Pr (ϵ) approaches zero as the length of encoded sequence approaches infinity. Because of the bandwidth requirement, the orthogonal signal technique is not efficient.

Estimation. 1. Linear estimation is a trivial modification of the detection problem. The optimum estimator is a simple correlator or matched filter followed by a gain.

2. The nonlinear estimation problem introduced several new ideas. The optimum receiver is sometimes difficult to realize exactly and an approximation is necessary. Above a certain energy-to-noise level we found that we could make the estimation error appreciably smaller than in the linear estimation case which used the same amount of energy. Specifically,

$$\operatorname{Var}[\hat{a} - A] \approx \frac{N_0/2}{\displaystyle\int_0^T \left[\frac{\partial s(t, A)}{\partial A}\right]^2 dt}. \tag{138}$$

As the noise level increased however, the receiver exhibited a *threshold* phenomenon and the error variance increased rapidly. Above the threshold we found that we had to consider the problem of a bandwidth constraint when we designed the system.

We now want to extend our model to a more general case. The next step in the direction of generality is to consider known signals in the presence of nonwhite additive Gaussian noise.

4.3 DETECTION AND ESTIMATION IN NONWHITE GAUSSIAN NOISE

Several situations in which nonwhite Gaussian interference can occur are of interest:

1. Between the actual noise source and the data-processing part of the receiver are elements (such as an antenna and RF filters) which shape the noise spectrum.
2. In addition to the desired signal at the receiver, there may be an interfering signal that can be characterized as a Gaussian process. In radar/sonar it is frequently an interfering target.

With this motivation we now formulate and solve the detection and estimation problem. As we have seen in the preceding section, a close coupling exists between detection and estimation. In fact, the development through construction of the likelihood ratio (or function) is identical. We derive the simple binary case in detail and then indicate how the results extend to other cases of interest. The first step is to specify the model.

When colored noise is present, we have to be more careful about our model. We assume that the transmitted signal on hypothesis 1 is

$$\sqrt{E}\, s(t) \triangleq \begin{cases} \sqrt{E}\, s_T(t), & 0 \le t \le T, \\ 0\quad, & \text{elsewhere.} \end{cases} \tag{139}$$

Observe that $s(t)$ is defined for all time. Before reception the signal is corrupted by additive Gaussian noise $n(t)$. The received waveform $r(t)$ is observed over the interval $T_i \le t \le T_f$. Thus

$$\begin{aligned} r(t) &= \sqrt{E}\, s(t) + n(t), \qquad & T_i \le t \le T_f : H_1 \\ &= n(t), & T_i \le t \le T_f : H_0. \end{aligned} \tag{140}$$

Sometimes T_i will equal zero and T_f will equal T. In general, however, we shall let T_i (≤ 0) and T_f $(\ge T)$ remain arbitrary. Specifically, we shall frequently examine the problem in which $T_i = -\infty$ and $T_f = +\infty$. A logical question is; why should we observe the received waveform when the signal component is zero? The reason is that the noise outside the interval is correlated with the noise inside the interval, and presumably the more knowledge available about the noise inside the interval the better we can combat it and improve our system performance. A trivial example can be used to illustrate this point.

Example. Let

$$\begin{aligned} \sqrt{E}\, s(t) &= 1, \qquad & 0 \le t \le 1 \\ &= 0, & \text{elsewhere.} \end{aligned} \tag{141}$$

Let

$$n(t) = n, \qquad 0 \le t \le 2, \tag{142}$$

where n is a Gaussian random variable. We can decide which hypothesis is true in the following way:

$$l = \int_0^1 r(t)\, dt - \int_1^2 r(t)\, dt. \tag{143}$$

If

$$\begin{aligned} l &= 0, \qquad & \text{say } H_0 \\ &\ne 0 & \text{say } H_1. \end{aligned}$$

Clearly, we can make error-free decisions. Here we used the extended interval to estimate the noise inside the interval where the signal was nonzero. Unfortunately, the actual situation is not so simple, but the idea of using an extended observation interval carries over to more realistic problems.

Initially, we shall find it useful to assume that the noise always contains an *independent* white component. Thus

$$n(t) = w(t) + n_c(t) \tag{144}$$

where $n_c(t)$ is the *colored* noise component. Then,

$$K_n(t, u) = \frac{N_0}{2}\, \delta(t - u) + K_c(t, u). \tag{145}$$

We assume the $n_c(t)$ has a finite mean-square value $[E(n_c^2(t)) < \infty$ for all $T_i \leq t \leq T_f]$ so $K_c(t, u)$ is a square-integrable function over $[T_i, T_f]$.

The white noise assumption is included for two reasons:

1. The physical reason is that regardless of the region of the spectrum used there will be a nonzero noise level. Extension of this level to infinity is just a convenience.
2. The mathematical reason will appear logically in our development. The white noise component enables us to guarantee that our operations will be meaningful. There are other ways to accomplish this objective but the white noise approach is the simplest.

Three logical approaches to the solution of the nonwhite noise problem are the following:

1. We choose the coordinates for the orthonormal expansion of $r(t)$ so that the coefficients are statistically independent. This will make the construction of the likelihood ratio straightforward. From our discussion in Chapter 3 we know how to carry out this procedure.
2. We operate on $r(t)$ to obtain a sufficient statistic and then use it to perform the detection.
3. We perform preliminary processing on $r(t)$ to transform the problem into a white Gaussian noise problem and then use the white Gaussian noise solution obtained in the preceding section. It is intuitively clear that if the preliminary processing is reversible it can have no effect on the performance of the system. Because we use the idea of reversibility repeatedly, however, it is worthwhile to provide a simple proof.

Reversibility. It is easy to demonstrate the desired result in a general setting. In Fig. 4.36*a* we show a system that operates on $r(u)$ to give an output that is optimum according to some desired criterion. (The problem of interest may be detection or estimation.) In system 2, shown in Fig. 4.36*b*, we first operate on $r(u)$ with a reversible operation $k[t, r(u)]$ to obtain $z(t)$. We then design a system that will perform an operation on $z(t)$ to obtain an output that is optimum according to the same criterion as in system 1. We now claim that the performances of the two systems are identical. Clearly, system 2 cannot perform better than system 1 or this would contradict our statement that system 1 is the optimum operation on $r(u)$. We now show that system 2 cannot be worse than system 1. Suppose that system 2 were worse than system 1. If this were true, we could design the system shown in Fig. 4.36*c*, which operates on $z(t)$ with the inverse of $k[t, r(u)]$ to give $r(u)$ and then passes it through system 1. This over-all system will work as well as system 1 (they are identical from the input-output standpoint). Because the result in Fig. 4.36*c* is obtained by operating on $z(t)$, it cannot be better than system 2 or it will contradict the statement that the second operation in system 2 is optimum. Thus system 2 cannot be worse than system 1.

Therefore *any* reversible operation can be included to facilitate the solution. We observe that linearity is not an issue, only the existence of an inverse. Reversibility is only *sufficient*, not *necessary*. (This is obvious from our discussion of sufficient statistics in Chapter 2.)

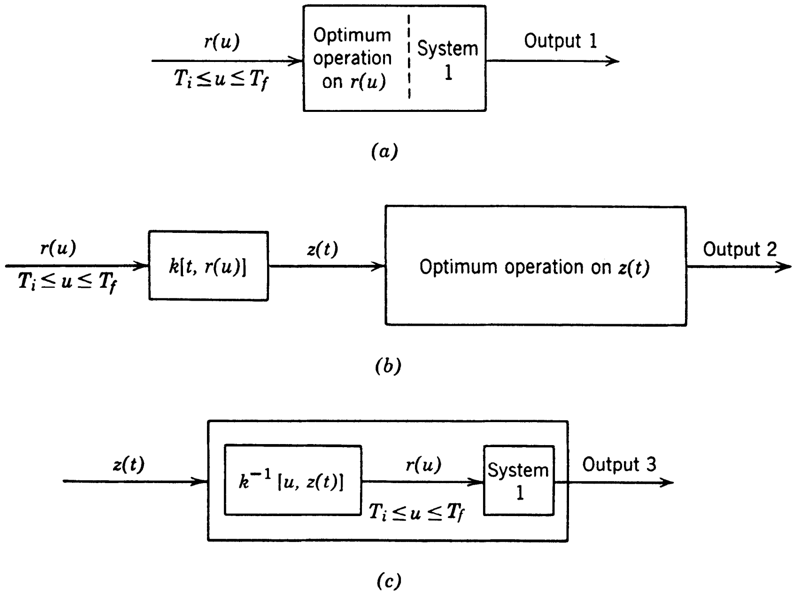

Fig. 4.36 Reversibility proof: (*a*) system 1; (*b*) system 2; (*c*) system 3.

We now return to the problem of interest. The first two of these approaches involve much less work and also extend in an easy fashion to more general cases. The third approach however, using reversibility, seems to have more intuitive appeal, so we shall do it first.

4.3.1 "Whitening" Approach

First we shall derive the structures of the optimum detector and estimator. In this section we require a nonzero white noise level.

Structures. As a preliminary operation, we shall pass $r(t)$ through a linear time-varying filter whose impulse response is $h_w(t, u)$ (Fig. 4.37). The impulse response is assumed to be zero for either t or u outside the interval $[T_i, T_f]$. For the moment, we shall not worry about realizability

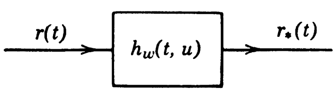

Fig. 4.37 "Whitening" filter.

and shall allow $h_w(t, u)$ to be nonzero for $u > t$. Later, in specific examples, we also look for realizable whitening filters. The output is

$$\begin{aligned} r_*(t) &\triangleq \int_{T_i}^{T_f} h_w(t, u)\, r(u)\, du \\ &= \int_{T_i}^{T_f} h_w(t, u)\sqrt{E}\, s(u)\, du + \int_{T_i}^{T_f} h_w(t, u)\, n(u)\, du \\ &\triangleq s_*(t) + n_*(t), \qquad T_i \le t \le T_f, \end{aligned} \tag{146}$$

when H_1 is true and

$$r_*(t) = n_*(t), \qquad T_i \le t \le T_f, \tag{147}$$

when H_0 is true. We want to choose $h_w(t, u)$ so that

$$K_{n_*}(t, u) = E[n_*(t)\, n_*(u)] = \delta(t - u), \qquad T_i \le t, u \le T_f. \tag{148}$$

Observe that we have arbitrarily specified a unity spectral height for the noise level at the output of the whitening filter. This is merely a convenient normalization.

The following logical question arises:

What conditions on $K_n(t, u)$ will guarantee that a reversible whitening filter exists? Because the whitening filter is linear, we can show reversibility by finding a filter $h_w^{-1}(t, u)$ such that

$$\int_{T_i}^{T_f} h_w^{-1}(t, z)\, h_w(z, u)\, dz = \delta(t - u), \qquad T_i \le t, u \le T_f. \tag{149}$$

For the moment we shall assume that we can find a suitable set of conditions and proceed with the development.

Because $n_*(t)$ is "white," we may use (22) and (23) directly ($N_0 = 2$):

$$\ln \Lambda[r_*(t)] = \int_{T_i}^{T_f} r_*(u)\, s_*(u)\, du - \tfrac{1}{2} \int_{T_i}^{T_f} s_*^2(u)\, du. \tag{150}$$

We can also write this directly in terms of the original waveforms and $h_w(t, u)$:

$$\begin{aligned} \ln \Lambda[r(t)] = &\int_{T_i}^{T_f} du \int_{T_i}^{T_f} h_w(u, z)\, r(z)\, dz \int_{T_i}^{T_f} h_w(u, v)\sqrt{E}\, s(v)\, dv \\ &- \tfrac{1}{2} \int_{T_i}^{T_f} du \int_{T_i}^{T_f} h_w(u, z)\sqrt{E}\, s(z)\, dz \int_{T_i}^{T_f} h_w(u, v)\sqrt{E}\, s(v)\, dv. \end{aligned} \tag{151}$$

This expression can be formally simplified by defining a new function:

$$\boxed{Q_n(z, v) = \int_{T_i}^{T_f} h_w(u, z)\, h_w(u, v)\, du, \qquad T_i < z, v < T_f.} \tag{152†}$$

For the moment we can regard it as a function that we accidentally stumbled on in an effort to simplify an equation. Later we shall see that it plays a fundamental role in many of our discussions. Rewriting (151), we have

$$\begin{aligned}\ln \Lambda[r(t)] = &\int_{T_i}^{T_f} r(z)\, dz \int_{T_i}^{T_f} Q_n(z, v)\sqrt{E}\, s(v)\, dv \\ &- \frac{E}{2}\int_{T_i}^{T_f} s(z)\, dz \int_{T_i}^{T_f} Q_n(z, v)\, s(v)\, dv.\end{aligned} \tag{153}$$

We can simplify (153) by writing

$$\boxed{g(z) = \int_{T_i}^{T_f} Q_n(z, v)\sqrt{E}\, s(v)\, dv, \qquad T_i < z < T_f.} \tag{154}$$

We have used a strict inequality in (154). Looking at (153), we see that $g(z)$ only appears inside an integral. Therefore, if $g(z)$ *does not* contain singularities at the endpoints, we can assign $g(z)$ any finite value at the endpoint and $\ln \Lambda[r(t)]$ will be unchanged. Whenever there is a white noise component, we can show that $g(z)$ is square-integrable (and therefore contains no singularities). For convenience we make $g(z)$ continuous at the endpoints.

$$g(T_f) = \lim_{z \to T_f^-} g(z),$$

$$g(T_i) = \lim_{z \to T_i^+} g(z).$$

We see that the construction of the likelihood function involves a correlation operation between the actual received waveform and a function $g(z)$. Thus, from the standpoint of constructing the receiver, the function $g(z)$ is the only one needed. Observe that the correlation of $r(t)$ with $g(t)$ is simply the reduction of the observation space to a single sufficient statistic.

Three canonical receiver structures for simple binary detection are

† Throughout this section we must be careful about the endpoints of the interval. The difficulty is with factors of 2 which arise because of the delta function in the noise covariance. We avoid this by using an open interval and then show that endpoints are not important in this problem. We suggest that the reader ignore the comments regarding endpoints until he has read through Section 4.3.3. This strategy will make these sections more readable.

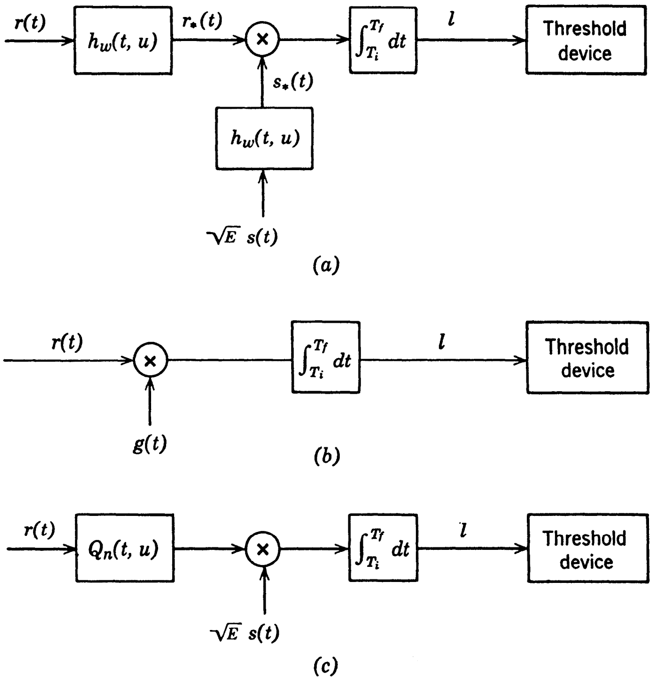

Fig. 4.38 Alternate structures for colored noise problem.

shown in Fig. 4.38. We shall see that the first two are practical implementations, whereas the third affords an interesting interpretation. The modification of Fig. 4.38*b* to obtain a matched filter realization is obvious. To implement the receivers we must solve (149), (152), or (154). Rather than finding closed-form solutions to these equations we shall content ourselves in this section with series solutions in terms of the eigenfunctions and eigenvalues of $K_c(t, u)$. These series solutions have two purposes:

1. They demonstrate that solutions exist.
2. They are useful in certain optimization problems.

After deriving these solutions, we shall look at the receiver performance and extensions to general binary detection, M-ary detection, and estimation problems. We shall then return to the issue of closed-form solutions. The advantage of this approach is that it enables us to obtain an integrated picture of the colored noise problem and many of its important features without getting lost in the tedious details of solving integral equations.

Construction of $Q_n(t, u)$ and $g(t)$. The first step is to express $Q_n(t, u)$ directly in terms of $K_n(t, u)$. We recall our definition of $h_w(t, u)$. It is a time-varying linear filter chosen so that when the input is $n(t)$ the output will be $n_*(t)$, a sample function from a white Gaussian process. Thus

$$n_*(t) = \int_{T_i}^{T_f} h_w(t, x)\, n(x)\, dx \qquad T_i \le t \le T_f \tag{155}$$

and

$$E[n_*(t) n_*(u)] = K_{n_*}(t, u) = \delta(t - u). \qquad T_i \le t \le T_f. \tag{156}$$

Substituting (155) into (156), we have

$$\delta(t - u) = E \iint_{T_i}^{T_f} h_w(t, x)\, h_w(u, z)\, n(x)\, n(z)\, dx\, dz. \tag{157}$$

By bringing the expectation inside the integrals, we have

$$\delta(t - u) = \iint_{T_i}^{T_f} h_w(t, x)\, h_w(u, z)\, K_n(x, z)\, dx\, dz, \qquad T_i < t, u < T_f. \tag{158}$$

In order to get (158) into a form such that we can introduce $Q_n(t, u)$, we multiply both sides by $h_w(t, v)$ and integrate with respect to t. This gives

$$h_w(u, v) = \int_{T_i}^{T_f} dz\, h_w(u, z) \int_{T_i}^{T_f} K_n(x, z)\, dx \int_{T_i}^{T_f} h_w(t, v)\, h_w(t, x)\, dt. \tag{159}$$

Looking at (152), we see that the last integral is just $Q_n(v, x)$. Therefore

$$h_w(u, v) = \int_{T_i}^{T_f} dz\, h_w(u, z) \int_{T_i}^{T_f} K_n(x, z)\, Q_n(v, x)\, dx. \tag{160}$$

This implies that the inner integral must be an impulse over the open interval,

$$\boxed{\delta(z - v) = \int_{T_i}^{T_f} K_n(x, z)\, Q_n(v, x)\, dx, \qquad T_i < z, v < T_f.} \tag{161}$$

This is the desired result that relates $Q_n(v, x)$ directly to the original covariance function. Because $K_n(x, z)$ is the kernel of many of the integral equations of interest to us, $Q_n(v, x)$ is frequently called the *inverse kernel.*

From (145) we know that $K_n(x, z)$ consists of an impulse and a well-behaved term. A logical approach is to try and express $Q_n(v, x)$ in a similar manner. We try a solution to (161) of the form

$$Q_n(v, x) = \frac{2}{N_0} [\delta(v - x) - h_o(v, x)] \qquad T_i < v, x < T_f. \tag{162}$$

Substituting (145) and (162) into (161) and rearranging terms, we obtain an equation that $h_0(v, x)$ must satisfy:

$$\frac{N_0}{2} h_o(v, z) + \int_{T_i}^{T_f} h_o(v, x) K_c(x, z)\, dx = K_c(v, z), \qquad T_i < z, v < T_f \quad (163)$$

This equation is familiar to us from the section on optimum linear filters in Chapter 3 [Section 3.4.5; particularly, (3-144)]. The significance of this similarity is seen by re-drawing the system in Fig. 4.38*c* as shown in Fig. 4.39. The function $Q_n(t, u)$ is divided into two parts. We see that the output of the filter in the bottom path is precisely the minimum mean-square error estimate of the colored noise component, assuming that H_0 is true. If we knew $n_c(t)$, it is clear that the optimum processing would consist of subtracting it from $r(t)$ and passing the result into a matched filter or correlation receiver. The optimum receiver does exactly that, except that it does not know $n_c(t)$; therefore it makes a MMSE estimate $\hat{n}_c(t)$ and uses it. This is an intuitively pleasing result of a type that we shall encounter frequently.†

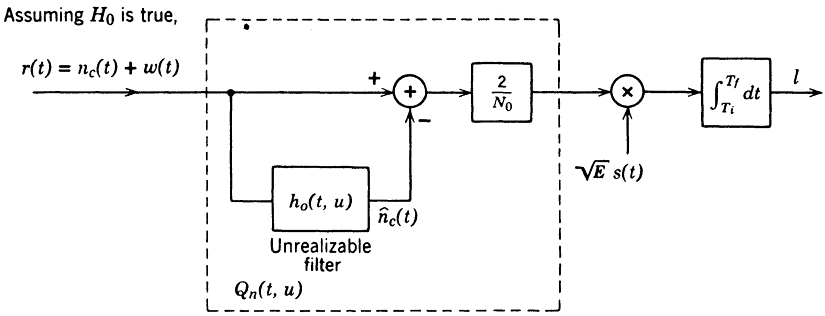

Fig. 4.39 Realization of detector using an optimum linear filter.

† The reader may wonder why we care whether a result is intuitively pleasing, if we know it is optimum. There are two reasons for this interest: (a) It is a crude error-checking device. For the type of problems of interest to us, when we obtain a mathematical result that is unintuitive it is usually necessary to go back over the model formulation and the subsequent derivation and satisfy ourselves that either the model omits some necessary feature of the problem or that our intuition is wrong. (b) In many cases the solution for the optimum receiver may be mathematically intractable. Having an intuitive interpretation for the solutions to the various Gaussian problems equips us to obtain a good receiver by using intuitive reasoning when we cannot get a mathematical solution.

From our results in Chapter 3 (3.154) we can write a formal solution for $h_o(t, u)$ in terms of the eigenvalues of $K_c(t, u)$. Using (3.154),

$$h_o(t, u) = \sum_{i=1}^{\infty} \frac{\lambda_i^c}{\lambda_i^c + N_0/2} \phi_i(t)\, \phi_i(u), \qquad T_i < t, u < T_f, \tag{164}$$

where λ_i^c and $\phi_i(t)$ are the eigenvalues and eigenfunctions, respectively, of $K_c(t, u)$. We can write the entire inverse kernel as

$$Q_n(t, u) = \frac{2}{N_0}\left[\delta(t - u) - \sum_{i=1}^{\infty} \frac{\lambda_i^c}{\lambda_i^c + N_0/2} \phi_i(t)\, \phi_i(u)\right]. \tag{165}$$

It is important to re-emphasize that our ability to write $Q_n(t, u)$ as an impulse function and a well-behaved function rests heavily on our assumption that there is a nonzero white noise level. This is the mathematical reason for the assumption.

We can also write $Q_n(t, u)$ as a single series. We express the impulse in terms of a series by using (3.128) and then combine the series to obtain

$$Q_n(t, u) = \sum_{i=1}^{\infty} \left(\frac{N_0}{2} + \lambda_i^c\right)^{-1} \phi_i(t)\, \phi_i(u) = \sum_{i=1}^{\infty} \frac{1}{\lambda_i^T} \phi_i(t)\, \phi_i(u), \tag{166}$$

where

$$\lambda_i^T \triangleq \frac{N_0}{2} + \lambda_i^c. \tag{167}$$

(T denotes total). The series in (166) does not converge. However, in most cases $Q_n(t, u)$ is inside an integral and the overall expression will converge.

As a final result, we want to find an equation that will specify $g(t)$ directly in terms of $K_n(t, z)$. We start with (154):

$$g(z) = \int_{T_i}^{T_f} Q_n(z, v)\sqrt{E}\, s(v)\, dv, \qquad T_i < z < T_f. \tag{168}$$

The technique that we use is based on the inverse relation between $K_n(t, z)$ and $Q_n(t, z)$, expressed by (161). To get rid of $Q_n(z, v)$ we simply multiply (168) by $K_n(t, z)$, integrate with respect to z, and use (161). The result is

$$\boxed{\int_{T_i}^{T_f} K_n(t, z)\, g(z)\, dz = \sqrt{E}\, s(t), \qquad T_i < t < T_f.} \tag{169a}$$

Substituting (145) into (169a), we obtain an equation for the open interval (T_i, T_f). Our continuity assumption after (154) extends the range to the closed interval $[T_i, T_f]$. The result is

$$\boxed{\frac{N_0}{2} g(t) + \int_{T_i}^{T_f} K_c(t, z)\, g(z)\, dz = \sqrt{E}\, s(t), \qquad T_i \le t \le T_f.} \tag{169b}$$

To implement the receiver, as shown in Fig. 4.38b, we would solve (169b)

directly. We shall develop techniques for obtaining closed-form solutions in 4.3.6. A series solution can be written easily by using (168) and (165):

$$g(z) = \frac{2}{N_0}\sqrt{E}\,s(z) - \frac{2}{N_0}\sum_{i=1}^{\infty}\frac{\lambda_i^c s_i}{\lambda_i^c + N_0/2}\,\phi_i(z), \tag{170}$$

where

$$s_i = \int_{T_i}^{T_f} \sqrt{E}\,s(t)\,\phi_i(t)\,dt. \tag{171}$$

The first term is familiar from the white noise case. The second term indicates the effect of nonwhite noise. Observe that $g(t)$ is *always* a square-integrable function over (T_i, T_f) when a white noise component is present. We defer checking the endpoint behavior until 4.3.3.

Summary

In this section we have derived the solution for the optimum receiver for the simple binary detection problem of a known signal in nonwhite Gaussian noise. Three realizations were the following:

1. Whitening realization (Fig. 4.38*a*).
2. Correlator realization (Fig. 4.38*b*).
3. Estimator-subtractor realization (Fig. 4.39).

Coupled with each of these realizations was an integral equation that must be solved to build the receiver: 1. (158). 2. (169). 3. (163).

We demonstrated that series solutions could be obtained in terms of eigenvalues and eigenfunctions, but we postponed the problem of actually finding a closed-form solution. The concept of an "inverse kernel" was introduced and a simple application shown. The following questions remain:

1. How well does the system perform?
2. How do we find closed-form solutions to the integral equations of interest?
3. What are the analogous results for the estimation problem?

Before answering these questions we digress briefly and rederive the results without using the idea of whitening. In view of these alternate derivations, we leave the proof that $h_w(t, u)$ is a reversible operator as an exercise for the reader (Problem 4.3.1).

4.3.2 A Direct Derivation Using the Karhunen-Loève Expansion†

In this section we consider a more fundamental approach. It is not only

† This approach to the problem is due to Grenander [30]. (See also: Kelly, Reed, and Root [31].)

more direct for this particular problem but extends easily to the general case. The derivation is analogous to the one on pp. 250–253.

The reason that the solution to the white noise detection problem in Section 4.2 was so straightforward was that regardless of the orthonormal set we chose, the resulting observables $r_1, r_2, \ldots, r_K$ were conditionally independent.

From our work in Chapter 3 we know that we can achieve the same simplicity if we choose an orthogonal set in a particular manner. Specifically, we want the orthogonal functions to be the eigenfunctions of the integral equation (3-46)

$$\lambda_i^c \, \phi_i(t) = \int_{T_i}^{T_f} K_c(t, u) \, \phi_i(u) \, du, \qquad T_i \leq t \leq T_f. \tag{172}$$

Observe that the λ_i^c are the eigenvalues of the colored noise process only. (If $K_c(t, u)$ is not positive-definite, we augment the set to make it complete.) Then we expand $r(t)$ in this coordinate system:

$$r(t) = \underset{K \to \infty}{\text{l.i.m.}} \sum_{i=1}^{K} r_i \, \phi_i(t) = \underset{K \to \infty}{\text{l.i.m.}} \sum_{i=1}^{K} s_i \, \phi_i(t) + \underset{K \to \infty}{\text{l.i.m.}} \sum_{i=1}^{K} n_i \, \phi_i(t), \qquad T_i \leq t \leq T_f, \tag{173}$$

where

$$r_i = \int_{T_i}^{T_f} r(t) \, \phi_i(t) \, dt, \tag{174}$$

$$s_i = \int_{T_i}^{T_f} \sqrt{E} \, s(t) \, \phi_i(t) \, dt, \tag{175}$$

and

$$n_i = \int_{T_i}^{T_f} n(t) \, \phi_i(t) \, dt. \tag{176}$$

From (3.42) we know

$$E(n_i) = 0, \qquad E(n_i n_j) = \lambda_i^T \delta_{ij}, \tag{177}$$

where

$$\lambda_i^T \triangleq \frac{N_0}{2} + \lambda_i^c. \tag{178}$$

Just as on p. 252 (20) we consider the first K coordinates. The likelihood ratio is

$$\Lambda\,[r_K(t)] = \frac{\displaystyle\prod_{i=1}^{K} \frac{1}{\sqrt{2\pi\lambda_i^T}} \exp\left[-\frac{1}{2}\frac{(R_i - s_i)^2}{\lambda_i^T}\right]}{\displaystyle\prod_{i=1}^{K} \frac{1}{\sqrt{2\pi\lambda_i^T}} \exp\left[-\frac{1}{2}\frac{R_i^2}{\lambda_i^T}\right]}. \tag{179}$$

Canceling common terms, letting $K \to \infty$, and taking the logarithm, we obtain

$$\ln \Lambda[r(t)] = \sum_{i=1}^{\infty} \frac{R_i s_i}{\lambda_i^T} - \frac{1}{2} \sum_{i=1}^{\infty} \frac{s_i^2}{\lambda_i^T}. \tag{180}$$

Using (174) and (175), we have

$$\ln \Lambda[r(t)] = \int_{T_i}^{T_f} dt \int_{T_i}^{T_f} du\, r(t) \sum_{i=1}^{\infty} \frac{\phi_i(t)\, \phi_i(u)}{\lambda_i^T} \sqrt{E}\, s(u)$$
$$- \frac{E}{2} \int_{T_i}^{T_f} dt \int_{T_i}^{T_f} du\, s(t) \sum_{i=1}^{\infty} \frac{\phi_i(t)\, \phi_i(u)}{\lambda_i^T} s(u). \tag{181}$$

From (166) we recognize the sum as $Q_n(t, u)$. Thus

$$\ln \Lambda[r(t)] = \int_{T_i}^{T_f} dt \int_{T_i}^{T_f} du\, r(t)\, Q_n(t, u) \sqrt{E}\, s(u)$$
$$- \frac{E}{2} \int_{T_i}^{T_f} dt \int_{T_i}^{T_f} du\, s(t)\, Q_n(t, u)\, s(u). \qquad (182)\dagger$$

This expression is identical to (153).

Observe that if we had not gone through the whitening approach we would have simply defined $Q_n(t, u)$ to fit our needs when we arrived at this point in the derivation. When we consider more general detection problems later in the text (specifically Chapter II.3), the direct derivation can easily be extended.

4.3.3 A Direct Derivation with a Sufficient Statistic‡

For convenience we rewrite the detection problem of interest (140):

$$\begin{aligned} r(t) &= \sqrt{E}\, s(t) + n(t), & T_i \le t \le T_f : H_1 \\ &= n(t), & T_i \le t \le T_f : H_0. \end{aligned} \tag{183}$$

In this section we will not require that the noise contain a white component.

From our work in Chapter 2 and Section 4.2 we know that if we can write

$$r(t) = r_1\, s(t) + y(t), \qquad T_i \le t \le T_f. \tag{184}$$

† To proceed rigorously from (181) to (182) we require $\sum_{i=1}^{\infty} (s_i^2/\lambda_i^{T^2}) < \infty$ (Grenander [30]; Kelly, Reed, and Root [31]). This is always true when white noise is present. Later, when we look at the effect of removing the white noise assumption, we shall see that the divergence of this series leads to an unstable test.

‡ This particular approach to the colored noise problem seems to have been developed independently by several people (Kailath [32]; Yudkin [39]). Although the two derivations are essentially the same, we follow the second.

where r_1 is a random variable obtained by operating on $r(t)$ and demonstrate that:

(a) r_1 and $y(t)$ are statistically independent on both hypotheses,
(b) the statistics of $y(t)$ do not depend on which hypothesis is true,

then r_1 is a sufficient statistic. We can then base our decision solely on r_1 and disregard $y(t)$. [Note that conditions (a) and (b) are sufficient, but not necessary, for r_1 to be a sufficient statistic (see pp. 35–36).]

To do this we hypothesize that r_1 can be obtained by the operation

$$r_1 = \int_{T_i}^{T_f} r(u)\, g(u)\, du \tag{185}$$

and try to find a $g(u)$ that will lead to the desired properties. Using (185), we can rewrite (184) as

$$\begin{aligned} r(t) &= (s_1 + n_1)\, s(t) + y(t) & :H_1 \\ &= n_1\, s(t) + y(t) & :H_0. \end{aligned} \tag{186}$$

where

$$s_1 \triangleq \int_{T_i}^{T_f} \sqrt{E}\, s(u)\, g(u)\, du \tag{187}$$

and

$$n_1 \triangleq \int_{T_i}^{T_f} n(u)\, g(u)\, du. \tag{188}$$

Because a sufficient statistic can be multiplied by any nonzero constant and remain a sufficient statistic we can introduce a constraint,

$$\int_{T_i}^{T_f} s(u) g(u)\, du = 1. \tag{189a}$$

Using (189*a*) in (187), we have

$$s_1 = \sqrt{E}. \tag{189b}$$

Clearly, n_1 is a zero-mean random variable and

$$n(t) = n_1\, s(t) + y(t), \qquad T_i \le t \le T_f. \tag{190}$$

This puts the problem in a convenient form and it remains only to find a condition on $g(u)$ such that

$$E[n_1\, y(t)] = 0, \qquad T_i \le t \le T_f, \tag{191}$$

or, equivalently,

$$E\{n_1[n(t) - n_1\, s(t)]\} = 0, \qquad T_i \le t \le T_f, \tag{192}$$

or

$$E[n_1 \cdot n(t)] = E[n_1^2]\, s(t), \qquad T_i \le t \le T_f. \tag{193}$$

Using (188)

$$\int_{T_i}^{T_f} K_n(t, u)\, g(u)\, du = s(t) \iint_{T_i}^{T_f} g(\sigma)\, K_n(\sigma, \beta)\, g(\beta)\, d\sigma\, d\beta, \qquad T_i \le t \le T_f. \tag{194}$$

Equations 189*a* and 194 will both be satisfied if

$$\int_{T_i}^{T_f} K_n(t, u)\, g(u)\, du = \sqrt{E}\, s(t), \qquad T_i \le t \le T_f. \tag{195}$$

[Substitute (195) into the right side of (194) and use (189*a*).] Our sufficient statistic r_1 is obtained by correlating $r(u)$ with $g(u)$. After obtaining r_1 we use it to construct a likelihood ratio test in order to decide which hypothesis is true.

We observe that (195) is over the closed interval $[T_i, T_f]$, whereas (169*a*) was over the open interval (T_i, T_f). The reason for this difference is that in the absence of white noise $g(u)$ may contain singularities at the endpoints. These singularities change the likelihood ratio so we can no longer arbitrarily choose the endpoint values. An advantage of our last derivation is that the correct endpoint conditions are included. We should also observe that if there is a white noise component (195) and (169*a*) will give different values for $g(T_i)$ and $g(T_f)$. However, because both sets of values are finite they lead to the same likelihood ratio.

In the last two sections we have developed two alternate derivations of the optimum receiver. Other derivations are available (a mathematically inclined reader might read Parzen [40], Hajek [41], Galtieri [43], or Kadota [45]). We now return to the questions posed on p. 297.

4.3.4 Detection Performance

The next question is: "How does the presence of colored noise affect performance?" In the course of answering it a number of interesting issues appear. We consider the simple binary detection case first.

Performance: Simple Binary Detection Problem. Looking at the receiver structure in Fig. 4.38*a*, we see that the performance is identical to that of a receiver in which the input signal is $s_*(t)$ and the noise is white with a spectral height of 2. Using (10) and (11), we have

$$d^2 = \int_{T_i}^{T_f} [s_*(t)]^2\, dt. \tag{196}$$

Thus the performance index d^2 is simply equal to the energy in the whitened signal. We can also express d^2 in terms of the original signal.

$$d^2 = \int_{T_i}^{T_f} dt \left[\int_{T_i}^{T_f} h_w(t, u)\sqrt{E}\, s(u)\, du\right]\left[\int_{T_i}^{T_f} h_w(t, z)\sqrt{E}\, s(z)\, dz\right]. \tag{197}$$

We use the definition of $Q_n(u, z)$ to perform the integration with respect to t. This gives

$$\boxed{\begin{aligned} d^2 &= E \iint_{T_i}^{T_f} du\, dz\, s(u)\, Q_n(u, z)\, s(z) \\ d^2 &= \sqrt{E} \int_{T_i}^{T_f} du\, s(u)\, g(u). \end{aligned}} \tag{198}$$

It is clear that the performance is no longer independent of the signal shape. The next logical step is to find the best possible signal shape. There are three cases of interest:

1. $T_i = 0, T_f = T$: the signal interval and observation interval coincide.
2. $T_i < 0, T_f > T$: the observation interval extends beyond the signal interval in one or both directions but is still finite.
3. $T_i = -\infty, T_f = \infty$: the observation interval is doubly infinite.

We consider only the first case.

Optimum Signal Design: Coincident Intervals. The problem is to constrain the signal energy E and determine how the detailed shape of $s(t)$ affects performance. The answer follows directly. Write

$$Q_n(t, u) = \sum_{i=1}^{\infty} \left(\frac{N_0}{2} + \lambda_i^c\right)^{-1} \phi_i(t)\, \phi_i(u). \tag{199}$$

Then

$$d^2 = \sum_{i=1}^{\infty} \frac{s_i^2}{N_0/2 + \lambda_i^c}, \tag{200}$$

where

$$s_i = \int_0^T \sqrt{E}\, s(t)\, \phi_i(t)\, dt. \tag{201}$$

Observe that

$$\sum_{i=1}^{\infty} s_i^2 = E, \tag{202}$$

because the functions are normalized.

Looking at (200), we see that d^2 is just a weighted sum of the s_i^2. Because (202) constrains the sum of the s_i^2, we want to distribute the energy so that those s_i with large weighting are large. If there exists a smallest eigenvalue, say $\lambda_j^c = \lambda_{\min}^c$, then d^2 will be maximized by letting $s_j = \sqrt{E}$ and all other $s_i = 0$. There are two cases of interest:

1. If $K_c(t, u)$ is positive-definite, the number of eigenvalues is infinite. There is no smallest eigenvalue. We let $s_j = \sqrt{E}$ and all other $s_i = 0$. Then, assuming the eigenvalues are ordered according to decreasing size,

$$d^2 \rightarrow \frac{2E}{N_0}$$

as we increase j. For many of the colored noises that we encounter in practice (e.g., the one-pole spectrum shown in Fig. 3.9), the frequency of the eigenfunction increases as the eigenvalues decrease. In other words, we increase the frequency of the signal until the colored noise becomes negligible. In these cases we obtain a more realistic signal design problem by including a bandwidth constraint.

2. If $K_c(t, u)$ is only nonnegative definite, there will be zero eigenvalues. If $s(t)$ is the eigenfunction corresponding to any one of these eigenvalues, then

$$d^2 = \frac{2E}{N_0}.$$

We see that the performance of the best signal is limited by the white noise.

Singularity. It is easy to see the effect of removing the white noise by setting N_0 equal to zero in (200). When the colored noise is positive-definite (Case 1), all eigenvalues are nonzero. We can achieve *perfect detection* ($d^2 = \infty$) if and only if the sum

$$d^2 = \sum_{i=1}^{\infty} \frac{s_i^2}{\lambda_i^c} \tag{203}$$

diverges.

It can be accomplished by choosing $s(t)$ so that s_i^2 is proportional to λ_i^c. We recall that

$$\sum_{i=1}^{\infty} \lambda_i^c = \int_{T_i}^{T_f} K_c(t, t)\, dt < M.$$

The right side is finite by our assumption below (145). Thus the energy in the signal ($E = \sum_{i=1}^{\infty} s_i^2$) will be finite. If there were a white noise component, we could not achieve this proportionality for all i with a finite energy signal. In (Case 2) there are zero eigenvalues. Thus we achieve $d^2 = \infty$ by choosing $s(t) = \phi_i(t)$ for any i that has a zero eigenvalue.

These two cases are referred to as *singular* detection. For arbitrarily small time intervals and arbitrarily small energy levels we achieve perfect detection. We know that this kind of performance cannot be obtained in an actual physical situation. Because the purpose of our mathematical model is to predict performance of an actual system, it is important that we make it realistic enough to eliminate singular detection. We have eliminated the possibility of singular detection by insisting on a nonzero white noise component. This accounts for the thermal noise in the receiver. Often it will appear to be insignificant. If, however, we design the signal to eliminate the effect of all other noises, it becomes the quantity that limits the performance and keeps our mathematical model from predicting results that would not occur in practice.

From (196) we know that d^2 is the energy in the whitened signal. Therefore, if the whitened signal has finite energy, the test is not singular. When the observation interval is infinite and the noise process is stationary with a rational spectrum, it is easy to check the finiteness of the energy of $s_*(t)$. We first find the transfer function of the whitening filter. Recall that

$$n_*(t) = \int_{-\infty}^{\infty} h_w(u)\, n(t - u)\, du. \tag{204}$$

We require that $n_*(t)$ be white with unity spectral height. This implies that

$$\iint_{-\infty}^{\infty} du\, dz\; h_w(u)\, h_w(z)\, K_n(t - u + z - v) = \delta(t - v), \qquad -\infty < t, v < \infty. \tag{205}$$

Transforming, we obtain

$$|H_w(j\omega)|^2 S_n(\omega) = 1 \tag{206a}$$

or

$$|H_w(j\omega)|^2 = \frac{1}{S_n(\omega)}. \tag{206b}$$

Now assume that $S_n(\omega)$ has a rational spectrum

$$S_n(\omega) = \frac{c_q\omega^{2q} + c_{q-1}\omega^{2q-2} + \cdots + c_0}{d_p\omega^{2p} + d_{p-1}\omega^{2p-2} + \cdots + d_0}. \tag{207a}$$

We define the difference between the order of denominator and numerator (as a function of ω^2) as r.

$$r \triangleq p - q \tag{207b}$$

If $n(t)$ has finite power then $r \geq 1$. However, if the noise consists of white noise plus colored noise with finite power, then $r = 0$. Using (207a)

in (206*b*), we see that we can write $H_w(j\omega)$ as a ratio of two polynomials in $j\omega$.

$$H_w(j\omega) = \frac{a_p(j\omega)^p + a_{p-1}(j\omega)^{p-1} + \cdots + a_0}{b_q(j\omega)^q + b_{q-1}(j\omega)^{q-1} + \cdots + b_0}. \tag{208a}$$

In Chapter 6 we develop an algorithm for finding the coefficients. For the moment their actual values are unimportant. Dividing the numerator by the denominator, we obtain

$$H_w(j\omega) = f_r(j\omega)^r + f_{r-1}(j\omega)^{r-1} + \cdots + f_0 + \frac{R(j\omega)}{b_q(j\omega)^q + \cdots + b_0}, \tag{208b}$$

where $f_r, \ldots, f_0$ are constants and $R(j\omega)$ is the remainder polynomial of order less than q. Recall that $(j\omega)^r$ in the frequency domain corresponds to taking the rth derivative in the time domain. Therefore, in order for the test to be nonsingular, the rth derivative must have finite energy. In other words, if

$$\int_{-\infty}^{\infty} \left(\frac{d^r s(t)}{dt^r}\right)^2 dt < M \tag{209}$$

the test is nonsingular; for example, if

$$S_n(\omega) = \frac{2\alpha\sigma_n^{\ 2}}{\omega^2 + \alpha^2} \tag{210a}$$

then

$$p - q = r = 1 \tag{210b}$$

and $s'(t)$ must have finite energy. If we had modeled the signal as an ideal rectangular pulse, then our model would indicate perfect detectability. We know that this perfect detectability will not occur in practice, so we must modify our model to accurately predict system performance. In this case we can eliminate the singular result by giving the pulse a finite rise time or by adding a white component to the noise. Clearly, whenever there is finite-power colored noise plus an independent white noise component, the integral in (209) is just the energy in the signal and singularity is never an issue.

Our discussion has assumed an infinite observation interval. Clearly, if the test is nonsingular on the infinite interval, it is nonsingular on the finite interval because the performance is related monotonically to the length of the observation interval. The converse is not true. Singularity on the infinite interval does not imply singularity on the finite interval. In this case we must check (203) or look at the finite-time whitening operation.

Throughout most of our work we retain the white noise assumption so singular tests never arise. Whenever the assumption is removed, it is necessary to check the model to ensure that it does not correspond to a singular test.

General Binary Receivers. Our discussion up to this point has considered only the simple binary detection problem. The extension to general binary receivers is straightforward. Let

$$\begin{aligned} r(t) &= \sqrt{E_1}\, s_1(t) + n(t), \qquad T_i \le t \le T_f : H_1,\\ r(t) &= \sqrt{E_0}\, s_0(t) + n(t), \qquad T_i \le t \le T_f : H_0, \end{aligned} \tag{211}$$

where $s_0(t)$ and $s_1(t)$ are normalized over the interval $(0, T)$ and are zero elsewhere. Proceeding in exactly the same manner as in the simple binary case, we obtain the following results. One receiver configuration is shown in Fig. 4.40*a*. The function $g_\Delta(t)$ satisfies

$$\begin{aligned} s_\Delta(t) &\triangleq \sqrt{E_1}\, s_1(t) - \sqrt{E_0}\, s_0(t)\\ &= \int_{T_i}^{T_f} g_\Delta(u)\, K_n(t, u)\, du, \qquad T_i \le t \le T_f. \end{aligned} \tag{212}$$

The performance is characterized by d^2:

$$d^2 = \iint_{T_i}^{T_f} s_\Delta(t)\, Q_n(t, u)\, s_\Delta(u)\, dt\, du. \tag{213}$$

The functions $K_n(t, u)$ and $Q_n(t, u)$ were defined in (145) and (161), respectively. As an alternative, we can use the whitening realization shown in Fig. 4.40*b*. Here $h_w(t, u)$ satisfies (158) and

$$s_{\Delta_*}(t) \triangleq \int_{T_i}^{T_f} h_w(t, u)\, s_\Delta(u)\, du, \qquad T_i \le t \le T_f. \tag{214}$$

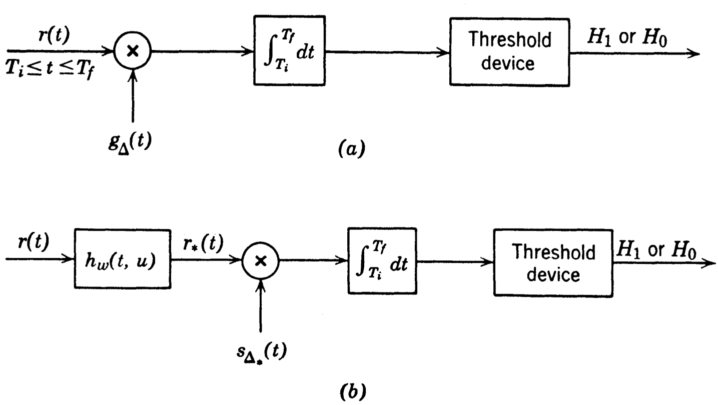

Fig. 4.40 (*a*) **Receiver configurations: general binary problem, colored noise;** (*b*) **alternate receiver realization.**

The performance is characterized by the energy in the whitened difference signal:

$$d^2 = \int_{T_i}^{T_f} s_{\Delta *}^2(t)\, dt. \tag{215}$$

The M-ary detection case is also a straightforward extension (see Problem 4.3.5). From our discussion of white noise we would expect that the estimation case would also follow easily. We discuss it briefly in the next section.

4.3.5 Estimation

The model for the received waveform in the parameter estimation problem is

$$r(t) = s(t, A) + n(t), \qquad T_i \le t \le T_f. \tag{216}$$

The basic operation on the received waveform consists of constructing the likelihood function, for which it is straightforward to derive an expression. If, however, we look at (98–101), and (146–153), it is clear that the answer will be:

$$\begin{aligned} \ln \Lambda_1[r(t), A] &= \int_{T_i}^{T_f} r(z)\, dz \int_{T_i}^{T_f} Q_n(z, v)\, s(v, A)\, dv \\ &\quad - \tfrac{1}{2} \int_{T_i}^{T_f} dz\, s(z, A) \int_{T_i}^{T_f} Q_n(z, v)\, s(v, A)\, dv. \end{aligned} \tag{217}$$

This result is analogous to (153) in the detection problem. If we define

$$g(z, A) = \int_{T_i}^{T_f} Q_n(z, v)\, s(v, A)\, dv, \qquad T_i < z < T_f, \tag{218}$$

or, equivalently,

$$s(v, A) = \int_{T_i}^{T_f} K_n(v, z)\, g(z, A)\, dz, \qquad T_i < v < T_f, \tag{219}$$

(217) reduces to

$$\begin{aligned} \ln \Lambda_1[r(t), A] &= \int_{T_i}^{T_f} r(z)\, g(z, A)\, dz \\ &\quad - \tfrac{1}{2} \int_{T_i}^{T_f} s(z, A)\, g(z, A)\, dz. \end{aligned} \tag{220}$$

The discussions in Sections 4.2.2 and 4.2.3 carry over to the colored noise case in an obvious manner. We summarize some of the important results for the linear and nonlinear estimation problems.

Linear Estimation. The received waveform is

$$r(t) = A\sqrt{E}\, s(t) + n(t), \qquad T_i \le t \le T_f, \tag{221}$$

where $s(t)$ is normalized $[0, T]$ and zero elsewhere. Substituting into (218), we see that

$$g(t, A) = A\, g(t), \tag{222}$$

where $g(t)$ is the function obtained in the simple binary detection case by solving (169).

Thus the linear estimation problem is essentially equivalent to simple binary detection. The estimator structure is shown in Fig. 4.41, and the estimator is completely specified by finding $g(t)$. If A is a nonrandom variable, the normalized error variance is

$$\sigma^2_{\hat{a}_\epsilon n} = (A^2 d^2)^{-1}, \tag{223}$$

where d^2 is given by (198). If A is a value of a random variable a with a Gaussian a priori density, $N(0, \sigma_a)$, the minimum mean-square error is

$$\sigma^2_{\hat{a}_\epsilon n} = (1 + \sigma_a{}^2 d^2)^{-1}. \tag{224}$$

(These results correspond to (96) and (97) in the white noise case) All discussion regarding singular tests and optimum signals carries over directly.

Nonlinear Estimation. In nonlinear estimation, in the presence of colored noise, we encounter all the difficulties that occur in the white noise case. In addition, we must find either $Q_n(t, u)$ or $g(t, A)$. Because all of the results are obvious modifications of those in 4.2.3, we simply summarize the results:

1. A necessary, but not sufficient, condition on $\hat{a}_{ml}$:

$$0 = \int_{T_i}^{T_f} dt \int_{T_i}^{T_f} du\, [r(t) - s(t, A)]\, Q_n(t, u) \frac{\partial s(u, A)}{\partial A}\bigg|_{A = \hat{a}_{ml}}. \tag{225}$$

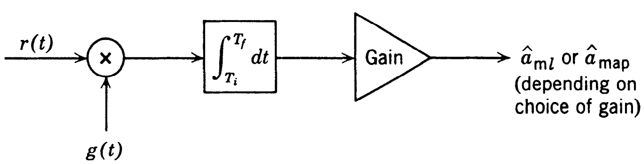

Fig. 4.41 Linear estimation, colored noise.

2. A necessary, but not sufficient, condition on $\hat{a}_{map}$ (assuming that a has a Gaussian a priori density):

$$\hat{a}_{map} = \sigma_a^2 \int_{T_i}^{T_f} dt\, [r(t) - s(t, A)] \int_{T_i}^{T_f} du\, Q_n(t, u) \left.\frac{\partial s(u, A)}{\partial A}\right|_{A = \hat{a}_{map}} \cdot \tag{226}$$

3. A lower bound on the variance of any *unbiased* estimate of the non-random variable A:

$$\operatorname{Var}(\hat{a} - A) \geq \left[\iint_{T_i}^{T_f} \frac{\partial s(t, A)}{\partial A}\, Q_n(t, u)\, \frac{\partial s(u, A)}{\partial A}\, dt\, du\right]^{-1}, \tag{227a}$$

or, equivalently,

$$\operatorname{Var}(\hat{a} - A) \geq \left[\int_{T_i}^{T_f} \frac{\partial s(t, A)}{\partial A} \frac{\partial g(t, A)}{\partial A}\, dt\right]^{-1}. \tag{227b}$$

4. A lower bound on the mean-square error in the estimate of a zero-mean Gaussian random variable a:

$$E[(\hat{a} - a)^2] \geq \left[\frac{1}{\sigma_a^2} + E_a\left(\int_{T_i}^{T_f} dt \int_{T_i}^{T_f} du\, \frac{s\partial(t, A)}{\partial A}\, Q_n(t, u)\, \frac{\partial s(u, A)}{\partial A}\right)\right]^{-1}. \tag{228}$$

5. A lower bound on the variance of any *unbiased* estimate of a non-random variable for the special case of an infinite observation interval and a stationary noise process:

$$\operatorname{Var}(\hat{a} - A) \geq \left[\int_{-\infty}^{\infty} \frac{\partial S^*(j\omega, A)}{\partial A}\, S_n^{-1}(\omega)\, \frac{\partial S(j\omega, A)}{\partial A} \frac{d\omega}{2\pi}\right]^{-1}, \tag{229}$$

where

$$S(j\omega, A) \triangleq \int_{-\infty}^{\infty} s(t, A) e^{-j\omega t}\, dt.$$

As we discussed in 4.2.3, results of this type are always valid, but we must always look at the over-all likelihood function to investigate their usefulness. In other words, we must not ignore the threshold problem.

The only remaining issue in the matter of colored noise is a closed form solution for $Q_n(t, u)$ or $g(t)$. We consider this problem in the next section.

4.3.6 Solution Techniques for Integral Equations

As we have seen above, to specify the receiver structure completely we must solve the integral equation for $g(t)$ or $Q_n(t, u)$.

In this section we consider three cases of interest:

1. Infinite observation interval; stationary noise process.
2. Finite observation interval; separable kernel.
3. Finite observation interval; stationary noise process.

Infinite Observation Interval; Stationary Noise. In this particular case $T_i = -\infty, T_f = \infty$, and the covariance function of the noise is a function only of the difference in the arguments. Then (161) becomes

$$\delta(z - v) = \int_{-\infty}^{\infty} Q_n(x - z)\, K_n(v - x)\, dx, \qquad -\infty < v, z < \infty, \quad (230)$$

where we assume that we can find a $Q_n(x, z)$ of this form. By denoting the Fourier transform of $K_n(\tau)$ by $S_n(\omega)$ and the Fourier transform of $Q_n(\tau)$ by $S_Q(\omega)$ and transforming both sides of (230) with respect to $\tau = z - v$, we obtain

$$\boxed{S_Q(\omega) = \frac{1}{S_n(\omega)}.} \quad (231)$$

We see that $S_Q(\omega)$ is just the inverse of the noise spectrum. Further, in the stationary case (152) can be written as

$$Q_n(z - v) = \int_{-\infty}^{\infty} h_w(u - z)\, h_w(u - v)\, du. \quad (232)$$

By denoting the Fourier transform of $h_w(\tau)$ by $H_w(j\omega)$, we find that (232) implies

$$\frac{1}{S_n(\omega)} = S_Q(\omega) = |H_w(j\omega)|^2. \quad (233)$$

Finally, for the detection and linear estimation cases (154) is useful. Transforming, we have

$$\boxed{G_\infty(j\omega) = \sqrt{E}\, S_Q(\omega)\, S(j\omega) = \frac{S(j\omega)\sqrt{E}}{S_n(\omega)},} \quad (234)$$

where the subscript ∞ indicates that we are dealing with an infinite interval.

To illustrate the various results, we consider some particular examples.

Example 1. We assume that the colored noise component has a rational spectrum. A typical case is

$$S_c(\omega) = \frac{2k\sigma_n^2}{\omega^2 + k^2}, \quad (235)$$

and

$$S_n(\omega) = \frac{N_0}{2} + \frac{2k\sigma_n^2}{\omega^2 + k^2}. \quad (236)$$

Then

$$S_Q(\omega) = \frac{\omega^2 + k^2}{\frac{N_0}{2}[\omega^2 + k^2(1 + \Lambda)]}, \tag{237}$$

where $\Lambda = 4\sigma_n^2/kN_0$. Writing

$$S_Q(\omega) = \frac{(j\omega + k)(-j\omega + k)}{(N_0/2)(j\omega + k\sqrt{1 + \Lambda})(-j\omega + k\sqrt{1 + \Lambda})}, \tag{238}$$

we want to choose an $H_w(j\omega)$ so that (233) will be satisfied. To obtain a realizable whitening filter we assign the term $(j\omega + k(1 + \Lambda)^{1/2})$ to $H_w(j\omega)$ and its conjugate to $H_w^*(j\omega)$. The term $(j\omega + k)$ in the numerator can be assigned to $H_w(j\omega)$ or $H_w^*(j\omega)$. Thus there are two equally good choices† for the whitening filter:

$$H_{w_1}(j\omega) = \left(\frac{2}{N_0}\right)^{1/2} \frac{j\omega + k}{j\omega + k(1 + \Lambda)^{1/2}} = \left(\frac{2}{N_0}\right)^{1/2}\left[1 - \frac{k(\sqrt{1 + \Lambda} - 1)}{j\omega + k\sqrt{1 + \Lambda}}\right] \tag{239}$$

and

$$H_{w_2}(j\omega) = \left(\frac{2}{N_0}\right)^{1/2} \frac{-j\omega + k}{j\omega + k(1 + \Lambda)^{1/2}} = \left(\frac{2}{N_0}\right)^{1/2}\left[-1 + \frac{k(\sqrt{1 + \Lambda} + 1)}{j\omega + k\sqrt{1 + \Lambda}}\right]. \tag{240}$$

Thus the optimum receiver (detector) can be realized in the whitening forms shown in Fig. 4.42. A sketch of the waveforms for the case in which $s(t)$ is a rectangular pulse is also shown. Three observations follow:

1. The whitening filter has an infinite memory. Thus it uses the entire past of $r(t)$ to generate the input to the correlator.
2. The signal input to the multiplier will start at $t = 0$, but even after time $t = T$ the input will continue.
3. The actual integration limits are $(0, \infty)$, because one multiplier input is zero before $t = 0$.

It is easy to verify that these observations are true whenever the noise consists of white noise plus an independent colored noise with a rational spectrum. It is also true, but less easy to verify directly, when the colored noise has a nonrational spectrum. Thus we conclude that under the above conditions an increase in observation interval will always improve the performance. It is worthwhile to observe that if we use $H_{w_1}(j\omega)$ as the whitening filter the output of the filter in the bottom path will be $\hat{n}_{c_r}(t)$, the minimum mean-square error *realizable* point estimate of $n_c(t)$. We shall verify that this result is always true when we study realizable estimators in Chapter 6.

Observe that we can just as easily (conceptually, at least) operate with $S_Q(\omega)$ directly. In this particular case it is *not* practical, but it does lead to an interesting interpretation of the optimum receiver. Notice that $S_Q(\omega)$ corresponds to an unrealizable filter. We see that we could pass $r(t)$ through this filter and then cross-correlate it with $s(t)$, as shown in Figure 4.43*a*. Observe that the integration is just over $[0, T]$ because $s(t)$ is zero elsewhere; $r_{**}(t)$, $0 \le t \le T$, however, is affected by $r(t)$, $-\infty < t < \infty$. We see that the receiver structure in Fig. 4.43*b* is the estimator-subtractor configuration shown in Fig. 4.39. Therefore the signal at the output of the bottom path must be $\hat{n}_{c_u}(t)$, the minimum mean-square error *unrealizable* estimate of

† There are actually an infinite number, for we can cascade $H_{w_1}(j\omega)$ with any filter whose transfer function has unity magnitude. Observe that we choose a realizable filter so that we can build it. Nothing in our mathematical model requires realizability.

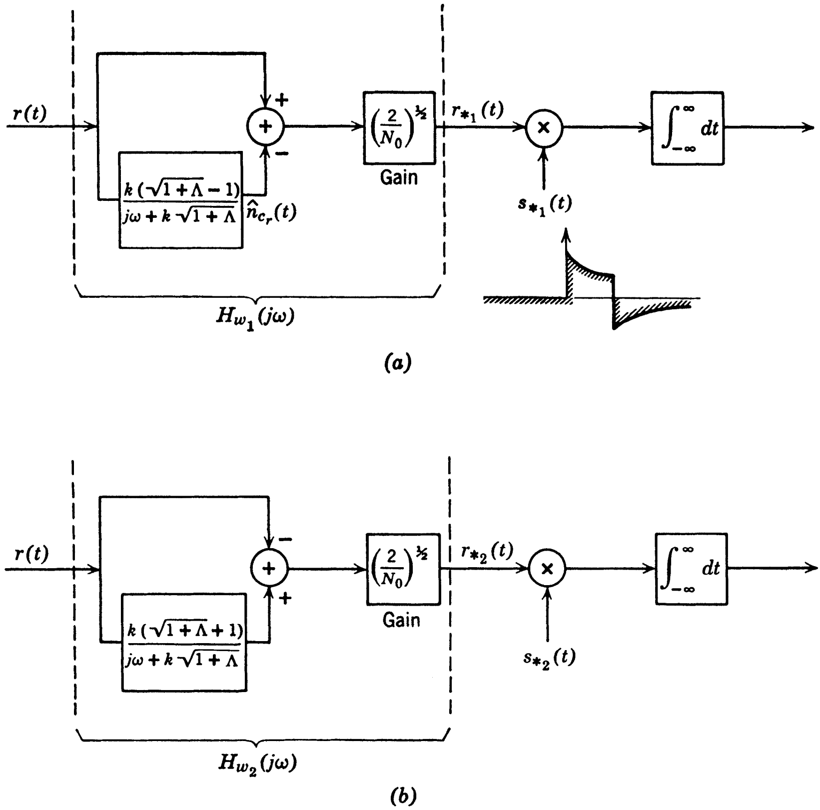

Fig. 4.42 "Optimum receiver": "whitening" realizations: (*a*) configuration 1; (*b*) configuration 2.

$n_c(t)$. This can be verified directly by substituting (235) and (236) into (3.239). We shall see that exactly the same result occurs in the general colored noise detection problem. Comparing Figs. 4.42 and 4.43, we see that they both contain estimates of colored noise but use them differently.

As a second example we investigate what happens when we *remove* the white noise component.

Example 2.

$$S_n(\omega) = \frac{2k\sigma_n^2}{\omega^2 + k^2}. \tag{241}$$

Then

$$S_Q(\omega) = \frac{\omega^2 + k^2}{2k\sigma_n^2}. \tag{242}$$

If we use a whitening realization, then one choice for the whitening filter is

$$H_w(j\omega) = \frac{1}{\sqrt{2k\sigma_n^2}}(j\omega + k). \tag{243}$$

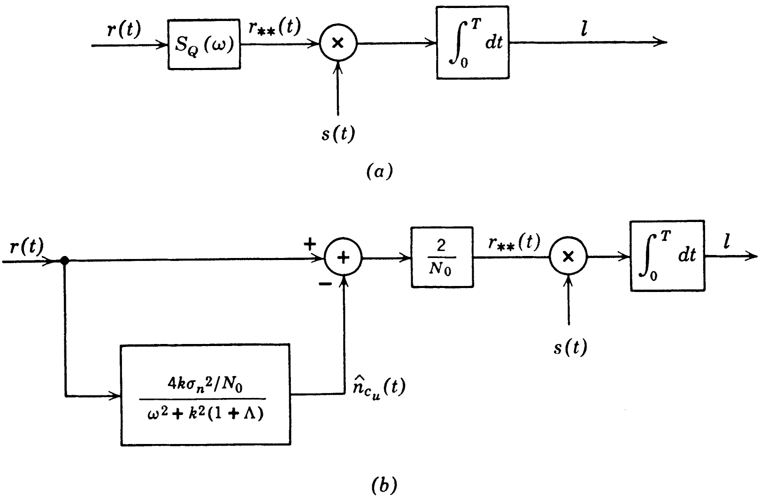

Fig. 4.43 Optimum receiver: estimator-subtractor interpretation.

Thus the whitening filter is a differentiator and gain in parallel (Fig. 4.44*a*). Alternately, using (234), we see that $G_\infty(j\omega)$ is,

$$G_\infty(j\omega) = \frac{\sqrt{E}\, S(j\omega)}{S_n(\omega)} = \frac{\sqrt{E}}{2k\sigma_n^2} (\omega^2 + k^2)\, S(j\omega). \tag{244}$$

Remembering that $j\omega$ in the frequency domain corresponds to differentiation in the time domain, we obtain

$$g_\infty(t) = \frac{\sqrt{E}}{2k\sigma_n^2} [-s''(t) + k^2\, s(t)], \tag{245}$$

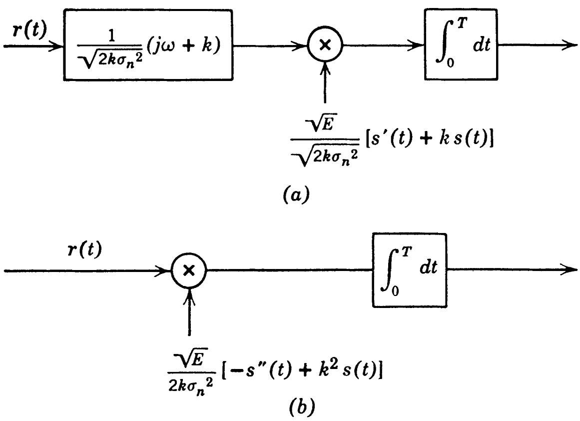

Fig. 4.44 Optimum receiver: no white noise component.

as shown in Fig. 4.44*b*. Observe that $s(t)$ must be differentiable everywhere in the interval $-\infty < t < \infty$; but we assume that $s(t) = 0$, $t < 0$, and $t > T$. Therefore $s(0)$ and $s(T)$ must also be zero. This restriction is intuitively logical. Recall the loose argument we made previously: if there *were* a step in the signal and it was differentiated formally, the result would be an impulse plus a white noise and lead to perfect detection. This is obviously not the actual physical case. By giving the pulse a finite rise time or including some white noise we avoid this condition.

We see that the receiver does not use any of the received waveform outside the interval $0 \le t \le T$, even though it is available. Thus we should expect the solution for $T_i = 0$ and $T_f = T$ to be identical. We shall see shortly that it is.

Clearly, this result will hold whenever the noise spectrum has *only poles*, because the whitening filter is a weighted sum of derivative operators. When the total noise spectrum has zeros, a longer observation time will help the detectability. Observe that when independent white noise is present the total noise spectrum will always have zeros.

Before leaving the section, it is worthwhile to summarize some of the important results.

1. For rational colored noise spectra and nonzero independent white noise, the infinite interval performance is better than any finite observation interval. Thus, the infinite interval performance which is characterized by d_∞^2 provides a simple bound on the finite interval performance. For the particular one-pole spectrum in Example 1 a realizable, stable whitening filter can be found. This filter is *not* unique. In Chapter 6 we shall again encounter whitening filters for rational spectra. At that time we demonstrate how to find whitening filters for arbitrary rational spectra.
2. For rational colored noise spectra with no zeros and no white noise the interval in which the signal is nonzero is the only region of importance. In this case the whitening filter is realizable but not stable (it contains differentiators).

We now consider stationary noise processes and a finite observation interval.

***Finite Observation Interval; Rational Spectra*†.** In this section we consider some of the properties of integral equations over a finite interval. Most of the properties have been proved in standard texts on integral equations (e.g., [33] and [34]). They have also been discussed in a clear manner in the detection theory context by Helstrom [14]. We now state some simple properties that are useful and work some typical examples.

The first equation of interest is (195),

$$\sqrt{E}\, s(t) = \int_{T_i}^{T_f} g(u)\, K_n(t, u)\, du; \qquad T_i \le t \le T_f, \tag{246}$$

† The integral equations in Section 3.4 are special cases of the equations studied in this section. Conversely, if the equation specifying the eigenfunctions and eigenvalues has already been solved, then the solutions to the equations in the section follow easily.

where $s(t)$ and $K_n(t, u)$ are known. We want to solve for $g(t)$. Two special cases should be considered separately.

Case 1. The kernel $K_n(t, u)$ *does not* contain singularities. Physically, this means that there is *no white noise* present. Here (246) is a *Fredholm* equation of the *first kind*, and we can show (see [33]) that if the range (T_i, T_f) is finite a continuous square-integrable solution will *not* exist in general. We shall find that we can always obtain a solution if we allow singularity functions (impulses and their derivatives) in $g(u)$ at the end points of the observation interval.

In Section 4.3.7 we show that whenever $g(t)$ is *not* square-integrable the test is unstable with respect to small perturbations in the model assumptions.

We have purposely excluded Case No. 1 from most of our discussion on physical grounds. In this section we shall do a simple exercise to show the result of letting the white noise level go to zero. We shall find that in the absence of white noise we must put additional restrictions on $s(t)$ to get physically meaningful results.

Case 2. The noise contains a *nonzero* white-noise term. We may then write

$$K_n(t, u) = \frac{N_0}{2}\,\delta(t - u) + K_c(t, u), \tag{247}$$

where $K_c(t, u)$ is a continuous square-integrable function. Then (169*b*) is the equation of interest,

$$\sqrt{E}\, s(t) = \frac{N_0}{2}\, g(t) + \int_{T_i}^{T_f} K_c(t, u)\, g(u)\, du, \qquad T_i \le t \le T_f. \tag{248}$$

This equation is called a *Fredholm* equation of the *second kind.* A continuous, square-integrable solution for $g(t)$ will always exist when $K_c(t, u)$ is a continuous square-integrable function.

We now discuss two types of kernels in which straightforward procedures for solving (246) and (248) are available.

Type A (Rational Kernels). The noise $n_c(t)$ is the steady-state response of a lumped, linear passive network excited with white Gaussian noise. Here the covariance function depends only on $(t - u)$ and we may write

$$K_c(t, u) = K_c(t - u) = K_c(\tau). \tag{249}$$

The transform is

$$S_c(\omega) = \int_{-\infty}^{\infty} K_c(\tau) e^{-j\omega\tau}\, d\tau \triangleq \frac{N(\omega^2)}{D(\omega^2)} \tag{250}$$

and is a ratio of two polynomials in ω^2. The numerator is of order q in ω^2 and the denominator is of order p in ω^2. We assume that $n_c(t)$ has finite power so $p - q \geq 1$. Kernels whose transforms satisfy (250) are called *rational* kernels.

Integral equations with this type of kernel have been studied in detail in [35–37], [47, pp. 1082–1102] [54, pp. 309–329], and [62]. We shall discuss a simple example that illustrates the techniques and problems involved.

Type B (Separable Kernels). The covariance function of the noise can be written as

$$K_c(t, u) = \sum_{i=1}^{K} \lambda_i \, \phi_i(t) \, \phi_i(u), \qquad T_i \leq t, u \leq T_f, \tag{251}$$

where K is *finite*. This type of kernel is frequently present in radar problems when there are multiple targets. As we shall see in a later section, the solution to (246) is straightforward. We refer to this type of kernel as *separable*. Observe that if we *had* allowed $K = \infty$ all kernels would be considered separable, for we can always write

$$K_c(t, u) = \sum_{i=1}^{\infty} \lambda_i \, \phi_i(t) \, \phi_i(u), \qquad T_i \leq t, u \leq T_f, \tag{252}$$

where the λ_i and $\phi_i(t)$ are the eigenvalues and eigenfunctions. Clearly, this is *not* a practical solution technique because we have to solve another integral equation to find the $\phi_i(t)$.

We consider rational kernels in this section and separable kernels in the next.

Fredholm Equations of the First Kind: Rational Kernels. The basic technique is to find a differential equation corresponding to the integral equation. Because of the form of the kernel, this will be a differential equation with constant coefficients whose solution can be readily obtained. In fact, the particular solution of the differential equation is precisely the $g_\infty(t)$ that we derived in the last section (234). An integral equation with a rational kernel corresponds to a differential equation *plus* a set of boundary conditions. To incorporate the boundary conditions, we substitute the particular solution plus a weighted sum of the homogeneous solutions back into the integral equation and try to adjust the weightings so that the equation will be satisfied. It is at this point that we may have difficulty. To illustrate the technique and the possible difficulties we may meet, we consider a simple example. The first step is to show how $g_\infty(t)$ enters the picture. Assume that

$$S_n(\omega) = \frac{N(\omega^2)}{D(\omega^2)} \tag{253}$$

and recall that

$$\delta(t - u) = \int_{-\infty}^{\infty} e^{j\omega(t-u)} \frac{d\omega}{2\pi}. \tag{254}$$

Differentiation with respect to t gives

$$p\,\delta(t - u) = \int_{-\infty}^{\infty} j\omega\, e^{j\omega(t-u)} \frac{d\omega}{2\pi}, \tag{255}$$

where $p \triangleq d/dt$. More generally,

$$N(-p^2)\,\delta(t - u) = \int_{-\infty}^{\infty} N(\omega^2)\, e^{j\omega(t-u)} \frac{d\omega}{2\pi}. \tag{256}$$

In an analogous fashion

$$D(-p^2)\,K_n(t - u) = \int_{-\infty}^{\infty} D(\omega^2)\, S_n(\omega)\, e^{j\omega(t-u)} \frac{d\omega}{2\pi}. \tag{257}$$

From (253) we see that the right sides of (256) and (257) are identical. Therefore the kernel satisfies the differential equation obtained by equating the left sides of (256) and (257):

$$N(-p^2)\,\delta(t - u) = D(-p^2)\,K_n(t - u). \tag{258}$$

Now the integral equation of interest is

$$\sqrt{E}\,s(t) = \int_{T_i}^{T_f} K_n(t - u)\,g(u)\,du, \qquad T_i \le t \le T_f. \tag{259}$$

Operating on both sides of this equation with $D(-p^2)$, we obtain

$$D(-p^2)\sqrt{E}\,s(t) = \int_{T_i}^{T_f} D(-p^2)\,K_n(t - u)\,g(u)\,du, \qquad T_i \le t \le T_f. \tag{260}$$

Using (258) on the right-hand side, we have

$$D(-p^2)\sqrt{E}\,s(t) = N(-p^2)\,g(t), \qquad T_i \le t \le T_f, \tag{261}$$

but from our previous results (234) we know that if the observation interval were infinite,

$$D(\omega^2)\sqrt{E}\,S(j\omega) = N(\omega^2)\,G_\infty(j\omega), \tag{262}$$

or

$$D(-p^2)\sqrt{E}\,s(t) = N(-p^2)\,g_\infty(t), \qquad -\infty < t < \infty. \tag{263}$$

Thus $g_\infty(t)$ corresponds to the *particular* solution of (261). There are also *homogeneous* solutions to (261):

$$0 = N(-p^2)\, g_{h_i}(t), \qquad i = 1, 2, \ldots, 2q. \tag{264}$$

We now add the particular solution $g_\infty(t)$ to a weighted sum of the $2q$ homogeneous solutions $g_{h_i}(t)$, substitute the result back into the integral equation, and adjust the weightings to satisfy the equation. At this point the discussion will be clearer if we consider a specific example.

Example. We consider (246) and use limits $[0, T]$ for algebraic simplicity.

$$K_n(t - u) = K_n(\tau) = \sigma_n^2 e^{-k|\tau|}, \qquad -\infty < \tau < \infty \tag{265}$$

or

$$S_n(\omega) = \frac{2k\sigma_n^2}{\omega^2 + k^2}. \tag{266}$$

Thus

$$N(\omega^2) = 2k\sigma_n^2 \tag{267}$$

and

$$D(\omega^2) = \omega^2 + k^2. \tag{268}$$

The differential equation (261) is

$$\sqrt{E}(-s''(t) + k^2 s(t)) = 2k\sigma_n^2\, g(t). \tag{269}$$

The particular solution is

$$g_\infty(t) = \frac{\sqrt{E}}{2k\sigma_n^2}\,[-s''(t) + k^2 s(t)] \tag{270}$$

and there is no homogeneous solution as

$$q = 0. \tag{271}$$

Substituting back into the integral equation, we obtain

$$\sqrt{E}\, s(t) = \sigma_n^2 \int_0^T \exp(-k|t - u|)\, g(u)\, du, \qquad 0 \le t \le T, \tag{272}$$

For $g(t)$ to be a solution, we require,

$$\begin{aligned} s(t) = \sigma_n^2 \Bigg\{ e^{-kt} \int_0^t e^{+ku} \left[\frac{-s''(u) + k^2 s(u)}{2k\sigma_n^2}\right] du \\ + e^{+kt} \int_t^T e^{-ku} \left[\frac{-s''(u) + k^2 s(u)}{2k\sigma_n^2}\right] du \Bigg\}, \qquad 0 \le t \le T. \end{aligned} \tag{273}$$

Because there are no homogeneous solutions, there are no weightings to adjust. Integrating by parts we obtain the equivalent requirement,

$$\begin{aligned} 0 = e^{-kt}\left\{\frac{1}{2k}\,[s'(0) - ks(0)]\right\} \\ - e^{+k(t-T)}\left\{\frac{1}{2k}\,[s'(T) + ks(T)]\right\}, \qquad 0 \le t \le T. \end{aligned} \tag{274}$$

Clearly, the two terms in brackets must vanish independently in order for $g_\infty(t)$ to satisfy the integral equation. *If* they do, then our solution is complete. Unfortunately, the signal behavior at the end points often will cause the terms in the brackets to be nonzero. We must add something to $g_\infty(t)$ to cancel the e^{-kt} and $e^{k(t-T)}$ terms. We denote this additional term by $g_\delta(t)$ and choose it so that

$$\sigma_n^2 \int_0^T \exp(-k|t-u|)g_\delta(u)\,du$$
$$= -\frac{1}{2k}[s'(0) - ks(0)]\,e^{-kt} + \frac{1}{2k}[s'(T) + ks(T)]\,e^{+k(t-T)}, \qquad 0 \le t \le T. \quad (275)$$

To generate an e^{-kt} term $g_\delta(u)$ must contain an impulse $c_1\,\delta(u)$. To generate an $e^{+k(t-T)}$ term $g_\delta(u)$ must contain an impulse $c_2\,\delta(u-T)$. Thus

$$g_\delta(u) = c_1\,\delta(u) + c_2\,\delta(u-T), \quad (276)$$

where

$$c_1 = \frac{k\,s(0) - s'(0)}{k\sigma_n^2},$$
$$c_2 = \frac{k\,s(T) + s'(T)}{k\sigma_n^2}, \quad (277)$$

to satisfy (274).† Thus the complete solution to the integral equation is

$$g(t) = g_\infty(t) + g_\delta(t), \qquad 0 \le t \le T. \quad (278)$$

From (153) and (154) we see that the output of the processor is

$$l = \int_0^T r(t)\,g(t)\,dt$$
$$= \frac{c_1}{2}r(0) + \frac{c_2}{2}r(T) + \int_0^T r(t)\left\{\sqrt{E}\left[\frac{k^2\,s(t) - s''(t)}{2k\sigma_n^2}\right]\right\}dt. \quad (279)$$

Thus the optimum processor consists of a *filter* and a *sampler.*

Observe that $g(t)$ will be square-integrable only when c_1 and c_2 are zero. We discuss the significance of this point in Section 4.3.7.

When the spectrum has more poles, higher order singularities must be added at the end points. When the spectrum has zeros, there will be homogeneous solutions, which we denote as $g_{h_i}(t)$. Then we can show that the general solution is of the form

$$g(t) = g_\infty(t) + \sum_{i=1}^{2q} a_i\,g_{h_i}(t) + \sum_{k=0}^{p-q-1}[b_k\,\delta^{(k)}(t) + c_k\,\delta^{(k)}(t-T)], \quad (280)$$

where $2p$ is the order of $D(\omega^2)$ as a function of ω and $2q$ is the order of $N(\omega^2)$ as a function of ω (e.g., [35]). The function $\delta^{(k)}(t)$ is the kth derivative of $\delta(t)$. A great deal of effort has been devoted to finding efficient methods of evaluating the coefficients in (280) (e.g., [63], [3]).

As we have pointed out, whenever we assume that white noise is present, the resulting integral equation will be a Fredholm equation of the second kind. For rational spectra the solution techniques are similar but the character of the solution is appreciably different.

† We assume that the impulse is symmetric. Thus only one half its area is in the interval.

Fredholm Equations of the Second Kind: Rational Kernels. The equation of interest is (248):

$$\sqrt{E}\, s(t) = \frac{N_0}{2}\, g(t) + \int_{T_i}^{T_f} K_c(t, u)\, g(u)\, du, \qquad T_i \le t \le T_f. \tag{281}$$

We assume that the noise is stationary with spectrum $S_n(\omega)$,

$$S_n(\omega) = \frac{N_0}{2} + S_c(\omega) \triangleq \frac{N(\omega^2)}{D(\omega^2)}. \tag{282}$$

[Observe that $N(\omega^2)$ and $D(\omega^2)$ are of the same order. (This is because $S_c(\omega)$ has finite power.)] Proceeding in a manner identical to the preceding section, we obtain a differential equation that has a particular solution, $g_\infty(t)$, and homogeneous solutions, $g_{h_i}(t)$. Substituting

$$g(t) = g_\infty(t) + \sum_{i=1}^{2q} a_i g_{h_i}(t), \tag{283}$$

into the integral equation, we find that by suitably choosing the a_i we can always obtain a solution to the integral equation. (No $g_\delta(t)$ is necessary because we have enough weightings (or degrees of freedom) to satisfy the boundary conditions.) A simple example illustrates the technique.

Example. Let

$$K_c(t, u) = \sigma_c^2 \exp(-k|t - u|); \tag{284}$$

the corresponding spectrum is

$$S_c(\omega) = \frac{\sigma_c^2 2k}{\omega^2 + k^2}. \tag{285}$$

Then

$$S_n(\omega) = \frac{N_0}{2} + \frac{\sigma_c^2 2k}{\omega^2 + k^2} = \frac{(N_0/2)[\omega^2 + k^2(1 + 4\sigma_c^2/kN_0)]}{\omega^2 + k^2}$$

$$\triangleq \frac{N(\omega^2)}{D(\omega^2)}. \tag{286}$$

The integral equation is (using the interval $(0, T)$ for simplicity)

$$\sqrt{E}\, s(t) = \frac{N_0}{2}\, g(t) + \sigma_c^2 \int_0^T e^{-k|t-u|}\, g(u)\, du, \qquad 0 \le t \le T. \tag{287}$$

The corresponding differential equation follows easily from (286),

$$\sqrt{E}(-s''(t) + k^2 s(t)) = \frac{N_0}{2}[-g''(t) + \gamma^2 g(t)], \tag{288}$$

where $\gamma^2 \triangleq k^2(1 + 4\sigma_c^2/kN_0)$. The particular solution is just $g_\infty(t)$. This can be obtained by solving the differential equation directly or by transform methods.

$$g_\infty(t) = \int_{-\infty}^{\infty} e^{+j\omega t} G_\infty(j\omega)\, \frac{d\omega}{2\pi}, \qquad 0 \le t \le T, \tag{289}$$

$$g_\infty(t) = \frac{2\sqrt{E}}{N_0} \int_{-\infty}^{\infty} e^{+j\omega t} \left(\frac{\omega^2 + k^2}{\omega^2 + \gamma^2}\right) S(j\omega)\, \frac{d\omega}{2\pi}, \qquad 0 \le t \le T. \tag{290}$$

The homogeneous solutions are

$$\begin{aligned} g_{h_1}(t) &= e^{\gamma t}, \\ g_{h_2}(t) &= e^{-\gamma t}. \end{aligned} \tag{291}$$

Then

$$g(t) = g_\infty(t) + a_1 e^{+\gamma t} + a_2 e^{-\gamma t}, \qquad 0 \le t \le T. \tag{292}$$

Substitution of (292) into (287) will lead to two simultaneous equations that a_1 and a_2 must satisfy. Solving for a_1 and a_2 explicitly gives the complete solution. Several typical cases are contained in the problems.

The particular property of interest is that a solution can always be found without having to add singularity functions. Thus the white noise assumption guarantees a square-integrable solution. (The convergence of the series in (164) and (170) implies that the solution is square-integrable.)

The final integral equation of interest is the one that specifies $h_o(t, u)$, (163). Rewriting it for the interval $[0, T]$, we have

$$h_o(t, z) + \frac{2}{N_0} \int_0^T K_c(t, u)\, h_o(u, z)\, du = \frac{2}{N_0} K_c(t, z), \qquad 0 \le t, z \le T. \tag{293}$$

We observe that this is identical to (281) in the preceding problem, except that there is an extra variable in each expression. Thus we can think of t as a fixed parameter and z as a variable or vice versa. In either case we have a Fredholm equation of the second kind.

For rational kernels the procedure is identical. We illustrate this with a simple example.

Example.

$$K_c(u, z) = \sigma_s^2 \exp(-k|u - z|), \tag{294}$$

$$h_o(t, z) + \frac{2}{N_0} \int_0^T h_o(t, u)\sigma_s^2 \exp(-k|u - z|)\, du = \frac{2}{N_0} \sigma_s^2 \exp(-k|t - z|), \qquad 0 \le t, z \le T. \tag{295}$$

Using the operator $k^2 - p^2$ and the results of (258) and (286), we have

$$\left(p \triangleq \frac{d}{dz}\right),$$

$$(k^2 - p^2)\, h_o(t, z) + \frac{2\sigma_s^2}{N_0} \cdot 2k\, h_o(t, z) = \frac{2\sigma_s^2}{N_0} 2k\, \delta(t - z), \tag{296}$$

or

$$(1 + \Lambda) h_o(t, z) - \frac{p^2}{k^2} h_o(t, z) = \Lambda\, \delta(t - z), \tag{297}$$

where

$$\Lambda = \frac{4\sigma_s^2}{kN_0}. \tag{298}$$

Let $\beta^2 = k^2(1 + \Lambda)$. The particular solution is

$$h_{op}(t, z) = \frac{2\sigma_s^2}{N_0\sqrt{1 + \Lambda}} \exp(-k\sqrt{1 + \Lambda}\, |t - z|), \qquad 0 \le t, z \le T. \tag{299}$$

Now add homogeneous solutions $a_1(t)e^{+\beta z}$ and $a_2(t)e^{-\beta z}$ to the particular solution in (299) and substitute the result into (295). We find that we require

$$a_1(t) = \frac{2k\sigma_s^2(\beta - k)[(\beta + k)e^{+\beta t} + (\beta - k)e^{-\beta t}]e^{-\beta T}}{N_0\beta[(\beta + k)^2 e^{\beta T} - (\beta - k)^2 e^{-\beta T}]} \tag{300}$$

and

$$a_2(t) = \frac{2k\sigma_s^2(\beta - k)[(\beta + k)e^{+\beta(T-t)} + (\beta - k)e^{-\beta(T-t)}]}{N_0\beta[(\beta + k)^2 e^{\beta T} - (\beta - k)^2 e^{-\beta T}]}. \tag{301}$$

The entire solution is

$$h_o(z, t) = \frac{2k\sigma_s^2[(\beta + k)e^{+\beta z} + (\beta - k)e^{-\beta z}][(\beta + k)e^{+\beta(T-t)} + (\beta - k)e^{-\beta(T-t)}]}{N_0\beta[(\beta + k)^2 e^{+\beta T} - (\beta - k)^2 e^{-\beta T}]},$$

$$0 \le z \le t \le T. \tag{302}$$

The solution is symmetric in z and t. This is clearly not a very appealing function to mechanize. An important special case that we will encounter later is the one in which the colored noise component is small. Then $\beta \simeq k$ and

$$h_o(z, t) \simeq \frac{2\sigma_s^2}{N_0} \exp - \beta|t - z|, \qquad 0 \le z, t \le T. \tag{303}$$

The important property to observe about (293) is that the extra variable complicates the algebra but the basic technique is still applicable.

This completes our discussion of integral equations with rational kernels and finite time intervals.

Several observations may be made:

1. The procedure is straightforward but tedious.
2. When there is no white noise, certain restrictions must be placed on $s(t)$ to guarantee that $g(t)$ will be square-integrable.
3. When white noise is present, increasing the observation interval always improves the performance.
4. The solution for $h_o(t, u)$ for arbitrary colored noise levels appears to be too complex to implement. We can use the d^2 derived from it (198) as a basis of comparison for simpler mechanizations. [In Section 6.7 we discuss an easier implementation of $h_o(t, u)$.]

Finite Observation Time: Separable Kernels. As a final category, we consider integral equations with separable kernels. By contrast with the tedium of the preceding section, the solution for separable kernels follows almost by inspection. In this case

$$K_c(t, u) = \sum_{i=1}^{K} \lambda_i\, \phi_i(t)\, \phi_i(u), \qquad T_i \le t, u \le T_f, \tag{304}$$

where λ_i and $\phi_i(t)$ are the eigenvalues and eigenfunctions of $K_c(t, u)$. Observe that (304) says that the noise has only K nonzero eigenvalues. Thus, unless we include a white noise component, we may have a singular problem. We include the white noise component and then observe that the solution for $h_o(t, u)$ is just a truncated version of the infinite series in (164). Thus

$$h_o(t, u) = \sum_{i=1}^{K} \frac{\lambda_i}{N_0/2 + \lambda_i} \phi_i(t)\, \phi_i(u), \qquad T_i \leq t, u \leq T_f, \tag{305}$$

The solution to (154) follows easily. Using (305) in (162) and the result in (154), we obtain

$$g(t) = \int_{T_i}^{T_f} du \sqrt{E}\, s(u) \frac{2}{N_0} \left[\delta(t - u) - \sum_{i=1}^{K} \frac{\lambda_i}{N_0/2 + \lambda_i} \phi_i(t)\, \phi_i(u)\right]$$
$$T_i < t < T_f. \tag{306}$$

Recalling the definition of s_i in (201) and recalling that $g(t)$ is continuous at the end-points, we have

$$\begin{aligned} g(t) &= \frac{2}{N_0}\left[\sqrt{E}\, s(t) - \sum_{i=1}^{K} \frac{s_i \lambda_i}{N_0/2 + \lambda_i} \phi_i(t)\right], && T_i \leq t \leq T_f, \\ g(t) &= 0, && \text{elsewhere.} \end{aligned} \tag{307}$$

This receiver structure is shown in Fig. 4.45. Fortunately, in addition to having a simple solution, the separable kernel problem occurs frequently in practice.

A typical case is shown in Fig. 4.46. Here we are trying to detect a target in the presence of an interfering target and white noise (Siebert [38]).

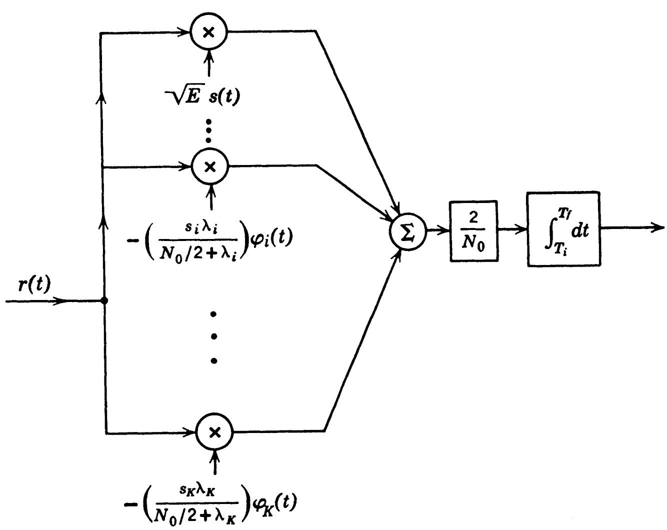

Fig. 4.45 Optimum receiver: separable noise process.

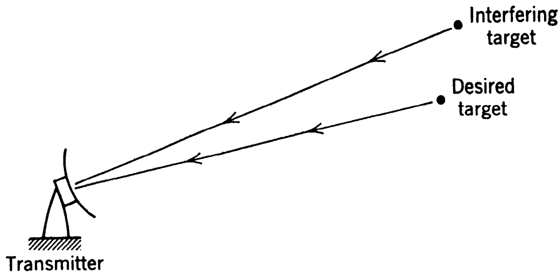

Fig. 4.46 Detection in presence of interfering target.

Let

$$r(t) = \sqrt{E}\, s(t) + a_I\, s_I(t) + w(t) \qquad T_i \le t \le T_f : H_1$$
$$= a_I\, s_I(t) + w(t) \qquad T_i \le t \le T_f : H_0. \qquad (308)$$

If we assume that a_I and $s_I(t)$ are known, the problem is trivial. The simplest nontrivial model is to assume that $s_I(t)$ is a known normalized waveform but a_I is a zero-mean Gaussian random variable, $N(0, \sigma_I)$. Then

$$K_n(t, u) = \sigma_I^2 s_I(t)\, s_I(u) + \frac{N_0}{2}\,\delta(t-u), \qquad T_i \le t, u \le T_f. \qquad (309)$$

This is a special case of the problem we have just solved. The receiver is shown in Fig. 4.47. The function $g(t)$ is obtained from (307). It can be redrawn, as shown in Fig. 4.47*b*, to illustrate the estimator-subtractor interpretation (this is obviously not an efficient realization). The performance index is obtained from (198),

$$d^2 = \frac{2E}{N_0}\left(1 - \frac{2\sigma_I^2/N_0}{1 + 2\sigma_I^2/N_0}\,\rho_I^2\right), \qquad (310)$$

where

$$\rho_I \triangleq \int_{T_i}^{T_f} s(t)\, s_I(t)\, dt. \qquad (311)$$

Rewriting (310), we have

$$d^2 = \frac{2E}{N_0}\left[\frac{1 + 2\sigma_I^2/N_0(1-\rho_I^2)}{1 + 2\sigma_I^2/N_0}\right] \qquad (312a)$$

as $\rho_I \to 0$, $d^2 \to 2E/N_0$. This result is intuitively logical. If the interfering signal is orthogonal to $s(t)$, then, regardless of its strength, it should not degrade the performance. On the other hand, as $\rho_I \to 1$,

$$d^2 \to \frac{2E/N_0}{1 + 2\sigma_I^2/N_0}. \qquad (312b)$$

Now the signals on the two hypotheses are equal and the difference in their amplitudes is the only basis for making a decision.

We have introduced this example for two reasons:

1. It demonstrates an important case of nonwhite noise in which the inverse kernel is particularly simple to calculate.
2. It shows all of the concepts (but not the detail) that is necessary to solve the problem of detection (or estimation) in the presence of clutter (radar) or reverberation (sonar). In Chapter II-4, after we have developed a detailed model for the reverberation problem, we shall see how these results can be extended to handle the actual problem.

Summary of Integral Equations. In this section we have developed techniques for solving the types of integral equation encountered in the detection and estimation problems in the presence of nonwhite noise. The character of the solution was determined by the presence or absence of a white noise component. The simplicity of the solution in the infinite-interval, stationary process case should be emphasized. Because the performance in this case always bounds the finite interval, stationary process case, it is a useful preliminary calculation.

As a final topic for the colored noise problem, we consider the sensitivity of the result to perturbations in the initial assumptions.

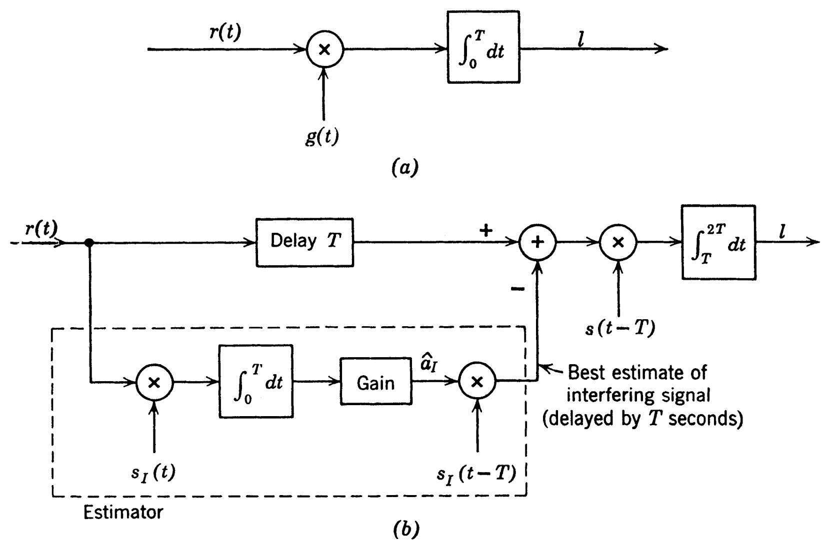

Fig. 4.47 Optimum receiver: interfering targets.

4.3.7 Sensitivity

Up to this point in our discussion we have assumed that all the quantities needed to design the optimum receiver were known exactly. We want to investigate the effects of imperfect knowledge of these quantities. In order to obtain some explicit results we shall discuss the sensitivity issue in the context of the simple binary decision problem developed in Section 4.3.1. Specifically, the model assumed is

$$\begin{aligned} r(t) &= \sqrt{E}\, s(t) + n(t), \qquad T_i \le t \le T_f : H_1, \\ r(t) &= n(t), \qquad T_i \le t \le T_f : H_0, \end{aligned} \tag{313}$$

where $s(t)$, the signal, and $K_n(t, u)$, the noise covariance function, are assumed known. Just as in the white noise case, there are two methods of sensitivity analysis: the parameter variation approach and the functional variation approach. In the white noise case we varied the signal. Now the variations can include both the signal and the noise.

Typical parameter variation examples are formulated below:

1. Let the assumed signal be

$$s(t) = \begin{cases} \left(\dfrac{2}{T}\right)^{1/2} \sin \omega_c t, & 0 \le t \le T, \\ 0, & \text{elsewhere}, \end{cases} \tag{314}$$

and the actual signal be

$$s_a(t) = \begin{cases} \left(\dfrac{2}{T}\right)^{1/2} \sin (\omega_c + \Delta\omega)t, & 0 \le t \le T, \\ 0, & \text{elsewhere}. \end{cases} \tag{315}$$

Find $\Delta d/d$ as a function of $\Delta\omega$.

2. Let the assumed noise covariance be

$$K_n(t, u) = \frac{N_0}{2}\, \delta(t - u) + K_c(t, u) \tag{316}$$

and the actual covariance be

$$K_{na}(t, u) = \left(\frac{N_0 + \Delta N_0}{2}\right) \delta(t - u) + K_c(t, u). \tag{317}$$

Find $\Delta d/d$ as a function of ΔN_0.

3. In the interfering target example of the last section (308) let the assumed interference signal be

$$s_I(t) = s(t - \tau). \tag{318}$$

In other words, it is a delayed version of the desired signal. Let the actual interference signal be

$$s_{Ia}(t) = s(t - \tau - \Delta\tau). \tag{319}$$

Find $\Delta d/d$ as a function of $\Delta\tau$.

These examples illustrate typical parameter variation problems. Clearly, the appropriate variations depend on the physical problem of interest. In almost all of them the succeeding calculations are straightforward. Some typical cases are included in the problems.

The functional variation approach is more interesting. As before, we do a "worst-case" analysis. Two examples are the following:

1. Let the actual signal be

$$s_a(t) = \sqrt{E}\, s(t) + \sqrt{E_\epsilon}\, s_\epsilon(t), \qquad T_i \le t \le T_f, \tag{320}$$

where

$$\int_{T_i}^{T_f} s_\epsilon^2(t)\, dt = 1. \tag{321}$$

To find the worst case we choose $s_\epsilon(t)$ to make Δd as negative as possible.

2. Let the actual noise be

$$n_a(t) = n(t) + n_\epsilon(t) \tag{322a}$$

whose covariance function is

$$K_{na}(t, u) = K_n(t, u) + K_{n\epsilon}(t, u), \tag{322b}$$

We assume that $n_\epsilon(t)$ has finite energy in the interval

$$E \int_{T_i}^{T_f} n_\epsilon^2(t)\, dt \le \Delta_n. \tag{323a}$$

This implies that

$$\iint_{T_i}^{T_f} K_{n\epsilon}^2(t, u)\, dt\, du \le \Delta_n. \tag{323b}$$

To find the worst case we choose $K_{n\epsilon}(t, u)$ to make Δd as negative as possible.

Various other perturbations and constraints are also possible. We now consider a simple version of the first problem. The second problem is developed in detail in [42].

We assume that the noise process is stationary with a spectrum $S_n(\omega)$ and that the observation interval is infinite. The optimum receiver, using a whitening realization (see Fig. 4.38*a*), is shown in Fig. 4.48*a*. The

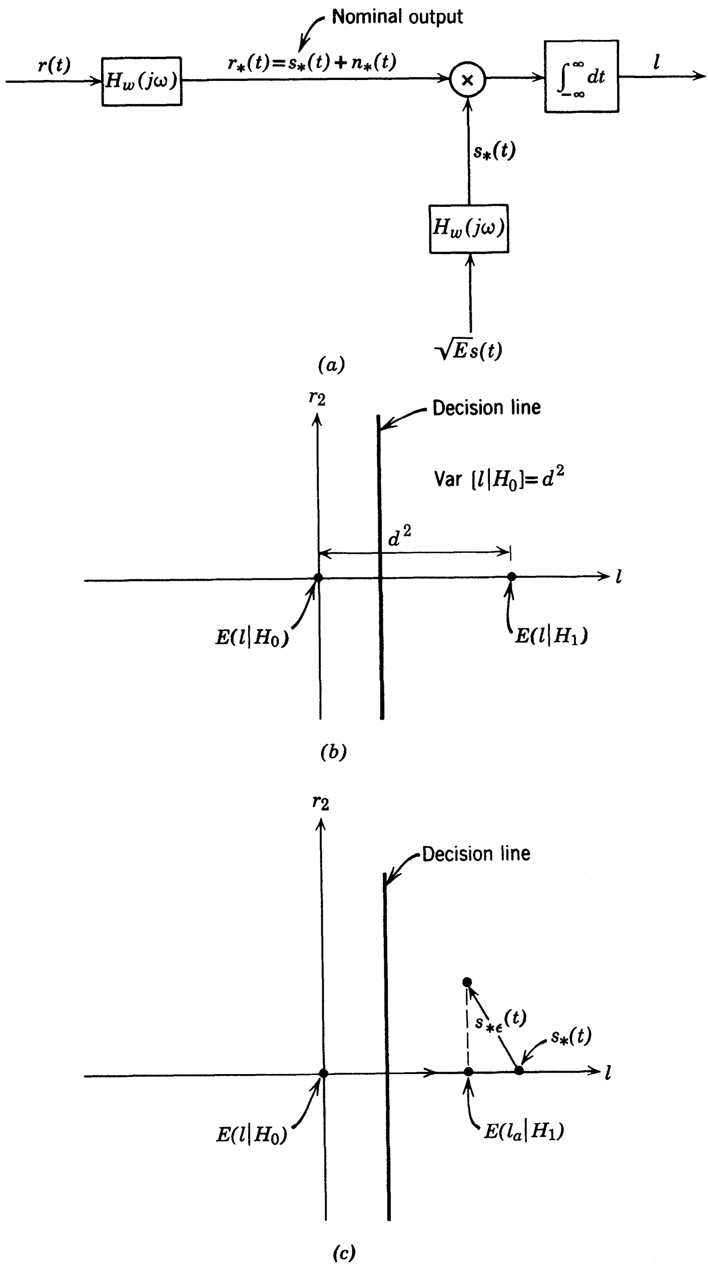

Fig. 4.48 Sensitivity analysis: (*a*) filter with nominal input; (*b*) nominal decision space; (*c*) actual design space.

corresponding decision space is shown in Fig. 4.48*b*. The nominal performance is

$$d = \frac{E(l|H_1) - E(l|H_0)}{[\mathrm{Var}\,(l|H_0)]^{1/2}} = \frac{\int_{-\infty}^{\infty} s_*^2(t)\,dt}{\left[\int_{-\infty}^{\infty} s_*^2(t)\,dt\right]^{1/2}}, \tag{324}$$

or

$$d = \left[\int_{-\infty}^{\infty} s_*^2(t)\,dt\right]^{1/2}. \tag{325}$$

We let the actual signal be

$$s_a(t) = \sqrt{E}\,s(t) + \sqrt{E_\epsilon}\,s_\epsilon(t), \qquad -\infty < t < \infty, \tag{326}$$

where $s(t)$ and $s_\epsilon(t)$ have unit energy. The output of the whitening filter will be

$$r_{*a}(t) \triangleq s_*(t) + s_{*\epsilon}(t) + n_*(t), \qquad -\infty < t < \infty, \tag{327}$$

and the decision space will be as shown in Fig. 4.48*c*. The only quantity that changes is $E(l_a|H_1)$. The variance is still the same because the noise covariance is unchanged. Thus

$$\Delta d = \frac{1}{d}\int_{-\infty}^{\infty} s_{*\epsilon}(t)\,s_*(t)\,dt. \tag{328}$$

To examine the sensitivity we want to make Δd as negative as possible. If we can make $\Delta d = -d$, then the actual operating characteristic will be the $P_D = P_F$ line which is equivalent to a random test. If $\Delta d < -d$, the actual test will be worse than a random test (see Fig. 2.9*a*). It is important to note that the constraint is on the energy in $s_\epsilon(t)$, *not* $s_{*\epsilon}(t)$. Using Parseval's theorem, we can write (328) as

$$\Delta d = \frac{1}{d}\int_{-\infty}^{\infty} S_{*\epsilon}(j\omega)\,S_*^*(j\omega)\,\frac{d\omega}{2\pi}. \tag{329}$$

This equation can be written in terms of the original quantities by observing that

$$S_{*\epsilon}(j\omega) = \sqrt{E_\epsilon}\,H_w(j\omega)\,S_\epsilon(j\omega) \tag{330}$$

and

$$S_*(j\omega) = \sqrt{E}\,H_w(j\omega)\,S(j\omega). \tag{331}$$

Thus

$$\begin{aligned}\Delta d &= \frac{\sqrt{EE_\epsilon}}{d}\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\,S_\epsilon(j\omega)|H_w(j\omega)|^2\,S^*(j\omega)\\ &= \frac{\sqrt{EE_\epsilon}}{d}\int_{-\infty}^{\infty} S_\epsilon(j\omega)\,\frac{S^*(j\omega)}{S_n(\omega)}\,\frac{d\omega}{2\pi}.\end{aligned} \tag{332}$$

The constraint in (321) can be written as

$$\int_{-\infty}^{\infty} |S_\epsilon(j\omega)|^2 \frac{d\omega}{2\pi} = 1. \tag{333}$$

To perform a worst-case analysis we minimize Δd subject to the constraint in (333) by using Lagrange multipliers. Let

$$F = \Delta d + \lambda \left[\int_{-\infty}^{\infty} |S_\epsilon(j\omega)|^2 \frac{d\omega}{2\pi} - 1\right]. \tag{334}$$

Minimizing with respect to $S_\epsilon(j\omega)$, we obtain

$$S_{\epsilon_o}(j\omega) = -\frac{\sqrt{EE_\epsilon}}{2\lambda d} \frac{S(j\omega)}{S_n(\omega)}, \tag{335}$$

(the subscript o denotes optimum). To evaluate λ we substitute into the constraint equation (333) and obtain

$$\frac{EE_\epsilon}{4\lambda^2 d^2} \int_{-\infty}^{\infty} \frac{|S(j\omega)|^2}{S_n^2(\omega)} \frac{d\omega}{2\pi} = 1. \tag{336}$$

If the integral exists, then

$$2\lambda = \frac{\sqrt{EE_\epsilon}}{d} \left[\int_{-\infty}^{\infty} \frac{|S(j\omega)|^2}{S_n^2(\omega)} \frac{d\omega}{2\pi}\right]^{1/2}. \tag{337}$$

Substituting into (335) and then (332), we have

$$\Delta d = -\left(\frac{EE_\epsilon}{d^2}\right)^{1/2} \left[\int_{-\infty}^{\infty} \frac{|S(j\omega)|^2}{S_n^2(\omega)} \frac{d\omega}{2\pi}\right]^{1/2}. \tag{338}$$

(Observe that we could also obtain (338) by using the Schwarz inequality in (332).) Using the frequency domain equivalent of (325), we have

$$\frac{\Delta d}{d} = -\left(\frac{E_\epsilon}{E}\right)^{1/2} \left\{\frac{\left[\int_{-\infty}^{\infty} \frac{|S(j\omega)|^2}{S_n^2(\omega)} \frac{d\omega}{2\pi}\right]^{1/2}}{\left[\int_{-\infty}^{\infty} \frac{|S(j\omega)|^2}{S_n(\omega)} \frac{d\omega}{2\pi}\right]}\right\}. \tag{339}$$

In the white noise case the term in the brace reduces to one and we obtain the same result as in (82). When the noise is not white, several observations are important:

1. If there is a white noise component, both integrals exist and the term in the braces is greater than or equal to one. (Use the Schwarz inequality on the denominator.) Thus in the colored noise case a small signal perturbation may cause a large change in performance.

2. If there is *no* white noise component *and* the nominal test is *not singular*, the integral in the denominator exists. Without further restrictions

on $S(j\omega)$ and $S_n(\omega)$ the integral in the numerator may not exist. If it does not exist, the above derivation is not valid. In this case we can find an $S_\epsilon(j\omega)$ so that Δd will be less than any desired Δd_x. Choose

$$S_\epsilon(j\omega) = \begin{cases} k\,\dfrac{S(j\omega)}{S_n(\omega)}, & \omega \text{ in } \Omega, \\ 0, & \omega \text{ not in } \Omega, \end{cases} \tag{340}$$

where Ω is a region such that

$$k\,\frac{\sqrt{EE_\epsilon}}{d}\left(\int_\Omega \frac{|S(j\omega)|^2}{S_n(\omega)}\,\frac{d\omega}{2\pi}\right)^{1/2} = \Delta d_x \tag{341}$$

and k is chosen to satisfy the energy constraint on $s_\epsilon(t)$. We see that in the absence of white noise a signal perturbation exists that will make the test performance arbitrarily bad. Such tests are referred to as *unstable* (or infinitely sensitive) tests. We see that stability is a stronger requirement than nonsingularity and that the white noise assumption guarantees a nonsingular, stable test. Clearly, even though a test is stable, it may be extremely sensitive.

3. Similar results can be obtained for a finite interval and nonstationary processes in terms of the eigenvalues. Specifically, we can show (e.g., [42]) that the condition

$$\sum_{i=1}^{\infty} \frac{s_i^2}{\lambda_i^2} < \infty$$

is necessary and sufficient for stability. This is identical to the condition for $g(t)$ to be square-integrable.

In this section we have illustrated some of the ideas involved in a sensitivity analysis of an optimum detection procedure. Although we have eliminated unstable tests by the white noise assumption, it is still possible to encounter sensitive tests. In any practical problem it is essential to check the test sensitivity against possible parameter and function variations. We can find cases in which the test is too sensitive to be of any practical value. In these cases we try to design a test that is nominally suboptimum but less sensitive. Techniques for finding this test depend on the problem of interest.

Before leaving the colored noise problem we consider briefly a closely related problem.

4.3.8 Known Linear Channels

There is an almost complete duality between the colored additive noise problem and the problem of transmitting through a known linear channel with memory. The latter is shown in Fig. 4.49*a*.

The received waveform on H_1 in the simple binary problem is

$$r(t) = \int_{T_i}^{T_f} h_{\rm ch}(t, u)\sqrt{E}\, s(u)du + w(t), \qquad T_i \le t \le T_f. \tag{342}$$

This is identical in form to (146). Thus $h_{\rm ch}(t, u)$ plays an analogous role to the whitening filter. The optimum receiver is shown in Fig. 4.49*b*. The performance index is

$$\begin{aligned} d^2 &= \frac{2}{N_0}\int_{T_i}^{T_f} s_*^2(t)\, dt \\ &= \frac{2E}{N_0}\int_{T_i}^{T_f} dt \left[\int_a^b h_{\rm ch}(t, u)\, s(u)\, du \int_a^b h_{\rm ch}(t, v)\, s(v)\, dv\right], \end{aligned} \tag{343}$$

where the limits (a, b) depend on the channel's impulse response and the input signal duration. We assume that $T_i \le a \le b \le T_f$. We can write this in a familiar quadratic form:

$$d^2 = \frac{2E}{N_0}\iint_a^b du\, dv\, s(u)\, Q_{\rm ch}(u, v)\, s(v) \tag{344}$$

by defining

$$Q_{\rm ch}(u, v) = \int_{T_i}^{T_f} h_{\rm ch}(t, u)\, h_{\rm ch}(t, v)\, dt, \qquad a \le u, v \le b. \tag{345}$$

The only difference is that now $Q_{\rm ch}(u, v)$ has the properties of a covariance function rather than an inverse kernel. A problem of interest is to choose

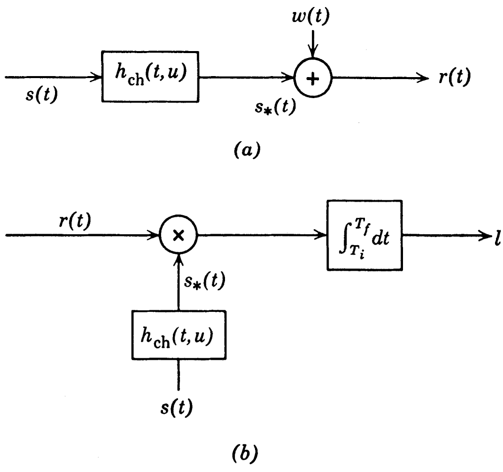

Fig. 4.49 Known dispersive channel.

$s(t)$ to maximize d^2. The solution follows directly from our earlier signal design results (p. 302). We can express d^2 in terms of the channel eigenvalues and eigenfunctions

$$d^2 = \frac{2}{N_0} \sum_{i=1}^{\infty} \lambda_i^{\text{ch}} s_i^2, \tag{346}$$

where

$$s_i \triangleq \int_a^b \sqrt{E}\, s(u)\, \phi_i(u)\, du \tag{347}$$

and λ_i^{ch} and $\phi_i(u)$ correspond to the kernel $Q_{\text{ch}}(u, v)$. To maximize d^2 we choose

$$s_1 = \sqrt{E},$$

and

$$s_i = 0, \qquad i \neq 1, \tag{348}$$

because λ_1^{ch} is defined as the largest eigenvalue of the channel kernel $Q_{\text{ch}}(u, v)$. Some typical channels and their optimum signals are developed in the problems.

When we try to communicate sequences of signals over channels with memory, another problem arises. Looking at the basic communications system in Fig. 4.1, we see that inside the basic interval $0 \leq t \leq T$ there is interference due to noise and the sequence of signals corresponding to previous data. This second interference is referred to as the intersymbol interference and it turns out to be the major disturbance in many systems of interest. We shall study effective methods of combatting intersymbol interference in Chapter II.4.

4.4 SIGNALS WITH UNWANTED PARAMETERS: THE COMPOSITE HYPOTHESIS PROBLEM

Up to this point in Chapter 4 we have assumed that the signals of concern were completely known. The only uncertainty was caused by the additive noise. As we pointed out at the beginning of this chapter, in many physical problems of interest this assumption is not realistic. One example occurs in the radar problem. The transmitted signal is a high frequency pulse that acquires a random phase angle (and perhaps a random amplitude) when it is reflected from the target. Another example arises in the communications problem in which there is an uncertainty in the oscillator phase. Both problems are characterized by the presence of an unwanted parameter.

Unwanted parameters appear in both detection and estimation problems. Because of the inherent similarities, it is adequate to confine our present