Even if the left side of (463) were not zero, however, the value *after* taking the expectation might be zero, which gives uncoupled MAP estimates.

Several interesting examples of multiple parameter estimation are included in the problems.

4.6.2 Extensions

The results can be modified in a straightforward manner to include other cases of interest.

1. Nonrandom variables, ML estimation.
2. Additive colored noise.
3. Random phase channels.
4. Rayleigh and Rician channels.
5. Multiple received signals.

Some of these cases are considered in the problems. One that will be used in the sequel is the additive colored noise case, discussed in Problem 4.6.7. The results are obtained by an obvious modification of (447) which is suggested by (226).

$$\hat{a}_i = \sigma_{a_i}^2 \int_{T_i}^{T_f} \left.\frac{\partial s(z, \mathbf{A})}{\partial A_i}\right|_{\mathbf{A}=\hat{\mathbf{a}}_{\text{map}}} [r_g(z) - g(z)]\, dz, \qquad i = 1, 2, \ldots, M, \tag{464}$$

where

$$r_g(z) - g(z) \triangleq \int_{T_i}^{T_f} Q_n(z, u)[r(u) - s(u, \hat{\mathbf{a}}_{\text{map}})]\, du, \qquad T_i \leq z \leq T_f. \tag{465}$$

4.7 SUMMARY AND OMISSIONS

4.7.1 Summary

In this chapter we have covered a wide range of problems. The central theme that related them was an additive Gaussian noise component. Using this theme as a starting point, we examined different types of problems and studied their solutions and the implications of these solutions. It turned out that the formal solution was the easiest part of the problem and that investigating the implications consumed most of our efforts. It is worthwhile to summarize some of the more general results.

The simplest detection problem was binary detection in the presence of white Gaussian noise. The optimum receiver could be realized as a matched filter or a correlation receiver. The performance depended only on the normalized distance between the two signal points in the decision space. This distance was characterized by the signal energies, their correlation coefficient, and the spectral height of the additive noise. For equal energy signals, a correlation coefficient of -1 was optimum. In all cases the signal shape was unimportant. The performance was insensitive to the detailed assumptions of the model.

The solution for the M signal problem followed easily. The receiver structure consisted of at most $M - 1$ matched filters or correlators. Except for a few special cases, performance calculations for arbitrary cost assignments and a priori probabilities were unwieldy. Therefore we devoted our attention to minimum probability of error decisions. For *arbitrary* signal sets the calculation of the probability of error was still tedious. For orthogonal and nonorthogonal equally-correlated signals simple expressions could be found and evaluated numerically. Simple bounds on the error probability were derived that were useful for certain ranges of parameter values. The question of the optimum signal set was discussed briefly in the text and in more detail in the problems. We found that for large M, orthogonal signals were essentially optimum.

The simple detection problem was then generalized by allowing a nonwhite additive Gaussian noise component. This generalization also included known linear channels. The formal extension by means of the whitening approach or a suitable set of observable coordinates was easy. As we examined the result, some issues developed that we had not encountered before. By including a nonzero white noise component we guaranteed that the matched filter would have a square-integrable impulse response and that perfect (or singular) detection would be impossible. The resulting test was stable, but its sensitivity depended on the white noise level. In the presence of a white noise component the performance could always be improved by extending the observation interval. In radar this was easy because of the relatively long time between successive pulses. Next we studied the effect of removing the white noise component. We saw that unless we put additional "smoothness" restrictions on the signal shape our mathematical model could lead us to singular and/or unstable tests.

The next degree of generalization was to allow for uncertainties in the signal even in the absence of noise. For the case in which these uncertainties could be parameterized by random variables with known densities, the desired procedure was clear. We considered in detail the random phase case and the random amplitude and phase case. In the random phase problem, we introduced the idea of a simple estimation system that measured the phase angle and used the measurement in the detector. This gave us a method of transition from the known signal case to situations, such as the radar problem, in which the phase is uniformly distributed. For binary signals we found that the optimum signal set depended on the quality of the phase measurement. As we expected, the optimum correlation coefficient ranged from $\rho = -1$ for perfect measurement to $\rho = 0$ for the uniform density.

The random amplitude and phase case enabled us to model a number of communication links that exhibited Rayleigh and Rician fading. Here we examined no-measurement receivers and perfect measurement receivers. We found that perfect measurement offered a 6-db improvement. However, even with perfect measurement, the channel fading caused the error probability to decrease linearly with $\bar{E}_r/N_0$ instead of exponentially as in a nonfading channel.

We next considered the problem of multiple channel systems. The vector Karhunen-Loève expansion enabled us to derive the likelihood ratio test easily. Except for a simple example, we postponed our discussion of vector systems to later chapters.

The *basic* ideas in the *estimation* problem were similar, and the entire formulation up through the likelihood function was identical. For linear estimation, the resulting receiver structures were identical to those obtained in the simple binary problem. The mean-square estimation error in white noise depended only on E/N_0.

The nonlinear estimation problem gave rise to a number of issues. The first difficulty was that a sufficient statistic did not exist, which meant that the mapping from the observation space to the estimation space depended on the parameter we were trying to estimate. In some cases this could be accommodated easily. In others approximate techniques were necessary. The resulting function in the estimation space had a number of local maxima and we had to choose the absolute maximum. Given that we were near the correct maximum, the mean-square error could be computed easily. The error could be reduced significantly over the linear estimation error by choosing a suitable signaling scheme. If we tried to reduce the error too far, however, a new phenomenon developed, which we termed threshold. In the cascade approximation to the optimum estimator the physical mechanism for the occurrence of a threshold was clear. The first stage chose the wrong interval in which to make its local estimate. In the continuous realization (such as range estimation) the occurrence was clear but a quantitative description was more difficult. Because the actual threshold level will depend on the signal structure, the quantitative results for the particular example discussed are less important than the realization that whenever we obtain an error decrease without an increase in signal energy or a decrease in noise level a threshold effect will occur at some signal-to-noise level.

The final problem of interest was multiple-parameter estimation. This served both to complete our discussion and as a starting point for the problem of waveform estimation. Here the useful results were relations that showed how estimation errors were coupled by the signal structure and the a priori densities.

In addition to summarizing what we have covered, it is equally important to point out some related issues that we have not.

4.7.2 TOPICS OMITTED

Digital Communications. We have done a great deal of the groundwork necessary for the study of modern digital systems. Except for a few cases, however, we have considered only single-digit transmission. (This is frequently referred to as the one-shot problem in the literature.) From the simple example in Section 4.2 it is clear that performance can be improved by transmitting and detecting blocks of digits. The study of efficient methods is one of the central problems of coding theory. Suitable references are given in [66] and [18]. This comment does not imply that all digital communication systems should employ coding, but it does imply that coding should always be considered as one of the possible alternatives in the over-all system design.

Non-Gaussian Interference. It is clear that in many applications the prime source of interference is non-Gaussian. Simple examples are man-made interference at lower frequencies, impulse noise, and galactic, solar, and atmospheric noise.

Our reason for the omission of non-Gaussian interferences is not because of a lack of interest in or appreciation of their importance. Neither is it because of our inability to solve a *particular* non-Gaussian problem. It is probable that if we can model or measure the pertinent statistics adequately a close-to-optimum receiver can be derived (e.g., [67], [68]). The reason is that it is too difficult to derive useful but general results.

Our goal with respect to the non-Gaussian problem is modest. First, it is to leave the user with an awareness that in any given situation we must verify that the Gaussian model is either valid or an adequate approximation to obtain useful results. Second, if the Gaussian model does not hold, we should be willing to try to solve the actual problem (even approximately) and not to retain the Gaussian solution because of its neatness.

In this chapter we have developed solutions for the problems of detection and finite parameter estimation. We now turn to waveform estimation.

4.8 PROBLEMS

The problems are divided according to sections in the text. Unless otherwise stated, all problems use the model from the corresponding section of the text; for example, the received signals are corrupted by additive zero-mean Gaussian noise which is independent of the hypotheses.

Section P4.2 Additive White Gaussian Noise

BINARY DETECTION

Problem 4.2.1. Derive an expression for the probability of detection P_D, in terms of d and P_F, for the known signal in the additive white Gaussian noise detection problem. [see (37) and (38)].

Problem 4.2.2. In a binary FSK system one of two sinusoids of different frequencies is transmitted; for example,

$$
\begin{aligned}
s_1(t) &= f(t)\cos 2\pi f_c t, && 0 \le t \le T,\\
s_2(t) &= f(t)\cos 2\pi(f_c + \Delta f)t, && 0 \le t \le T,
\end{aligned}
$$

where $f_c \gg 1/T$ and Δf. The correlation coefficient is

$$\rho = \frac{\int_0^T f^2(t)\cos(2\pi \Delta f t)\,dt}{\int_0^T f^2(t)\,dt}.$$

The transmitted signal is corrupted by additive white Gaussian noise ($N_0/2$).

1. Evaluate ρ for a rectangular pulse; that is,

$$
\begin{aligned}
f(t) &= \left(\frac{2E}{T}\right)^{1/2}, && 0 \le t \le T,\\
&= 0, && \text{elsewhere.}
\end{aligned}
$$

Sketch the result as a function of ΔfT.

2. Assume that we require $\Pr(\epsilon) = 0.01$. What value of E/N_0 is necessary to achieve this if $\Delta f = \infty$? Plot the *increase* in E/N_0 over this asymptotic value that is necessary to achieve the same $\Pr(\epsilon)$ as a function of ΔfT.

Problem 4.2.3. The risk involved in an experiment is

$$\mathcal{R} = C_F P_F P_0 + C_M P_M P_1.$$

The applicable ROC is Fig. 2.9. You are given (a) $C_M = 2$; (b) $C_F = 1$; (c) P_1 may vary between 0 and 1. Sketch the line on the ROC that will minimize your maximum possible risk (i.e., assume P_1 is chosen to make $\mathcal{R}$ as *large* as possible. Your line should be a locus of the thresholds that will cause the maximum to be as small as possible).

Problem 4.2.4. Consider the linear feedback system shown below

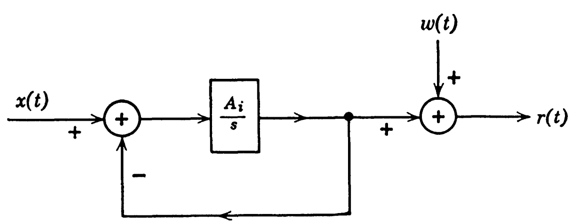

Fig. P4.1

The function $x(t)$ is a known deterministic function that is zero for $t < 0$. Under H_1, $A_i = A_1$. Under H_0, $A_i = A_0$. The noise $w(t)$ is a sample function from a white Gaussian process of spectral height $N_0/2$. We observe $r(t)$ over the interval $(0, T)$. All initial conditions in the feedback system are zero.

1. Find the likelihood ratio test.
2. Find an expression for P_D and P_F for the special case in which $x(t) = \delta(t)$ (an impulse) and $T = \infty$.

Problem 4.2.5. Three commonly used methods for transmitting binary signals over an additive Gaussian noise channel are on-off keying (ASK), frequency-shift keying (FSK), and phase-shift keying (PSK):

$$\begin{aligned} H_0 &: r(t) = s_0(t) + w(t), \qquad 0 \le t \le T, \\ H_1 &: r(t) = s_1(t) + w(t), \qquad 0 \le t \le T, \end{aligned}$$

where $w(t)$ is a sample function from a white Gaussian process of spectral height $N_0/2$. The signals for the three cases are as follows:

	ASK	FSK	PSK
$s_0(t)$	0	$\sqrt{2E/T}\sin\omega_1 t$	$\sqrt{2E/T}\sin\omega_0 t$
$s_1(t)$	$\sqrt{2E/T}\sin\omega_1 t$	$\sqrt{2E/T}\sin\omega_0 t$	$\sqrt{2E/T}\sin(\omega_0 t + \pi)$

,

where $\omega_0 - \omega_1 = 2\pi n/T$ for some nonzero integer n and $w_0 = 2\pi mT$ for some nonzero integer m.

1. Draw appropriate signal spaces for the three techniques.
2. Find d^2 and the resulting probability of error for the three schemes (assume that the two hypotheses are equally likely).
3. Comment on the relative efficiency of the three schemes (a) with regard to utilization of transmitter energy, (b) with regard to ease of implementation.
4. Give an example in which the model of this problem does not accurately describe the actual physical situation.

Problem 4.2.6. Suboptimum Receivers. In this problem we investigate the degradation in performance that results from using a filter other than the optimum receiver filter. A reasonable performance comparison is the increase in transmitted energy required to overcome the decrease in d^2 that results from the mismatching. We would hope that for many practical cases the equipment simplification that results from using other than the matched filter is well worth the required increase in transmitted energy. The system of interest is shown in Fig. P4.2, in which

$$\int_0^T s^2(t)\,dt = 1 \quad \text{and} \quad E[w(t)\,w(\tau)] = \frac{N_0}{2}\,\delta(t-\tau).$$

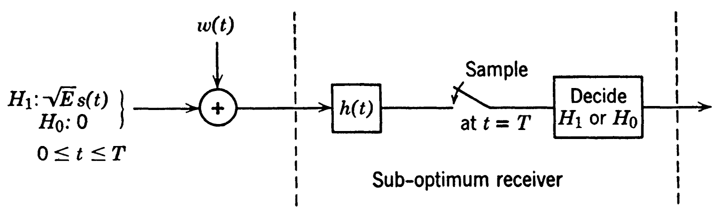

Fig. P4.2

The received waveform is

$$H_1: r(t) = \sqrt{E}\, s(t) + w(t), \qquad -\infty < t < \infty,$$
$$H_0: r(t) = \qquad\qquad w(t), \qquad -\infty < t < \infty.$$

We know that

$$h_{\mathrm{opt}}(t) = \begin{cases} s(T-t), & 0 \le t \le T, \\ 0, & \text{elsewhere,} \end{cases}$$

and

$$d_{\mathrm{opt}}^2 = \frac{2E}{N_0}.$$

Suppose that

$$h(t) = e^{-at}u_{-1}(t), \qquad -\infty < t < \infty,$$

$$s(t) = \begin{cases} \left(\dfrac{1}{T}\right)^{1/2}, & 0 \le t \le T, \\ 0, & \text{elsewhere.} \end{cases}$$

1. Choose the parameter a to maximize the output signal-to-noise ratio d^2.
2. Compute the resulting d^2 and compare with d_{opt}^2. How many decibels must the transmitter energy be increased to obtain the same performance?

M-ARY SIGNALS

Problem 4.2.7. Gram-Schmidt. In this problem we go through the details of the geometric representation of a set of M waveforms in terms of $N (N \le M)$ orthogonal signals.

Consider the M signals $s_1(t), \ldots, s_M(t)$ which are either linearly independent or linearly dependent. If they are linearly dependent, we can write (by definition)

$$\sum_{i=1}^{M} a_i s_i(t) = 0.$$

1. Show that if M signals are linearly dependent, then $s_M(t)$ can be expressed in terms of $s_i(t): i = 1, \ldots, M-1$.
2. Continue this procedure until you obtain N-linearly independent signals and M-N signals expressed in terms of them. N is called the *dimension* of the signal set.
3. Carry out the details of the Gram-Schmidt procedure described on p. 258.

Problem 4.2.8. Translation/Simplex Signals [18]. For maximum a posteriori reception the probability of error is not affected by a linear translation of the signals in the decision space; for example, the two decision spaces in Figs. P4.3*a* and P4.3*b* have the same Pr (ϵ). Clearly, the sets do not require the same energy. Denote the average energy in a signal set as

$$\bar{E} \triangleq \sum_{i=1}^{M} \Pr(H_i)\, |\mathbf{s}_i|^2 = \sum_{i=1}^{M} \Pr(H_i) E_i \int_0^T s_i^2(t)\, dt.$$

1. Find the linear translation that minimizes the average energy of the translated signal set; that is, minimize

$$\bar{E} \triangleq \sum_{i=1}^{M} \Pr(H_i) |\mathbf{s}_i - \mathbf{m}|^2.$$

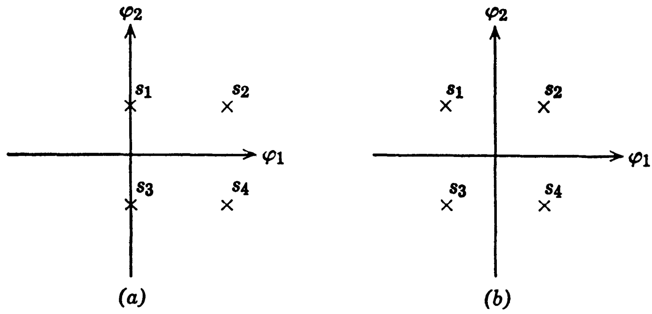

Fig. P4.3

2. Explain the geometric meaning of the result in part 1.
3. Apply the result in part 1 to the case of M orthogonal equal-energy signals representing equally likely hypotheses. The resulting signals are called *Simplex* signals. Sketch the signal vectors for $M = 2, 3, 4$.
4. What is the energy required to transmit each signal in the Simplex set?
5. Discuss the energy reduction obtained in going from the orthogonal set to the Simplex set while keeping the same Pr (ϵ).

Problem 4.2.9. Equally correlated signals. Consider M equally correlated signals

$$E\int_0^T s_i(t)s_j(t)\,dt = \begin{cases} E, & i = j, \\ \rho E, & i \neq j. \end{cases}$$

1. Prove

$$-\frac{1}{M-1} \le \rho \le 1.$$

2. Verify that the left inequality is given by a Simplex set.
3. Prove that an equally-correlated set with energy E has the same Pr (ϵ) as an orthogonal set with energy $E_{\text{orth}} = E(1 - \rho)$.
4. Express the Pr (ϵ) of the Simplex set in terms of the Pr (ϵ) for the orthogonal set and M.

Problem 4.2.10. M Signals, Arbitrary Correlation. Consider an M-ary system used to transmit equally likely messages. The signals have equal energy and may be correlated:

$$\rho_{ij} = \int_0^T s_i(t)\, s_j(t)\, dt, \qquad i, j = 1, 2, \ldots, M.$$

The channel adds white Gaussian noise with spectral height $N_0/2$. Thus

$$r(t) = \sqrt{E}\, s_i(t) + w(t), \qquad 0 \le t \le T : H_i, \qquad i = 1, \ldots, M.$$

1. Draw a block diagram of an optimum receiver containing M matched filters. What is the minimum number of matched filters that can be used?
2. Let $\boldsymbol{\rho}$ be the signal correlation matrix. The ij element is ρ_{ij}. If $\boldsymbol{\rho}$ is nonsingular, what is the dimension of the signal space?
3. Find an expression for Pr $(\epsilon|H_1)$, the probability of error, assuming H_1 is true. Assume that $\boldsymbol{\rho}$ is nonsingular.
4. Find an expression for Pr (ϵ).
5. Is this error expression valid for Simplex signals? (Is $\boldsymbol{\rho}$ singular?)

Problem 4.2.11 (continuation). Error Probability [69]. In this problem we derive an alternate expression for the Pr (ϵ) for the system in Problem 4.2.10. The desired expression is

$$1 - \Pr(\epsilon) = \frac{1}{M}\exp\left(-\frac{E}{N_0}\right)\int_{-\infty}^{\infty}\exp\left[\left(\frac{2E}{N_0}\right)^{1/2}x\right] \times \left[\frac{d}{dx}\int_{-\infty}^{x}\cdots\int_{-\infty}^{x}\frac{\exp\left(-\frac{1}{2}y^T\rho^{-1}y\right)}{(2\pi)^{M/2}|\rho|^{1/2}}\,dy\right]dx. \quad \text{(P.1)}$$

Develop the following steps:

1. Rewrite the receiver in terms of M orthonormal functions $\phi_i(t)$. Define

$$s_i(t) = \sum_{k=1}^{M} s_{ik}\,\phi_k(t), \qquad i = 1, 2, \ldots, M,$$

$$r(t) = \sum_{k=1}^{\infty} r_k\,\phi_k(t).$$

Verify that the optimum receiver forms the statistics

$$l_i = \int_0^T r(t)\,s_i(t)\,dt = \sum_{k=1}^{M} s_{ik}R_k$$

and chooses the greatest.

2. Assume that $s_m(t)$ is transmitted. Show

$$\Pr(\bar{\epsilon}|m) \triangleq \Pr(\mathbf{R} \text{ in } Z_m) = \Pr\left(\sum_{k=1}^{M} s_{mk}R_k = \max_j \sum_{k=1}^{M} s_{jk}R_k\right).$$

3. Verify that

$$\Pr(\bar{\epsilon}) = \frac{1}{M}\exp\left(-\frac{E}{N_0}\right)\sum_{m=1}^{M}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{\exp\left[-(1/N_0)\sum_{k=1}^{M}R_k^2\right]}{(\pi N_0)^{M/2}} \times \exp\left(\frac{2}{N_0}\max_j \sum_{k=1}^{M} R_k s_{jk}\right). \quad \text{(P.2)}$$

4. Define

$$f(\mathbf{R}) = \exp\left\{\max_j\left[\left(\frac{2}{EN_0}\right)^{1/2}\sum_{k=1}^{M} s_{jk}R_k\right]\right\}$$

and observe that (P.2) can be viewed as the expectation of $f(\mathbf{R})$ over a set of statistically independent zero-mean Gaussian variables, R_k, with variance $N_0/2$. To evaluate this expectation, define

$$z_j \triangleq \left(\frac{2}{EN_0}\right)^{1/2}\sum_{k=1}^{M} s_{jk}R_k, \qquad j = 1, 2, \ldots, M,$$

and

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_M \end{bmatrix}.$$

Find $p_z(z)$. Define

$$x = \max_j z_j.$$

Find $p_x(x)$.

5. Using the results in (4), we have

$$1 - \Pr(\epsilon) = \frac{1}{M}\exp\left(-\frac{E}{N_0}\right)\int_{-\infty}^{\infty}\exp\left[\left(\frac{2E}{N_0}\right)^{1/2} X\right]p_x(X)\,dX.$$

Use $p_x(X)$ from (4) to obtain the desired result.

Problem 4.2.12 (continuation).

1. Using the expression in (P.1) of Problem 4.2.11, show that $\partial \Pr(\epsilon)/\partial\rho_{12} > 0$. Does your derivation still hold if $1 \to i$ and $2 \to j$?
2. Use the results of part 1 and Problem 4.2.9 to develop an intuitive argument that the Simplex set is locally optimum.

Comment. The proof of local optimality is contained in [70]. The proof of global optimality is contained in [71].

Problem 4.2.13. Consider the system in Problem 4.2.10. Define

$$\rho_{\max} = \max_{i \neq j} \rho_{ij}.$$

1. Prove that $\Pr(\epsilon)$ on any signal set is less than the $\Pr(\epsilon)$ for a set of equally correlated signals with correlation equal to $\rho_{\max}$.
2. Express this in terms of the error probability for a set of orthogonal signals.
3. Show that the $\Pr(\epsilon)$ is upper bounded by

$$\Pr(\epsilon) \leq (M-1)\left\{\operatorname{erfc}_*\left(\left[\frac{E}{N_0}(1-\rho_{\max})\right]^{1/2}\right\}.$$

Problem 4.2.14 [72]. Consider the system in Problem 4.2.10. Define

d_i: distance between the ith message point and the nearest neighbor.

Observe

$$d_i = \min_j 2\sqrt{(1-\rho_{ij})E/N_0}$$

$$\bar{d} = \frac{1}{M}\sum_{i=1}^{M} d_i,$$

$$d_{\min} = \min_i d_i.$$

Prove

$$\operatorname{erfc}_*(\bar{d}) \leq \Pr(\epsilon) \leq (M-1)\operatorname{erfc}_*(d_{\min})$$

Note that this result extends to signals with unequal energies in an obvious manner.

Problem 4.2.15. In (68) of the text we used the limit

$$\lim_{M\to\infty}\frac{\ln \operatorname{erf}_*\left[y + \left(\dfrac{2PT\log_2 M}{N_0}\right)^{1/2}\right]}{1/(M-1)}.$$

Use l'Hospital's rule to verify the limits asserted in (69) and (70).

Problem 4.2.16. The error probability in (66) is the probability of error in deciding which signal was sent. Each signal corresponds to a sequence of digits; for example, if $M = 8$,

$$\begin{array}{ll} 000 \to s_0(t) & 100 \to s_4(t) \\ 001 \to s_1(t) & 101 \to s_5(t) \\ 010 \to s_2(t) & 110 \to s_6(t) \\ 011 \to s_3(t) & 111 \to s_7(t). \end{array}$$

Therefore an error in the signal decision does not necessarily mean that all digits will be in error. Frequently the digit (or bit) error probability [$\Pr_B(\epsilon)$] is the error of interest.

1. Verify that if an error is made any of the other $M - 1$ signals are equally likely to be chosen.
2. Verify that the expected number of bits in error, given a signal error is made, is

$$\left[\frac{\sum_{i=1}^{\log_2 M} i\binom{\log_2 M}{i}}{\sum_{i=1}^{\log_2 M} \binom{\log_2 M}{i}}\right] = \frac{(\log_2 M)M}{2(M-1)}.$$

3. Verify that the bit error probability is

$$\Pr_B(\epsilon) = \frac{M}{2(M-1)} \Pr(\epsilon).$$

4. Sketch the behavior of the bit error probability for $M = 2$, 4, and 8 (use Fig. 4.25).

Problem 4.2.17. Bi-orthogonal Signals. Prove that for a set of M bi-orthogonal signals with energy E and equally likely hypotheses the $\Pr(\epsilon)$ is

$$\Pr(\epsilon) = 1 - \int_0^\infty \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{1}{N_0}(x - \sqrt{E})^2\right]\left[\int_{-x}^{x} \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{y^2}{N_0}\right) dy\right]^{M/2-1} dx.$$

Verify that this $\Pr(\epsilon)$ approaches the error probability for orthogonal signals for large M and d^2. What is the advantage of the bi-orthogonal set?

Problem 4.2.18. Consider the following digital communication system. There are four equally probable hypotheses. The signals transmitted under the hypotheses are

$$\begin{aligned}
&H_1: \left(\frac{2}{T}\right)^{1/2} A \sin \omega_c t, && 0 \le t \le T, \\
&H_2: \frac{1}{3}\left(\frac{2}{T}\right)^{1/2} A \sin \omega_c t, && 0 \le t \le T, \qquad \omega_c = \frac{2\pi n}{T} \\
&H_3: -\frac{1}{3}\left(\frac{2}{T}\right)^{1/2} A \sin \omega_c t, && 0 \le t \le T, \\
&H_4: -\left(\frac{2}{T}\right)^{1/2} A \sin \omega_c t, && 0 \le t \le T.
\end{aligned}$$

The signal is corrupted by additive Gaussian white noise $w(t)$, $(N_0/2)$.

1. Draw a block diagram of the minimum probability of error receiver and the decision space and *compute* the resulting probability of error.
2. How does the probability of error behave for large A^2/N_0?

Problem 4.2.19. M-ary ASK [72]. An ASK system is used to transmit equally likely messages

$$s_i(t) = \sqrt{E_i}\, \phi(t), \qquad i = 1, 2, \ldots, M,$$

where

$$\sqrt{E_i} = (i-1)\Delta, \qquad \int_0^T \phi^2(t)\, dt = 1.$$

The received signal under the ith hypothesis is

$$r(t) = s_i(t) + w(t), \qquad 0 \le t \le T: H_i, \qquad i = 1, 2, \ldots, M,$$

where $w(t)$ is a white noise with spectral height $N_0/2$.

1. Draw a block diagram of the optimum receiver.
2. Draw the decision space and compute the Pr (ϵ).
3. What is the average transmitted energy?

Note. $\sum_{j=1}^{n-1} j^2 = \dfrac{(n-1)\,n(2n-1)}{6}$.

4. What translation of the signal set in the decision space would maintain the Pr (ϵ) while minimizing the average transmitted energy?

Problem 4.2.20 (continuation). Use the sequence transmission model on pp. 264–265 with the ASK system in part 4 of Problem 4.2.19. Consider specifically the case in which $M = 4$. How should the digit sequence be mapped into signals to minimize the bit error probability? Compute the signal error probability and the bit error probability.

Problem 4.2.21. M-ary PSK [72]. A communication system transmitter sends one of M messages over an additive white Gaussian noise channel (spectral height $N_0/2$) using the signals

$$s_i(t) = \begin{cases} \left(\dfrac{2E}{T}\right)^{1/2} \cos\left(2\pi \dfrac{n}{T} t + \dfrac{2\pi i}{M}\right), & 0 \le t \le T, \\ 0, & \text{elsewhere}, \qquad i = 0, 1, 2, \ldots, M-1, \end{cases}$$

where n is an integer. The messages are equally likely. This type of system is called an M-ary phase-shift-keyed (PSK) system.

1. Draw a block diagram of the optimum receiver. Use the minimum number of filters.
2. Draw the decision-space and decision lines for various M.
3. Prove

$$\alpha \le \Pr(\epsilon) \le 2\alpha,$$

where

$$\alpha = \operatorname{erfc}_*\left(\left(\frac{2E}{N_0}\right)^{1/2} \sin\frac{\pi}{M}\right).$$

Problem 4.2.22 (continuation). Optimum PSK [73]. The basic system is shown in Fig. 4.24. The possible signaling strategies are the following:

1. Use a binary PSK set with the energy in each signal equal to PT.
2. Use an M-ary PSK set with the energy in each signal equal to PT $\log_2$ M.

Discuss how you would choose M to minimize the digit error probability. Compare bi-phase and four phase PSK on this basis.

Problem 4.2.23 (continuation). In the context of an M-ary PSK system discuss qualitatively the effect of an incorrect phase reference. In other words, the nominal signal set is given in Problem 4.2.22 and the receiver is designed on that basis. The actual signal set, however, is

$$s_i(t) = \begin{cases} \left(\frac{2E}{T}\right)^{1/2} \cos\left(\frac{2\pi n}{T}t + \frac{2\pi i}{M} + \theta\right), & 0 \le t \le T, \quad i = 1, 2, \dots, M, \\ & n \text{ is an integer}, \\ 0, & \text{elsewhere}, \end{cases}$$

where θ is a random phase angle. How does the importance of a phase error change as M increases?

ESTIMATION

Problem 4.2.24. Bhattacharyya Bound. Let

$$r(t) = s(t, A) + w(t), \qquad 0 \le t \le T,$$

where $s(t, A)$ is differentiable k times with respect to A. The noise has spectral height $N_0/2$.

1. Extend the Bhattacharyya bound technique developed in Problem 2.4.23 to the waveform for the $n = 2$ case. Assume that A is nonrandom variable.
2. Repeat for the case in which A is a Gaussian random variable; $N(0, \sigma_a)$.
3. Extend the results in parts 1 and 2 to the case in which $n = 3$.

Problem 4.2.25. Consider the problem in Example 1 on p. 276. In addition to the unknown time of arrival, the pulse has an unknown amplitude. Thus

$$r(t) = b\, s(t - a) + w(t), \qquad -T \le t \le T,$$

where a is a uniformly distributed random variable (see Fig. 4.29*b*) and b is Gaussian, $N(0, \sigma_b)$.

Draw a block diagram of a receiver to generate the joint MAP estimates, $\hat{a}_{map}$ and $\hat{b}_{map}$.

Problem 4.2.26. The known signal $s(t)$, $0 \le t \le T$, is transmitted over a channel with unknown *nonnegative* gain A and additive Gaussian noise $n(t)$:

$$\int_0^T s^2(t)\, dt = E,$$

$$K_n(t, \tau) = \frac{N_0}{2}\,\delta(t - \tau).$$

1. What is the maximum likelihood estimate of A?
2. What is the bias in the estimate?
3. Is the estimate asymptotically unbiased?

Problem 4.2.27. Consider the stationary Poisson random process $x(t)$. A typical sample function is shown in Fig. P4.4.

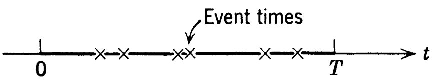

Fig. P4.4

The probability of n events in any interval τ is

$$\Pr(n, \tau) = \frac{(k\tau)^n}{n!} e^{-k\tau}.$$

The parameter k of the process is an unknown nonrandom variable which we want to estimate. We observe $x(t)$ over an interval $(0, T)$.

1. Is it necessary to record the event times or is it adequate to count the number of events that occur in the interval? Prove that n^*, the number of events that occur in the interval $(0, T)$ is a sufficient statistic.
2. Find the Cramér-Rao inequality for any unbiased estimate of k.
3. Find the maximum-likelihood estimate of k. Call this estimate $\hat{k}$.
4. Prove that $\hat{k}$ is unbiased.
5. Find

$$\operatorname{Var}(\hat{k} - k).$$

6. Is the maximum-likelihood estimate efficient?

Problem 4.2.28. When a signal is transmitted through a particular medium, the amplitude of the output is inversely proportional to the murkiness of the medium. Before observation the output of the medium is corrupted by additive, white Gaussian noise. (Spectral height $N_0/2$, double-sided.) Thus

$$r(t) = \frac{1}{M} f(t) + w(t), \qquad 0 \le t \le T,$$

where $f(t)$ is a known signal and

$$\int_0^T f^2(t)\, dt = E.$$

We want to design an optimum Murky-Meter.

1. Assume that M is a nonrandom variable. *Derive* the block diagram of a system whose output is the maximum-likelihood estimate of M (denoted by $\hat{m}_{ml}$).
2. Now assume that M is a Gaussian random variable with zero mean and variance σ_M^2. Find the *equation* that specifies the maximum a posteriori estimate of M (denoted by $\hat{m}_{\text{map}}$).
3. Show that

$$\hat{m}_{\text{map}} \to \hat{m}_{ml}$$

as

$$\sigma_M^2 \to \infty.$$

Section 4.3 Nonwhite Additive Gaussian Noise

MATHEMATICAL PRELIMINARIES

Problem 4.3.1. Reversibility. Prove that $h_w(t, u)$ [defined in (157)] is a reversible operation by demonstrating an $h_w^{-1}(t, u)$ such that

$$\int_{T_i}^{T_f} h_w(t, u)\, h_w^{-1}(u, z)\, du = \delta(t - z).$$

What restrictions on the noise are needed?

Problem 4.3.2. We saw in (163) that the integral equation

$$\frac{N_0}{2} h_o(z, v) + \int_{T_i}^{T_f} h_o(v, x) K_c(x, z)\, dx = K_c(z, v), \qquad T_i \le z, v \le T_f,$$

specifies the inverse kernel

$$Q_n(t, \tau) = \frac{2}{N_0} [\delta(t - \tau) - h_o(t, \tau)].$$

Show that an equivalent equation is

$$\frac{N_0}{2} h_o(z, v) + \int_{T_i}^{T_f} h_o(z, v) K_c(x, v)\, dx = K_c(z, v), \qquad T_i \le z, v \le T_f.$$

Problem 4.3.3 [74] We saw in Problem 4.3.2 that the inverse kernel $Q_n(t, \tau)$ can be obtained from the solution to an integral equation:

$$\frac{N_0}{2} h_o(t, \tau) + \int_{T_i}^{T_f} h_o(t, u) K_c(u, \tau)\, du = K_c(t, \tau), \qquad T_i \le t, \tau \le T_f,$$

where

$$Q_n(t, \tau) = \frac{2}{N_0} [\delta(t - \tau) - h_o(t, \tau)].$$

Suppose we let T_f, the end point of the interval, be a variable. We indicate this by writing $h_o(t, \tau{:}T_f)$ instead of $h_o(t, \tau)$:

$$\frac{N_0}{2} h_o(t, \tau{:}T_f) + \int_{T_i}^{T_f} h_o(t, u{:}T_f) K_c(u, \tau)\, du = K_c(t, \tau), \qquad T_i \le t, \tau \le T_f.$$

Now differentiate this equation with respect to T_f and show that

$$\frac{\partial h_o(t, \tau{:}T_f)}{\partial T_f} = -h_o(t, T_f{:}T_f)\, h_o(T_f, \tau{:}T_f).$$

Hint.

$$\int_{T_i}^{T_f} f(\tau) K_c(t, \tau) d\tau = \lambda f(t), \qquad T_i \le t \le T_f$$

has no solution for $\lambda < 0$.

Problem 4.3.4. Realizable Whitening Filters [91] In the text, two equivalent realizations of the optimum receiver for the colored noise problem were given in Figs. 4.38*a* and *b*. We also saw that $Q_n(t, u)$ was an unrealizable filter specified by (162) and (163). Furthermore, we found one solution for $h_w(t, \tau)$, the whitening filter, in terms of eigenfunctions that was an unrealizable filter. We want to investigate the possibility of finding a *realizable* whitening filter. Recall that we were able to do so in the simple example on p. 311.

1. Write down the log-likelihood ratio in terms of $h_o(t, \tau) = h_o(t, \tau{:}T_f)$ (see Problem 4.3.3).
2. Write

$$\ln \Lambda(r(t)) = \int_{T_i}^{T_f} dt \left[\int_{T_i}^{T_f} h_{wr}(t, u) \sqrt{E}\, s(u)\, du\right]\left[\int_{T_i}^{T_f} h_{wr}(t, z)\, r(z)\, dz\right] \triangleq L(T_f) = \int_{T_i}^{T_f} \frac{\partial L(t)}{\partial t}\, dt.$$

The additional subscript r denotes realizable.

3. Use the result from Problem 4.3.3 that

$$\frac{\partial h_o(u, v:t)}{\partial t} = -h_o(t, u:t)\, h_o(t, v:t)$$

to show that

$$h_{wr}(t, u) = \left(\frac{2}{N_0}\right)^{1/2} [\delta(t - u) - h_o(t, u:t)].$$

Observe that $h_o(t, u:t)$ is a *realizable* filter.

4. Write down the integral equation satisfied by $h_o(t, \tau:t)$. In Chapter 6 we discuss techniques for solving this equation.

Problem 4.3.5. M-ary Signals, Colored Noise. Let the received signal on the ith hypothesis be

$$r(t) = \sqrt{E_i}\, s_i(t) + n_c(t) + w(t), \qquad T_i \le t \le T_f : H_i, \qquad i = 1, 2, \dots, M,$$

where $w(t)$ is zero-mean white Gaussian noise with spectral height $N_0/2$ and $n_c(t)$ is independent zero-mean colored noise with covariance function $K_c(t, u)$. The signals $s_i(t)$ are normalized over $(0, T)$ and are zero outside that interval. Assume that the hypotheses are equally likely and that the criterion is minimum Pr (ϵ). Draw a block diagram of the optimum receiver.

Estimation

Problem 4.3.6. Consider the following estimation problem:

$$r(t) = A\, s(t) + \sum_{i=1}^{3} b_i s_i(t) + w(t), \qquad 0 \le t \le T,$$

where A is a nonrandom variable, b_i are independent, zero-mean, Gaussian random variables $[E(b_i^2) = \sigma_i^2]$, $w(t)$ is white noise $(N_0/2)$, $s(t) = \sum_{i=1}^{3} c_i s_i(t)$,

$$\int_0^T s_i(t)\, s_j(t)\, dt = \delta_{ij}, \quad \text{and} \quad \int_0^T = s^2(t)\, dt = 1.$$

1. Draw a block diagram of the maximum-likelihood estimator of A, $\hat{a}_{ml}$.
2. Choose c_1, c_2, c_3 to minimize the variance of the estimate.

Integral Equation Solutions

Problem 4.3.7. In this problem we solve a simple Fredholm equation of the second kind,

$$\sqrt{E}\, s(t) = \frac{N_0}{2} g(t) + \int_{T_i}^{T_f} K_c(t, u)\, g(u)\, du, \qquad T_i \le t \le T_f,$$

where

$$K_c(t, u) = \sigma_c^2 \exp[-k|t - u|],$$

$$s(t) = \frac{1}{\sqrt{T}}, \qquad 0 \le t \le T,$$

$$T_i = 0,$$
$$T_f = T.$$

1. Find $g(t)$.
2. Evaluate the performance index d^2.

Problem 4.3.8 (continuation). Solve Problem 4.3.7 for the case in which $T_i = -\infty$ and $T_f = \infty$. Compare the value of d^2 that you obtain with the value obtained in that problem.

Problem 4.3.9. Solve the Fredholm equation of the first kind,

$$\int_0^T K(t, u)\, g(u) = s(t), \qquad 0 \le t \le T,$$

for the triangular kernel

$$K(t, u) = \begin{cases} 1 - |t - u| & \text{for } |t - u| < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Assume that $s(t)$ is twice differentiable and that $T < 1$.

Now apply this result to the problem of detecting a known signal $s(t)$, $0 \le t \le T$, which is observed in additive Gaussian noise with covariance

$$K_n(t, u) = \begin{cases} 1 - |t - u| & \text{for } |t - u| \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

1. What is the optimum receiver? Note that we cannot physically generate impulses so that correlation with $g(t)$ is not a satisfactory answer.
2. Calculate d^2.

What is a necessary and sufficient condition for singular detection in this problem if $s(t)$ is bounded?

Problem 4.3.10.

1. Evaluate d^2 for the example given on p. 318.
2. Provided that $s(t)$ is bounded and has finite energy, what is a necessary and sufficient condition on $s(t)$ for a nonsingular test?

Problem 4.3.11 (continuation). The optimum receiver for Problem 4.3.10 includes a matched filter plus a sampler. Find d^2 for the suboptimum receiver that has the matched filter but not the sampler.

Problem 4.3.12. The opposition is using a binary communication system to transmit data. The two signals used are the following:

$$s_1(t) = \sin^2 \frac{2\pi}{T} t, \qquad 0 \le t \le T,$$

$$s_0(t) = -\sin^2 \frac{2\pi}{T} t, \qquad 0 \le t \le T.$$

The received signal is either

$$\begin{aligned} H_1: r(t) &= s_1(t) + n(t), \qquad 0 \le t \le T, \\ H_0: r(t) &= s_0(t) + n(t), \qquad 0 \le t \le T, \end{aligned}$$

where $n(t)$ is a sample function from a zero-mean Gaussian random process with covariance function

$$K_n(\tau) = e^{-\alpha|\tau|}.$$

Assume that he knows α and builds a min Pr (ϵ) receiver. Choose α to minimize his performance.

SENSITIVITY AND SINGULARITY

Problem 4.3.13. Singularity [2]. Consider the simple binary detection problem shown in Fig. P4.5. On H_1 the transmitted signal is a finite energy signal $x(t)$. On H_0 there is no transmitted signal. The additive noise $w(t)$ is a sample function from a white process $1\ v^2/\text{cps}$). The received waveform $r(t)$ is passed through a filter whose transfer function is $H(j\omega)$. The output $y(t)$, $0 \le t \le T$ is the signal available for processing. Let λ_k and $\phi_k(t)$ be the eigenvalues and eigenfunctions, respectively, of $n(t)$, $0 \le t \le T$. To have singular detection, we require

$$\sum_{k=1}^{\infty} \frac{s_k^2}{\lambda_k} = \infty.$$

We want to prove that this cannot happen in this case.

1. From

$$s_k = \int_0^T \phi_k(t)\, s(t)\, dt$$

show that

$$s_k = \int_{-\infty}^{\infty} X(f)\, H(j2\pi f)\, \Phi_k^*(f)\, df,$$

where

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft}\, dt$$

and

$$\Phi_k(f) = \int_{-\infty}^{\infty} \phi_k(t) e^{-j2\pi ft}\, dt = \int_0^T \phi_k(t) e^{-j2\pi ft}\, dt.$$

2. Show that

$$\int_{-\infty}^{\infty} [H^*(j2\pi f)\, \Phi_m(f)][H(j2\pi f)\, \Phi_k^*(f)]\, df = \begin{cases} \lambda_k & \text{if } m = k, \\ 0 & \text{if } m \ne k. \end{cases}$$

3. Observe from part 2 that for some set of numbers c_k

$$X(f) = \sum_{k=1}^{\infty} \frac{c_k}{\lambda_k} H^*(j2\pi f)\, \Phi_k(f) + U(f),$$

where

$$\int_{-\infty}^{\infty} U(f)\, H(j2\pi f)\, \Phi_k^*(f)\, df = 0.$$

4. Using (1), (2), and (3), show that

$$\sum_{k=1}^{\infty} \frac{s_k^2}{\lambda_k} \le \int_{-\infty}^{\infty} x^2(t)\, dt,$$

hence that perfect detection is impossible in this situation.

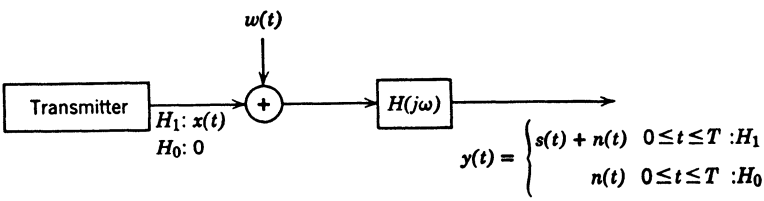

Fig. P4.5

Problem 4.3.14. Consider the following system:

$$H_1: r(t) = s_1(t) + n(t), \qquad 0 \le t \le T,$$
$$H_0: r(t) = n(t), \qquad 0 \le t \le T.$$

It is given that

$$n(t) = \sum_{n=1}^{6} a_n \cos n \frac{2\pi}{T} t,$$

where a_n are zero-mean random variables. The signal energy is

$$\int_0^T s_1^2(t)\, dt = E.$$

Choose $s_1(t)$ and the corresponding receiver so that perfect decisions can be made with probability 1.

Problem 4.3.15. Because white noise is a mathematical fiction (it has infinite energy, which is physically impossible), we sometimes talk about band-limited white noise; that is,

$$S_n(\omega) = \begin{cases} \dfrac{N_0}{2}, & \text{for } \omega_1 \le |\omega| \le \omega_2, \\ 0, & \text{otherwise.} \end{cases}$$

Now suppose that we wish to detect a strictly time-limited signal

$$s(t) = \begin{array}{ll} \left(\dfrac{E}{T}\right)^{1/2}, & 0 \le t \le T, \\ 0, & \text{otherwise.} \end{array}$$

Is this a good mathematical model for a physical problem? Justify your answer.

Problem 4.3.16. Sensitivity to White Noise Level. The received waveforms under the two hypotheses are

$$r(t) = s(t) + n_c(t) + w(t), \qquad -\infty < t < \infty : H_1,$$
$$r(t) = n_c(t) + w(t), \qquad -\infty < t < \infty : H_0.$$

The signal waveform $s(t)$ and the colored noise spectrum $S_n(\omega)$ are known exactly. The white noise level is

$$\frac{N_a}{2} = \frac{N_0}{2}(1 + x),$$

where $N_0/2$ is the nominal value and x is a small variation. Assume that the receiver is designed on the basis of the nominal white noise level.

1. Find an expression for $\left.\dfrac{\partial d/\partial x}{d}\right|_{x=0} = \left.\dfrac{\partial \ln d}{\partial x}\right|_{x=0} \triangleq \Delta.$

2. Assume that

$$s(t) = \begin{array}{ll} \sqrt{2kP}\, e^{-kt}, & t \ge 0, \\ 0, & t < 0 \end{array}$$

and

$$S_{n_c}(\omega) = \frac{2k\sigma_c^2}{\omega^2 + k^2}.$$

Evaluate Δ as a function of $\Lambda \triangleq 4\sigma_c^2/kN_0$.

Problem 4.3.17. Sensitivity to Noise Spectrum. Assume the same nominal model as in Problem 4.3.16.

1. Now let

$$\frac{N_a}{2} = \frac{N_0}{2}$$

and

$$S_{n_c}(\omega) = \frac{2k_a\sigma_a^2}{\omega^2 + k_a^2},$$

where

$$k_a = k\,(1 + y)$$
$$\sigma_a^2 = \sigma_c^2\,(1 + z)$$

Find

$$\left.\frac{\partial d/\partial y}{d}\right|_{\substack{z=0\\y=0}} \triangleq \Delta_y \quad \text{and} \quad \left.\frac{\partial d/\partial z}{d}\right|_{\substack{z=0\\y=0}} \triangleq \Delta_z.$$

2. Evaluate Δ_y and Δ_z for the signal shape in Problem 4.3.16.

Problem 4.3.18. Sensitivity to Delay and Gain. The received waveforms under the two hypotheses are

$$r(t) = \sqrt{E}\,s(t) + b_I s(t - \tau) + w(t), \qquad -\infty < t < \infty : H_1,$$
$$r(t) = b_I s(t - \tau) + w(t), \qquad -\infty < t < \infty : H_0,$$

where b_I is $N(0, \sigma_I)$ and $w(t)$ is white with spectral height $N_0/2$. The signal is

$$s(t) = \left(\frac{1}{T}\right)^{1/2}, \qquad 0 \le t \le T,$$
$$= 0, \qquad \text{elsewhere.}$$

1. Design the optimum receiver, assuming that τ is known.
2. Evaluate d^2 as a function of τ and σ_I.
3. Now assume

$$\tau_a = \tau\,(1 + x).$$

Find an expression for d^2 of the nominal receiver as a function of x. Discuss the implications of your results.

4. Now we want to study the effect of changing σ_I. Let

$$\sigma_{Ia}^2 = \sigma_I^2\,(1 + y)$$

and find an expression for d^2 as a function of y.

Linear Channels

Problem 4.3.19. Optimum Signals. Consider the system shown in Fig. 4.49*a*. Assume that the channel is time-invariant with impulse response $h(\tau)$ [or transfer function $H(f)$]. Let

$$H(f) = 1, \qquad |f| < W,$$
$$= 0, \qquad \text{otherwise.}$$

The output observation interval is infinite. The signal input is $s(t)$, $0 \le t \le T$ and is normalized to have unity energy. The additive white Gaussian noise has spectral height $N_0/2$.

1. Find the optimum receiver.
2. Choose $s(t)$, $0 \le t \le T$, to maximize d^2.