}\{H(\omega)\}$ and $H_i(\omega) = \text{Im}\{H_i(\omega)\}$.

For the discrete case we have the relation

$$y(n) = e^{j\omega n} H(e^{j\omega})$$

TABLE 3.2.1 Convolution Properties

1. **Commutative**

$$g(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} f(t-\tau)h(\tau)\,d\tau$$

2. **Distributive**

$$g(t) = f(t) * [h_1(t) + h_2(t)] = f(t) * h_1(t) + f(t) * h_2(t)$$

3. **Associative**

$$[f(t) * h_1(t)] * h_2(t) = f(t) * [h_1(t) * h_2(t)]$$

4. **Shift Invariance**

$$g(t) = f(t) * h(t)$$

$$g(t-t_0) = f(t-t_0) * h(t) = \int_{-\infty}^{\infty} f(\tau - t_0)h(t-\tau)\,d\tau$$

5. **Area Property**

$$A_f = \text{area of } f(t),$$

$$m_f = \int_{-\infty}^{\infty} t f(t)\,dt = \text{first moment}$$

$$K_f = \frac{m_f}{A_f} = \text{center of gravity}$$

$$A_g = A_f A_h, \qquad K_g = K_f + K_h$$

6. **Scaling**

$$g(t) = f(t) * h(t)$$

$$f\left(\frac{t}{a}\right) * h\left(\frac{t}{a}\right) = |a| g\left(\frac{t}{a}\right)$$

7. **Complex Valued Functions**

$$g(t) = f(t) * h(t) = [f_r(t) * h_r(t) - f_i(t) * h_i(t)] + j[f_r(t) * h_i(t) + f_i(t) * h_r(t)]$$

8. **Derivative**

$$g(t) = f(t) * \frac{d\delta(t)}{dt} = \frac{df(t)}{dt}$$

9. **Moment Expansion**

$$g(t) = m_{h0} f(t) - m_{h1} f^{(1)}(t) + \frac{m_{h2}}{2!} f^{(1)}(t) + \cdots + \frac{(-1)^{n-1}}{n-1!} m_{h(n-1)} f^{(n-1)}(t) + E_n$$

$$m_{hk} = \int_{-\infty}^{\infty} \tau^k h(\tau) d\tau$$

$$E_n = \frac{(-1)^n m_{hn}}{n!} f^{(n)}(t - \tau_0), \qquad \tau_0 = \text{constant in the interval of integration}$$

10. **Fourier Transform**

$$\mathcal{F}\{f(t) * h(t)\} = F(\omega)H(\omega)$$

11. **Inverse Fourier Transform**

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)H(\omega)e^{j\omega t}\,d\omega = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau$$

12. **Band-limited Function**

$$g(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau = \sum_{n=-\infty}^{\infty} T f(nT) h_\sigma(t - nT)$$

$$h_\sigma(t) = \frac{1}{2\pi} \int_{-\sigma}^{\sigma} H(\omega)e^{j\omega t}\,d\omega, \qquad f(t) = \sigma - \text{band limited} = 0, \qquad |t| > \sigma$$

TABLE 3.2.1 Convolution Properties (Continued)

13. Cyclical Convolution	$$x(n) \otimes y(n) = \sum_{m=0}^{N-1} x((n-m) \bmod N) y(m)$$
14. Discrete-Time	$$x(n) * y(n) = \sum_{m=-\infty}^{\infty} x(n-m) y(m)$$
15. Sampled	$$x(nT) * y(nT) = T \sum_{m=-\infty}^{\infty} x(nT - mT) y(mT)$$

where

$$H\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n}$$

1.4 Correlation

The **cross-correlation** of two different functions is defined by the relation

$$R_{fh}(t) \doteq f(t) \lozenge h(t) = \int_{-\infty}^{\infty} f(\tau) h(\tau - t) d\tau = \int_{-\infty}^{\infty} f(\tau + t) h(\tau) d\tau \tag{4.1}$$

When $f(t) = h(t)$ the correlation operation is called **autocorrelation**.

$$R_{ff}(t) \doteq f(t) \lozenge f(t) = \int_{-\infty}^{\infty} f(\tau) f(\tau - t) d\tau = \int_{-\infty}^{\infty} f(\tau + t) f(\tau) d\tau \tag{4.2}$$

For complex functions the correlation operations are given by

$$R_{fh}(t) \doteq f(t) \lozenge h^*(t) = \int_{-\infty}^{\infty} f(\tau) h^*(\tau - t) d\tau \tag{4.3}$$

$$R_{ff}(t) \doteq f(t) \lozenge f^*(t) = \int_{-\infty}^{\infty} f(\tau) f^*(\tau - t) d\tau \tag{4.4}$$

The two basic properties of correlation are

$$f(t) \lozenge h(t) \neq h(t) \lozenge f(t) \tag{4.5}$$

$$\begin{aligned} \left|R_{ff}(t)\right| &\doteq \left|f(t) \lozenge f^*(t)\right| = \left|\int_{-\infty}^{\infty} f(\tau) f^*(\tau - t) d\tau\right| \\ &\leq \left[\int_{-\infty}^{\infty} |f(\tau)|^2 d\tau\right]^{1/2} \left[\int_{-\infty}^{\infty} |f(\tau - t)|^2 d\tau\right]^{1/2} \\ &= \int_{-\infty}^{\infty} |f(\tau)|^2 d\tau \leq R_{ff}(0) \end{aligned} \tag{4.6}$$

Example

The cross-correlation of the following two functions, $f(t) = p(t)$ and $h(t) = e^{-(t-3)}\, u(t-3)$, is given by

$$R_{fh}(t) = \int_{-\infty}^{\infty} p(\tau) e^{-(\tau-t-3)} u(\tau - t - 3)\, d\tau$$

The ranges of t are

1. $t > -2$: $R_{fh}(t) = 0$ (no overlap of function)
2. $-4 < t < -2$: $R_{fh}(t) = \int_{3+t}^{1} e^{-(\tau-t-3)}\, d\tau = 1 - e^{2}e^{t}$
3. $-\infty < t < -4$: $R_{fh}(t) = \int_{-1}^{1} e^{-(\tau-t-3)}\, d\tau = e^{t}e^{2}\,(e^{2} - 1)$

The discrete form of correlation is given by

$$x(n) \Diamond y(n) = \sum_{m=-\infty}^{\infty} x(m-n) y^{*}(m) \equiv \text{ crosscorrelation} \tag{4.7}$$

$$x(n) \Diamond x(n) = \sum_{m=-\infty}^{\infty} x(m-n) x^{*}(m) \equiv \text{ autocorrelation} \tag{4.8}$$

$$x(nT) \Diamond y(nT) = T \sum_{m=-\infty}^{\infty} x(mT - nT) y^{*}(mT) \equiv \text{ sampled cross-correlation} \tag{4.9}$$

1.5 Orthogonality of Signals

1.5.1 Introduction

Modern analysis regards some classes of functions as multidimensional vectors introducing the definition of inner products and expansion in term of orthogonal functions (base functions). In this section, functions $\Phi(t)$, $f(t)$, $F(x)$, … symbolize either functions of one independent variable t, or, for brevity, a function of a set n independent variables $t^1, t^2, \ldots, t^n$. Hence, $dt = dt^1 \ldots dt^n$.

A real or complex function $f(t)$ defined on the measurable set E of elements $\{r\}$ is **quadratically integrable** on E if and only if

$$\int_E \left| f(\tau) \right|^2 d\tau$$

exists in the sense of Lebesque. The class L_2 of all real or complex functions is quadratically integrable on a given interval if one regards the functions $f(t)$, $h(t)$, … as vectors and defines

Vector sum of $f(t)$ and $h(t)$ as $f(t) + h(t)$

Product of $f(t)$ by a scalar α as $\alpha f(t)$

The **inner product** of $f(t)$ and $h(t)$ is defined as

$$\langle f, h \rangle \doteq \int_I \gamma(\tau) f^{*}(\tau) h(\tau)\, d\tau \tag{5.1.1}$$

where $\gamma(\tau)$ is a real nonnative function (**weighting function**) quadratically integrable on I.

Norm

The norm in L_2 is the quantity

$$\|f\| = \left[\langle f, f \rangle\right]^{1/2} \doteq \left[\int_I \gamma(\tau)\left|f(\tau)\right|^2 d\tau\right]^{1/2} \tag{5.1.2}$$

If $\|f\|$ exists and is different from zero, the function is normalizable.

Normalization

$$\frac{f(t)}{\|f\|} = \text{unit norm}$$

Inequalities

If $f(t)$, $h(t)$, and the nonnegative weighting function $\gamma(t)$ are quadratically integrable on I, then

Cauchy–Schwarz Inequality

$$\left|\langle f(t), h(t) \rangle\right| \doteq \left|\int_I \gamma(\tau) f^* h \, d\tau\right|^2 \le \int_I \gamma|f|^2 d\tau \int_I \gamma|h|^2 d\tau \doteq \langle f, f \rangle \langle h, h \rangle \tag{5.1.3}$$

Minkowski Inequality

$$\begin{aligned} \|f+h\| &\doteq \left(\int_I \gamma|f+h|^2 d\tau\right)^{1/2} \\ &\le \left(\int_I \gamma|f|^2 d\tau\right)^{1/2} + \left(\int_I \gamma|h|^2 d\tau\right)^{1/2} \\ &= \|f\| + \|h\| \end{aligned} \tag{5.1.4}$$

Convergence in Mean

The space L_2 admits the **distance function** (matric)

$$d\langle f, h \rangle \doteq \|f-h\| \doteq \left[\int_I \gamma(\tau)\left|f(\tau)-h(\tau)\right|^2 d\tau\right]^{1/2} \tag{5.1.5}$$

The root-mean-square difference of the above equation between the two functions $f(t)$ and $h(t)$ is equal to zero if and only if $f(t) = h(t)$ for almost all t in I.

Every sequence in I of functions $r_0(t), r_1(t), r_2(t), \ldots$ **converges in the mean** to the limit $r(t)$ if and only if

$$d^2\langle r_n, r \rangle \doteq \|r_n - r\|^2 \doteq \int_I \gamma(\tau)\left|r_n(\tau) - r(\tau)\right|^2 d\tau \to 0 \quad \text{as } n \to \infty \tag{5.1.6}$$

Therefore we define limit in the mean

$$\underset{n\to\infty}{\text{l.i.m}}\; r_n(t) = r(t) \tag{5.1.7}$$

Convergence in the mean does not necessarily imply convergence of the sequence at every point, nor does convergence of all points on I imply convergence in the mean.

Riess–Fischer Theorem

The L_2 space with a given interval I is **complete**; every sequence of quadratically integrable functions $r_0(t)$, $r_1(t)$, $r_2(t)$, … such that $\text{l.i.m.}_{m\to\infty,\, n\to\infty} |r_m - r_n| = 0$ (**Cauchy sequence**), converges in the mean to a quadratically integrable function $r(t)$ and defines $r(t)$ uniquely for almost all t in I.

Orthogonality

Two quadratically integrable functions $f(t)$, $h(t)$ are **orthogonal** on I if and only if

$$\langle f, h\rangle = \int_I \gamma(\tau) f^*(\tau) h(\tau)\, d\tau = 0 \tag{5.1.8}$$

Orthonormal

A set of function $r_i(t)$, $i = 1, 2, \ldots$ is an **orthonormal** set if and only if

$$\langle r_i, r_j\rangle \doteq \int_I \gamma(\tau) r_i^*(\tau) r_j(\tau)\, d\tau = \delta_{ij} = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases} \quad (i, j = 1, 2, \ldots) \tag{5.1.9}$$

Every set of normalizable mutually orthogonal functions is linearly independent.

Bessel's Inequalities

Given a finite or infinite orthonormal set $\varphi_1(t)$, $\varphi_2(t)$, $\varphi_3(t)$, … and any function $f(t)$ quadratically integrable over I

$$\sum_i \left|\langle \varphi_i\, f\rangle\right|^2 \le \langle f, f\rangle \tag{5.1.10}$$

The equal sign applies if and only if $f(t)$ belongs to the space spanned by all $\varphi_i(t)$.

Complete Orthonormal Set of Functions (Orthonormal Bases)

A set of functions $\{\varphi_i(t)\}$, $i = 1, 2, \ldots$, in L_2 is a complete orthonormal set if and only if the set satisfies the following conditions:

1. Every quadratically integrable function $f(t)$ can be expanded in the form

$$f(t) = \langle f, \varphi_1\rangle\varphi_1 + \langle f, \varphi_2\rangle\varphi_2 + \cdots + \langle f, \varphi_i\rangle\varphi_i + \cdots, \qquad i = 1, 2, \ldots$$

2. If (1) above is true, then

$$\langle f, f\rangle = |\langle f, \varphi_1\rangle|^2 + |\langle f, \varphi_2\rangle|^2 + \cdots$$

   which is the completeness relation (Parseval's identity).
3. For any pair of functions $f(t)$ and $h(t)$ in L_2, the relation holds

$$\langle f, h\rangle = \langle f, \varphi_1\rangle\langle h, \varphi_1\rangle + \langle f, \varphi_2\rangle\langle h, \varphi_2\rangle + \cdots$$

4. The orthonormal set $\varphi_1(t)$, $\varphi_2(t)$, $\varphi_3(t)$, … is not contained in any other orthonormal set in L_2.

The above conditions imply the following: given a complete orthonormal set $\{\varphi_i(t)\}$, $i = 1, 2, \ldots$ in L_2 and a set of complex numbers $\langle f, \varphi_1\rangle, \langle f, \varphi_2\rangle + \cdots$ such that $\sum_{i=1}^{\infty} |\langle f, \varphi_i\rangle|^2 < \infty$, there exists a quadratically integrable function $f(t)$ such that $\langle f, \varphi_1\rangle\varphi_1 + \langle f, \varphi_2\rangle\varphi_2 + \cdots$ converges in the mean of $f(t)$.

Gram–Schmidt Orthonormalization Process

Given any countable (finite or infinite) set of linear independent functions $r_1(t), r_2(t), \ldots$ normalizable in I, there exists an orthogonal set $\varphi_1(t), \varphi_2(t), \ldots$ spanning the same space of functions. Hence

$$\varphi_1 = r_1, \quad \varphi_2 = r_2 - \frac{\int_I \varphi_1 r_2\, dt}{\int_I \varphi_1^2\, dt}, \quad \varphi_3 = r_3 - \frac{\int_I \varphi_1 r_3\, dt}{\int_I \varphi_1^2\, dt}\varphi_1 - \frac{\int_I \varphi_2 r_3\, dt}{\int_I \varphi_2^2\, dt}\varphi_2, \quad \text{etc.} \tag{5.1.11}$$

For creating an orthonormal set, we proceed as follows:

$$\varphi_i(t) = \frac{\upsilon_i(t)}{\|\upsilon_i(t)\|} = \frac{\upsilon_i(t)}{+\sqrt{\langle \upsilon_i, \upsilon_i\rangle}}$$

$$\upsilon_1(t) = r_1(t), \quad \upsilon_{i+1}(t) = r_{i+1}(t) - \sum_{k=1}^{i} \langle \varphi_k, r_{i+1}\rangle \varphi_k(t), \quad i = 1, 2, \ldots \tag{5.1.12}$$

Series Approximation

If $f(t)$ is a quadratically integrable function, then

$$\int_I |f_n(t) - f(t)|^2\, dt$$

yields the **least mean square error**. The set $\{\varphi_i(t)\}$, $i = 1, 2, \ldots$ is orthonormal and the approximation to $f(t)$ is

$$f_n(t) = a_1\varphi_1(t) + a_2\varphi_2(t) + \cdots + a_n\varphi_n(t), \qquad n = 1, 2, \ldots \tag{5.1.13}$$

1.5.2 Legendre Polynomials

1.5.2.1 Relations of Legendre Polynomials

Legendre polynomials are closely associated with physical phenomena for which spherical geometry is important. The polynomials $P_n(t)$ are called Legendre polynomials in honor of their discoverer, and they are given by

$$P_n(t) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!\, t^{n-2k}}{2^n k!(n-k)!(n-2k)!} \tag{5.2.1}$$

$$[n/2] = \begin{cases} n/2 & n \text{ even} \\ (n-1)/2 & n \text{ odd} \end{cases}$$

$$\frac{1}{\sqrt{1-2st+s^2}} = \begin{cases} \sum_{n=0}^{\infty} P_n(t)s^n & |s|<1 \\ \sum_{n=0}^{\infty} P_n(t)s^{-n-1} & |s|>1 \text{ generating function} \end{cases} \tag{5.2.1a}$$

Table 5.2.1 gives the first eight Legendre polynomials. Figure 5.2.1 shows the first six Legendre polynomials.

TABLE 5.2.1 Legendre Polynomials

$P_0 = 1$
$P_1 = t$
$P_2 = \frac{3}{2}t^2 - \frac{1}{2}$
$P_3 = \frac{5}{2}t^3 - \frac{3}{2}t$
$P_4 = \frac{35}{8}t^4 - \frac{30}{8}t^2 + \frac{3}{8}$
$P_5 = \frac{63}{8}t^5 - \frac{70}{8}t^3 + \frac{15}{8}t$
$P_6 = \frac{231}{16}t^6 - \frac{315}{16}t^4 + \frac{105}{16}t^2 - \frac{5}{16}$
$P_7 = \frac{429}{16}t^7 - \frac{693}{16}t^5 + \frac{315}{16}t^3 - \frac{35}{16}t$

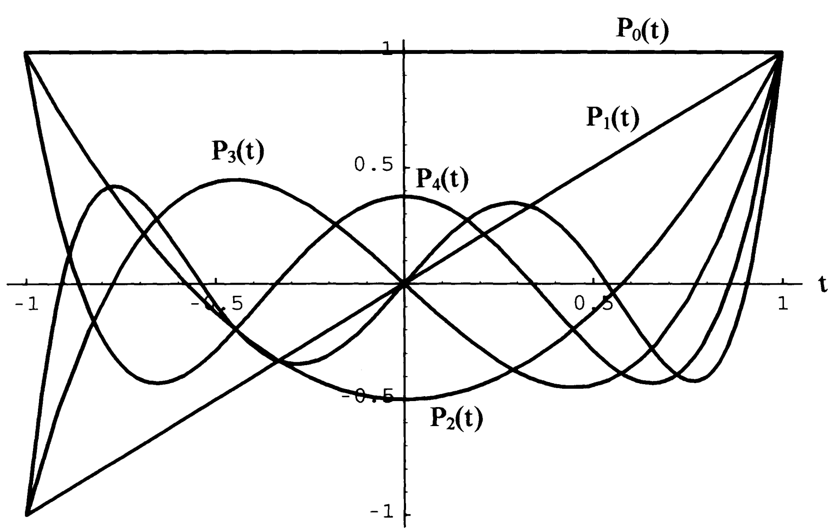


FIGURE 5.2.1

Rodrigues Formula

$$P_n(t) = \frac{1}{2^n n!}\frac{d^n}{dt^n}(t^2-1)^n, \quad n = 0, 1, 2, \ldots \tag{5.2.2}$$

Recursive Formulas

$$(n+1)P_{n+1}(t) - (2n+1)tP_n(t) + nP_{n-1}(t) = 0, \quad n = 1, 2, \ldots \tag{5.2.3}$$

$$P'_{n+1}(t) - tP'_n(t) = (n+1)P_n(t), \quad \left(P'(t) \doteq \text{ derivative of } P(t)\right) \quad n = 0, 1, 2, \ldots \tag{5.2.4}$$

$$tP'_n(t) - P'_{n-1}(t) = nP_n(t) \quad n = 1, 2, \ldots \tag{5.2.5}$$

$$P'_{n+1}(t) - P'_{n-1}(t) = (2n+1)P_n(t) \quad n = 1, 2, \ldots \tag{5.2.6}$$

$$(t^2 - 1)P_n'(t) = ntP_n(t) - nP_{n-1}(t) \tag{5.2.7}$$

$$P_0(t) = 1, \quad P_1(t) = t \tag{5.2.8}$$

Example

From (5.2.1), when n is even, implies $P_n(-t) = P_n(t)$ and when n is odd, $P_n(-t) = -P_n(t)$. Therefore

$$P_n(-t) = (-1)^n\, P_n(t) \tag{5.2.9}$$

Example

From (5.2.7) $t = 1$ implies $0 = nP_{n-1}(1) - nP_{n-1}(1)$ or $P_n(1) = P_{n-1}(1)$. For $n = 1$ it implies $P_1(1) = P_0(1) = 1$. For $n = 2$ $P_2(1) = P_1(1) = 1$, and so forth. Hence, $P_n(1) = 1$. From (5.2.9) $P_n(-1)^n$. Hence

$$P_n(1) = 1, \qquad P_n(-1) = (-1)^n \tag{5.2.10}$$

$$P_n(t) < 1 \qquad \text{for } -1 < t < 1 \tag{5.2.11}$$

Example

From (5.2.7) we get

$$\frac{d}{dt}\left[(1-t^2)P'_n(t)\right] = nP'_{n-1}(t) - nP_n(t) - ntP'_n(t)$$

Use (5.2.5) to find

$$\frac{d}{dt}\left[(1-t^2)P'_n(t)\right] + n(n+1)P_n(t) = 0$$

or

$$(1 - t^2)P_n''(t) - 2tP_n'(t) + n(n+1)P_n(t) = 0 \tag{5.2.12}$$

We have deduced the Legendre polynomials $y = P_n(t)$ $(n = 0, 1, 2, \ldots)$ as the solution of the linear second-order ordinary differential equation

$$(1 - t^2)y''(t) - 2ty'(t) + n(n + 1)y(t) = 0 \tag{5.2.12a}$$

called the **Legendre differential equation**.

If we let $x = \cos \varphi$ then the above equation transforms to the trigonometric form

$$y'' + (\cot \varphi)y' + n(n + 1)y = 0 \tag{5.2.12b}$$

It can be shown that (5.2.12a) has solutions of a first kind

$$\begin{aligned} y = {} & C_0\left[1 - \frac{n(n+1)}{2!}t^2 + \frac{n(n+1)(n-2)(n+3)}{4!}t^4 - \ldots\right] \\ & + C_1\left[1 - \frac{(n-1)(n+2)}{3!}t^3 + \frac{(n-1)(n+2)(n-3)(n+4)}{5!}t^5 - \ldots\right] \end{aligned} \tag{5.2.12c}$$

valid for $|t| < 1$, C_0 and C_1 being arbitrary constants.

Schläfli's Integral Formula

$$P_n(t) = \frac{1}{2\pi j}\int_C \frac{(z^2 - 1)^n}{2^n (z - t)^{n+1}} dz \tag{5.2.13}$$

where C is any regular, simple, closed curve surrounding t.

1.5.2.2 Complete Orthonormal System, $\left\{\left[\frac{1}{2}(2n+1)\right]^{1/2} P_n(t)\right\}$

The Legendre polynomials are orthogonal in [–1, 1]

$$\int_{-1}^{1} P_n(t)P_m(t)dt = 0 \tag{5.2.14}$$

$$\int_{-1}^{1} \left[P_n(t)\right]^2 dt = \frac{2}{2n+1} \qquad n = 0, 1, 2, \cdots \tag{5.2.15}$$

and therefore the set

$$\varphi_n(t) = \sqrt{\frac{2n+1}{2}} P_n(t) \qquad n = 0, 1, 2, \cdots \tag{5.2.16}$$

is orthonormal.

Series Expansion

If $f(t)$ is integrable in [–1, 1], then

$$f(t) = \sum_{n=0}^{\infty} a_n P_n(t) \qquad -1 < t < 1 \tag{5.2.16a}$$

$$a_n = \frac{2n+1}{2} \int_{-1}^{1} f(t) P_n(t) dt \qquad n = 0, 1, 2, \cdots \quad (5.2.16b)$$

For even $f(t)$, the series will contain term $P_n(t)$ of even index; if $f(t)$ is odd, the term of odd index only.

If the real function $f(t)$ is piecewise smooth in $(-1, 1)$ and if it is square integrable in $(-1, 1)$, then the series (5.2.16a) converges to $f(t)$ at every continuity point of $f(t)$.

Change of Range

If a function $f(t)$ is defined in $[a, b]$, it is sometimes necessary in the application to expand the function in a series of orthogonal polynomials in this interval. Clearly the substitution

$$t = \frac{2}{b-a}\left[x - \frac{b+a}{2}\right], \quad a < b, \quad \left[x = \frac{b-a}{2} t + \frac{b+a}{2}\right] \qquad (5.2.17)$$

transform the interval $[a, b]$ of the x-axis into the interval $[-1, 1]$ of the t-axis. It is, therefore, sufficient to consider the expansion in series of Legendre polynomials of

$$f\left[\frac{b-a}{2} t + \frac{b+a}{2}\right] = \sum_{n=0}^{\infty} a_n P_n(t) \qquad (5.2.18a)$$

$$a_n = \frac{2n+1}{2} \int_{-1}^{1} f\left[\frac{b-a}{2} t + \frac{b+a}{2}\right] P_n(t) dt \qquad (5.2.18b)$$

The above equation can also be accomplished as follows:

$$f(t) = \sum_{n=0}^{\infty} a_n X_n(t) \qquad (5.2.19a)$$

$$X_n(t) = \frac{1}{n!(b-a)^n} \frac{d^n (t-a)^n (t-b)^n}{dt^n} \qquad (5.2.19b)$$

$$a_n = \frac{2n+1}{b-a} \int_{a}^{b} f(t) X_n(t) dt \qquad (5.2.19c)$$

Example

Suppose $f(t)$ is given by

$$f(t) = \begin{cases} 0 & -1 \le t < a \\ 1 & a < t \le 1 \end{cases}$$

Then from (5.2.16b)

$$a_n = \frac{2n+1}{2} \int_{a}^{1} P_n(t) dt$$

Using (5.2.6), and noting that $P_n(1) = 1$, we obtain

$$a_n = -\frac{1}{2}\left[P_{n+1}(a) - P_{n-1}(a)\right], \quad a_0 = \frac{1}{2}(1-a)$$

which leads to the expansion

$$f(t) \cong \frac{1}{2}(1-a) - \frac{1}{2}\sum_{n=1}^{\infty}\left[P_{n+1}(a) - P_{n-1}(a)\right]P_n(t), \quad -1 < t < 1$$

Example

Suppose $f(t)$ is given by

$$f(t) = \begin{cases} -1 & -1 \le t < 0 \\ 1 & 0 < t \le 1 \end{cases}$$

The function is an odd function and, therefore, $f(t)P_n(t)$ is an odd function of $P_n(t)$ with even index. Hence, a_n are zero for $n = 0, 2, 4, \ldots$ For odd index n, the product $f(t)P_n(t)$ is even and hence

$$a_n = \left(n + \frac{1}{2}\right)\int_{-1}^{1} f(t)P_n(t)dt = 2\left(n + \frac{1}{2}\right)\int_{0}^{1} P_n(t)dt, \quad n = 1, 3, 5, \ldots$$

Using (5.2.6) and setting $n = 2k + 1$, $k = 0, 1, 2, \ldots$ we obtain

$$\begin{aligned} a_{2k+1} &= (4k+3)\int_0^1 P_{2k+1}(t)dt = \int_0^1 \left[P'_{2k+2}(t) - P'_{2k}(t)\right]dt \\ &= \left[P_{2k+2}(t) - P_{2k}(t)\right]\Big|_0^1 = P_{2k}(0) - P_{2k+2}(0) \end{aligned}$$

where we have used the property $P_n(1) = 1$ for all n. But

$$P_{2n}(0) = \binom{-\frac{1}{2}}{n} = \frac{(-1)^n(2n)!}{2^{2n}(n!)^2} \qquad (5.2.20)$$

and, thus, we have

$$\begin{aligned} a_{2k+1} &= \frac{(-1)^k(2k)!}{2^{2k}(k!)^2} - \frac{(-1)^{k+1}(2k+2)!}{2^{2k+2}\left[(k+1)!\right]^2} = \frac{(-1)^k(2k)!}{2^{2k}(k!)^2}\left[1 + \frac{2k+1}{2k+2}\right] \\ &= \frac{(-1)^k(2k)!(4k+3)}{2^{2k+1}k!(k+1)!} \end{aligned}$$

The expansion is

$$f(t) = \sum_{k=0}^{\infty} \frac{(-1)^k(2k)!(4k+3)}{2^{2k+1}k!(k+1)!} P_{2k+1}(t), \quad -1 \le t \le 1 \qquad (5.2.21)$$

1.5.2.3 Associated Legendre Polynomials

If m is a positive integer and $-1 \le t \le 1$, then

$$P_n^m(t) = \left(1-t^2\right)^{m/2} \frac{d^m P_n(t)}{dt^m}, \quad m = 1, 2, \ldots, n \tag{5.2.22}$$

where $P_n^m(t)$ is known as the **associated Legendre function** or **Ferrer's functions**.

Rodrigues Formula

$$P_n^m(t) = \frac{\left(1-t^2\right)^{m/2}}{2^n n!} \frac{d^{n+m}}{dt^{n+m}}\left(t^2-1\right)^n, \quad m = 1, 2, \ldots, n;\ n+m \ge 0 \tag{5.2.23}$$

Properties

$$P_n^{-m}(t) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(t) \tag{5.2.24}$$

$$P_n^0(t) = P_n(t) \tag{5.2.25}$$

$$(n-m+1)P_{n+1}^m(t) - (2n+1)tP_n^m(t) + (n+m)P_{n-1}^m(t) = 0 \tag{5.2.26}$$

$$\left(1-t^2\right)^{1/2} P_n^m(t) = \frac{1}{2n+1}\left[P_{n+1}^{m+1}(t) - P_{n-1}^{m+1}(t)\right] \tag{5.2.27}$$

$$\begin{aligned} \left(1-t^2\right)^{1/2} P_n^m(t) = \frac{1}{2n+1}\Big[&(n+m)(n+m-1)P_{n-1}^{m-1}(t) \\ &-(n-m+1)(n-m+2)P_{n+1}^{m-1}(t)\Big] \end{aligned} \tag{5.2.28}$$

$$P_n^{m+1}(t) = 2mt\left(1-t^2\right)^{-1/2} P_n^m(t) - \left[n(n+1) - m(m-1)\right]P_n^{m-1}(t) \tag{5.2.29}$$

$$\int_{-1}^{1} P_n^m(t) P_k^m(t)\,dt = 0, \quad k \ne n \tag{5.2.30}$$

$$\int_{-1}^{1} \left[P_n^m(t)\right]^2 dt = \frac{2}{(2n+1)} \frac{(n+m)!}{(n-m)!} \tag{5.2.31}$$

Example

To evaluate the integral $\int_{-1}^{1} t^m P_n(t)\,dt$, we use the Rodrigues formula and proceed as follows:

$$\begin{aligned} \int_{-1}^{1} t^m P_n(t)\,dt &= \frac{1}{2^n n!}\int_{-1}^{1} t^m D^n\left[\left(t^2-1\right)^n\right]dt, \quad \left(D^n = \frac{d^n}{dt^n}\right) \\ &= \frac{1}{2^n n!}\left[\left[t^m D^{n-1}\left(t^2-1\right)^n\right]\Big|_{t=-1}^{t=1} - m\int_{-1}^{1} t^{m-1} D^{n-1}\left[\left(t^2-1\right)^n\right]dt\right] \end{aligned}$$

where integration by parts was used. The left expression is zero because of the presence of the expression $(t^2 - 1)^n$.

(a) For $m < n$ and after m integrations by parts we obtain

$$\int_{-1}^{1} t^m P_n(t)dt = \frac{(-1)^m m!}{2^n n!}\int_{-1}^{1} D^{n-m}\left[\left(t^2-1\right)^n\right]dt$$

$$= \frac{(-1)^m m!}{2^n n!}\left[D^{n-m-1}\left(t^2-1\right)^n\right]\Big|_{t=-1}^{t=1} = 0, \quad m<n$$

(b) $m \ge n$. Integrate n times by parts to find the following expression:

$$\int_{-1}^{1} t^m P_n(t)dt = C_{mn}\int_{-1}^{1} t^{m-n}\left(2^2-1\right)^n dt$$

where

$$C_{mn} = \frac{(-1)^m m(m-1)(m-2)\cdots(m-[n-1])}{2^n n!}$$

Multiplying numerator and denominator by $(m - n)!$ and incorporating the $(-1)^n$ in the integrand, we obtain

$$\int_{-1}^{1} t^m P_n(t)dt = \frac{m!}{2^n n!(m-n)!}\int_{-1}^{1} t^{m-n}\left(1-t^2\right)^n dt, \quad m \ge n$$

If $m - n$ is odd the integrand is an odd function and hence is equal to zero. If $m - n$ is even then the integrand is even and hence

$$\int_{-1}^{1} t^m P_n(t)dt = \frac{m!2}{2^n n!(m-n)!}\int_{0}^{1} t^{m-n}\left(1-t^2\right)^n dt$$

$$= \frac{m!\Gamma\left(\frac{m-n+1}{2}\right)}{2^{n-1}(m-n)!(m+n+1)\Gamma\left(\frac{m+n+1}{2}\right)}, \quad m \ge n,\ m-n \text{ is even}$$

If $m = n$

$$\int_{-1}^{1} t^m P_n(t)dt = \frac{n!\Gamma\left(\frac{1}{2}\right)}{2^{n-1}(2n+1)\left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\cdots\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}$$

$$= \frac{n!2^n}{2^{n-1}(2n+1)(2n-1)(2n-3)\cdots(3)(1)}$$

$$= \frac{n!2^n 2^n n!}{2^{n-1}(2n+1)(2n)(2n-1)(2n-2)(2n-3)\cdots(3)(2)(1)}$$

$$= \frac{2^{n+1}(n!)^2}{(2n+1)!}$$

Hence,

$$\int_{-1}^{1} t^m P_n(t)\,dt = \begin{cases} 0 & m < n \\ 0 & m \ge n,\ m-n \text{ is odd} \\ \dfrac{m!\,\Gamma\left(\dfrac{\mathrm{m-n+1}}{2}\right)}{2^{n-1}(m-n)!\,(m+n+1)\,\Gamma\left(\dfrac{m+n+1}{2}\right)} & m > n,\ m-n \text{ is even} \\ \dfrac{2^{n+1}(n!)^2}{(2n+1)!} & m = n \end{cases}$$

Example

To find $P_{2n}(0)$ we use the summation

$$P_{2n}(t) = \frac{(-1)^n}{2^{2n-1}} \sum_{k=0}^{n} \frac{(-1)^k (2n+2k-1)!}{(2k)!\,(n+k-1)!\,(n-k)!} t^{2k}$$

with $k = 0$. Hence

$$P_{2n}(0) = \frac{(-1)^n (2n-1)!}{2^{2n-1}(n-1)!\,n!} = \frac{(-1)^n 2n\big[(2n-1)!\big]}{2^{2n} n\big[(n-1)!\big] n!} = \frac{(-1)^n (2n)!}{2^{2n}(n!)^2}$$

Example

To evaluate $\int_0^1 P_m(t)\,dt$ for $m \neq 0$, we must consider the two cases: m being odd and m being even.

(a) m is even and $m \neq 0$

$$\int_0^1 P_m(t)\,dt = \frac{1}{2}\int_{-1}^{1} P_m(t)\,dt = \frac{1}{2}\int_{-1}^{1} P_m(t)\cdot 1\,dt = \frac{1}{2}\int_{-1}^{1} P_m(t) P_0(t)\,dt = 0$$

The result is due to the orthogonality principle.

(b) m is odd and $m \neq 0$. From the relation (see Table 5.2.2)

$$\int_t^1 P_m(t)\,dt = \frac{1}{2m+1}\big[P_{m-1}(t) - P_{m+1}(t)\big]$$

with $t = 0$ we obtain

$$\int_0^1 P_m(t)\,dt = \frac{1}{2m+1}\big[P_{m-1}(0) - P_{m+1}(0)\big]$$

TABLE 5.2.2 Properties of Legendre and Associate Legendre Functions

1.	$\dfrac{1}{\sqrt{1-2tx+x^2}} = \sum_{n=0}^{\infty} P_n(t)x^n,$	$\lvert t\rvert \le 1,\ \lvert x\rvert < 1$
2.	$P_n(t) = \sum_{k=0}^{[n/2]} \dfrac{(-1)^k(2n-2k)!t^{n-2k}}{2^n k!(n-k)!(n-2k)!},$	$[n/2] = \dfrac{n}{2}$, n is even; $[n/2] = (n-1)/2$, n is odd
3.	$P_0(t) = 1$	
4.	$P_{2n}(0) = \begin{pmatrix} -\frac{1}{2} \\ n \end{pmatrix} = \dfrac{(-1)^n(2n)!}{2^{2n}(n!)^2},$	$n = 1, 2, \ldots$
5.	$P_{2n+1}(0) = 0,$	$n = 0, 1, 2, \ldots$
6.	$P_{2n}(-t) = P_{2n}(t), \qquad P_{2n+1}(-t) = -P_{2n+1}(t),$	$n = 0, 1, 2, \ldots$
7.	$P_n(-t) = (-1)^n P_n(t),$	$n = 0, 1, 2, \ldots$
8.	$P_n(1) = 1,$	$n = 0, 1, 2, \ldots;$
	$P_n(-1) = (-1)^n,$	$n = 0, 1, 2, \ldots$
9.	$P_n(t) = \dfrac{1}{2^n n!}\dfrac{d^n}{dt^n}(t^2-1)^n =$ Rodrigues formula,	$n = 0, 1, 2, \ldots$
10.	$(n+1)P_{n+1}(t) - (2n+1)tP_n(t) + nP_{n-1}(t) = 0,$	$n = 1, 2, \ldots$
11.	$P'_{n+1}(t) - 2tP'_n(t) + P'_{n-1}(t) - P_n(t) = 0,$	$n = 1, 2, \ldots$
12.	$P'_{n-1}(t) = P_n(t) + 2tP'_n(t) - P'_{n+1}(t)$	$n = 1, 2, \ldots$
13.	$P'_{n+1}(t) = P_n(t) + 2tP'_n(t) - P'_{n-1}(t)$	$n = 1, 2, \ldots$
14.	$P'_{n+1}(t) - tP'_n(t) = (n+1)P_n(t)$	$n = 0, 1, 2, \ldots$
15.	$t\,P'_n(t) - P'_{n-1}(t) = nP_n(t)$	$n = 1, 2, \ldots$
16.	$P'_{n+1}(t) - P'_{n-1}(t) = (2n+1)P_n(t)$	$n = 1, 2, \ldots$
17.	$(1-t^2)P'_n(t) = nP_{n-1}(t) - ntP_n(t)$	$n = 1, 2, \ldots$
18.	$\lvert P_n(t)\rvert < 1,$	$-1 < t < 1$
19.	$P_{2n}(t) = \dfrac{(-1)^n}{2^{2n-1}} \sum_{k=0}^{n} \dfrac{(-1)^k(2n+2k-1)!}{(2k)!(n+k-1)!(n-k)!}t^{2k},$	$n = 0, 1, 2, \ldots$
20.	$(1-t^2)P'_n(t) = (n+1)[tP_n(t) - P_{n+1}(t)],$	$n = 0, 1, 2, \ldots$
21.	$\int_{-1}^{1} P_n(t)\,dt = 0,$	$n = 1, 2, \ldots$
22.	$\lvert P_n(t)\rvert \le 1,$	$\lvert t\rvert \le 1$
23.	$\int_{-1}^{1} P_n(t)P_m(t)\,dt = 0,$	$n \ne m$

24.	$\int_{-1}^{1} [P_n(t)]^2\,dt = \dfrac{2}{2n+1},$	$n = 0, 1, 2, \ldots$
25.	$\dfrac{1}{2}\int_{-1}^{1} t^m P_s(t)\,dt = \dfrac{m(m-2)\cdots(m-s+2)}{(m+s+1)(m+s-1)\cdots(m+1)},$	m, s are even
26.	$\dfrac{1}{2}\int_{-1}^{1} t^m P_s(t)\,dt = \dfrac{(m-1)(m-3)\cdots(m-s+2)}{(m+s+1)(m+s-1)\cdots(m+2)},$	m, s are odd
27.	$\int_{-1}^{1} t P_n(t) P_{n-1}(t)\,dt = \dfrac{2n}{4n^2-1},$	$n = 1, 2, \ldots$
28.	$\int_{-1}^{1} P_n(t) P'_{n+1}(t)\,dt = 2,$	$n = 0, 1, 2, \ldots$
29.	$\int_{-1}^{1} t P'_n(t) P_n(t)\,dt = \dfrac{2n}{2n+1},$	$n = 0, 1, 2, \ldots$
30.	$\int_{-1}^{1} (1-t^2) P'_n(t) P'_k(t)\,dt = 0,$	$k \neq n$
31.	$\int_{-1}^{1} (1-t)^{-1/2} P_n(t)\,dt = \dfrac{2\sqrt{2}}{2n+1},$	$n = 0, 1, 2, \ldots$
32.	$\int_{-1}^{1} t^2 P_{n+1}(t) P_{n-1}(t)\,dt = \dfrac{2n(n+1)}{(4n^2-1)(2n+3)},$	$n = 1, 2, \ldots$
33.	$\int_{-1}^{1} (t^2-1) P_{n+1}(t) P'_n(t)\,dt = \dfrac{2n(n+1)}{(2n+1)(2n+3)},$	$n = 1, 2, \ldots$
34.	$\int_{-1}^{1} t^n P_n(t)\,dt = \dfrac{2^{n+1}(n!)^2}{(2n+1)!},$	$n = 0, 1, 2, \ldots$
35.	$\int_{-1}^{1} t^2 [P_n(t)]^2\,dt = \dfrac{2}{(2n+1)^2}\left[\dfrac{(n+1)^2}{2n+3} + \dfrac{n^2}{2n-1}\right]$	$n = 0, 1, 2, \ldots$
36.	$P_n^m(t) = (1-t^2)^{m/2} \dfrac{d^m}{dt^m} P_n(t),$	$m > 0$
37.	$P_n^m(t) = \dfrac{1}{2^n n!}(1-t^2)^{m/2} \dfrac{d^{n+m}}{dt^{n+m}}[(t^2-1)^n],$	$m + n \geq 0$
38.	$P_n^{-m}(t) = (-1)^m \dfrac{(n-m)!}{(n+m)!} P_n^m(t)$	
39.	$P_n^0(t) = P_n(t)$	
40.	$(n-m+1) P_{n+1}^m(t) - (2n+1) t P_n^m(t) + (n+m) P_{n-1}^m(t) = 0$	
41.	$(1-t^2)^{1/2} P_n^m(t) = \dfrac{1}{2n+1}\left[P_{n+1}^{m+1}(t) - P_{n-1}^{m+1}(t)\right]$	
42.	$(1-t^2)^{1/2} P_n^m(t) = \dfrac{1}{2n+1}\left[(n+m)(n+m-1) P_{n-1}^{m-1}(t) - (n-m+1)(n-m+2) P_{n+1}^{m-1}(t)\right]$	
43.	$P_n^{m+1}(t) = 2mt(1-t^2)^{-1/2} P_n^m(t) - [n(n+1) - m(m-1)] P_n^{m-1}(t)$	

44.	$\int_{-1}^{1} P_n^m(t) P_k^m(t)\, dt = 0\,,$	$k \neq n$
45.	$\int_{-1}^{1} \left[P_n^m(t)\right]^2 dt = \dfrac{2}{2n+1}\dfrac{(n+m)!}{(n-m)!}$	
46.	$P_n^m(-t) = (-1)^{n+m} P_n^m(t)$	
47.	$P_n^m(\pm 1) = 0\,,$	$m > 0$
48.	$P_{2n}^1(0) = 0\,, \quad P_{2n+1}^1(0) = \dfrac{(-1)^n (2n+1)!}{2^{2n}(n!)^2}$	
49.	$P_n^m(0) = 0\,,$	$n + m$ is odd
	$P_n^m(0) = (-1)^{(n-m)/2} \dfrac{(n+m)!}{2^n[(n-m)/2]![(n+m)/2]!}\,,$	$n + m$ is even
50.	$\int_{-1}^{1} P_n^m(t) P_n^k(t)(1-t^2)^{-1} dt = 0\,,$	$k \neq m$
51.	$\int_{-1}^{1} (1-t^2)^{-1/2} P_{2m}(t)\, dt = \left[\dfrac{\Gamma\left(\frac{1}{2}+m\right)}{m!}\right]^2$	
52.	$\int_{-1}^{1} t(1-t^2)^{-1/2} P_{2m+1}(t)\, dt = \dfrac{\Gamma\left(\frac{1}{2}+m\right)\Gamma\left(\frac{3}{2}+m\right)}{m!(m+1)!}$	
53.	$\int_{t}^{1} P_n(t)\, dt = \dfrac{1}{2n+1}\left[P_{n-1}(t) - P_{n+1}(t)\right]$	
54.	$\int_{0}^{1} t^q P_n(t)\, dt = \Gamma(q+1)\sum_{k=0}^{n} \dfrac{(-1)^k \Gamma(n+k+1)}{2^k k!\Gamma(n-k+1)\Gamma(q+k+2)}\,,$	$q > -1$
55.	$\int_{0}^{1} t^{-1/2} P_n(t)\, dt = \begin{cases} \dfrac{2(-1)^{n/2}}{2n+1} & n \text{ is even} \\ \dfrac{2(-1)^{(n-1)/2}}{2n+1} & n \text{ is odd} \end{cases}$	
56.	$\int_{0}^{1} t^{1/2} P_n(t)\, dt = \begin{cases} \dfrac{2(-1)^{(n+2)/2}}{(2n-1)(2n+3)} & n \text{ is even} \\ \dfrac{2(-1)^{(n+3)/2}}{(2n-1)(2n+3)} & n \text{ is odd} \end{cases}$	

Using the results of the previous example, we obtain

$$\int_0^1 P_m(t)dt = \frac{1}{2m+1}\left[\frac{(-1)^{\frac{m-1}{2}}(m-1)!}{2^{m-1}\left[\left(\frac{m-1}{2}\right)!\right]^2} - \frac{(-1)^{\frac{m+1}{2}}(m+1)!}{2^{m+1}\left[\left(\frac{m+1}{2}\right)!\right]^2}\right]$$

$$= \frac{(-1)^{\frac{m-1}{2}}(m-1)!(2m+1)(m+1)}{(2m+1)2^{m+1}\left(\frac{m+1}{2}\right)!\left(\frac{m+1}{2}\right)\left(\frac{m-1}{2}\right)!} = \frac{(-1)^{\frac{m-1}{2}}(m-1)!}{2^m\left(\frac{m+1}{2}\right)!\left(\frac{m-1}{2}\right)!}.$$

m is odd

Example

One hemisphere of a homogeneous spherical solid is maintained at 300°C while the other half is kept at 75°C. To find the temperature distribution we must use the equation for heat conduction

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c}\left(\nabla^2 T + \frac{\partial Q}{\partial t}\right)$$

where T is temperature, t is time, k is the thermal conductivity, ρ is the density, c is specific heat, and $\partial Q/\partial t$ is the rate of heat generation. Because of the steady-state condition of the problem, $\partial T/\partial t = \partial Q/\partial t = 0$. Hence, the equation becomes

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial T}{\partial\varphi}\right) = 0$$

where T is independent of θ.

Assuming a solution of the form

$$T = FG = f(r)g(\varphi)$$

we obtain

$$\frac{\partial T}{\partial r} = G\frac{dF}{dr}, \quad \frac{\partial^2 T}{\partial r^2} = G\frac{d^2F}{dr^2}$$

Similarly, we obtain

$$\frac{\partial T}{\partial\varphi} = F\frac{dG}{d\varphi}, \quad \frac{\partial^2 T}{\partial\varphi^2} = F\frac{d^2G}{d\varphi^2}$$

Introducing these relations in the Laplacian, we obtain

$$2rG\frac{dF}{dr} + r^2G\frac{d^2F}{dr^2} + F\frac{dG}{d\varphi}\cot\varphi + F\frac{d^2G}{d\varphi^2} = 0$$

or

$$\frac{2r\dfrac{dF}{dr}+r^2\dfrac{d^2F}{dr^2}}{F}=-\frac{\dfrac{dG}{d\varphi}\cot\varphi+\dfrac{d^2G}{dr^2}}{G}$$

Setting the above ratios equal to positive constant k^2, $k \neq 0$, we obtain

$$r^2\frac{d^2F}{dr^2}+2r\frac{dF}{dr}-k^2F=0$$

$$\frac{d^2G}{d\varphi^2}+(\cot\varphi)\frac{dG}{d\varphi}+k^2G=0$$

For $k^2 = n(n + 1)$, we recognize that the above equation is the Legendre equation with G playing the role of y. Thus, a particular solution is

$$G = C_n P_n(\cos\ \varphi)$$

where C_n is an arbitrary constant. With $k^2 = n(n + 1)$ the general solution for F is given by

$$F=S_n r^n+\frac{B_n}{r^{n+1}}$$

where S_n and B_n are arbitrary constants. Because for $r = 0$ the second term becomes infinity, we set B_n = 0. Hence, the product solution is

$$T = FG = S_n C_n r^n P_n(\cos\ \varphi) = D_n r^n P_n(\cos\ \varphi)$$

Because Legendre polynomials are continuous we must create a procedure to alleviate this problem. We denote the excess of the temperature T on the upper half of the surface over that of T on the lower half. On the bounding great circle between these halves, we arbitrarily set it equal to (300 – 75)/2. We then have

$$T_E(\varphi)=\begin{cases}225 & 0\le\varphi<\pi/2\\ 0 & \pi/2<\varphi\le\pi\\ 225/2 & \varphi=\pi/2\end{cases}$$

If we let $x = \text{cost}\ \varphi$, then $T_E(\varphi)$ becomes $f(x)$

$$f(x)=\begin{cases}225 & 0<x\le 1\\ 0 & -1\le x<0\\ 225/2 & x=0\end{cases}$$

Next we expand $f(x)$ in the form

$$f(x)=\sum_{n=0}^{\infty}a_nP_n(x),\quad a_n=\frac{2n+1}{2}\int_0^1 f(x)P_n(x)dx$$

$$=225\left[\frac{1}{2}+\frac{3}{4}P_1(x)-\frac{7}{16}P_3(x)+\frac{11}{32}P_5(x)-\cdots\right]$$

Setting $D_n = a_n/R^n$, where a_n is the coefficient of $P_n(x)$ and R is the radius of the solid, the solution is given by

$$T(r,\varphi) = 75 + \sum_{n=0}^{\infty} a_n \left(\frac{r}{R}\right)^n P_n(\cos\varphi)$$

$$= 75 + 225\left[\frac{1}{2} + \frac{3}{4}\left(\frac{r}{R}\right) P_1(\cos\varphi) - \frac{7}{16}\left(\frac{r}{R}\right)^2 P_3(\cos\varphi)\right.$$

$$\left. + \frac{11}{32}\left(\frac{r}{R}\right)^5 P_5(\cos\varphi) - \cdots\right]$$

Table 5.2.2 gives relationships of Legendre and associated Legendre functions.

1.5.3 Hermite Polynomials

1.5.3.1 Generating Function

If we define the Hermite polynomial by the Rodrigues formula

$$H_n(t) = (-1)^n e^{t^2} \frac{d^n e^{-t^2}}{dt^n}, \quad n = 0, 1, 2, \ldots, -\infty < t < \infty \tag{5.3.1}$$

The first few Hermite polynomials are

$$\begin{aligned} H_0(t) &= 1, \\ H_1(t) &= 2t, \\ H_2(t) &= 4t^2 - 2, \\ H_3(t) &= 8t^3 - 12t, \\ H_4(t) &= 16r^4 - 48t + 12, \\ H_5(t) &= 32t^5 - 160t^3 + 120t \end{aligned}$$

and therefore

$$H_n(t) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{k!(n-2k)!} (2t)^{n-2k} \tag{5.3.2}$$

$$[n/2] \equiv \text{largest integer} \le n/2$$

The Hermite polynomials are orthogonal with weight $\gamma(t) = e^{-t^2}$ on the interval $(-\infty, \infty)$.

The relation between Hermite polynomial and the generating function is

$$w(t,x) = e^{2tx - x^2} = \sum_{n=0}^{\infty} \frac{H_n(t)}{n!} x^n, \quad |x| < \infty \tag{5.3.3}$$

Because $w(t, x)$ is the entire function in x it can be expanded in Taylor's series at $x = 0$ with $|x| < \infty$. Hence the derivatives of the expansion are

$$\left.\left(\frac{\partial^n w}{\partial x^n}\right)\right|_{x=0} = e^{t^2}\left[\frac{\partial^n}{\partial x^n}e^{-(t-x)^2}\right]_{x=0} = (-1)^n e^{t^2}\left[\frac{d^n e^{-u^2}}{du^n}\right]_{u=t} \doteq H_n(t)$$

Figure 5.3.1 shows several Hermite polynomials.

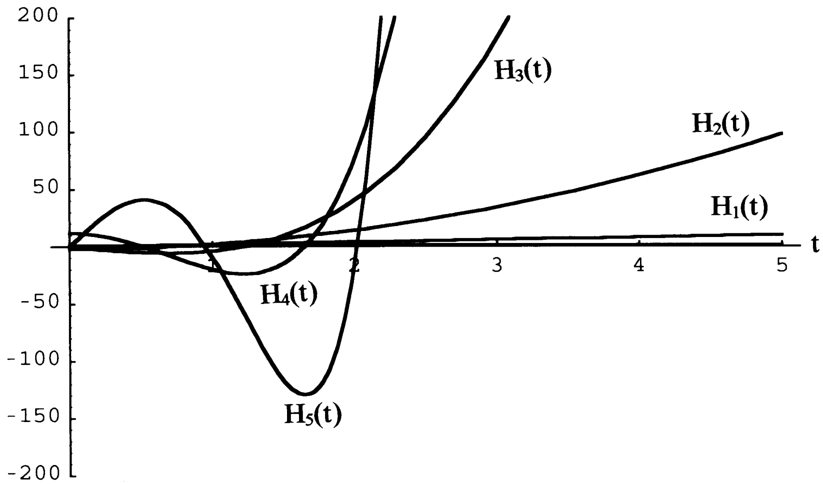


FIGURE 5.3.1

Example

Let $t = 0$ in (5.3.3) and expand e^{-x^2} in power series. Comparing equal powers of both sides we find that

$$H_{2n}(0) = (-1)^n \frac{(2n)!}{n!}$$

Hermite polynomials are even for even n and odd for n odd. Hence,

$$H_n(-t) = (-1)^n H_n(t) \tag{5.3.4}$$

1.5.3.2 Recurrence Relation

If we substitute $w(t, x)$ of (5.3.3) into identity

$$\frac{\partial w}{\partial x} - 2(t-x)w = 0$$

we obtain

$$\sum_{n=0}^{\infty} \frac{H_{n+1}(t)}{n!}x^n - 2t\sum_{n=0}^{\infty} \frac{H_n(t)}{n!}x^n + 2\sum_{n=0}^{\infty} \frac{H_n(t)}{n!}x^{n+1} = 0$$

or

$$\sum_{n=1}^{\infty} \left[H_{n+1}(t) - 2tH_n(t) + 2nH_{n-1}(t)\right]\frac{x^n}{n!} + H_1(t) - 2tH_0(t) = 0$$

But $H_1(t) - 2tH_0(t) = 0$ and hence

$$H_{n+1}(t) - 2tH_n(t) + 2nH_{n-1}(t) = 0, \quad n = 1, 2, \ldots \tag{5.3.5}$$

If we use

$$\frac{\partial w}{\partial x} - 2xw = 0$$

we obtain

$$H_n'(t) = 2nH_{n-1}(t), \quad n = 1, 2, \ldots \tag{5.3.6}$$

Eliminating $H_{n-1}(t)$ from (5.3.6) and (5.3.5), we obtain

$$H_{n+1}(t) - 2tH_n(t) + H_n'(t) = 0, \quad n = 0, 1, 2, \ldots \tag{5.3.7}$$

Differentiate (5.3.6), combine with (5.3.5), and use the relation $H'_{n+1} = 2(n+1)H_{(n+1)-1}$, we obtain

$$H_n'' - 2tH_n'(\mathrm{t}) + 2nH_n(t) = 0, \quad n = 0, 1, 2, \ldots \tag{5.3.8}$$

From the above equation, with $y = H_n(t)$ $(n = 0, 1, 2, \ldots)$, we observe that the Hermite polynomials are the solution to the second-order ordinary differential equation known as the **Hermite equation**

$$y'' - 2\mathrm{t}y' + 2ny = 0 \tag{5.3.9}$$

1.5.3.3 Integral Representation and Integral Equation

The integral representation of Hermite polynomials is given by

$$H_n(t) = \frac{(-j)^n 2^n e^{t^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2 + j2tx} x^n dx, \quad n = 0, 1, 2, \ldots \tag{5.3.10}$$

The integral equation satisfied by the Hermite polynomials is

$$e^{-t^2/2} H_n(t) = \frac{1}{j^n \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{jty} e^{-y^2/2} H_n(y) dy, \quad n = 0, 1, 2, \ldots \tag{5.3.10a}$$

Also, because $H_{2m}(t)$ is an even function and $H_{2m+1}(t)$ is an odd function, then the above equation implies the following two integrals:

$$e^{-t^2/2} H_{2m}(t) = (-1)^m \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-y^2\ 2} H_{2m}(y) \cos ty\, dy$$

$$e^{-t^2/2} H_{2m+1}(t) = (-1)^m \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-y^2\ 2} H_{2m+1}(y) \sin ty\, dy, \quad m = 0, 1, 2, \ldots \tag{5.3.11}$$

1.5.3.4 Orthogonality Relation: Hermite Series

The **orthogonality property** of the Hermite polynomials is given by

$$\int_{-\infty}^{\infty} e^{-t^2} H_m(t) H_n(t) dt = 0 \quad \text{if } m \neq n \tag{5.3.12}$$

and

$$\int_{-\infty}^{\infty} e^{-t^2} H_n^2(t)dt = 2^n n! \sqrt{\pi}, \quad n = 0, 1, 2, \ldots \tag{5.3.13}$$

Therefore, the **orthonormal** Hermite polynomials are

$$\varphi_n(t) = \left(2^n n! \sqrt{\pi}\right)^{-1/2} e^{-t^2/2} H_n(t), \quad n = 0, 1, 2, \ldots, \quad -\infty < t < \infty \tag{5.3.14}$$

Theorem 5.3.1
If $f(t)$ is piecewise smooth in every finite interval $[-a, a]$ and

$$\int_{-\infty}^{\infty} e^{-t^2} f^2(t)dt < \infty$$

then the Hermite series

$$f(t) = \sum_{n=0}^{\infty} C_n H_n(t), \quad -\infty < t < \infty \tag{5.3.15}$$

$$C_n = \frac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} f(t) H_n(t) dt \quad n = 0, 1, 2, \ldots \tag{5.3.16}$$

converges pointwise to $f(t)$ at every continuity point and converges at $[f(t+) - f(t-)]/2$ at points of discontinuity.

Example
The function $f(t) = t^{2p}$, $p = 1, 2, \ldots$ satisfies Theorem 5.3.1 and it is even. Hence,

$$t^{2p} = \sum_{n=0}^{p} C_{2n} H_{2n}(t)$$

where

$$\begin{aligned} C_{2n} &= \frac{1}{2^{2n}(2n)!\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} t^{2p} H_{2n}(t)dt \\ &= \frac{1}{2^{2n}(2n)!\sqrt{\pi}} \int_{-\infty}^{\infty} t^{2p} \frac{d^{2n}}{dt^{2n}}\left(e^{-t^2}\right)dt = \frac{1}{2^{2n}(2n)!\sqrt{\pi}} \frac{(2p)!}{(2p-2n)!} \int_{-\infty}^{\infty} e^{-t^2} t^{2p-2n} dt \\ &= \frac{1}{2^{2n}(2n)!\sqrt{\pi}} \frac{(2p)!}{(2p-2n)!} \Gamma\left(p - n + \frac{1}{2}\right) \end{aligned}$$

to find C_{2n}, integration by parts was performed n times.

Example
The function e^{at}, where a is an arbitrary number, satisfies Theorem 5.3.1. Hence

$$e^{at} = \sum_{n=0}^{\infty} C_n H_n(t)$$

where

$$C_n = \frac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{at} e^{-t^2} H_n(t) dt = \frac{(-1)^n}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{at} \frac{d^n}{dt^n}\left(e^{-t^2}\right) dt$$

$$= \frac{a^n}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{at - t^2} dt = \frac{a^n}{2^n n!} e^{-a^2 4}$$

Example

The $\operatorname{sgn}(t)$ function is odd and hence its expansion takes the form

$$\operatorname{sgn}(t) = \sum_{n=0}^{\infty} C_{2n+1} H_{2n+1}(t)$$

where

$$C_{2n+1} = \frac{1}{2^{2n+1}(2n+1)! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} H_{2n+1}(t) \operatorname{sgn}(t) dt$$

$$= \frac{1}{2^{2n}(2n+1)! \sqrt{\pi}} \int_{0}^{\infty} e^{-t^2} H_{2n+1}(t) dt$$

Use the identity

$$e^{-t^2} H_n(t) = -\frac{d}{dt}\left[e^{-t^2} H_{n-1}(t)\right]$$

which results from (5.3.5) and (5.3.6), to find that

$$C_{2n+1} = \frac{H_{2n}(0)}{2^{2n}(2n+1)! \sqrt{\pi}} = \frac{(-1)^n}{2^{2n}(2n+1)! \, n! \sqrt{\pi}}$$

Table 5.3.1 gives the Hermite relationships.

1.5.4 Laguerre Polynomials

Generating Function and Rodrigues Formula

The generating function for the Laguerre polynomials is given by

$$w(t, x) = (1-x)^{-1} \exp\left[-\frac{tx}{1-x}\right] = \sum_{n=0}^{\infty} L_n(t) x^n, \quad |x| < 1, \quad 0 \le t < \infty \tag{5.4.1}$$

By expressing the exponential function in a series, realizing that

TABLE 5.3.1 Properties of the Hermite Polynomials

1. $H_n(t) = (-1)^n e^{t^2} \dfrac{d^n e^{-t^2}}{dt^n}$

2. $H_n(t) = \sum_{k=0}^{[n/2]} \dfrac{(-1)^k n!}{k!(n-2k)!}(2t)^{n-2k}$

   $[n/2] =$ largest integer $\leq n/2$

3. $e^{2tx-x^2} = \sum_{n=0}^{\infty} H_n(t) \dfrac{x^n}{n!}$

4. $H_{2n}(0) = (-1)^n \dfrac{(2n)!}{n!}$

5. $H_{2n+1}(0) = 0,\ H'_{2n}(0) = 0,\ H'_{2n+1}(0) = (-1)^n \dfrac{(2n+2)!}{(n+1)!}$

6. $H_n(-t) = (-1)^n H_n(t)$

7. $H_{2n}(t)$ are even functions, $\qquad H_{2n+1}(t)$ are odd functions

8. $H_{n+1}(t) - 2tH_n(t) + 2nH_{n-1}(t) = 0, \qquad n = 1, 2, \ldots$

9. $H'_n(t) = 2nH_{n-1}(t), \qquad n = 1, 2, \ldots$

10. $H_{n+1}(t) - 2tH_n(t) + H'_n(t) = 0 \qquad n = 0, 1, 2, \ldots$

11. $H''_n(t) - 2tH'_n(t) + 2nH_n(t) = 0 \qquad n = 0, 1, 2, \ldots$

12. $H_n(t) = \dfrac{(-j)^n 2^n e^{t^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2+j2tx} x^n \, dx \qquad n = 0, 1, 2, \ldots$

13. $e^{-t^2/2} H_n(t) = \dfrac{1}{j^n \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{jty} e^{-y^2/2} H_n(y) \, dy =$ integral equation

14. $e^{-t^2/2} H_{2m}(t) = (-1)^m \sqrt{\dfrac{2}{\pi}} \int_0^{\infty} e^{-y^2/2} H_{2m}(y) \cos ty \, dy$

15. $e^{-t^2/2} H_{2m+1}(t) = (-1)^m \sqrt{\dfrac{2}{\pi}} \int_0^{\infty} e^{-y^2/2} H_{2m+1}(y) \sin ty \, dy$

16. $\int_{-\infty}^{\infty} e^{-t^2} H_m(t) H_n(t) \, dt = 0, \qquad$ if $m \neq n$

17. $\int_{-\infty}^{\infty} e^{-t^2} H_n^2(t) \, dt = 2^n n! \sqrt{\pi} \qquad n = 0, 1, 2, \ldots$

18. $f(t) = \sum_{n=0}^{\infty} C_n H_n(t) \qquad -\infty < t < \infty$

    $C_n = \dfrac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} f(t) H_n(t) \, dt$

19. $\int_{-\infty}^{\infty} t^k e^{-t^2} H_n(t) \, dt = 0, \qquad k = 0, 1, 2, \ldots, n-1$

TABLE 5.3.1 Properties of the Hermite Polynomials (Continued)

20.	$\int_{-\infty}^{\infty} t^2 e^{-t^2} H_n^2(t)\,dt = \sqrt{\pi}2^n n!\left(n+\frac{1}{2}\right)$
21.	$\int_{-\infty}^{\infty} x^n e^{-x^2} H_n(tx)\,dx = \frac{\sqrt{\pi}n!}{2} P_n(t)$
22.	$\int_{-\infty}^{\infty} e^{-2t^2} H_n^2(t)\,dt = 2^{n-\frac{1}{2}}\Gamma\left(n+\frac{1}{2}\right)$
23.	$\frac{d^m H_n(t)}{dt^m} = \frac{2^m n!}{(n-m)!} H_{n-m}(t), \qquad m < n$
24.	$\int_{-\infty}^{\infty} e^{-a^2t^2} H_{2n}(t)\,dt = \frac{(2n)!}{n!}\frac{\sqrt{\pi}}{a}\left(\frac{1-a^2}{a^2}\right)^n, \qquad a > 0$

$$\binom{-k-1}{m} = (-1)^m \binom{k+m}{m}$$

and finally making the change of index $m = n - k$, (5.4.1) leads to

$$L_n(t) = \sum_{k=0}^{n} \frac{(-1)^k n!\,t^k}{(k!)^2 (n-k)!} \qquad n = 0, 1, 2, \ldots, \; 0 \le t < \infty \tag{5.4.2}$$

The Rodrigues formula for creating Laguerre polynomials is given by

$$L_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}\left(t^n e^{-t}\right), \quad n = 0, 1, 2, \ldots \tag{5.4.3}$$

which can be verified by application of the Leibniz formula

$$\frac{d^n}{dt^n}(fg) = \sum_{k=0}^{n} \binom{n}{k} \frac{d^{n-k}}{dt^{n-k}} \frac{d^k g}{dt^k}, \quad n = 1, 2, \ldots \tag{5.4.4}$$

For a real $a > -1$ the general Laguerre polynomials are defined by the formula

$$L_n^a(t) = e^t \frac{t^{-a}}{n!}\frac{d^n}{dt^n}\left(e^{-t} t^{n+a}\right), \quad n = 0, 1, 2, \ldots \tag{5.4.5a}$$

Using Leibniz's formula

$$L_n^a(t) = \sum_{k=0}^{n} \frac{\Gamma(n+a+1)}{\Gamma(k+a+1)} \frac{(-t)^k}{k!(n-k)!} \tag{5.4.5b}$$

Table 5.4.1 gives a few Laguerre polynomials. Figure 5.4.1 shows several Laguerre polynomials.

Recurrence Relations

The generating function $w(t, x)$, (5.4.1) satisfies the identity

TABLE 5.4.1 Laguerre Polynomials

$L_0(t) = 1$
$L_1(t) = -t + 1$
$L_2(t) = \frac{1}{2!}(t^2 - 4t + 2)$
$L_3(t) = \frac{1}{3!}(-t^3 + 9t^2 - 18t + 6)$
$L_4(t) = \frac{1}{4!}(t^4 - 16t^3 + 72t^2 - 96t + 24)$

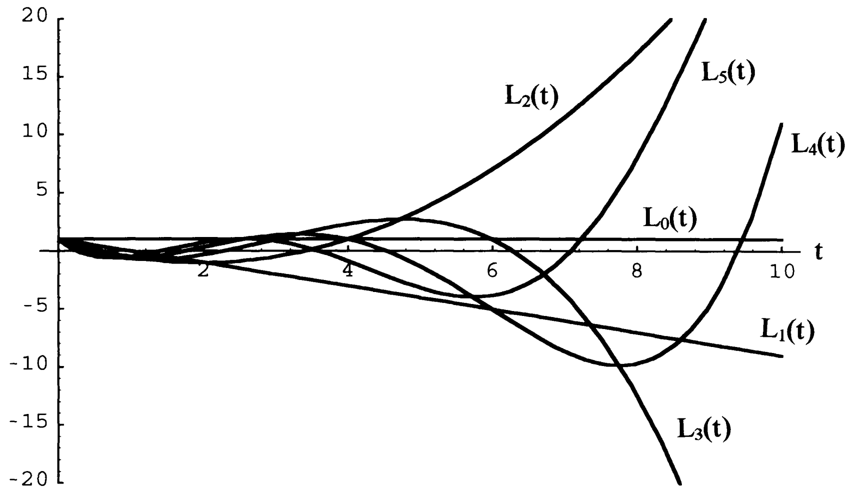


FIGURE 5.4.1

$$\left(1-x^2\right)\frac{\partial w}{\partial x}+\left(t-1+\right)w=0 \qquad (5.4.6)$$

Substituting (5.4.1) in (5.4.6) and equating the coefficients of x^n to zero, we obtain

$$(n+1)L_{n+1}(t)+(t-1-2n)L_n(t)+nL_{n-1}(\mathrm{t})=0, \quad n=1,2,\ldots \qquad (5.4.7)$$

Similarly substituting (5.4.1) into

$$\left(1-x\right)\frac{\partial w}{\partial t}+xw=0 \qquad (5.4.8)$$

we obtain the relation

$$L_n'(t)-L_{n-1}'(t)+L_{n-1}(t)=0, \quad n=1,2,\ldots \qquad (5.4.9)$$

From this we obtain

$$L'_{n+1}(t) = L'_n(t) - L_n(t) \tag{5.4.10}$$

$$L'_{n-1}(t) = L'_n(t) + L_{n-1}(t) \tag{5.4.11}$$

From (5.4.7) by differentiation we find

$$(n+1)L'_{n+1}(t) + (t-1-2n)L'_n(t) + L_n(t) + nL'_{n-1}(t) = 0 \tag{5.4.12}$$

Eliminating $L'_{n+1}(t)$ and $L'_{n-1}(t)$ by using (5.4.10), (5.4.11), and (5.4.12), we obtain

$$tL_n'(t) = nL_n(t) - nL_{n-1}(t) \tag{5.4.13}$$

By differentiating (5.4.13) and using (5.4.9), we obtain

$$tL_n''(t) + L_n'(t) = -nL_{n-1}(t)$$

Next, eliminating $L_{n-1}(t)$ using (5.4.13) we obtain

$$tL_n''(t) + (1-t)L_n'(t) + nL_n(t) = 0 \tag{5.4.14}$$

Setting $y = L_n(t)$ $(n = 0, 1, 2, \ldots)$, we conclude that all $L_n(t)$ are the solution to the Laguerre equation

$$ty'' + (1-t)y' + ny = 0 \tag{5.4.15}$$

Orthogonality, Laguerre Series

The orthogonality relations for Laguerre polynomials are

$$\int_0^\infty e^{-t} L_n(t) L_m(t)\,dt = 0, \quad n \neq m \tag{5.4.16}$$

$$\int_0^\infty e^{-t}\left[L_n(t)\right]^2 dt = \frac{\Gamma(n+1)}{n!} = 1, \quad n = 0, 1, 2, \ldots \tag{5.4.17}$$

For the generalized Laguerre polynomials, the orthogonality relations

$$\int_0^\infty e^{-t} t^a L_m^a(t) L_n^a(t)\,dt = 0, \quad n \neq m,\ a > -1$$

$$\int_0^\infty e^{-t} t^a \left[L_n^a(t)\right]^2 dt = \frac{\Gamma(n+a+1)}{n!}, \quad a > -1,\ n = 0, 1, 2, \ldots \tag{5.4.18}$$

The orthogonal system for the generalized polynomials on the interval $0 \le t < \infty$ is

$$\varphi_n^a(t) = \left[\frac{n!}{\Gamma(n+a+1)}\right]^{1/2} e^{-t/2} t^{a/2} L_n^a(t), \quad n = 0, 1, 2, \ldots \tag{5.4.19}$$

The Laguerre series is given by

$$f(t)=\sum_{n=0}^{\infty} C_n L_n(t), \quad 0\le t<\infty \tag{5.4.20}$$

where

$$C_n=\int_0^{\infty} e^{-t} f(t) L_n(t)\,dt, \quad n=0,1,2,\dots \tag{5.4.21}$$

Theorem 5.4.1

If $f(t)$ is piecewise smooth in every finite interval $t_1 \le t \le t_2$, $0 < t_1 < t_2 < \infty$ and

$$\int_0^{\infty} e^{-t} f^2(t)\,dt<\infty$$

then the Laguerre series converges pointwise to $f(t)$ at every continuity point of $f(t)$, and at the points of discontinuity the series converges to $[f(t+) - f(t-)]/2$.

If we set $a = m =$ integer ($m = 0, 1, 2, \dots$), then (5.4.5b) becomes

$$L_n^m(t)=\sum_{k=0}^{n} \frac{(-1)^k (n+m)!\,t^k}{(n-k)!\,(m+k)!\,k!}, \quad m=0,1,2,\dots \tag{5.4.22}$$

The Rodrigues formula is

$$L_n^m(t)=\frac{1}{n!} e^t t^{-m} \frac{d^n}{dt^n}\left(e^{-t} t^{n+m}\right) \tag{5.4.23}$$

Example

The function t^b can be expanded in series

$$t^b=\sum_{n=0}^{\infty} C_n L_n^a(t), \quad b>-\frac{1}{2}(a+1)$$

$$\begin{aligned}
C_n &= \frac{n!}{\Gamma(n+a+1)} \int_0^{\infty} t^{b+a} e^{-t} L_n(t)\,dt \\
&= \frac{n!}{\Gamma(n+a+1)} \int_0^{\infty} e^{-t} t^{b+a} \frac{e^t t^{-a}}{n!} \frac{d^n}{dt^n}\left(t^{n+a} e^{-t}\right) dt \\
&= \frac{1}{\Gamma(n+a+1)} \int_0^{\infty} t^b \frac{d^n}{dt^n}\left(t^{n+a} e^{-t}\right) dt \\
&= \frac{(-1)^n b(b-1)\cdots(b-n+1)}{\Gamma(n+a+1)} \int_0^{\infty} e^{-t} t^{b+a}\,dt \\
&= (-1)^n \frac{\Gamma(b+1)}{\Gamma(n+b+1)\Gamma(b-n+1)} \int_0^{\infty} e^{-t} t^{(b+a+1)-1}\,dt \\
&= (-1)^n \frac{\Gamma(b+1)\Gamma(b+a+1)}{\Gamma(n+b+1)\Gamma(b-n+1)}
\end{aligned}$$

The steps to find C_n were: a) substitution of (5.4.5), b) integration by parts n times, and c) multiplication of numerator and denominator by $\Gamma(b - n + 1)$. In particular if $b = m =$ positive integer

$$t^m = \Gamma(m+a+1)m!\sum_{n=0}^{m}\frac{(-1)^n L_n^a(t)}{\Gamma(n+a+1)(m-n)!},$$

$$0 < t < \infty,\ a > -1, \text{ and } m = 0, 1, 2, \ldots$$

If $a = 0$, we obtain the expansion

$$t^m = \Gamma(m+1)m!\sum_{n=0}^{m}\frac{(-1)^m L_n(t)}{n!(m-n)!}$$

Example

The function $f(t) = e^{-bt}$, with $b > -1/2$ and $t > 0$, is expanded as follows

$$C_n = \frac{n!}{\Gamma(n+a+1)}\int_0^\infty e^{-(b+1)t} t^a L_n^a(t)dt = \frac{1}{\Gamma(n+a+1)}\int_0^\infty e^{-bt}\frac{d^n}{dt^n}\left(e^{-t}t^{n+a}\right)dt$$

$$= \frac{b^b}{\Gamma(n+a+1)}\int_0^\infty e^{-(b+1)t} t^{n+a}dt = \frac{b^n}{(b+1)^{n+a+1}}, \quad n = 0, 1, 2, \ldots$$

and thus

$$e^{-bt} = (b+1)^{-a-1}\sum_{n=0}^{\infty}\left(\frac{b}{b+1}\right)^n L_n^a(t), \quad 0 \le t < \infty$$

For $a = 0$

$$e^{-bt} = (b+1)^{-1}\sum_{n=0}^{\infty}\left(\frac{b}{b+1}\right)^n L_n(t), \quad 0 \le t < \infty$$

Table 5.4.2 gives relationships of Laguerre polynomials.

1.5.5 Chebyshev Polynomials

The Chebyshev polynomials can be derived from the Gegenbauer polynomials, and are given

$$T_n(t) = \frac{n}{2}\sum_{k=0}^{[n/2]}\frac{(-1)^k(n-k-1)!}{k!(n-2k)!}(2t)^{n-2k}, \quad -1 < t < 1 \tag{5.5.1}$$

The Chebyshev polynomials of the second kind are simply

$$U_n(t) = C_n^1(t), \quad n = 0, 1, 2, \ldots \tag{5.5.2a}$$

TABLE 5.4.2 Properties of the Laguerre Polynomials

1.	$L_n(t)=\sum_{k=0}^{n}\frac{(-1)^k n!t^k}{(k!)^2(n-k)!}=\sum_{k=0}^{n}(-1)^k\frac{1}{k!}\binom{n}{k}t^k$	$0\le t<\infty,\ n=0,1,2,\ldots$
2.	$L_n(t)=\frac{e^t}{n!}\frac{d^n}{dt^n}\left(t^n e^{-t}\right)$	$n=0,1,2,\ldots$
3.	$(n+1)L_{n+1}(t)+(t-1-2n)L_n(t)+nL_{n-1}(t)=0$	$n=1,2,3,\ldots$
4.	$L_n'(t)-L_{n-1}'(t)+L_{n-1}(t)=0$,	$n=1,2,3,\ldots$
5.	$(n+1)L_{n+1}'(t)+(t-1-2n)L_n'(t)+L_n(t)+nL_{n-1}'(t)=0$,	$n=1,2,3,\ldots$
6.	$L_{n+1}'(t)=L_n'(t)-L_n(t)$	
7.	$tL_n'(t)=nL_n(t)-nL_{n-1}(t)$,	$n=1,2,3,\ldots$
8.	$tL_n''(t)+(1-t)L_n'(t)+nL_n(t)=0$, Laguerre differential equation	
9.	$w(t,x)=(1-x)^{-1}\exp\left[-\frac{tx}{1-x}\right]=\sum_{n=0}^{\infty}L_n(t)x^n$, generating function	
10.	$\int_0^{\infty}e^{-t}L_n(t)L_k(t)dt=0$,	$k\ne n$
11.	$\int_0^{\infty}e^{-t}\left[L_n(t)\right]^2 dt=1$	
12.	$f(t)=\sum_{n=0}^{\infty}C_nL_n(t)$, $C_n=\int_0^{\infty}e^{-t}f(t)L_n(t)dt$	$0\le t<\infty$
13.	$L_n(0)=1,\ L_n'(0)=-n,\ L_n''(0)=\frac{1}{2}n(n-1)$	
14.	$L_n^m(t)=(-1)^m\frac{d^m}{dt^m}\left[L_{n+m}(t)\right]$,	$m=0,1,2,\ldots$
15.	$L_n^m(t)=\sum_{k=0}^{n}\frac{(-1)^k(n+m)!t^k}{(n-k)!(m+k)!k!}$,	$m=0,1,2,\ldots$
16.	$(n+1)L_{n+1}^m(t)+(t-1-2n-m)L_n^m(t)+(n+m)L_{n-1}^m(t)=0$	
17.	$tL_n^{m\prime}(t)-nL_n^m(t)+(n+m)L_{n-1}^m(t)=0$	
18.	$L_n^m(t)=\frac{1}{n!}e^t t^{-m}\frac{d^n}{dt^n}\left(e^{-t}t^{n+m}\right)=$ Rodrigues formula	
19.	$L_{n-1}^m(t)+L_n^{m-1}(t)-L_n^m(t)=0$	
20.	$L_n^{m\prime}(t)=-L_{n-1}^{m+1}(t)$	
21.	$L_n^m(0)=\frac{(n+m)!}{n!m!}$	

TABLE 5.4.2 Properties of the Laguerre Polynomials (Continued)

22.	$\int_0^\infty e^{-t}t^k L_n(t)\,dt = \begin{cases} 0 & k<n \\ (-1)^n n! & k=n \end{cases}$	
23.	$\int_0^t L_k(x)L_n(t-x)\,dx = \int_0^t L_{n+k}(x)\,dx = L_{n+k}(t) - L_{n+k+1}(t)$	
24.	$\int_t^\infty e^{-x}L_n^m(x)\,dx = e^{-t}[L_n^m(t) - L_{n-1}^m(t)],$	$m = 0, 1, 2, \ldots$
25.	$\int_0^t (t-x)^m L_n(x)\,dx = \frac{m!n!}{(m+n+1)!}t^{m+1}L_n^{m+1}(t),$	$m = 0, 1, 2, \ldots$
26.	$\int_0^1 x^a(1-x)^{b-1}L_n^a(tx)\,dx = \frac{\Gamma(b)\Gamma(n+a+1)}{\Gamma(n+a+b+1)}L_n^{a+b}(t),$	$a > -1,\ b > 0$
27.	$\int_0^\infty e^{-t}t^a L_n^a(t)L_k^a(t)\,dt = 0,$	$k \neq n, a > -1$
28.	$\int_0^\infty e^{-t}t^a[L_n^a(t)]^2\,dt = \frac{\Gamma(n+a+1)}{n!},$	$a > -1$
29.	$\int_0^\infty e^{-t}t^{a+1}[L_n^a(t)]^2\,dt = \frac{\Gamma(n+a+1)}{n!}(2n+a+1),$	$a > -1$
30.	$L_n^{-1/2}(t) = \frac{(-1)^n}{2^{2n}n!}H_{2n}(\sqrt{t})$	
31.	$L_n^{1/2}(t) = \frac{(-1)^n}{2^{2n+1}n!}\frac{H_{2n+1}(\sqrt{t})}{\sqrt{t}}$	
32.	$f(t) = \sum_{n=0}^\infty C_n L_n^m(t)$ $C_n = \frac{n!}{\Gamma(n+m+1)}\int_0^\infty e^{-t}t^m f(t)L_n^m(t)\,dt$	
33.	$t^p = p!\sum_{n=0}^p \binom{p}{n}(-1)^n L_n(t)$	
34.	$e^{-at} = (a+1)^{-1}\sum_{n=0}^\infty \left(\frac{a}{a+1}\right)^n L_n(t),$	$a > -\frac{1}{2}$
35.	$\int_0^\infty \frac{e^{-tx}}{x+1}\,dx = \sum_{n=0}^\infty \frac{L_n(t)}{n+1}$	

where $C_n^1(t)$ is the Gegenbauer polynomial with $\lambda = 1$

$$C_n^\lambda(t) = (-1)^n \sum_{k=0}^{[n/2]} \binom{-\lambda}{n-k}\binom{n-k}{k}(2t)^{n-2k} \tag{5.5.2b}$$

Hence, the second kind Chebyshev polynomials are

$$U_n(t) = \sum_{k=0}^{[n/2]} \binom{n-k}{k}(-1)^k(2t)^{n-2k} \tag{5.5.3}$$

The recurrence are

$$T_{n+1}(t) - 2tT_n(t) + T_{n-1}(t) = 0 \tag{5.5.4}$$

$$U_{n+1}(t) - 2tU_n(t) + U_{n-1}(t) = 0 \tag{5.5.5}$$

The orthogonality properties are

$$\int_{-1}^{1}(1-t^2)^{-1/2}T_n(t)T_k(t)dt = 0, \quad k \neq n \tag{5.5.6}$$

$$\int_{-1}^{1}(1-t^2)^{1/2}U_n(t)U_k(t)dt = 0, \quad k \neq n \tag{5.5.7}$$

The governing differential equations for $T_n(t)$ and $U_n(t)$ are, respectively,

$$(1 - t^2)y'' - ty' + n^2y = 0 \tag{5.5.8}$$

$$(1 - t^2)y'' - 3ty' + n(n + 2)y = 0 \tag{5.5.9}$$

The following are relationships between the two Chebyshev types:

$$T_n(t) = U_n(t) - tU_{n-1}(t) \tag{5.5.10}$$

$$(1 - t^2)U_n(t) = tT_n(t) - T_{n+1}(t) \tag{5.5.11}$$

Table 5.5.1 gives relationships for the Chebyshev polynomials.

TABLE 5.5.1 Properties of the Chebyshev Polynomials

1. $(1-t^2)\dfrac{d^2y}{dt^2} - t\dfrac{dy}{dt} + n^2y = 0;\ y(t) = T_n(t)$
2. $T_n(t) = \dfrac{n}{2}\sum_{k=0}^{[n/2]}\dfrac{(-1)^k(n-k-1)!}{k!(n-2k)!}(2t)^{n-2k}, \qquad n = 1, 2, \ldots, \ [n/2] = \text{largest integer} \leq n/2$
3. $T_n(t) = \dfrac{(-2)^n n!}{(2n)!}\sqrt{1-t^2}\dfrac{d^n}{dt^n}(1-t^2)^{n-\frac{1}{2}}$, Rodrigues formula
4. $T_n(t) = \cos(n\cos^{-1}t)$
5. $\dfrac{1-st}{1-2st+s^2} = \sum_{n=0}^{\infty}T_n(t)s^n$, generating function
6. $T_{n+1}(t) = 2tT_n(t) - T_{n-1}(t)$
7. $\displaystyle\int_{-1}^{1}\frac{T_n(t)T_m(t)}{\sqrt{(1-t^2)}}dt = \begin{cases} 0 & n \neq m \\ \pi/2 & n = m \neq 0 \\ \pi & n = m = 0 \end{cases}$
8. $T_n(1) = 1,\ T_n(-1) = (-1)^n,\ T_{2n}(0) = (-1)^n,\ T_{2n+1}(0) = 0$

If we set $t = \cos\theta$ in (5.5.8), we find that it reduces to

$$\frac{d^2y}{d\theta^2} + n^2y = 0$$

with solution $\cos n\theta$ and $\sin n\theta$. Therefore, if we set $T_n(\cos\theta) = C_n \cos n\theta$, we find that $C_n = 1$ for all n because $T_n(1) = 1$ for all n. Hence

$$T_n(t) = \cos n\theta = \cos(n \cos^{-1} t) \tag{5.5.12}$$

Similarly

$$U_n(t) = \frac{\sin\left[(n+1)\cos^{-1} t\right]}{\sqrt{1-t^2}} \qquad (5.5.13$$

The generating function for the Chebyshev polynomial is

$$\frac{1-st}{1-2st+s^2} = \sum_{n=0}^{\infty} T_n(t)s^n \tag{5.5.14}$$

The generalized Rodrigues formula is

$$T_n(t) = \frac{(-2)^n n!}{(2n)!}\sqrt{1-t^2}\,\frac{d^n}{dt^n}\left(1-t^2\right)^{n-\frac{1}{2}} \tag{5.5.15}$$

Figure 5.5.1 shows several Chebyshev polynomials.

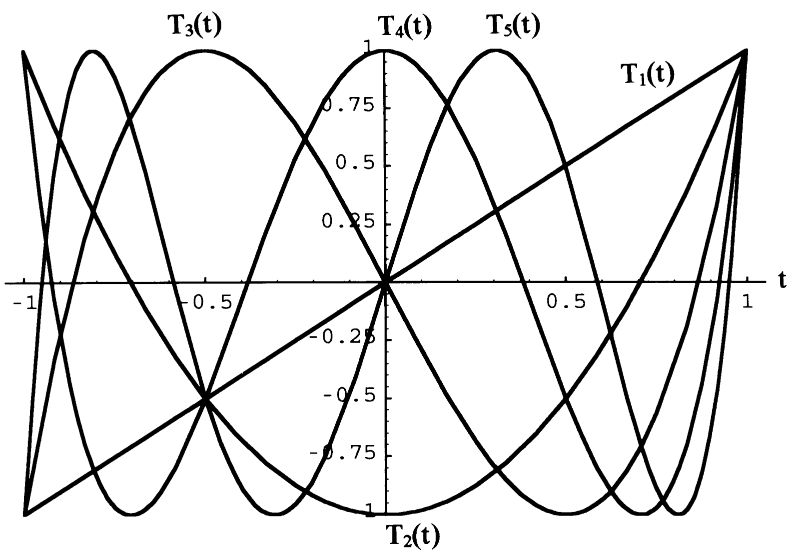


FIGURE 5.5.1

1.5.6 Bessel Functions

Bessel Functions of the First Kind

General relations The solution of Bessel's equation

$$y'' + \frac{1}{t}y' + \left(1 - \frac{n^2}{t^2}\right)y = 0, \quad n = 0, 1, 2, \ldots \tag{5.6.1}$$

is the function $y = J_n(t)$, known as the **Bessel function of the first kind and order** n. The Bessel function is defined by the series

$$J_n(t) = \sum_{k=0}^{\infty} \frac{(-1)^k (t/2)^{n+2k}}{k!(n+k)!}, \quad -\infty < t < \infty \tag{5.6.2}$$

We can find (5.6.2) by expanding the function $w(t, x)$ in series of the two exponentials $\exp(tx/2)$ and $\exp(-t/2x)$ in the form

$$w(t, x) \doteq e^{\frac{1}{2}t\left(x - \frac{1}{x}\right)} = \sum_{n=-\infty}^{\infty} J_n(t) x^n, \quad x \neq 0 \tag{5.6.3}$$

By setting $n = -n$ in (5.6.2) we obtain

$$J_{-n}(t) = \sum_{k=0}^{\infty} \frac{(-1)^k (t/2)^{2k-n}}{k!(k-n)!} = \sum_{k=n}^{\infty} \frac{(-1)^k (t/2)^{2k-n}}{k!(k-n)!}$$

because $1/[(k-n)!] = 0$ for $k = 0, 1, 2, \ldots, n-1$ ($\Gamma(n) = \infty$ for negative n). Setting $k = m + n$, we obtain

$$J_{-n}(t) = \sum_{m=0}^{\infty} \frac{(-1)^{m+n} (t/2)^{2m+n}}{m!(m+n)!} \tag{5.6.4}$$

from which it follows that

$$J_{-n}(t) = (-1)^n J_n(t), \quad n = 0, 1, 2, \ldots \tag{5.6.5}$$

Equating like terms in the expanded form of (5.6.3), we obtain

$$J_0(0) = 1, \quad J_n(0) = 0, \quad n \neq 0 \tag{5.6.6}$$

Figure 5.6.1 shows several Bessel functions of the first kind and zero order.

Bessel Functions of Nonintegral Order

The Bessel functions of a noninteger number are given by (υ = noninteger number)

$$J_\upsilon(t) = \sum_{k=0}^{\infty} \frac{(-1)^k (t/2)^{2k+\upsilon}}{k!\Gamma(k+\upsilon+1)}, \quad \upsilon \geq 0 \tag{5.6.7a}$$

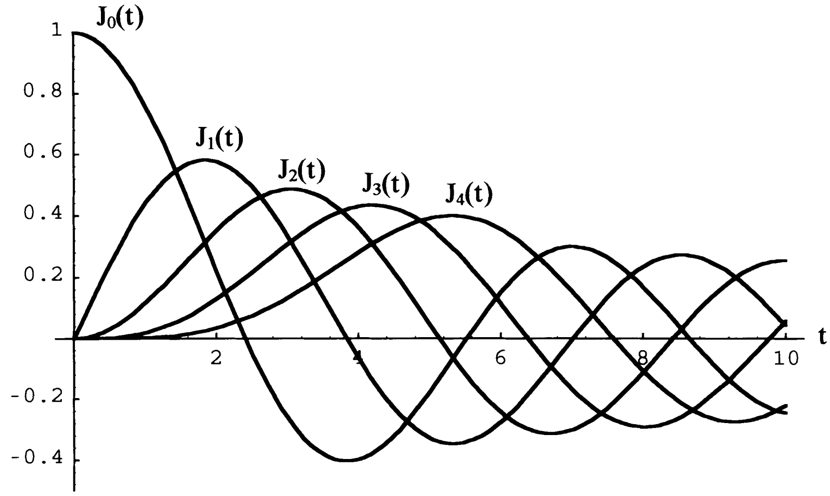


FIGURE 5.6.1

$$J_{-\upsilon}(t)=\sum_{k=0}^{\infty}\frac{(-1)^k (t/2)^{2k-\upsilon}}{k!\Gamma(k-\upsilon+1)}, \quad \upsilon \geq 0 \tag{5.6.7b}$$

The two functions $J_{-\upsilon}(t)$ and $J_{\upsilon}(t)$ are linear independent for noninteger values of υ and they do not satisfy any generating-function relation. The functions $J_{-\upsilon}(0) = \infty$ and $J_{\upsilon}(0)$ remain finite. Both share most of the properties of $J_n(t)$ and $J_{-n}(t)$.

Recurrence Relation

$$\begin{aligned}\frac{d}{dt}\left[t^{\upsilon}J_{\upsilon}(t)\right] &= \frac{d}{dt}\sum_{k=0}^{\infty}\frac{(-1)^k (t)^{2k+2\upsilon}}{2^{2k+\upsilon}k!\Gamma(k+\upsilon+1)} = t^{\upsilon}\sum_{k=0}^{\infty}\frac{(-1)^k (t/2)^{2k+(\upsilon-1)}}{k!\Gamma(k+\upsilon)} \\ &= t^{\upsilon}J_{\upsilon-1}(t)\end{aligned} \tag{5.6.8}$$

Similarly

$$\frac{d}{dt}\left[t^{-\upsilon}J_{\upsilon}(t)\right] = -t^{-\upsilon}J_{\upsilon+1}(t) \tag{5.6.9}$$

Differentiate (5.6.8) and (5.6.9) and dividing by t^{υ} and $t^{-\upsilon}$, respectively, we find

$$J_{\upsilon}'(t)+\frac{\upsilon}{t}J_{\upsilon}(t) = J_{\upsilon-1}(t) \tag{5.6.10}$$

$$J_{\upsilon}'(t)-\frac{\upsilon}{t}J_{\upsilon}(t) = -J_{\upsilon+1}(t) \tag{5.6.11}$$

Set $\upsilon = 0$ in (5.6.11) to obtain

$$J_0'(t) = -J_1(t) \tag{5.6.12}$$

Add and subtract (5.6.10) and (5.6.11) to find, respectively, the relations

$$2J_\upsilon'(t) = J_{\upsilon-1}(t) - J_{\upsilon+1}(t) \tag{5.6.13}$$

$$\frac{2\upsilon}{t}J_\upsilon(t) = J_{\upsilon-1}(t) + J_{\upsilon+1}(t) \tag{5.6.14}$$

The last relation is known as the **three-term recurrence formula**. Repeated operations result in

$$\left(\frac{d}{tdt}\right)^m \left[t^\upsilon J_\upsilon(t)\right] = t^{\upsilon-m} J_{\upsilon-m}(t) \tag{5.6.15}$$

$$\left(\frac{d}{tdt}\right)^m \left[t^{-\upsilon} J_\upsilon(t)\right] = (-1)^m t^{-\upsilon-m} J_{\upsilon+m}(t) \quad m = 1, 2, \ldots \tag{5.6.16}$$

Example

We proceed to find the following derivative

$$\frac{d}{dt}\left[t^\upsilon J_\upsilon(at)\right] = \frac{d}{dt}\left[\left(\frac{u}{a}\right)^\upsilon J_\upsilon(u)\right] = \frac{d}{du}\left[\frac{u^\upsilon}{a^\upsilon} J_\upsilon(u)\right]\frac{du}{dt}$$

$$= a^{-\upsilon}\frac{d}{du}\left[u^\upsilon J_\upsilon(u)\right]a = a^{1-\upsilon}\left[u^\upsilon J_{\upsilon-1}(u)\right]$$

$$= a^{1-\upsilon}\left[(at)^\upsilon J_{\upsilon-1}(at)\right] = at^\upsilon J_{\upsilon-1}(at)$$

where (5.6.8) was used.

Example

Differentiate (5.6.13) to find

$$\frac{d^2 J_\upsilon(t)}{dt^2} = \frac{1}{2}\left(\frac{dJ_{\upsilon-1}(t)}{dt} - \frac{dJ_{\upsilon+1}(t)}{dt}\right)$$

Then apply the same equation to each derivative on the right side to find

$$\frac{d^2 J_\upsilon(t)}{dt^2} = \frac{1}{2}\left[\frac{1}{2}\left[J_{\upsilon-2}(t) - J_\upsilon(t)\right] - \frac{1}{2}\left[J_\upsilon(t) - J_{\upsilon+2}(t)\right]\right]$$

$$= \frac{1}{2^2}\left[J_{\upsilon-2}(t) - 2J_\upsilon(t) + J_{\upsilon+2}(t)\right]$$

Similarly we find

$$\frac{d^3 J_\upsilon(t)}{dt^3} = \frac{1}{2^3}\left[J_{\upsilon-3}(t) - 3J_{\upsilon-1}(t) + 3J_{\upsilon+1}(t) - J_{\upsilon+3}(t)\right]$$

Integral Representation

Set $x = \exp(-j\varphi)$ in (5.6.3), multiply both sides by $\exp(jn\varphi)$, and integrate the results from 0 to π. Hence

$$\int_0^{\pi} e^{j(n\varphi - t\sin\varphi)}\,d\varphi = \sum_{k=-\infty}^{\infty} J_k(t)\int_0^{\pi} e^{j(n-k)\varphi}\,d\varphi \tag{5.6.17}$$

Expand on both sides the exponentials in Eauler's formula; equate the real and imaginary parts and use the relation

$$\int_0^{\pi} \cos(n-k)\varphi\,d\varphi = \begin{cases} 0 & k \neq 0 \\ \pi & k = n \end{cases}$$

to find that all terms of the infinite sum vanish except for $k = n$. Hence, we obtain

$$J_n(t) = \frac{1}{\pi}\int_0^{\pi} \cos(n\varphi - t\sin\varphi)\,d\varphi, \quad n = 0, 1, 2, \ldots \tag{5.6.18}$$

When $n = 0$, we find

$$J_0(t) = \frac{1}{\pi}\int_0^{\pi} \cos(t\sin\varphi)\,d\varphi \tag{5.6.19}$$

For a Bessel function with nonintegral order, the Poisson formula is

$$J_\upsilon(t) = \frac{(t/2)^\upsilon}{\sqrt{\pi}\,\Gamma\left(\upsilon + \frac{1}{2}\right)}\int_{-1}^{1}\left(1 - x^2\right)^{\upsilon - \frac{1}{2}} e^{jtx}\,dx, \quad \upsilon > -\frac{1}{2}, t > 0 \tag{5.6.20}$$

Set $x = \cos\theta$ to obtain

$$J_\upsilon(t) = \frac{(t/2)^\upsilon}{\sqrt{\pi}\,\Gamma\left(\upsilon + \frac{1}{2}\right)}\int_0^{\pi} \cos(t\cos\theta)\sin^{2\upsilon}\theta\,d\theta, \quad \upsilon > -\frac{1}{2}, t > 0 \tag{5.6.21}$$

Integrals Involving Bessel Functions

Start with the identities

$$\frac{d}{dt}\left[t^\upsilon J_\upsilon(t)\right] = t^\upsilon J_{\upsilon-1}(t) \tag{5.6.22}$$

$$\frac{d}{dt}\left[t^{-\upsilon} J_\upsilon(t)\right] = -t^{-\upsilon} J_{\upsilon+1}(t) \tag{5.6.23}$$

and directly integrate to find

$$\int t^\upsilon J_{\upsilon-1}(t)\,dt = t^\upsilon J_\upsilon(t) + C \tag{5.6.24}$$

$$\int t^{-\upsilon} J_{\upsilon+1}(t)dt = -t^{-\upsilon} J_{\upsilon}(t) + C \tag{5.6.25}$$

where C is the constant of integration.

Example
We apply the integration procedure to find

$$\begin{aligned}\int t^2 J_2(t)dt &= \int t^3\left[t^{-1}J_2(t)\right]dt = -\int t^3 \frac{d}{dt}\left[t^{-1}J_1(t)\right]dt \\ &= -t^2 J_1(t) + 3\int tJ_1(t)dt = -t^2 J_1(t) - 3\int t\left[-J_1(t)\right]dt \\ &= -t^2 J_1(t) - 3\int t\left[\frac{d}{dt}J_0(t)\right]dt = -t^2 J_1(t) - 3tJ_0(t) + 3\int J_0(t)dt\end{aligned}$$

The last integral has no closed solution.

Example
If $a >$ and $b > 0$, then (see [5.6.19])

$$\begin{aligned}\int_0^\infty e^{-at} J_0(bt)dt &= \int_0^\infty e^{-at}dt \frac{2}{\pi}\int_0^{\pi/2} \cos(bt\sin\varphi)d\varphi \\ &= \frac{2}{\pi}\int_0^{\pi/2} d\varphi \int_0^\infty e^{-at}\cos(bt\sin\varphi)dt \\ &= \frac{2}{\pi}\int_0^{\pi/2} \frac{ad\varphi}{a^2 + b^2\sin^2\varphi} = \frac{1}{\sqrt{a^2+b^2}}\end{aligned}$$

Example
For $a > 0$, $b > 0$, and $\upsilon > -1$ (υ is real), then

$$\begin{aligned}\int_0^\infty e^{-a^2t^2} J_\upsilon(bt)t^{\upsilon+1}dt &= \int_0^\infty e^{-a^2t^2} t^{\upsilon+1}dt \sum_{k=0}^\infty \frac{(-1)^k (bt/2)^{\upsilon+2k}}{k!\Gamma(k+\upsilon+1)} \\ &= \sum_{k=0}^\infty \frac{(-1)^k}{k!\Gamma(k+\upsilon+1)}\left(\frac{b}{2}\right)^{\upsilon+2k} \int_0^\infty e^{-a^2t^2} t^{2\upsilon+2k+1}dt \\ &= \sum_{k=0}^\infty \frac{(-1)^k}{k!\Gamma(k+\upsilon+1)}\left(\frac{b}{2}\right)^{\upsilon+2k} \frac{1}{2a^{2\upsilon+2k+2}}\int_0^\infty e^{-r} r^{\upsilon+k}dr \\ &= \frac{b^\upsilon}{(2a^2)^{\upsilon+1}} \sum_{k=0}^\infty \frac{\left(-\frac{b^2}{4a^2}\right)^k}{k!} = \frac{b^\upsilon}{(2a^2)^{\upsilon+1}} e^{-b^2/4a^2}\end{aligned} \tag{5.6.26}$$

where the last integral is the gamma function and the summation is the exponential expression.

The usual method to find definite integrals involving Bessel functions is to replace the Bessel function by its series representation. To illustrate the technique, let us find the value of the integral

$$
\begin{aligned}
I &= \int_0^\infty e^{-at} t^p J_p(bt)\,dt, \quad p > -\frac{1}{2}, \quad a>0, b>0 \\
&= \sum_{k=0}^\infty \frac{(-1)^k (b/2)^{2k+p}}{k!\,\Gamma(k+p+1)} \int_0^\infty e^{-at} t^{2k+2p} dt \qquad (5.6.27)\\
&= b^p \sum_{k=0}^\infty \frac{(-1)^k \Gamma(2k+2p+1)}{2^{2k+p} k!\,\Gamma(k+p+1)} (a^2)^{-\left(p+\frac{1}{2}\right)-k} (b^2)^k
\end{aligned}
$$

where the last integral is in the form of a gamma function. But we know that

$$
\binom{-r}{k} = (-1)^k \binom{r+k-1}{k}, \quad \binom{n}{k} = \binom{n}{n-k}
$$

$$
\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}, \quad 0 \le k \le n-1
$$

and thus we obtain

$$
\begin{aligned}
\frac{(-1)^k \Gamma(2k+2p+1)}{2^{2k+p} k!\,\Gamma(k+p+1)} &= \frac{(-1)^k 2^p \Gamma\left(p+k+\frac{1}{2}\right)}{\sqrt{\pi}\,k!} \\
&= \frac{(-1)^k}{\sqrt{\pi}} 2^p \Gamma\left(p+\frac{1}{2}\right) \binom{p+k-\frac{1}{2}}{k} \qquad (5.6.28)\\
&= \frac{2^p \Gamma\left(p+\frac{1}{2}\right)}{\sqrt{\pi}} \binom{-\left(p+\frac{1}{2}\right)}{k}
\end{aligned}
$$

Therefore, (5.6.27) becomes

$$
\begin{aligned}
I &= \int_0^\infty e^{-at} t^p J_p(bt)\,dt \\
&= \frac{(2b)^p \Gamma\left(p+\frac{1}{2}\right)}{\sqrt{\pi}} \sum_{k=0}^\infty \binom{-\left(p+\frac{1}{2}\right)}{k} (a^2)^{-(p+(1\ 2))-k} (b^2)^k \qquad (5.6.29)\\
&= \frac{(2b)^p \Gamma\left(p+\frac{1}{2}\right)}{\sqrt{\pi}\left(a^2+b^2\right)^{p+\frac{1}{2}}}, \quad p > -\frac{1}{2}, a>0, b>0
\end{aligned}
$$

Setting $p = 0$ in this equation we find

$$\int_0^\infty e^{-at} J_0(bt) = \frac{1}{\left[a^2+b^2\right]^{1/2}}, \quad a>0, b>0 \tag{5.6.30}$$

Set a = 0+ in this equation to obtain

$$\int_0^\infty J_0(bt)dt = \frac{1}{b}, \quad b>0 \tag{5.6.31}$$

By assuming the real approaches zero and writing a as pure imaginary, (5.6.30) becomes

$$\int_0^\infty e^{-jat} J_0(bt)dt = \begin{cases} \dfrac{1}{\left(b^2-a^2\right)^{1/2}} & b>a \\ \dfrac{-j}{\left(a^2-b^2\right)^{1/2}} & b<a \end{cases} \tag{5.6.32}$$

The above integral, by equating real and imaginary parts, becomes

$$\int_0^\infty \cos(at) J_0(bt)dt = \frac{1}{\left(b^2-a^2\right)^{1/2}}, \quad b>a \tag{5.6.33}$$

$$\int_0^\infty \sin(at) J_0(bt)dt = \frac{1}{\left(a^2-b^2\right)^{1/2}}, \quad b<a \tag{5.6.34}$$

Example

To evaluate the integral $\int_0^b tJ_0(at)dt$, we proceed as follows:

$$\begin{aligned} \int_0^\infty tJ_0(at)dt &= \int_0^\infty \frac{1}{a}\frac{d}{dt}\left[tJ_1(at)\right]dt \\ &= \frac{1}{a}\left[tJ_1(at)\right]\Big|_{t=0}^{b} = \frac{b}{a}J_1(ab), \quad a \neq 0 \end{aligned} \tag{5.6.35}$$

where (5.6.8) with υ = 1 was used.

Example

To evaluate the integral $I = \int_0^b t^2 J_0(at)dt$, where a is a constant and nonequal to zero, we proceed as follows (set $at = r$):

$$\begin{aligned} I &= \frac{1}{a^3}\int_0^{ab} r^2 J_0(r)dr = \frac{1}{a^3}\int_0^{ab} rrJ_0(r)dr \\ &= \frac{1}{a^3}\int_0^{ab} r\frac{d}{dr}\left[rJ_1(r)\right]dr = \frac{1}{a^3}\left[a^2b^2J_1(ab) - \int_0^{ab} rJ_1(r)dr\right] \end{aligned}$$

But (see [5.6.23])

$$\int_0^{ab} rJ_1(r)\,dr = -\int_0^{ab} r\frac{d}{dr}\left[J_0(r)\right]dr = -rJ_0(r)\Big|_{r=0}^{ab} + \int_0^{ab} J_0(r)\,dr = -abJ_0(ab) + \int_0^{ab} J_0(r)\,dr$$

and therefore

$$I = \frac{1}{a^3}\left[a^2b^2J_1(ab) + abJ_0(ab) - \int_0^{ab} J_0(r)\,dr\right] \tag{5.6.36}$$

The integral can be approximately evaluated with any desired accuracy by termwise integration of the series of $J_0(t)$. Hence, we write

$$\int_0^{ab} J_0(t)\,dt = ab - \frac{a^3b^3}{3\cdot 2^2} + \frac{a^5b^5}{5\cdot 2^4\cdot (2!)^2} - \frac{a^7b^7}{7\cdot 2^6\cdot (3!)^2} + \cdots$$

Fourier Bessel Series

A Bessel series is a member of the class of generalized Fourier series. It is defined by

$$f(t) = \sum_{n=1}^{\infty} c_n J_\upsilon(t_n t), \quad 0 < t < a, \quad \upsilon > -\frac{1}{2} \tag{5.6.37}$$

where c's are the expansion coefficient constants and t_n's (n = 1, 2, 3, …) are the zeros (positive roots) of the function

$$J_\upsilon(t_n t), \quad n = 1, 2, 3, \ldots \tag{5.6.38}$$

The orthogonality property is defined as follows ($\upsilon > -1$):

$$\int_0^a tJ_\upsilon(t_m t)J_\upsilon(t_n t)\,dt = 0, \quad m \neq n \tag{5.6.39}$$

with weight t. It can also be shown that

$$\int_0^a t\left[J_\upsilon(t_n t)\right]^2 dt = \frac{a^2}{2}\left[J_{\upsilon+1}(t_n a)\right]^2 \tag{5.6.40}$$

Theorem 5.6.1

If a real function $f(t)$ is piecewise continuous on $(0, a)$ and is of bounded variation in every subinterval $[t_1, t_2]$ where $0 < t_1 < t_2 < a$, then if the integral

$$\int_0^a \sqrt{t}\,|f(t)|\,dt$$

is finite, the Fourier–Bessel series converges to $f(t)$ at every continuity point of $f(t)$ and to $[f(t+) - f(t-)]/2$ at every discontinuity point.

To begin, multiply (5.6.37) by $tJ_\upsilon(t_m t)$ and integrate from 0 to a. Assuming that termwise integration is permitted, we obtain

$$\begin{aligned}\int_0^a tf(t)J_\upsilon(t_m t)dt &= \sum_{n=1}^{\infty} c_n \int_0^a tJ_\upsilon(t_m t)J_\upsilon(t_n t)dt \\ &= c_m \int_0^a \left[J_\upsilon(t_m t)\right]^2 dt\end{aligned} \tag{5.6.41}$$

because the integral is zero if $n \neq m$ (see [5.6.39]). Hence, from this equation we obtain

$$c_n = \frac{2}{a^2\left[J_{\upsilon+1}(t_n a)\right]^2}\int_0^a tf(t)J_\upsilon(t_n t)dt, \quad n = 1, 2, 3, \ldots \tag{5.6.42}$$

Example

Find the Fourier–Bessel series for the function

$$f(t) = \begin{cases} t & 0 < t < 1 \\ 0 & 1 < t < 2 \end{cases}$$

corresponding to the set of functions $\{J_1(t_n t)\}$ where t_n satisfies $J_1(2t_n) = 0$ ($n = 1, 2, 3, \ldots$).

Solution We write the solution

$$f(t) = \sum_{n=1}^{\infty} c_n J_1(t_n t), \quad 0 < t < 2$$

where

$$\begin{aligned}c_n &= \frac{1}{2\left[J_2(2t_n)\right]^2}\int_0^2 tf(t)\,J_1(t_n t)\,dt \\ &= (\cdot)\int_0^1 t^2 J_1(t_n t)\,dt \quad (\text{let } r = t_n t) \\ &= (\cdot)\frac{1}{t_n^3}\int_0^{t_n} r^2 J_1(r)\,dr \quad (\text{apply } [5.6.22]) \\ &= (\cdot)\frac{1}{t_n^3}\int_0^{t_n} \frac{d}{dt}\left[r^2 J_2(r)\right]dr \\ &= \frac{1}{2\left[J_2(2t_n)\right]^2 t_n^3}\, t_n^2 J_2(t_n) = \frac{J_2(t_n)}{2t_n\left[J_2(2t_n)\right]^2}, \quad n = 1, 2, 3, \ldots\end{aligned}$$

Example

To express the function $f(t) = 1$ on the open interval $0 < t < a$ as an infinite series of Bessel functions of zero order, we proceed as follows (see [5.6.42]):

$$c_n = \frac{2}{a^2\left[J_1(t_n a)\right]^2}\int_0^a t\cdot 1\cdot J_0(t_n t)\,dt$$

$$= \frac{2}{a^2\left[J_1(t_n a)\right]^2}\int_0^a \frac{1}{t_n}\frac{d}{dt}\left[tJ_1(t_n t)\right]dt \quad \left(\text{see } [5.6.22]\right)$$

$$= \frac{2}{a^2 t_n\left[J_1(t_n a)\right]^2}\left[tJ_1(t_n t)\right]\Big|_{t=0}^{a} = \frac{2}{a t_n J_1(t_n a)}$$

Hence the expression is

$$1 = 2\sum_{n=1}^{\infty}\frac{J_n(t_n t)}{t_n J_1(t_n a)}, \quad 0 < t < a$$

Example
Let us expand the function $f(t) = t^2$, $0 \le t \le 1$, in a series of the form

$$c_1 J_0(t_1 t) + c_2 J_0(t_2 t) + c_3 J_0(t_3 t) + \ldots$$

where t_n denotes the nth positive zero of $J_0(t)$. From (5.6.42) we obtain ($a = 1$)

$$c_n = \frac{2}{\left[J_1(t_n)\right]^2}\int_0^1 t^3 J_0(t_n t)\,dt$$

$$= \frac{2}{\left[J_1(t_n)\right]^2}\int_0^{t_n}\frac{r^3}{t_n^3}J_0(r)\frac{dr}{t_n} = \frac{2}{t_n^4\left[J_1(t_n)\right]^2}\int_0^{t_n} r^2\frac{d}{dr}\left[rJ_1(r)\right]$$

$$= \frac{2}{t_n^4\left[J_1(t_n)\right]^2}\left[r^3 J_1(r)\Big|_{r=0}^{t_n} - 2\int_0^{t_n} r^2 J_1(r)\,dr\right]$$

$$= \frac{2}{t_n^4\left[J_1(t_n)\right]^2}\left[t_n^3 J_1(t_n) - 2\int_0^{t_n}\frac{d}{dr}\left[r^2 J_2(r)\right]dr\right]$$

$$= \frac{2}{t_n^4\left[J_1(t_n)\right]^2}\left[t_n^3 J_1(t_n) - 2t_n^2 J_2(t_n)\right]$$

Table 5.6.1 gives Bessel function relationships. Tables 5.6.2 and 5.6.3 give numerical values for Bessel functions and Table 5.6.4 gives the zeros of several Bessel functions.

1.5.7 Zernike Polynomials

Zernike polynomials are a set of complex exponentials that form a complete orthogonal set over the interior of the unit circle. Polynomial representation of optical wave fronts is essential in the analysis of interferometric test data, for example, to assess optical system performance. One such set, which is attractive for its simple rotational properties, is the **circle polynomials or Zernike polynomials**. The set of these polynomials is denoted by

$$V_{nl}(x, y) = V_{nl}(r\cos\theta, r\sin\theta) = V_{nl}(r, \theta) = R_{nl}(r)e^{jl\theta} \tag{5.7.1}$$

TABLE 5.6.1 Properties of Bessel Functions of the First Kind

1. $J_n(t)=\sum_{k=0}^{\infty}\frac{(-1)^k(t/2)^{n+2k}}{k!(n+k)!}$,	$-\infty<t<\infty,\ n=0,1,2,3,\ldots$
2. $J_{-n}(t)=\sum_{m=0}^{\infty}\frac{(-1)^{m+n}(t/2)^{2m+n}}{m!(m+n)!}$,	$-\infty<t<\infty,\ n=0,1,2,3,\ldots$
3. $J_{-n}(t)=(-1)^nJ_n(t)$,	$n=0,1,2,3,\ldots$
4. $J_0(0)=1,\qquad J_n(0)=0,$	$n\neq 0$
5. $J_\nu(t)=\sum_{k=0}^{\infty}\frac{(-1)^k(t/2)^{2k+\nu}}{k!\Gamma(k+\nu+1)}$,	$\nu\geq 0$, ν is noninteger
6. $J_{-\nu}(t)=\sum_{k=0}^{\infty}\frac{(-1)^k(t/2)^{2k-\nu}}{k!\Gamma(k-\nu+1)}$,	$\nu\geq 0$, ν is noninteger
7. $\frac{d}{dt}[t^\nu J_\nu(t)]=t^\nu J_{\nu-1}(t)$	
8. $\frac{d}{dt}[t^\nu J_\nu(at)]=at^\nu J_{\nu-1}(at)$	
9. $\frac{d}{dt}[t^{-\nu}J_\nu(t)]=-t^{-\nu}J_{\nu+1}(t)$	
10. $\frac{d^2J_\nu(t)}{dt^2}=\frac{1}{2^2}[J_{\nu-2}(t)-2J_\nu(t)+J_{\nu+2}(t)]$	
11. $\frac{d^3J_\nu(t)}{dt^3}=\frac{1}{2^3}[J_{\nu-3}(t)-3J_{\nu-1}(t)+3J_{\nu+1}(t)-J_{\nu+3}(t)]$	
12. $J_\nu'(t)+\frac{\nu}{t}J_\nu(t)=J_{\nu-1}(t)$	
13. $J_\nu'(t)-\frac{\nu}{t}J_\nu(t)=-J_{\nu+1}(t)$	
14. $J_0'(t)=-J_1(t)$	
15. $2J_\nu'(t)=J_{\nu-1}(t)-J_{\nu+1}(t)$	
16. $\frac{2\nu}{t}J_\nu(t)=J_{\nu-1}(t)+J_{\nu+1}(t)$	
17. $\left(\frac{d}{tdt}\right)^m[t^\nu J_\nu(t)]=t^{\nu-m}J_{\nu-m}(t)$	$m=1,2,3,\ldots$
18. $\left(\frac{d}{tdt}\right)^m[t^{-\nu}J_\nu(t)]=(-1)^mt^{-\nu-m}J_{\nu+m}(t)$	$m=1,2,3,\ldots$
19. $J_1'(0)=\frac{1}{2},\qquad J_n'(0)=0,$	$n>1$
20. $J_n(t+r)=\sum_{k=-\infty}^{\infty}J_k(t)J_{n-k}(r)$	
21. $J_0(2t)=[J_0(t)]^2+2\sum_{k=1}^{\infty}(-1)^k[J_k(t)]^2$	

TABLE 5.6.1 Properties of Bessel Functions of the First Kind (Continued)

22.	$\lvert J_0(t)\rvert \le 1, \qquad \lvert J_n(t)\rvert \le \frac{1}{\sqrt{2}},$	$n = 1, 2, 3, \ldots$
23.	$e^{jt\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(t) e^{jn\theta}$	
24.	$\cos(t\sin\theta) = J_0(t) + 2\sum_{n=1}^{\infty} J_{2n}(t)\cos(2n\theta)$	
25.	$\cos(t\cos\theta) = J_0(t) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(t)\cos(2n\theta)$	
26.	$\sin(t\sin\theta) = 2\sum_{n=1}^{\infty} J_{2n-1}(t)\sin[(2n-1)\theta]$	
27.	$\sin(t\cos\theta) = 2\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(t)\cos[(2n+1)\theta]$	
28.	$\cos t = J_0(t) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(t)$	
29.	$\sin t = 2\sum_{n=1}^{\infty}(-1)^n J_{2n-1}(t)$	
30.	$J_\nu(t)J_{1-\nu}(t) + J_{-\nu}(t)J_{\nu-1}(t) = \frac{2\sin\nu\pi}{\pi t}$	Lommel's formula
31.	$\frac{d}{dt}[tJ_\nu(t)J_{\nu+1}(t)] = t[[J_\nu(t)]^2 - [J_{\nu+1}(t)]^2]$	
32.	$\frac{d}{dt}[t^2 J_{\nu-1}(t)J_{\nu+1}(t)] = 2t^2 J_\nu(t)J'_\nu(t)$	
33.	$J_{1/2}(t) = \sqrt{\frac{2}{\pi t}}\sin t, \qquad J_{-1/2}(t) = \sqrt{\frac{2}{\pi t}}\cos t$	
34.	$J_{1/2}(t)J_{-1/2}(t) = \frac{\sin 2t}{\pi t}, \qquad [J_{1/2}(t)]^2 + [J_{-1/2}(t)]^2 = \frac{2}{\pi t}$	
35.	$[J_0(t)]^2 = \sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{(n!)^4}\left(\frac{t}{2}\right)^{2n}$	
36.	$J_n(t) = \frac{1}{\pi}\int_0^\pi \cos(n\varphi - t\sin\varphi)\,d\varphi$	
37.	$J_0(t) = \frac{1}{\pi}\int_0^\pi \cos(t\sin\varphi)\,d\varphi$	
38.	$J_\nu(t) = \frac{(t/2)^\nu}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\int_{-1}^{1}(1-x^2)^{\nu-\frac{1}{2}}e^{jtx}\,dx,$	$\nu > -\frac{1}{2},\ t > 0$
39.	$J_\nu(t) = \frac{(t/2)^\nu}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^\pi \cos(t\cos\theta)\sin^{2\nu}\theta\,d\theta,$	$\nu > -\frac{1}{2},\ t > 0$
40.	$\int t^\nu J_{\nu-1}(t)\,dt = t^\nu J_\nu(t) + C,$	$C =$ constant
41.	$\int t^{-\nu} J_{\nu+1}(t)\,dt = -t^{-\nu}J_\nu(t) + C,$	$C =$ constant

42.	$[1+(-1)^n]J_n(t) = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} \cos n\varphi \cos(t\sin\varphi)\,d\varphi,$	$n = 0, 1, 2, \ldots$
43.	$J_{2k}(t) = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \cos 2k\varphi \cos(t\sin\varphi)\,d\varphi,$	$k = 0, 1, 2, \ldots$
44.	$J_{2k+1}(t) = \dfrac{1}{\pi}\displaystyle\int_0^{\infty} \sin[(2k+1)\varphi]\sin(t\sin\varphi)\,d\varphi,$	$k = 0, 1, 2, \ldots$
45.	$\displaystyle\int_0^{\pi} \cos[(2k+1)\varphi]\cos(t\sin\varphi)\,d\varphi = 0,$	$k = 0, 1, 2, \ldots$
46.	$\displaystyle\int_0^{\pi} \sin 2k\varphi \sin(t\sin\varphi)\,d\varphi = 0,$	$k = 0, 1, 2, \ldots$
47.	$J_0(t) = \dfrac{2}{\pi}\displaystyle\int_0^{1} \frac{\cos tx}{\sqrt{1-x^2}}\,dx$	
48.	$\dfrac{2\sin t}{t} = \sqrt{\dfrac{2\pi}{t}}\,J_{1/2}(t)$	
49.	$\displaystyle\int tJ_0(t)\,dt = tJ_1(t) + C$	
50.	$\displaystyle\int t^2J_0(t)\,dt = t^2J_1(t) + tJ_0(t) - \int J_0(t)\,dt + C$	
51.	$\displaystyle\int t^3J_0(t)\,dt = (t^3-4t)J_1(t) + 2t^2J_0(t) + C$	
52.	$\displaystyle\int J_1(t)\,dt = -J_0(t) + C$	
53.	$\displaystyle\int tJ_1(t)\,dt = -tJ_0(t) + \int J_0(t)\,dt + C$	
54.	$\displaystyle\int t^2J_1(t)\,dt = 2tJ_1(t) - t^2J_0(t) + C$	
55.	$\displaystyle\int t^3J_1(t)\,dt = 3t^2J_1(t) - (t^3-3t)J_0(t) - 3\int J_0(t)\,dt + C$	
56.	$\displaystyle\int J_3(t)\,dt = -J_2(t) - 2t^{-1}J_1(t) + C$	
57.	$\displaystyle\int t^{-1}J_1(t)\,dt = -J_1(t) + \int J_0(t)\,dt + C$	
58.	$\displaystyle\int t^{-2}J_2(t)\,dt = -\frac{2}{3t^2}J_1(t) - \frac{1}{3}J_1(t) + \frac{1}{3t}J_0(t) + \frac{1}{3}\int J_0(t)\,dt + C$	
59.	$\displaystyle\int J_0(t)\cos t\,dt = tJ_0(t)\cos t + tJ_1(t)\sin t + C$	
60.	$\displaystyle\int J_0(t)\sin t\,dt = tJ_0(t)\sin t - tJ_1(t)\cos t + C$	
61.	$\displaystyle\int_0^{\infty} e^{-at}t^pJ_p(bt)\,dt = \frac{(2b)^p\Gamma\left(p+\frac{1}{2}\right)}{\sqrt{\pi}(a^2+b^2)^{p+\frac{1}{2}}},$	$p > -\frac{1}{2},\ a > 0,\ b > 0$

TABLE 5.6.1 Properties of Bessel Functions of the First Kind (Continued)

62.	$\int_0^\infty e^{-at} J_0(bt)\,dt = \frac{1}{(a^2+b^2)^{1/2}}$,	$a > 0,\ b > 0$
63.	$\int_0^\infty J_0(bt)\,dt = \frac{1}{b}$,	$b > 0$
64.	$\int_0^\infty J_{n+1}(t) = \int_0^\infty J_{n-1}(t)\,dt$,	$n = 1, 2, \ldots$
65.	$\int_0^\infty J_n(at)\,dt = \frac{1}{a}$	$a > 0$,
66.	$\int_0^\infty t^{-1} J_n(t)\,dt = \frac{1}{n}$,	$n = 1, 2, \ldots$
67.	$\int_0^\infty e^{-at} t^{p+1} J_p(bt)\,dt = \frac{2^{p+1}\Gamma\left(p+\frac{3}{2}\right)}{\sqrt{\pi}} \frac{ab^p}{(a^2+b^2)^{p+\frac{3}{2}}}$,	$p > -1,\ a > 0,\ b > 0$
68.	$\int_0^\infty t^2 e^{-at} J_0(bt)\,dt = \frac{2a^2-b^2}{(a^2+b^2)^{5/2}}$,	$a > 0, b > 0$
69.	$\int_0^\infty e^{-at^2} t^{p+1} J_p(bt)\,dt = \frac{b^p e^{-b^2/4a}}{(2a)^{p+1}}$,	$p > -1,\ a > 0,\ b > 0$
70.	$\int_0^\infty e^{-at^2} t^{p+3} J_p(bt)\,dt = \frac{b^p}{2^{p+1}a^{p+2}} \left(p+1-\frac{b^2}{4a}\right) e^{-b^2/4a}$,	$p > -1,\ a > 0,\ b > 0$
71.	$\int_0^\infty t^{-1} \sin t\, J_0(bt)\,dt = \arcsin\left(\frac{1}{b}\right)$,	$b > 1$
72.	$\int_0^{\pi/2} J_0(t\cos\varphi)\cos\varphi\,d\varphi = \frac{\sin t}{t}$	
73.	$\int_0^{\pi/2} J_1(t\cos\varphi)\,d\varphi = \frac{1-\cos t}{t}$	
74.	$\int_0^\infty e^{-t\cos\varphi} J_0(t\sin\varphi) t^n dt = n!P_n(\cos\varphi)$,	$0 \le \varphi < \pi$
	$P_n(t) = n$th Legendre polynomial	
75.	$\int_0^\infty t(t^2+a^2)^{-1/2} J_0(bt)\,dt = \frac{e^{-ab}}{b}$,	$a \ge 0,\ b > 0$
76.	$\int_0^\infty \frac{J_p(t)}{t^m}\,dt = \frac{\Gamma((p+1-m)/2)}{2^m\Gamma((p+1+m)/2)}$,	$m > \frac{1}{2},\ p - m > -1$
77.	$\frac{1}{8}(1-t^2) = \sum_{n=1}^\infty \frac{J_0(k_n t)}{k_n^3 J_1(k_n)}$,	$0 \le t \le 1,\ J_0(k_n) = 0,$ $n = 1, 2, \ldots$
78.	$t^p = 2\sum_{n=1}^\infty \frac{J_p(k_n t)}{k_n J_{p+1}(k_n)}$,	$0 < t < 1,\ J_p(k_n) = 0,$ $n = 1, 2, \ldots$
79.	$t^{p+1} = 2^2(p+1)\sum_{n=1}^\infty \frac{J_{p+1}(k_n t)}{k_n^2 J_{p+1}(k_n)}$,	$0 < t < 1,\ p > -1/2,$

TABLE 5.6.2

$J_0(x)$

x	0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	1.0000	.9975	.9900	.9776	.9604	.9385	.9120	.8812	.8463	.8075
1	.7652	.7196	.6711	.6201	.5669	.5118	.4554	.3980	.3400	.2818
2	.2239	.1666	.1104	.0555	.0025	−.0484	−.0968	−.1424	−.1850	−.2243
3	− .2601	−.2921	−.3202	−.3443	−.3643	−.3801	−.3918	−.3992	−.4026	−.4018
4	− .3971	−.3887	−.3766	−.3610	−.3423	−.3205	−.2961	−.2693	−.2404	−.2097
5	− .1776	−.1443	−.1103	−.0758	−.0412	−.0068	.0270	.0599	.0917	.1220
6	.1506	.1773	.2017	.2238	.2433	.2601	.2740	.2851	.2931	.2981
7	.3001	.2991	.2951	.2882	.2786	.2663	.2516	.2346	.2154	.1944
8	.1717	.1475	.1222	.0960	.0692	.0419	.0146	−.0125	−.0392	−.0653
9	− .0903	−.1142	−.1367	−.1577	−.1768	−.1939	−.2090	−.2218	−.2323	−.2403
10	− .2459	−.2490	−.2496	−.2477	−.2434	−.2366	−.2276	−.2164	−.2032	−.1881
11	− .1712	−.1528	−.1330	−.1121	−.0902	−.0677	−.0446	−.0213	.0020	.0250
12	.0477	.0697	.0908	.1108	.1296	.1469	.1626	.1766	.1887	.1988
13	.2069	.2129	.2167	.2183	.2177	.2150	.2101	.2032	.1943	.1836
14	.1711	.1570	.1414	.1245	.1065	.0875	.0679	.0476	.0271	.0064
15	− .0142	−.0346	−.0544	−.0736	−.0919	−.1092	−.1253	−.1401	−.1533	−.1650

When $x > 15.9$,

$$J_0(x) \simeq \sqrt{\left(\frac{2}{\pi x}\right)}\left\{\sin\left(x+\frac{1}{4}\pi\right)+\frac{1}{8x}\sin\left(x-\frac{1}{4}\pi\right)\right\}$$

$$\simeq \frac{.7979}{\sqrt{x}}\left\{\sin(57.296x+45^\circ)+\frac{1}{8x}\sin(57.296x-45^\circ)\right\}$$

$J_1(x)$

x	0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	.0000	.0499	.0995	.1483	.1960	.2423	.2867	.3290	.3688	.4059
1	.4401	.4709	.4983	.5220	.5419	.5579	.5699	.5778	.5815	.5812
2	.5767	.5683	.5560	.5399	.5202	.4971	.4708	.4416	.4097	.3754
3	.3391	.3009	.2613	.2207	.1792	.1374	.0955	.0538	.0128	−.0272
4	−.0660	−.1033	−.1386	−.1719	−.2028	−.2311	−.2566	−.2791	−.2985	−.3147
5	−.3276	−.3371	−.3432	−.3460	−.3453	−.3414	−.3343	−.3241	−.3110	−.2951
6	−.2767	−.2559	−.2329	−.2081	−.1816	−.1538	−.1250	−.0953	−.0652	−.0349
7	−.0047	.0252	.0543	.0826	.1096	.1352	.1592	.1813	.2014	.2192
8	.2346	.2476	.2580	.2657	.2708	.2731	.2728	.2697	.2641	.2559
9	.2453	.2324	.2174	.2004	.1816	.1613	.1395	.1166	.0928	.0684
10	.0435	.0184	−.0066	−.0313	−.0555	−.0789	−.1012	−.1224	−.1422	−.1603
11	−.1768	−.1913	−.2039	−.2143	−.2225	−.2284	−.2320	−.2333	−.2323	−.2290
12	−.2234	−.2157	−.2060	−.1943	−.1807	−.1655	−.1487	−.1307	−.1114	−.0912
13	−.0703	−.0489	−.0271	−.0052	.0166	.0380	.0590	.0791	.0984	.1165
14	.1334	.1488	.1626	.1747	.1850	.1934	.1999	.2043	.2066	.2069
15	.2051	.2013	.1955	.1879	.1784	.1672	.1544	.1402	.1247	.1080

When $x > 15.9$,

$$J_1(x) \simeq \sqrt{\left(\frac{2}{\pi x}\right)}\left\{\sin\left(x-\frac{1}{4}\pi\right)+\frac{3}{8x}\sin\left(x+\frac{1}{4}\pi\right)\right\}$$

$$\simeq \frac{.7979}{\sqrt{x}}\left\{\sin(57.296x-45^\circ)+\frac{3}{8x}\sin(57.296x+45^\circ)\right\}$$

TABLE 5.6.3

					$J_2(x)$					
x	0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	.0000	.0012	.0050	.0112	.0197	.0306	.0437	.0588	.0758	.0946
1	.1149	.1366	.1593	.1830	.2074	.2321	.2570	.2817	.3061	.3299
2	.3528	.3746	.3951	.4139	.4310	.4461	.4590	.4696	.4777	.4832
3	.4861	.4862	.4835	.4780	.4697	.4586	.4448	.4283	.4093	.3879
4	.3641	.3383	.3105	.2811	.2501	.2178	.1846	.1506	.1161	.0813

When $0 \le x < 1$, $J_2(x) \simeq \frac{x^2}{8}\left(1-\frac{x^2}{12}\right)$.

					$J_3(x)$					
x	0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	.0000	.0000	.0002	.0006	.0013	.0026	.0044	.0069	.0102	.0144
1	.0196	.0257	.0329	.0411	.0505	.0610	.0725	.0851	.0988	.1134
2	.1289	.1453	.1623	.1800	.1981	.2166	.2353	.2540	.2727	.2911
3	.3091	.3264	.3431	.3588	.3734	.3868	.3988	.4092	.4180	.4250
4	.4302	.4333	.4344	.4333	.4301	.4247	.4171	.4072	.3952	.3811

When $0 \le x < 1$, $J_3(x) \simeq \frac{x^3}{48}\left(1-\frac{x^2}{16}\right)$.

					$J_4(x)$					
x	0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	.0000	.0000	.0000	.0000	.0001	.0002	.0003	.0006	.0010	.0016
1	.0025	.0036	.0050	.0068	.0091	.0118	.0150	.0188	.0232	.0283
2	.0340	.0405	.0476	.0556	.0643	.0738	.0840	.0950	.1067	.1190
3	.1320	.1456	.1597	.1743	.1891	.2044	.2198	.2353	.2507	.2661
4	.2811	.2958	.3100	.3236	.3365	.3484	.3594	.3693	.3780	.3853

When $0 \le x < 1$, $J_4(x) \simeq \frac{x^4}{384}\left(1-\frac{x^2}{20}\right)$.

TABLE 5.6.4 Zeros of $J_0(x)$, $J_1(x)$, $J_2(x)$, $J_3(x)$, $J_4(x)$, $J_5(x)$

m	$j_{0,m}$	$j_{1,m}$	$j_{2,m}$	$j_{3,m}$	$j_{4,m}$	$j_{5,m}$
1	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178
6	18.0711	19.6159	21.1170	22.5827	24.0190	25.4303
7	21.2116	22.7601	24.2701	25.7482	27.1991	28.6266
8	24.3525	25.9037	27.4206	28.9084	30.3710	31.8117
9	27.4935	29.0468	30.5692	32.0649	33.5371	34.9888
10	30.6346	32.1897	33.7165	35.2187	36.6990	38.1599

where

n is a nonnegative integer, $n \ge 0$
l is an integer subject to constraints: $n - |l|$ is even and $|l| \le n$
r is the length of vector from origin to (x, y) point
θ is the angle between r- and x-axis in the counterclockwise direction

The orthogonality property is expressed by the formula

$$\iint_{x^2+y^2\le 1} V_{nl}^*(r,\theta)V_{mk}(r,\theta)r\,dr\,d\theta = \frac{\pi}{n+1}\delta_{mn}\delta_{kl} \tag{5.7.2}$$

where δ_{ij} is the Kronecker symbol. The real-valued radial polynomials satisfy the orthogonality relation

$$\int_0^1 R_{nl}(r)R_{ml}(r)r\,dr = \frac{1}{2(n+1)}\delta_{mn} \tag{5.7.3}$$

The radial polynomials are given by

$$\begin{aligned} R_{n\pm|l|}(r) &= \frac{1}{\left(\frac{n-|l|}{2}\right)!\,r^m}\left[\frac{d}{d(r^2)}\right]^{\frac{n-|l|}{2}}\left[\left(r^2\right)^{\frac{n+|l|}{2}}\left(r^2-1\right)^{\frac{n-|l|}{2}}\right] \\ &= \sum_{s=0}^{\frac{n-|l|}{2}}(-1)^2\frac{(n-s)!}{s!\left(\frac{n+|l|}{2}-s\right)!\left(\frac{n-|l|}{2}-s\right)!}r^{n-2s} \end{aligned} \tag{5.7.4}$$

For all permissible values of n and $|l|$

$$R_{n\pm|l|}(1) = 1, \quad R_{n|l|}(r) = R_{n(-|l|)}(r) \tag{5.7.5}$$

Table 5.7.1 gives the explicit form of the function $R_{n|l|}(r)$.

A relation between radial Zernike polynomials and Bessel functions of the first kind is given by

$$\int_0^1 R_{n|l|}(r)J_n(\upsilon r)r\,dr = (-1)^{\frac{n-|l|}{2}}\frac{J_{n+1}(\upsilon)}{\upsilon} \tag{5.7.6}$$

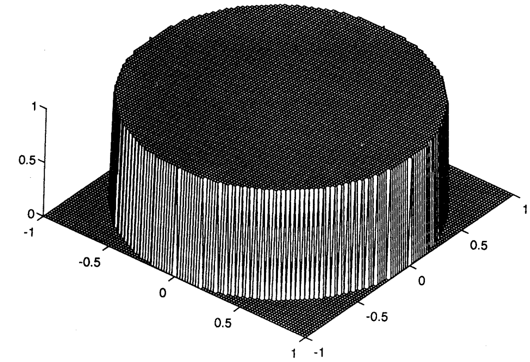
From (5.7.1) we obtain the following real Zernike polynomials:

$$\begin{aligned} U_{nl} &= \frac{1}{2}\left[V_{nl} + V_{n(-l)}\right] = R_{nl}(r)\cos l\theta, \quad l \ne 0 \\ U_{n(-l)} &= \frac{1}{2j}\left[V_{nl} - V_{n(-l)}\right] = R_{nl}(r)\sin l\theta, \quad l \ne 0 \\ V_{n0} &= R_{n0}(r) \end{aligned} \tag{5.7.7}$$

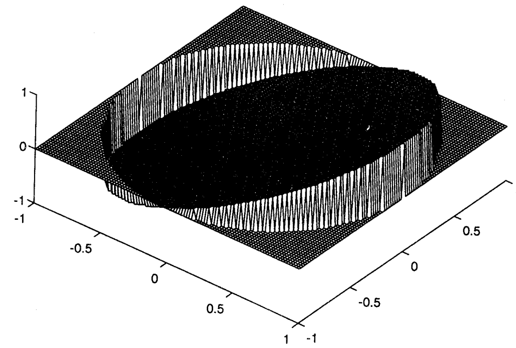
Figure 5.7.1 shows the function U_{nl} for a few radial modes.

TABLE 5.7.1 The Radial Polynomials $R_{n|l|}(r)$ for $|l| \le 8$, $n \le 8$

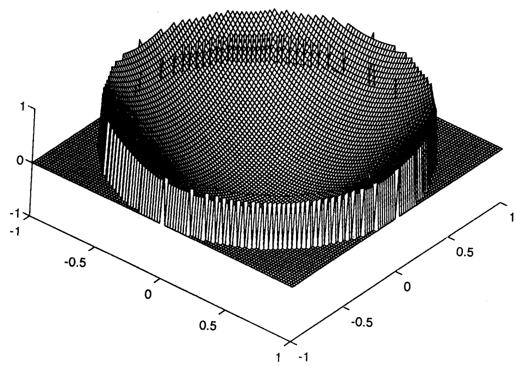
$\frac{n}{\lvert l\rvert}$	0	1	2	3	4	5	6	7	8
0	1		$2r^2 - 1$		$6r^4 - 6r^2 + 1$		$20r^6 - 30r^4 + 12r^2 - 1$		$70r^8 - 140r^6 + 90r^4 - 20r^2 + 1$
1		r		$3r^3 - 2r$		$10r^5 - 12r^3 + 3r$		$35r^7 - 60r^5 + 30r^3 - 4r$	
2			r^2		$4r^4 - 3r^2$		$15r^6 - 20r^4 + 6r^2$		$56r^8 - 105r^6 + 60r^4 - 10r^2$
3				r^3		$5r^5 - 4r^3$		$21r^7 - 30r^5 + 10r^3$	
4					r^4		$6r^6 - 5r^4$		$28r^8 - 42r^6 + 15r^4$
5						r^5		$7r^7 - 6r^5$	
6							r^6		$8r^8 - 7r^6$
7								r^7	
8									r^8



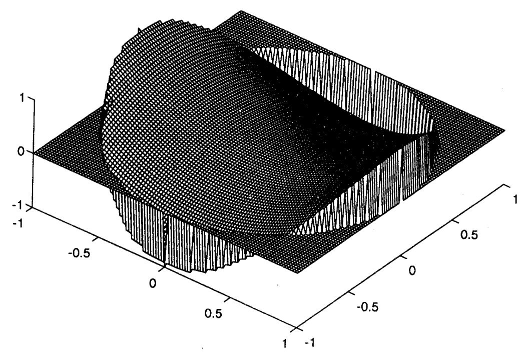
$n = 0, \ell = 0$



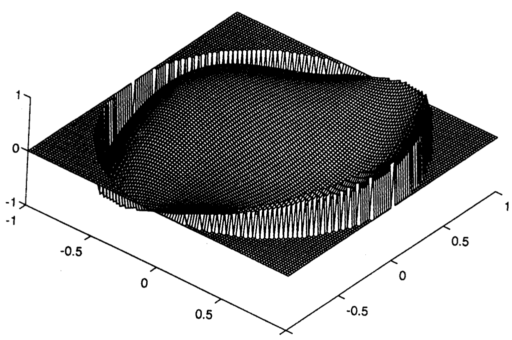
$n = 1, \ell = 1$



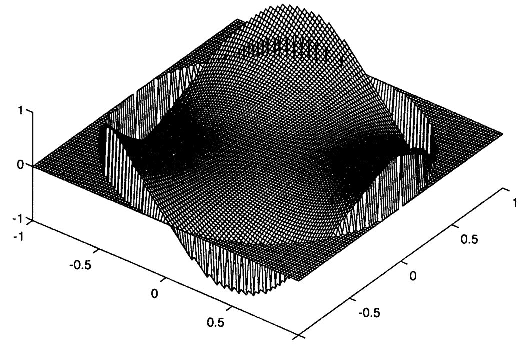
$n = 2, \ell = 0$



$n = 2, \ell = 2$

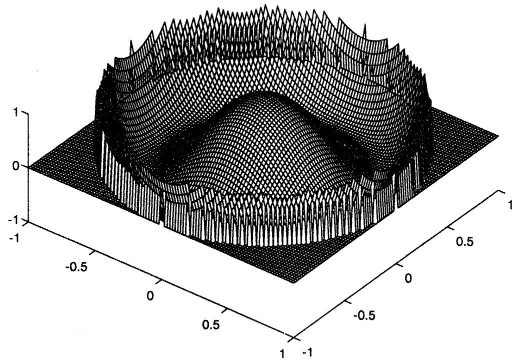


$n = 3, \ell = 1$

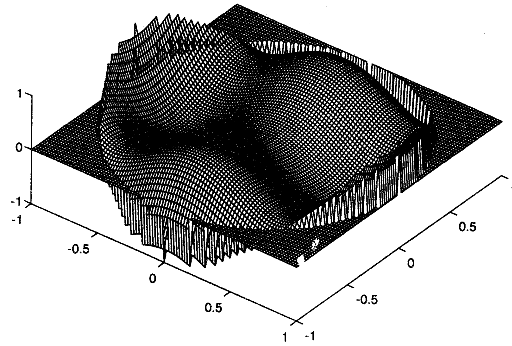


$n = 3, \ell = 3$

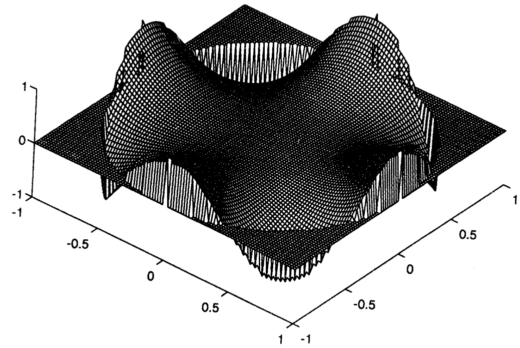
FIGURE 5.7.1



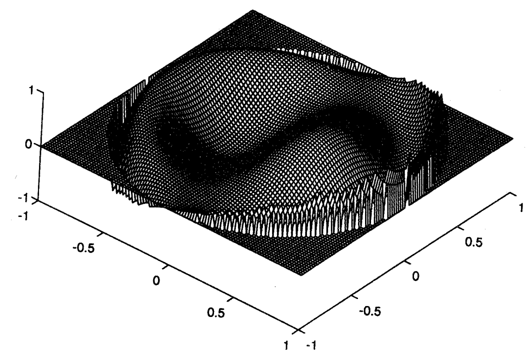
$n = 4, \ell = 0$



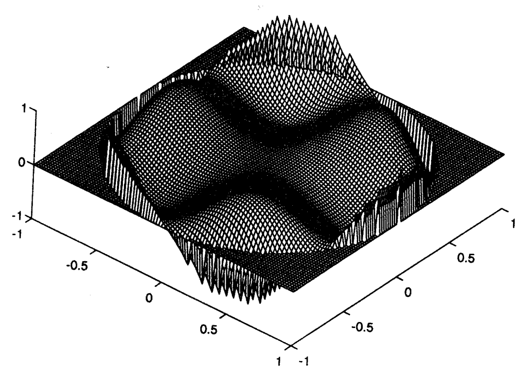
$n = 4, \ell = 2$



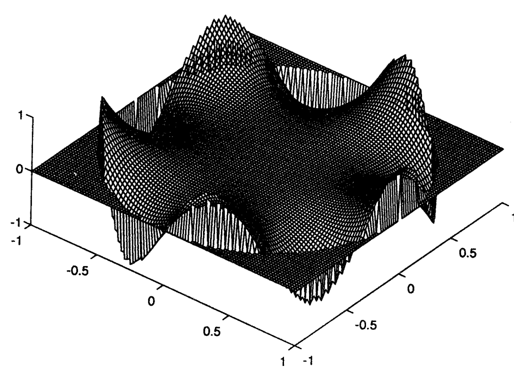
$n = 4, \ell = 4$



$n = 5, \ell = 1$



$n = 5, \ell = 3$



$n = 5, \ell = 5$

FIGURE 5.7.1 (Continued)

Expansion in Zernike Polynomials

If $f(x, y)$ is a piecewise continuous function, we can expand this function in Zernike polynomials in the form

$$f(x,y)=\sum_{n=0}^{\infty}\sum_{l=-\infty}^{\infty}A_{nl}V_{nl}(x,y),\quad n-|l| \text{ is even, } |l|\le n \tag{5.7.8}$$

Multiplying by $V_{nl}^{*}(x, y)$, integrating over the unit circle, and taking into consideration the orthogonality property we obtain

$$\begin{aligned} A_{nl} &= \frac{n+1}{\pi}\int_0^{2\pi}\int_0^1 V_{nl}^{*}(r,\theta)f(r\cos\theta,\, r\sin\theta)r\,dr\,d\theta \\ &= \frac{n+1}{\pi}\iint\limits_{x^2+y^2\le 1} V_{nl}^{*}(x,y)f(x,y)dx\,dy = A_{n(-l)}^{*} \end{aligned} \tag{5.7.9}$$

with restrictions of the values of n and l as shown above. A_{nl}'s are also known as **Zernike moments**.

Example

Expand the function $f(x, y) = x$ in Zernike polynomials.

Solution We write $f(r\cos\theta, r\sin\theta) = r\cos\theta$ and observe that r has exponent (degree) one. Therefore, the values of n will be 0, 1 and because $n - |l|$ must be even, l will take 0, 1 and –1 values. We then write

$$\begin{aligned} f(x,y) &= \sum_{n=0}^{\infty}\sum_{l=-\infty}^{\infty}A_{nl}R_{nl}(r)e^{jl\theta} \\ &= \sum_{n=0}^{1}\left(A_{n(-1)}R_{n(-1)}(r)e^{-j\theta}+A_{n0}R_{n0}(r)+A_{n1}R_{n1}(r)e^{j\theta}\right) \\ &= A_{00}R_{00}(r)+A_{1(-1)}R_{1(-1)}(r)e^{-j\theta}+A_{11}R_{11}(r)e^{j\theta} \end{aligned} \tag{5.7.10}$$

where three terms were dropped because they did not obey the condition that $n - |l|$ is even. From (5.7.5) $R_{1(-1)}(r) = R_{11}(r)$ and hence we obtain

$$A_{00}=\frac{1}{\pi}\int_0^{2\pi}\int_0^1 R_{00}(r)r\cos\theta r\,dr\,d\theta=0$$

$$A_{1(-1)}=\frac{2}{\pi}\int_0^{2\pi}\int_0^1 R_{11}(r)r\cos\theta e^{-j\theta}r\,dr\,d\theta=\frac{1}{2}$$

$$A_{11}=\frac{2}{\pi}\int_0^{2\pi}\int_0^1 R_{11}(r)r\cos\theta e^{j\theta}r\,dr\,d\theta=\frac{1}{2}$$

Therefore, the expansion becomes

$$f(x,y)=\frac{1}{2}re^{j\theta}+\frac{1}{2}re^{-j\theta}=r\cos\theta=R_{11}(r)\cos\theta=x$$

as was expected.

The radial polynomials $R_{nl}(r)$ are real valued and if $f(x, y)$ is real, that is, image intensity, it is often convenient to expand in real-values series. The real expansion corresponding to (5.7.8)

$$f(x, y) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \left(C_{nl} \cos l\theta + S_{nl} \sin l\theta \right) R_{nl}(r) \tag{5.7.11}$$

where $n - l$ is even and $l < n$. Observe that l takes only positive value. The unknown constants are found from

$$\begin{bmatrix} C_{nl} \\ S_{nl} \end{bmatrix} = \frac{2n+2}{\pi} \int_0^1 \int_0^{2\pi} r\,dr\,d\theta\, f(r\cos\theta, r\sin\theta) R_{nl}(r) \begin{bmatrix} \cos l\theta \\ \sin l\theta \end{bmatrix}, \quad l \neq 0 \tag{5.7.12}$$

$$C_{n0} = A_{n0} = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} r\,dr\,d\theta\, f(r\cos\theta, r\sin\theta) R_{nl}(r), \quad l = 0 \tag{5.7.13a}$$

$$S_{n0} = 0, \quad l = 0 \tag{5.7.13b}$$

If the function is axially symmetric only the cosine terms are needed. The connection between real and complex Zernike coefficients are

$$C_{nl} = 2\mathrm{Re}\{A_{nl}\} \tag{5.7.14a}$$

$$S_{nl} = -2\mathrm{Im}\{A_{nl}\} \tag{5.7.14b}$$

$$A_{nl} = (C_{nl} - jS_{nl})/2 = (A_{n(-l)})^* \tag{5.7.14c}$$

Figure 5.7.2 shows the reconstruction of the letter Z using different orders of Zernike moments.

1.6 Sampling of Signals

Two critical questions in signal sampling are: First, do the sampled values of a function adequately represent the system? Second, what must the sampling interval be in order that an optimum recovery of the signal can be accomplished from the sampled values?

The value of the function at the sampling points is the **sampled value**, the time that separates the sampling points is the **sampling interval**, and the reciprocal of the sampling interval is the **sampling frequency** or **sampling rate**.

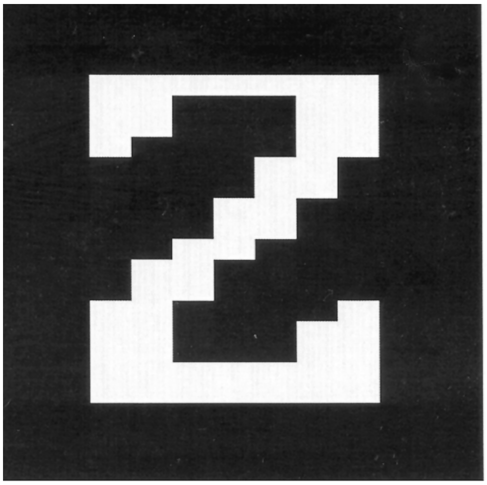
If the sampling interval T_s is chosen to be constant, and $n = 0 \pm 1, \pm 2, \ldots$, the sampled signal is

$$f_s(t) = f(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} f(nT_s)\, \delta(t - nT_s) \tag{6.1}$$

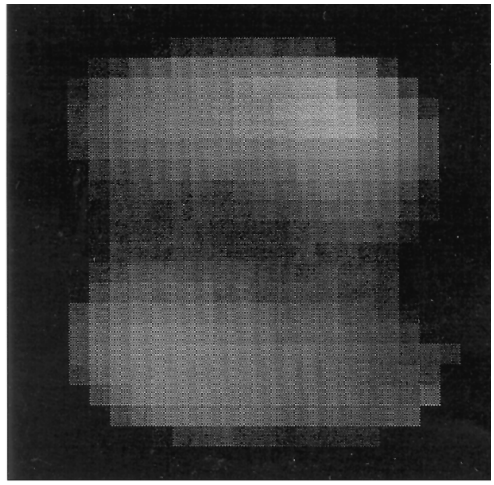
Its Fourier transform is

$$F_s(\omega) \doteq \mathcal{F}\{f_s(t)\} = \sum_{n=-\infty}^{\infty} f(nT_s)\, \mathcal{F}\{\delta(t - nT_s)\} = \sum_{n=-\infty}^{\infty} f(nT_s) e^{-jn\omega T_s} \tag{6.2}$$

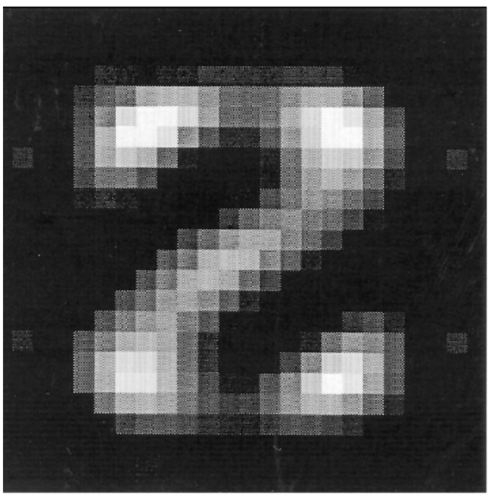
We can also represent the Fourier transform of a sampled function as follows:



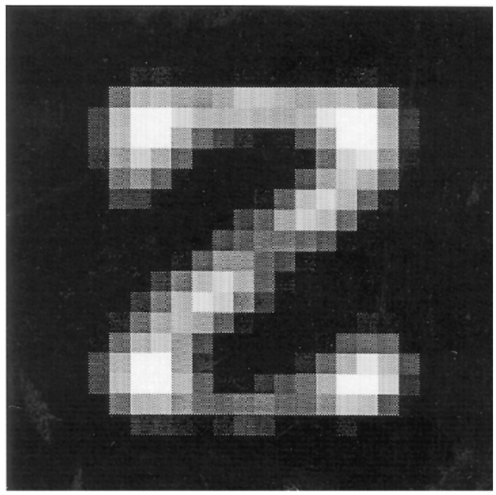
Original



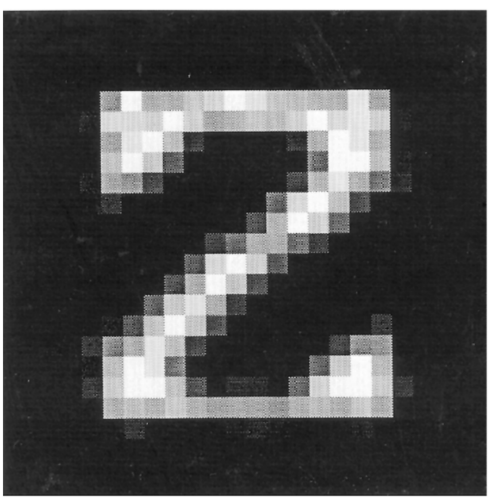
n up to 5



n up to 10



n up to 15



n up to 20

FIGURE 5.7.2

$$
\begin{aligned}
F_s(\omega) &\doteq \mathcal{F}\left\{f(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_s)\right\} = \frac{1}{2\pi}\mathcal{F}\{f(t)\} * \mathcal{F}\left\{\sum_{n=-\infty}^{\infty}\delta(t-nT_s)\right\} \\
&= \frac{1}{2\pi}F(\omega) * \left[\frac{2\pi}{T_s}\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_s)\right] \\
&= \frac{1}{T_s}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}F(x)\delta(\omega-n\omega_s-x)\,dx = \frac{1}{T_s}\sum_{n=-\infty}^{\infty}F(\omega-n\omega_s) \\
&= \frac{1}{T_s}\sum_{n=-\infty}^{\infty}F(\omega+n\omega_s), \quad \omega_s = \frac{2\pi}{T_s}
\end{aligned}
\tag{6.3}
$$

$F_s(\omega)$ is periodic with period ω_s in the frequency domain.

Example

$$
\mathcal{F}\left\{e^{-|t|}\sum_{n=-\infty}^{\infty}\delta(t-nT_s)\right\} \doteq \mathcal{F}_s(\omega) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty}\frac{2}{1+(\omega-n\omega_s)^2}
$$

1.6.1 The Sampling Theorem

It can be shown that it is possible for a **band-limited** signal $f(t)$ to be exactly specified by its sampled valued provided that the time distance between sample values does not exceed a critical sampling interval.

Theorem 6.1.1

A finite energy function $f(t)$ having a band-limited Fourier transform, $F(\omega) = 0$ for $|\omega| \geq \omega_N$, can be completely reconstructed from its sampled values $f(nT_s)$ (see Figure 6.1.1), with

$$
f(t) = \sum_{n=-\infty}^{\infty} T_s f(nT_s)\left\{\frac{\sin\left[\frac{\omega_s(t-nT_s)}{2}\right]}{\pi(t-nT_s)}\right\}, \quad \omega_s = \frac{2\pi}{T_s} \tag{6.1.1}
$$

provided that

$$
\frac{2\pi}{\omega_s} = T_s \leq \frac{\pi}{\omega_N} = \frac{1}{2f_N} = \frac{T_N}{2}
$$

The function within the braces, which is the sinc function, is often called the interpolation function to indicate that it allows an interpolation between the sampled values to find $f(t)$ for all t.

Proof Employ (6.3) and Figure 6.1.1c to write

$$
F(\omega) = p_{\omega_s/2}(\omega)\,T_s F_s(\omega) \tag{6.1.2}
$$

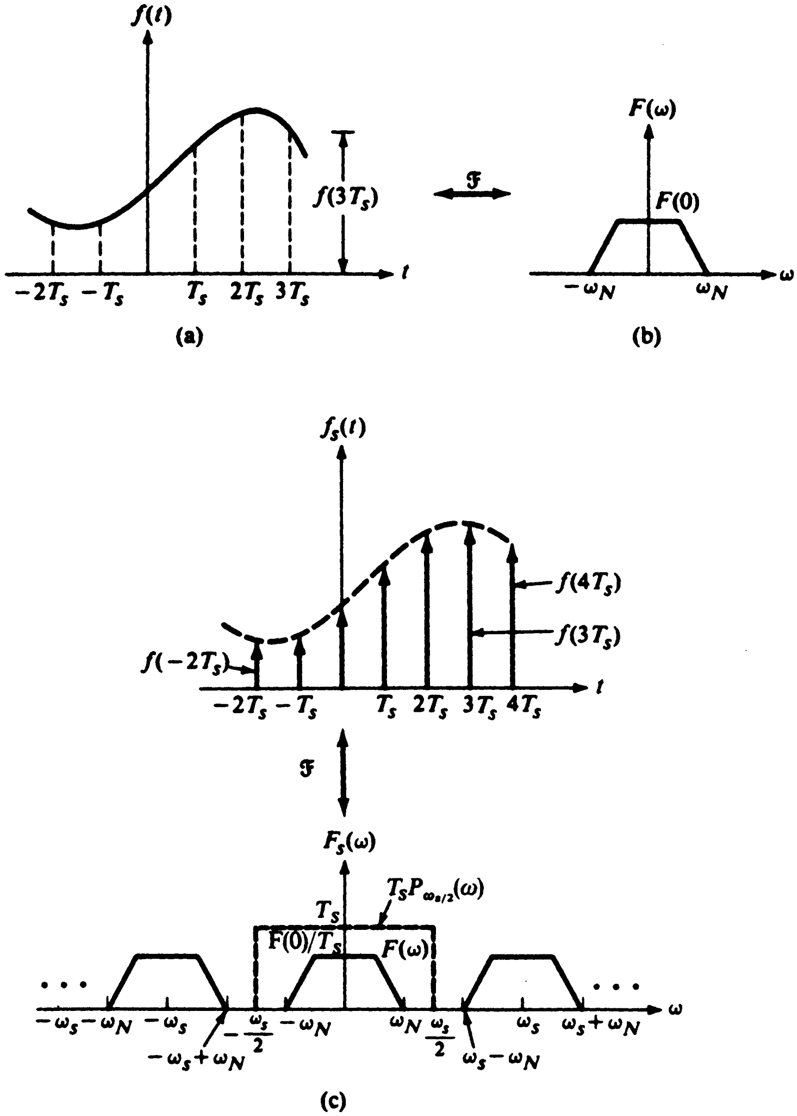


FIGURE 6.1.1

By (6.1.2), the above equation becomes

$$f(t) = \mathscr{F}^{-1}\{F(\omega)\} = \mathscr{F}^{-1}\left\{p_{\omega_s/2}(\omega)T_s \sum_{n=-\infty}^{\infty} f(nT_s)e^{-jn\omega T_s}\right\}$$

$$= T_s \sum_{n=-\infty}^{\infty} f(nT_s)\mathscr{F}^{-1}\left\{p_{\omega_s/2}(\omega)e^{-jn\omega T_s}\right\}$$

By application of the frequency-shift property of the Fourier transform, this equation proves the theorem.

The sampling time

$$T_s = \frac{T_N}{2} = \frac{1}{2f_N} \tag{6.1.3}$$

is related to the **Nyquist interval.** It is the largest time interval that can be used for sampling of a band-limited signal and still allows recovering of the signal without distortion. If, however, the sampling time is larger than the Nyquist interval, overlap of spectra takes place, known as **aliasing**, and no perfect reconstruction of the band-limited signal is possible. Figure 6.1.2 shows the delta sampling representation and recovery of a band-limited signal. The following definitions have been used in the figure:

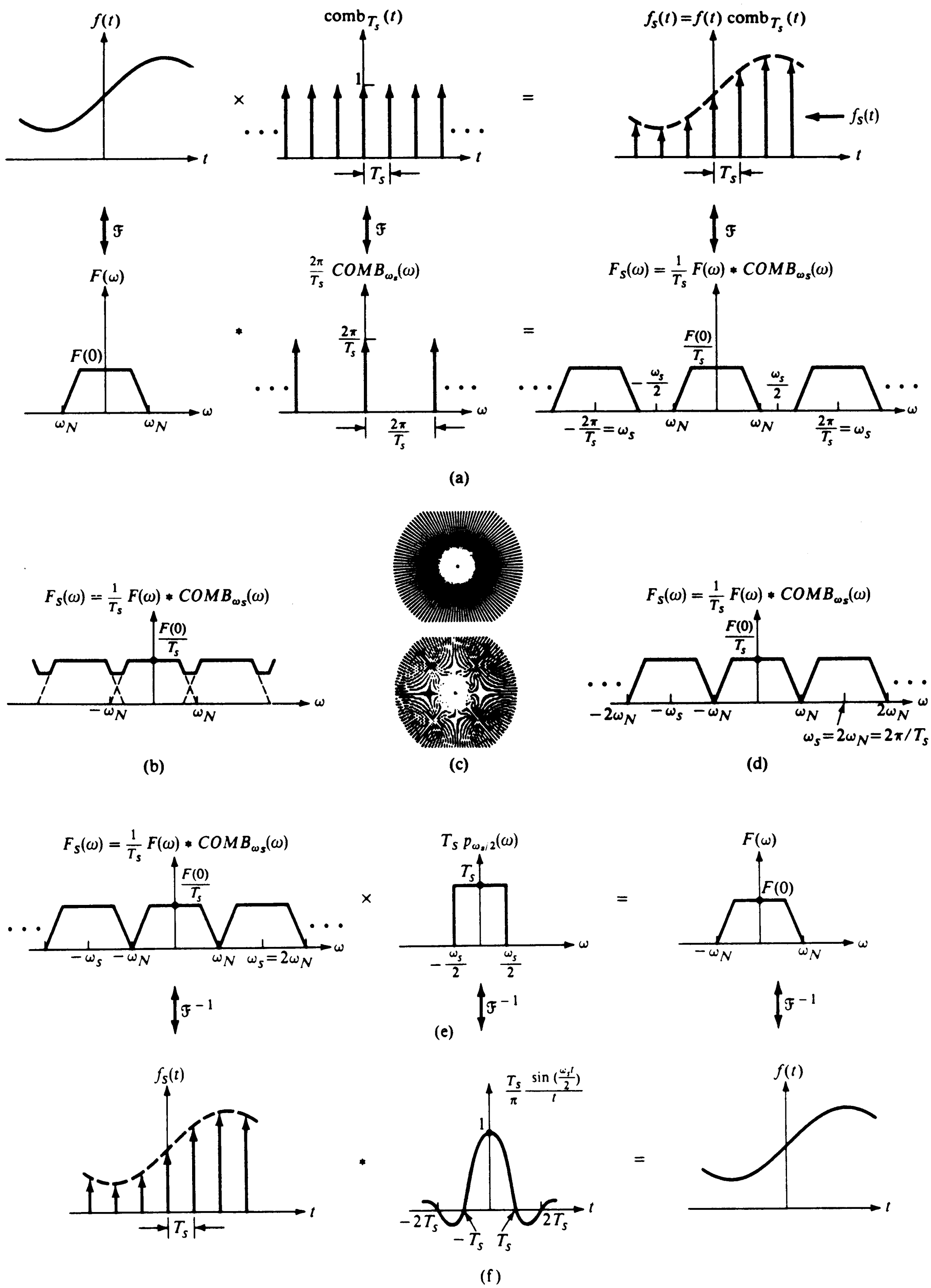


FIGURE 6.1.2

$$\mathrm{comb}_{T_s}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \tag{6.1.4}$$

$$\mathrm{COMB}_{\omega_s}(\omega) = \sum_{n=-\infty}^{\infty} \delta(\omega - \omega_s) \tag{6.1.5}$$

Frequency Sampling

Analogous to the time-sampling theorem, a frequency-sampling equivalent also exits.

Theorem 6.1.2

A time function $f(t)$ that is time limited so that

$$f(t) = 0, \quad |t| > T_N \tag{6.1.6}$$

possesses a Fourier transform that can be uniquely determined from its samples at distances $n\pi/T_N$, and is given by

$$F(\omega) = \sum_{n=-\infty}^{\infty} F\left(n\frac{\pi}{T_N}\right)\frac{\sin(\omega T_N - n\pi)}{\omega T_N - n\pi} \tag{6.1.7}$$

where the sampling is at the Nyquist rate.

Sampling With a Train of Rectangular Pulses

The Fourier transform of a band-limited function sampled with periodic pulses is given by (see Figure 6.1.3)

$$\begin{aligned} F_s(\omega) &= \mathcal{F}\{f(t)f_p(t)\} = \frac{1}{2\pi}F(\omega) * F_p(\omega) \\ &= \frac{1}{2\pi}F(\omega) * \left\{\sum_{n=-\infty}^{\infty} 2\pi \frac{\sin\left(\frac{n\omega_s\tau}{2}\right)}{\frac{n\omega_s\tau}{2}}\delta(\omega - n\omega_s)\right\} \\ &= \sum_{n=-\infty}^{\infty} \frac{\sin\left(\frac{n\omega_s\tau}{2}\right)}{\frac{n\omega_s\tau}{2}} \int_{-\infty}^{\infty} \delta(x - n\omega_s)F(\omega - x)\,dx \\ &= \sum_{n=-\infty}^{\infty} \frac{\sin\left(\frac{n\omega_s\tau}{2}\right)}{\frac{n\omega_s\tau}{2}} F(\omega - n\omega_s) \end{aligned} \tag{6.1.8}$$

where τ is the width of the pulse. The above expression indicates that as long as $\omega_s > 2\omega_N$, the spectrum of the sampled signal contains no overlapping spectra of $f(t)$ and can be recovered using a low-pass filter.

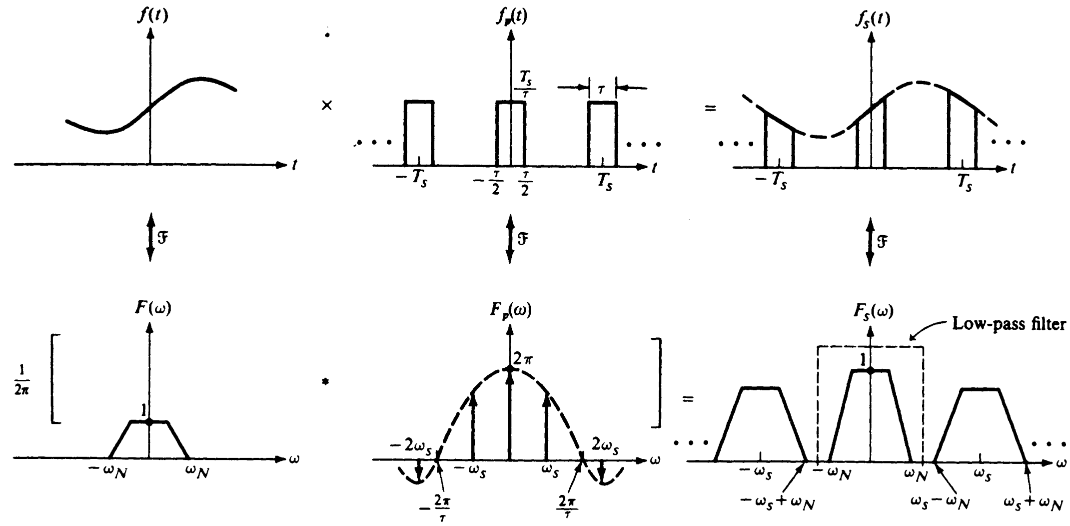


FIGURE 6.1.3

1.6.2 Extensions of the Sampling Theorem

The sampling theorem of a band-limited function of n variables is given by the following theorem:

Theorem 6.2.1

Let $f(t_1, t_2, \ldots, t_n)$ be a function of n real variables, whose n-dimensional Fourier integral exists and is identically zero outside an n-dimensional rectangle and is symmetrical about the origin, that is,

$$g(y_1, y_2, \ldots, y_n)\ 0, \quad |y_k| > |\omega_k|, \quad k = 1, 2, \ldots, n \tag{6.2.1}$$

Then

$$f(t_1, t_2, \ldots, t_n) = \sum_{m_1=-\infty}^{\infty} \cdots \sum_{m_n=-\infty}^{\infty} f\left(\frac{\pi m_1}{\omega_1}, \ldots, \frac{\pi m_n}{\omega_n}\right) \times \frac{\sin(\omega_1 t_1 - m_1\pi)}{\omega_1 t_1 - m_1\pi} \cdots \frac{\sin(\omega_n t_n - m_n\pi)}{\omega_n t_n - m_n\pi} \tag{6.2.2}$$

An additional theorem on the sampling of band-limited signals follows.

Theorem 6.2.2

Let $f(t)$ be a continuous function with finite Fourier transform $F(\omega)[F(\omega) = 0$ for $|\omega| > 2\pi f_N]$. Then

$$f(t) = \sum_{k=-\infty}^{\infty} \left[\xi(kh) + (t-kh)\xi^{(1)}(kh) + \cdots + \frac{(t-kh)^R}{R!}\xi^{(R)}(kh)\right] \times \left[\frac{\sin\frac{\pi}{h}(t-kh)}{\frac{\pi}{h}(t-kh)}\right]^{R+1} \tag{6.2.3}$$

where:

R is the highest derivative order

$h = (R + 1)/(2f_N)$

$\xi^{(R)}(kh)$ is the Rth derivative of the function $\xi(.)$

$$\xi^{(j)}(kh) = \sum_{i=0}^{j} \binom{j}{i} \left(\frac{\pi}{h}\right)^{j-1} \Gamma_{R+1}^{(j-1)} f^{(i)}(kh)$$

$$\Gamma_{a}^{(\beta)} = \frac{d^{\beta}}{dt^{\beta}} \left[\left(\frac{t}{\sin t} \right)^{\alpha} \right]_{t=0}$$

$$\Gamma_{\alpha}^{(0)} = 1, \Gamma_{\alpha}^{(2)} = \frac{\alpha}{3}, \Gamma_{\alpha}^{(4)} = \frac{\alpha(5\alpha+2)}{15}, \Gamma_{\alpha}^{(6)} = \frac{\alpha(35\alpha^2+42\alpha+16)}{63}, \ldots, \Gamma_{\alpha}^{(\beta)} = 0 \text{ for odd } \beta$$

Papoulis Extensions

The band-limited signal

$$f(t) = \frac{1}{2\pi} \int_{-w_1}^{w_1} F(\omega) e^{j\omega t} d\omega \tag{6.2.4}$$

can be represented by

$$f(t) = \sum_{n=-\infty}^{\infty} f(nT) \frac{\sin w_0(t-nT)}{w_2(t-nT)} \tag{6.2.5}$$

where

$$w_2 = \frac{\pi}{T} \geq w_1, \quad w_1 \leq w_0 \leq 2w_2 - w_1$$

Theorem 6.2.3

Given an arbitrary sequence of numbers $\{a_n\}$, if we form the sum

$$x(t) = \sum_{n=-\infty}^{\infty} a_n \frac{\sin w_0(t-nT)}{w_2(t-nT)} \tag{6.2.6}$$

then $x(t)$ is band limited by w_0.

The sampling expansion of $f^2(t)$ is given by

$$f^2(t) = \sum_{n=-\infty}^{\infty} f^2(nT) \frac{\sin w_0(t-nT)}{w_2(t-nT)} \tag{6.2.7}$$

where

$$w_2 = \frac{\pi}{T}, w_2 \geq 2w_1, \quad 2w_1 \leq w_0 \leq 2w_2 - 2w_1, T \leq \frac{\pi}{2w_1}$$

The band-limited signal given in (6.2.4) can be expressed in terms of the sample values $g(nT)$ of the output

$$g(t) = \frac{1}{2\pi} \int_{-w_1}^{w_1} F(\omega) H(\omega) e^{jwt} d\omega \tag{6.2.8}$$

of a system with transfer function $H(\omega)$ driven by $f(t)$. The sampling expansion of $f(t)$ is

$$f(t) = \sum_{n=-\infty}^{\infty} g(nT) y(t - nT) \tag{6.2.9}$$

where

$$y(t) = \frac{1}{2w_1} \int_{-w_1}^{w_1} \frac{e^{j\omega t}}{H(\omega)} d\omega \tag{6.2.10}$$

1.7 Asymptotic Series

Functions such as $f(z)$ and $\varphi(z)$ are defined on a set R in the complex plane. By a neighborhood of z_0 we mean an open disc $|z - z_0| < \delta$ if z_0 is at a finite distance, and a region $|z| > \delta$ if z_0 is the point at infinity.

$f = O(\varphi)$ and $f = o(\varphi)$ Notation

We write $f = O(\varphi)$ if there exists a constant A such that $|f| \le A|\varphi|$ for all z in R.

We also write $f = O(\varphi)$ as $z \to z_0$ if there exists a constant A and a neighborhood U of z_0 such that $|f| \le A|\varphi|$ for all points in the intersection of U and R.

We write $f = o(\varphi)$ as $z \to z_0$ if, for any positive number ε, there exists a neighborhood U of z_0 such that $|f| \le \varepsilon|\varphi|$ for all points z of the intersection of U and R.

More simply, if φ does not vanish on R, $f = O(\varphi)$ means that f/φ is bounded, $f = o(\varphi)$ means that f/φ tends to zero as $z \to z_0$.

Asymptotic Sequence

A sequence of functions $\{\varphi_n(z)\}$ is called an **asymptotic sequence** as $z \to z_0$ if there is a neighborhood of z_0 in which none of the functions vanish (except the point z_0) and if for all n

$$\varphi_{n+1} = o(\varphi_n) \qquad \text{as } z \to z_0$$

For example, if z_0 is finite $\{(z - z_0)^n\}$ is an asymptotic sequence as $z \to z_0$, and $\{z^{-n}\}$ is as $z \to \infty$.

Poincaré Sense Asymptotic Sequence

The formal series

$$f(z) \cong \sum_{n=0}^{\infty} a_n \varphi_n(z) \tag{7.1}$$

which is not necessarily convergent, is an asymptotic expansion of $f(z)$ in the Poincaré sense with respect to the asymptotic sequence $\{\varphi_n(z)\}$, if for every value of m,

$$f(z) - \sum_{n=0}^{\infty} a_n \varphi_n(z) = o(\varphi_m(z)) \tag{7.2}$$

as $z \to z_0$.

Because

$$f(z) - \sum_{n=0}^{m-1} a_n \varphi_n(z) = a_m \varphi_m(z) + o(\varphi_m(z)) \tag{7.3}$$

in partial sum

$$\sum_{n=0}^{m-1} a_n \varphi_n(z) \tag{7.4}$$

is an approximation to $f(z)$ with an error $O(\varphi_m)$ as $z \to z_0$; this error is of the same order of magnitude as the first term omitted. If such an asymptotic expansion exists, it is unique, and the coefficients are given successively by

$$a_m = \frac{\lim_{z \to z_0} \left\{ f(z) - \sum_{n=0}^{m-1} a_n \varphi_n(z) \right\}}{\varphi_m(z)} \tag{7.5}$$

Hence, for a function $f(z)$ we write

$$f(z) \cong \sum_{n=0}^{\infty} a_n \varphi_n(z) \tag{7.6}$$

Asymptotic Approximation

A partial sum of (7.6) is called an **asymptotic approximation** to $f(z)$. The first term is called the **dominant term**.

The above definition applies equally well for a real variable z.

Asymptotic Power Series

We shall assume that the transformation $z' = 1/(z - z_0)$ has been done for limit points z_0 located at a finite distance. Hence we can always consider expansions as z approaches infinity in a sector $a < ph\ z < \beta$; or, for real value x, as x approaches infinity or as x approaches negative infinity.

The divergence series

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{z_n} = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots \frac{a_n}{z^n} + \cdots \tag{7.7}$$

in which the sum of the first $(n + 1)$ terms is $S_n(z)$, is said to be an **asymptotic expansion** of a function $f(z)$ for a given range of values of arg z, if the expansion $R_n(z) = z^n\{f(z) - S_n(z)\}$ satisfies the condition

$$\lim_{|z| \to \infty} R_n(z) = 0 \quad (n \text{ is fixed}) \tag{7.8}$$

even though

$$\lim_{n\to\infty}\left|R_n(z)\right| = \infty \quad (z \text{ is fixed})$$

When this is true, we can make

$$\left|z^n\{f(z) - S_n(z)\}\right| < \varepsilon \tag{7.9}$$

where ε is arbitrarily small, by making $|z|$ sufficiently large. This definition is due to Poincaré.

Example

For real x, integration on the real axis and repeated integration by parts, we obtain

$$f(x) = \int_x^\infty t^{-1}e^{x-t}\,dt = \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \cdots + \frac{(-1)^{n-1}(n-1)!}{x^n} + (-1)^n n!\int_x^\infty \frac{e^{x-t}\,dt}{t^{n+1}}$$

If we consider the expansion

$$u_{n-1} = \frac{(-1)^{n-1}(n-1)!}{x^n}$$

we can write

$$\sum_{m=0}^{n} u_m = \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \cdots + \frac{(-1)^n n!}{x^{n+1}} = S_n(x)$$

But $\left|u_m/u_{m-1}\right| = mx^{-1} \to \infty$ as $m \to \infty$. The Series Σu_m is divergent for all values of x. However, the series can be used to calculate $f(x)$.

For a fixed n, we can calculate S_n from the relation

$$f(x) - S_n(x) = (-1)^{n+1}(n+1)!\int_x^\infty \frac{e^{x-t}\,dt}{t^{n+2}}$$

Because $\exp(x - t) \le 1$,

$$\left|f(x) - S_n(x)\right| = (n+1)!\int_x^\infty \frac{e^{x-t}\,dt}{t^{n+2}} < (n+1)!\int_x^\infty \frac{dt}{t^{n+2}} = \frac{n!}{x^{n+1}}$$

For large values of x the right-hand member of the above relation is very small. This shows that the value of $f(x)$ can be calculated with great accuracy for large values of x, by taking the sum of a suitable number of terms of the series Σu_m. From the last relation we obtain

$$\left|x^n\{ f(x) - S_n(x)\}\right| < n!x^{-1} \to 0 \quad \text{as } x \to \infty$$

which satisfies the asymptotic expansion condition.

Operation of Asymptotic Power Series

Let the following two functions possess asymptotic expansions:

$$f(x) \approx \sum_{n=0}^{\infty} \frac{a_n}{x^n}, \quad g(x) \approx \sum_{n=0}^{\infty} \frac{b_n}{x^n} \quad \text{as } x \to \infty$$

on the real axis.

(a) If A is constant

$$Af(x) \approx \sum_{n=0}^{\infty} \frac{Aa_n}{x^n} \tag{7.10}$$

(b)

$$f(x) + g(x) \approx \sum_{n=0}^{\infty} \frac{a_n + b_n}{x^n} \tag{7.11}$$

(c)

$$\begin{gathered} f(x)g(x) \approx \sum_{n=0}^{\infty} \frac{c_n}{x^n} \\ c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0 \end{gathered} \tag{7.12}$$

(d) If $a_0 \neq 0$, then

$$\frac{1}{f(x)} \approx \frac{1}{a_0} + \sum_{n=1}^{\infty} \frac{d_n}{x^n}, \quad x \to \infty \tag{7.13}$$

The function $1/f(x)$ tends to a finite limit $1/a_0$ as x approaches infinity. Hence,

$$\begin{aligned} \left(\frac{1}{f(x)} - \frac{1}{a_0} \right) \Big/ (1/x) &= x \left(\frac{1}{a_0 + (a_1/x) + O(1/x^2)} - \frac{1}{a_0} \right) \\ &= \frac{-a_1 + O\left(\frac{1}{x}\right)}{a_0 \left[a_0 + (a_1/x) + O(1/x^2) \right]} \to -\frac{a_1}{a_0^2} = d_1 \end{aligned}$$

Similarly we obtain

$$\left(\frac{1}{f(x)} - \frac{1}{a_0} + \frac{a_1}{a_0^2 x} \right) \Big/ \left(\frac{1}{x^2} \right) \to \frac{a_1^2 - a_0 a_2}{a_0^3} = d_2$$

and so on.

In general, any rational function of $f(x)$ has an asymptotic power series expansion provided that the denominator does not tend to zero as x approaches infinity.

(e) If $f(x)$ is continuous for $x > a > 0$ and if $x > a$, then

$$F(x) = \int_x^\infty \left(f(t) - a_0 - \frac{a_1}{t} \right) dt \approx \frac{a_2}{x} + \frac{a_3}{2x^2} + \cdots + \frac{a_{n+1}}{nx^n} + \cdots \tag{7.14}$$

(f) If $f(x)$ has a continuous derivative $f'(x)$, and if $f'(x)$ possess an analytic power series expansion as x approaches infinity, the expression is

$$f'(x) \approx -\sum_{n=2}^{\infty} \frac{(n-1)a_{n-1}}{x^n} \tag{7.15}$$

(g) It is permissible to integrate an asymptotic expansion term-by-term. The resulting series is the expansion of the integral of the function represented by the original series.
Let

$$f(x) \approx \sum_{m=2}^{\infty} a_m x^{-m} \quad \text{and} \quad S_n = \sum_{m=2}^{n} a_m x^{-m}$$

Then, given any positive number e, we can find x_0 such that

$$|f(x) - S_n(x)| < \varepsilon |x|^{-n} \quad \text{for } x > x_0$$

Hence

$$\left| \int_x^\infty f(x)dx - \int_x^\infty S_n(x)dx \right| \le \int_x^\infty |f(x) - S_n(x)| dx < \frac{\varepsilon}{(n-1)x^{n-1}}$$

However,

$$\int_x^\infty S_n(x)dx = \frac{a_2}{x} + \frac{a_3}{2x^2} + \cdots + \frac{a_n}{(n-1)x^{n-1}}$$

and therefore

$$\int_x^\infty f(x)dx \approx \sum_{m=2}^{\infty} \frac{a_m}{(m-1)x^{m-1}}$$

Example
The Fresnel integrals

$$\int_u^\infty \cos(\theta^2)d\theta, \quad \int_u^\infty \sin(\theta^2)d\theta \tag{7.16}$$

can be written in the form

$$\int_{u^2}^\infty \frac{\cos t}{\sqrt{t}}dt, \quad \int_{u^2}^\infty \frac{\sin t}{\sqrt{t}}dt$$

These are particular cases of the real and imaginary parts of the integral

$$f(x,a)=\int_x^{\infty}\frac{e^{jt}}{t^a}\,dt \tag{7.17}$$

Integrating by parts we obtain

$$\begin{aligned} f(x,a) &= \frac{je^{jx}}{x^a}-jaf(x,a+1)=\frac{je^{jx}}{x^a}\sum_{r=0}^{n}\frac{\Gamma(a+r)}{\Gamma(a)(jx)^r} \\ &+\frac{1}{j^{n+1}}\frac{\Gamma(a+n+1)}{\Gamma(a)}f(x,a+n+1)\end{aligned} \tag{7.18}$$

Hence

$$f(x,a)\approx\frac{je^{jx}}{x^a}\sum_{r=0}^{\infty}\frac{\Gamma(a+r)}{\Gamma(a)(jx)^r} \tag{7.19}$$

as x approaches infinity. The absolute value of the remainder after $n + 1$ terms is

$$\frac{\Gamma(a+n+1)}{\Gamma(a)}\left|\int_x^{\infty}\frac{e^{jt}}{t^{a+n+1}}\,dt\right|\le\frac{\Gamma(a+n+1)}{\Gamma(a)}\int_x^{\infty}\frac{dt}{t^{a+n+1}}=\frac{\Gamma(a+n)}{\Gamma(a)x^{a+n}}$$

Hence, the remainder after n terms does not exceed in absolute value the absolute value of the $(n + 1)$th term, which proves the result.

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