Problem 4.3.20. Repeat Problem 4.3.19 for the $h(t, \tau)$ given below:

$$h(t, \tau) = \delta(t - \tau), \qquad 0 \le \tau \le \frac{T}{4}, \frac{T}{2} \le \tau \le \frac{3T}{4}, \; -\infty < t < \infty,$$
$$= 0, \qquad \text{elsewhere.}$$

Problem 4.3.21. The system of interest is shown in Fig. P4.6. Design an optimum *binary* signaling system subject to the constraints:

1. $\int_0^T s^2(t)\, dt = E_t.$
2. $s(t) = 0, \quad t < 0,$
   $\quad t < T.$
3. $h(\tau) = e^{-k\tau}, \quad \tau \ge 0,$
   $= 0, \quad \tau < 0.$

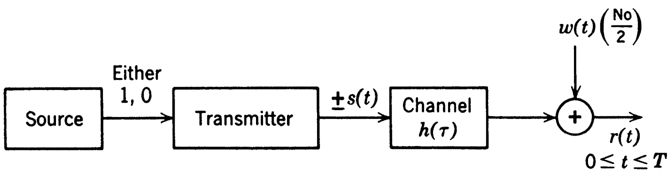


Fig. P4.6

Section P.4.4 Signals with Unwanted Parameters

MATHEMATICAL PRELIMINARIES

Formulas. Some of the problems in this section require the manipulation of Bessel functions and Q functions. A few convenient formulas are listed below. Other relations can be found in [75] and the appendices of [47] and [92]

I. ***Modified Bessel Functions***

$$I_n(z) \triangleq \frac{1}{2\pi}\int_0^{2\pi} \exp(\pm jn\theta)\exp(z\cos\theta)\, d\theta, \tag{F.1.1}$$

$$I_n(z) = I_{-n}(z), \tag{F.1.2}$$

$$I_v(z) \simeq \frac{(\frac{1}{2}z)^v}{\Gamma(v + 1)}, \qquad v \ne -1, -2, \ldots, z \ll 1, \tag{F.1.3}$$

$$I_v(z) \simeq \frac{e^z}{\sqrt{2\pi z}}\left[1 - \frac{4v^2 - 1}{8z}\right], \qquad z \gg 1, \tag{F.1.4}$$

$$\frac{1}{z^k}\frac{d^k}{dz^k}(z^{-v} I_v(z)) = z^{-v-k} I_{v+k}(z), \tag{F.1.5}$$

$$\frac{1}{z^k}\frac{d^k}{dz^k}(z^{v} I_v(z)) = z^{v-k} I_{v-k}(z). \tag{F.1.6}$$

II. ***Marcum's Q-function*** [92]

$$Q(\sqrt{2a}, \sqrt{2b}) = \int_b^\infty \exp(-a + x)\, I_0(2\sqrt{ax})\, dx, \tag{F.2.1}$$

$$Q(a, a) = \tfrac{1}{2}[1 + I_0(a^2)\exp(-a^2)], \tag{F.2.2}$$

$$1 + Q(a, b) - Q(b, a)$$
$$= \frac{b^2 - a^2}{b^2 + a^2} \int_{a^2 + b^2/2}^{\infty} \exp(-x)\, I_0\left(\frac{2abx}{a^2 + b^2}\right) dx, \qquad b > a > 0, \tag{F.2.3}$$

$$\int_0^{\infty} Q\left(\frac{\alpha_2}{\sigma_2}, \frac{R_1}{\sigma_2}\right) \frac{R_1}{\sigma_1^2} \exp\left[-\frac{\alpha_1^2 + R_1^2}{2\sigma_1^2}\right] I_0\left(\frac{\alpha_1 R_1}{\sigma_1^2}\right) dR_1$$
$$= \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}\left[1 - Q\left(\sqrt{\frac{\alpha_1^2}{\sigma_1^2 + \sigma_2^2}}, \sqrt{\frac{\alpha_2^2}{\sigma_1^2 + \sigma_2^2}}\right)\right]$$
$$+ \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} Q\left(\sqrt{\frac{\alpha_2^2}{\sigma_1^2 + \sigma_2^2}}, \sqrt{\frac{\alpha_1^2}{\sigma_1^2 + \sigma_2^2}}\right), \tag{F.2.4}$$

$$Q(a, b) \simeq \operatorname{erfc}_*(b - a), \qquad b \gg 1,\ b \gg b - a. \tag{F.2.5}$$

III. ***Rician variables*** [76]

Consider the two statistically independent Rician variables, x_1 and x_2 with probability densities,

$$p_{x_k}(X_k) = \frac{X_k}{\sigma_k^2} \exp\left(-\frac{a_k^2 + X_k^2}{2\sigma_k^2}\right) I_0\left(\frac{a_k X_k}{\sigma_k^2}\right), \qquad \begin{array}{l} 0 < a_k < \infty, \\ 0 < X_k < \infty, \\ k = 1, 2. \end{array} \tag{F.3.1}$$

The probability of interest is

$$P_* = \Pr[x_2 > x_1].$$

Define the constants

$$a = \frac{a_2^2}{\sigma_1^2 + \sigma_2^2}, \qquad b = \frac{a_1^2}{\sigma_1^2 + \sigma_2^2}, \qquad c = \frac{\sigma_1}{\sigma_2}.$$

Then

$$P_* = Q(\sqrt{a}, \sqrt{b}) - \frac{c^2}{1 + c^2} \exp\left(-\frac{a + b}{2}\right) I_0(\sqrt{ab}), \tag{F.3.2}$$

or

$$P_* = \frac{c^2}{1 + c^2}[1 - Q(\sqrt{b}, \sqrt{a})] + \frac{1}{1 + c^2} Q(\sqrt{a}, \sqrt{b}), \tag{F.3.3}$$

or

$$P_* = \tfrac{1}{2}[1 - Q(\sqrt{b}, \sqrt{a}) + Q(\sqrt{a}, \sqrt{b})] - \frac{1}{2}\frac{c^2 - 1}{c^2 + 1} \exp\left(-\frac{a + b}{2}\right) I_0(\sqrt{ab}). \tag{F.3.4}$$

Problem 4.4.1. Q-function Properties. Marcum's Q-function appears frequently in the calculation of error probabilities:

$$Q(\alpha, \beta) = \int_{\beta}^{\infty} x \exp\left[-\tfrac{1}{2}(x^2 + \alpha^2)\right] I_0(\alpha x)\, dx.$$

Verify the following properties:

1. $Q(\alpha, 0) = 1,$

2. $Q(0, \beta) = e^{-\beta^2/2}.$

3. $$Q(\alpha, \beta) = e^{-(\alpha^2 + \beta^2)/2} \sum_{n=0}^{\infty} \left(\frac{\alpha}{\beta}\right)^n I_n(\alpha\beta), \qquad \alpha < \beta,$$
$$= 1 - e^{-(\alpha^2 + \beta^2)/2} \sum_{n=1}^{\infty} \left(\frac{\beta}{\alpha}\right)^n I_n(\alpha\beta), \qquad \beta < \alpha.$$

4. $Q(\alpha, \beta) + Q(\beta, \alpha) = 1 + (e^{-(\alpha^2+\beta^2)/2})\, I_0(\alpha\beta)$.

5. $Q(\alpha, \beta) \simeq 1 - \dfrac{1}{\alpha - \beta}\left(\dfrac{\beta}{2\pi\alpha}\right)^{1/2}(e^{-(\alpha-\beta)^2/2}), \qquad \alpha \gg \beta \gg 1.$

6. $Q(\alpha, \beta) \simeq \dfrac{1}{\beta - \alpha}\left(\dfrac{\beta}{2\pi\alpha}\right)^{1/2}(e^{-(\beta-\alpha)^2/2}), \qquad \beta \gg \alpha \gg 1.$

Problem 4.4.2. Let x be a Gaussian random variable $N(m_x, \sigma_x)$.

1. Prove that

$$M_{x^2}(jv) \triangleq E[\exp(+jvx^2)] = \frac{\exp[jvm_x^2/(1 - 2jv\sigma_x^2)]}{(1 - 2jv\sigma_x^2)^{1/2}}.$$

Hint.

$$M_{x^2}(jv) = [M_{x^2}(jv)\, M_{y^2}(jv)]^{1/2},$$

where y is an independent Gaussian random variable with identical statistics.

2. Let z be a complex number. Modify the derivation in part 1 to show that

$$E[\exp(+zx^2)] = \frac{\exp[zm_x^2/(1 - 2z\sigma_x^2)]}{(1 - 2z\sigma_x^2)^{1/2}}, \qquad \operatorname{Re}(z) < \frac{1}{2\sigma_x^2}.$$

3. Let

$$y^2 = \sum_{i=1}^{2M} \lambda_i x_i^2,$$

where the x_i are statistically independent Gaussian variables, $N(m_i, \sigma_i)$.

Find $M_{y^2}(jv)$ and $E[\exp(+zy^2)]$. What condition must be imposed on $\operatorname{Re}(z)$ in order for the latter expectation to exist.

4. Consider the special case in which $\lambda_i = 1$ and $\sigma_i^2 = \sigma^2$. Verify that the probability density of y^2 is

$$\begin{aligned} p_{y^2}(Y) &= \frac{1}{2\sigma^2}\left(\frac{Y}{S\sigma^2}\right)^{\frac{M-1}{2}} \exp\left(-\frac{Y + S\sigma^2}{2\sigma^2}\right) I_{M-1}\left[\left(\frac{YS}{\sigma^2}\right)^{1/2}\right], \qquad && Y \geq 0, \\ &= 0, && \text{elsewhere,} \end{aligned}$$

where $S = \sum_{i=1}^{2M} m_i^2$. (See Erdelyi [75], p. 197, eq. 18.)

Problem 4.4.3. Let $Q(\mathbf{x})$ be a quadratic form of correlated Gaussian random variables,

$$Q(\mathbf{x}) \triangleq \mathbf{x}^T\mathbf{A}\mathbf{x}.$$

1. Show that the characteristic function of Q is

$$M_Q(jv) \triangleq E(e^{jvQ}) = \frac{\exp\{-\frac{1}{2}\mathbf{m_x}^T\mathbf{\Lambda}^{-1}[\mathbf{I} - (\mathbf{I} - 2jv\mathbf{\Lambda}\mathbf{A})^{-1}]\mathbf{m_x}\}}{|\mathbf{I} - 2jv\mathbf{\Lambda}\mathbf{A}|^{1/2}}.$$

2. Consider the special case in which $\mathbf{\Lambda}^{-1} = \mathbf{A}$ and $\mathbf{m_x} = \mathbf{0}$. What is the resulting density?

3. Extend the result in part 1 to find $E(e^{zQ})$, where z is a complex number. What restrictions must be put on $\operatorname{Re}(z)$?

Problem 4.4.4. [76] Let x_1, x_2, x_3, x_4 be statistically independent Gaussian random variables with identical variances. Prove

$$\Pr(x_1^2 + x_2^2 \geq x_3^2 + x_4^2) = \tfrac{1}{2}[1 - Q(\beta, \alpha) + Q(\alpha, \beta)],$$

where

$$\alpha = \left(\frac{\overline{x_1}^2 + \overline{x_2}^2}{2\sigma^2}\right)^{1/2}, \quad \beta = \left(\frac{\overline{x_3}^2 + \overline{x_4}^2}{2\sigma^2}\right)^{1/2}.$$

RANDOM PHASE CHANNELS

Problem. 4.4.5. On-Off Signaling: Partially Coherent Channel. Consider the hypothesis testing problem stated in (357) and (358) with the probability density given by (364). From (371) we see that an equivalent test statistic is

$$(\beta + L_c)^2 + L_s^2 \underset{H_0}{\overset{H_1}{\gtrless}} \gamma,$$

where

$$\beta \triangleq \frac{N_0}{2} \frac{\Lambda_m}{\sqrt{E_r}}.$$

1. Express P_F as a Q-function.
2. Express P_D as an integral of a Q-function.

Problem 4.4.6. M-orthogonal Signals: Partially Coherent Channel. Assume that each of the M hypotheses are equally likely. The received signals at the output of a random phase channel are

$$r(t) = \sqrt{2E_r}\, f_i(t) \cos [\omega_c t + \phi_i(t) + \theta] + w(t), \quad 0 \le t \le T : H_i, \quad i = 1, 2, \ldots, M,$$

where $p_\theta(\theta)$ satisfies (364) and $w(t)$ is white with spectral height $N_0/2$. Find the LRT and draw a block diagram of the minimum probability of error receiver.

Problem 4.4.7 (continuation). Error Probability; Uniform Phase. [18] Consider the special case of the above model in which the signals are orthogonal and θ has a uniform density.

1. Show that

$$\Pr(\epsilon|\theta) = 1 - E\left\{\left[1 - \exp\left(-\frac{x^2 + y^2}{2}\right)\right]^{M-1}\right\},$$

where x and y are statistically independent Gaussian random variables with unit variance.

$$E[x|\theta] = \sqrt{2E_r/N_0} \cos \theta,$$
$$E[y|\theta] = \sqrt{2E_r/N_0} \sin \theta.$$

The expectation is over x and y, given θ.

2. Show that

$$\Pr(\epsilon) = \sum_{k=1}^{M-1} \binom{M-1}{k} (-1)^{k+1} \left(\frac{\exp[-(E_r/N_0)k/(k+1)]}{k+1}\right).$$

Problem 4.4.8. In the binary communication problem on pp. 345–348 we assumed that the signals on the two hypotheses were *not* phase-modulated. The general binary problem in white noise is

$$r(t) = \sqrt{2E_r}\, f_1(t) \cos [\omega_c t + \phi_1(t) + \theta] + w(t), \qquad 0 \le t \le T : H_1,$$
$$r(t) = \sqrt{2E_r}\, f_0(t) \cos [\omega_c t + \phi_0(t) + \theta] + w(t), \qquad 0 \le t \le T : H_0,$$

where E_r is the energy received in the signal component. The noise is white with spectral height $N_0/2$, and $p_\theta(\theta)$ satisfies (364). Verify that the optimum receiver structure is as shown in Fig. P4.7 for $i = 0, 1$, and that the minimum probability of error test is

$$z_1^2 \underset{H_0}{\overset{H_1}{\gtrless}} z_0^2.$$

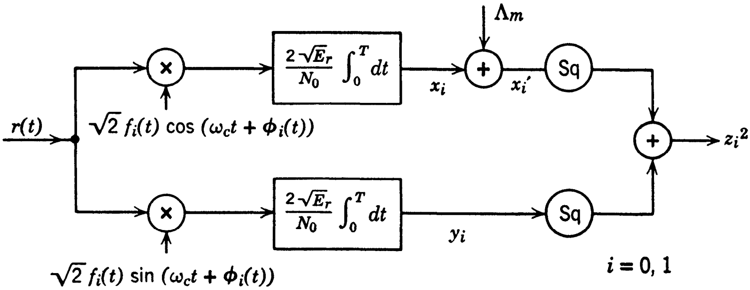


Fig. P4.7

Problem 4.4.9 (continuation) [44]. Assume that the signal components on the two hypotheses are orthogonal.

1. Assuming that H_0 is true, verify that

$$\begin{aligned} E(x_0') &= \Lambda_m + d^2 \cos\theta, \\ E(y_0) &= d^2 \sin\theta, \\ E(x_1') &= \Lambda_m, \\ E(y_1) &= 0, \end{aligned}$$

where $d^2 \triangleq 2E_r/N_0$ and

$$\operatorname{Var}(x_0') = \operatorname{Var}(y_0) = \operatorname{Var}(x_1) = \operatorname{Var}(y_1) = d^2.$$

2. Prove that

$$p_{z_0|H_0,\theta}(Z_0|H_0,\theta) = \frac{Z_0}{d^2}\exp\left(-\frac{Z_0^2 + \Lambda_m^2 + d^4 + 2\Lambda_m d^2\cos\theta}{2d^2}\right) \times \left\{I_0\left(\frac{[(\Lambda_m^2 + d^4 + 2\Lambda_m d^2\cos\theta)^{1/2} Z_0]}{d^2}\right)\right\}$$

and

$$p_{z_1|H_0,\theta}(Z_1|H_0,\theta) = \frac{Z_1}{d^2}\exp\left(-\frac{z_1^2 + \Lambda_m^2}{2d^2}\right) I_0\left(\frac{\Lambda_m}{d^2} Z_1\right).$$

3. Show that

$$\Pr(\epsilon) = \Pr(\epsilon|H_0) = \Pr(z_0 < z_1|H_0) = \int_{-\pi}^{\pi} p_\theta(\theta)\,d\theta \int_0^\infty p_{z_0 H_0,\theta}(Z_0|H_0,\theta)\,dZ_0 \times \int_{z_0}^\infty p_{z_1|H_1,\theta}(Z_1|H_1,\theta)\,dZ_1,$$

4. Prove that the inner two integrals can be rewritten as

$$\Pr(\epsilon|\theta) = Q(a,b) - \tfrac{1}{2}\exp\left(\frac{a^2+b^2}{2}\right) I_0(ab),$$

where

$$a = \frac{\sqrt{2}\,\Lambda_m}{d},$$

$$b = \frac{[2(\Lambda_m^2 + d^4 + 2\Lambda_m d^2\cos\theta)]^{1/2}}{d}.$$

5. Check your result for the two special cases in which $\Lambda_m \to 0$ and $\Lambda_m \to \infty$. Compare the resulting $\Pr(\epsilon)$ for these two cases in the region where d is large.

Problem 4.4.10 (continuation). Error Probability, Binary Nonorthogonal Signals [77]. When bandpass signals are not orthogonal, it is conventional to define their correlation in the following manner:

$$\tilde{f}_1(t) \triangleq f_1(t)e^{j\phi_1(t)}$$

$$\tilde{f}_1(t) \triangleq f_0(t)e^{j\phi_0(t)}$$

$$\tilde{\rho} \triangleq \int_0^T \tilde{f}_0(t)\,\tilde{f}_1^*(t)\,dt,$$

which is a complex number.

1. Express the actual signals in terms of $\tilde{f}_i(t)$.
2. Express the actual correlation coefficient of two signals in terms of $\tilde{\rho}$.
3. Assume $\Lambda_m = 0$ (this corresponds to a uniform density) and define the quantity

$$\lambda = (1 - |\tilde{\rho}|^2)^{1/2}.$$

Show that

$$\Pr(\epsilon) = Q\left(\frac{d}{2}\sqrt{1-\lambda}, \frac{d}{2}\sqrt{1+\lambda}\right) - \tfrac{1}{2}\exp\left(-\frac{d^2}{2}\right) I_0\left(\frac{d^2}{4}|\tilde{\rho}|\right).$$

Problem 4.4.11 (continuation). When $p_\theta(\theta)$ is nonuniform and the signals are nonorthogonal, the calculations are much more tedious. Set up the problem and then refer to [44] for the detailed manipulations.

Problem 4.4.12. M-ary PSK. Consider the M-ary PSK communication system in Problem 4.2.21. Assume that

$$p_\theta(\theta) = \frac{\exp(\Lambda_m \cos\theta)}{2\pi\, I_0(\Lambda_m)}, \qquad -\pi \le \theta \le \pi.$$

1. Find the optimum receiver.
2. Write an expression for the $\Pr(\epsilon)$.

Problem 4.4.13. ASK: Incoherent Channel [72]. An ASK system transmits equally likely messages

$$s_i(t) = \sqrt{2E_i}\, f(t)\cos\omega_c t, \qquad i = 1, 2, \ldots, M, \qquad 0 \le t \le T,$$

where

$$\sqrt{E_i} = (i-1)\,\Delta,$$

$$\int_0^T f^2(t)\,dt = 1,$$

and

$$(M-1)\Delta \triangleq E.$$

The received signal under the ith hypothesis is

$$r(t) = \sqrt{2E_i}\, f(t)\cos(\omega_c t + \theta) + w(t), \qquad 0 \le t \le T : H_i, \qquad i = 1, 2, \ldots, M,$$

where $w(t)$ is white noise $(N_0/2)$. The phase θ is a random variable with a uniform density $(0, 2\pi)$.

1. Find the minimum $\Pr(\epsilon)$ receiver.
2. Draw the decision space and compute the $\Pr(\epsilon)$.

Problem 4.4.14. Asymptotic Behavior of Incoherent M-ary Systems [78]. In the text we saw that the probability of error in a communication system using M orthogonal signals approached zero as $M \to \infty$ as long as the rate in digits per second was less than $P/N_0 \ln 2$ (the channel capacity) (see pp. 264–267). Use exactly the same model as in Example 4 on pp. 264–267. Assume, however, that the channel adds a random phase angle. Prove that exactly the same results hold for this case. (*Comment.* The derivation is somewhat involved. The result is due originally to Turin [78]. A detailed derivation is given in Section 8.10 of [69].)

Problem 4.4.15 [79]. Calculate the moment generating function, mean, and variance of the test statistic $G = L_c^2 + L_s^2$ for the random phase problem of Section 4.4.1 under the hypothesis H_1.

Problem 4.4.16 (continuation) [79]. We can show that for $d \gtrsim 3$ the equivalent test statistic

$$R = \sqrt{L_c^2 + L_s^2}$$

is approximately Gaussian (see Fig. 4.73*a*). Assuming that this is true, find the mean and variance of R.

Problem 4.4.17 (continuation) [79]. Now use the result of Problem 4.4.16 to derive an approximate expression for the probability of detection. Express the result in terms of d and P_F and show that P_D is approximately a straight line when plotted versus d on probability paper. Compare a few points with Fig. 4.59. Evaluate the increase in d over the known signal case that is necessary to achieve the same performance.

Problem 4.4.18. Amplitude Estimation. We consider a simple estimation problem in which an unwanted parameter is present. The received signal is

$$r(t) = A\sqrt{E}\, s(t, \theta) + w(t),$$

where A is a nonrandom parameter we want to estimate (assume that it is nonnegative):

$$s(t, \theta) = f(t) \cos [\omega_c t + \phi(t) + \theta],$$

where $f(t)$ and $\phi(t)$ are slowly varying known functions of time and θ is a random variable whose probability density is,

$$p_\theta(\theta) = \begin{cases} \dfrac{1}{2\pi}, & -\pi < \theta < \pi, \\ 0, & \text{otherwise}, \end{cases}$$

$w(t)$ is white Gaussian noise of spectral height $N_0/2$.

Find the transcendental equation satisfied by the maximum-likelihood estimate of A.

Problem 4.4.19. Frequency Estimation: Random Phase Channel. The received signal is

$$r(t) = \sqrt{2E}\, f(t) \cos (\omega_c t + \phi(t) + \omega t + \theta) + w(t), \qquad 0 \le t \le T,$$

where $\int_0^T f^2(t)\, dt = 1$ and $f(t)$, $\phi(t)$, and E are known. The noise $w(t)$ is a sample function from a white Gaussian noise process $(N_0/2)$. The frequency shift ω is an unknown nonrandom variable.

1. Find $\Lambda[r(t)|\omega]$.
2. Find the likelihood equation.
3. Draw a receiver whose output is a good approximation to $\hat{\omega}_{ml}$.

Problem 4.4.20 (continuation). Estimation Errors.

1. Compute a bound on the variance of any unbiased estimate of ω.
2. Compare the variance of $\hat{\omega}_{ml}$ under the assumption of a small error. Compare the result with the bound in part 1.
3. Compare the result of this problem with example 2 on p. 278.

RANDOM AMPLITUDE AND PHASE

Problem 4.4.21. Consider the detection problem in which

$$r(t) = \sum_{i=1}^{M} a_i s_i(t) + n(t), \qquad 0 \le t \le T : H_1,$$
$$= n(t), \qquad 0 \le t \le T : H_0.$$

The a_i are jointly Gaussian variables which we denote by the vector $\mathbf{a}$. The signals are denoted by the vector $s(t)$.

$$E(\mathbf{a}) \triangleq \mathbf{m_a},$$

$$E[(\mathbf{a} - \mathbf{m_a})(\mathbf{a}^T - \mathbf{m_a}^T)] \triangleq \mathbf{\Lambda}_a,$$

and

$$\boldsymbol{\rho} = \int_0^T \mathbf{s}(t)\, \mathbf{s}^T(t)\, dt,$$

$$E[n(t)\, n(u)] = \frac{N_0}{2}\, \delta(t - u)$$

1. Find the optimum receiver structure. Draw the various interpretations analogous to Figs. 4.66 and 4.67. *Hint.* Find a set of sufficient statistics and use (2.326).
2. Find $\mu(s)$ for this system. (See Section 2.7 of Chapter 2.)

Problem 4.4.22 (continuation). Extend the preceding problem to the case in which

$$E[n(t)\, n(u)] = \frac{N_0}{2}\, \delta(t - u) + K_c(t, u).$$

Problem 4.4.23. Consider the ASK system in Problem 4.4.13, operating over a Rayleigh channel. The received signal under the kth hypothesis is

$$r(t) = \sqrt{2E_k}\, v\, f(t) \cos(\omega_c t + \theta) + w(t), \qquad 0 \le t \le T : H_k, \qquad k = 1, 2, \ldots, M.$$

All quantities are described in Problem 4.4.13 except v:

$$p_v(V) = \begin{cases} V \exp\left(-\dfrac{V^2}{2}\right), & V \ge 0, \\ 0, & \text{elsewhere}, \end{cases}$$

and is independent of H_k. The hypotheses are equally likely.

1. Find the minimum $\Pr(\epsilon)$ receiver.
2. Find the $\Pr(\epsilon)$.

Problem 4.4.24. M-Orthogonal Signals: Rayleigh Channel [80]. One of M-orthogonal signals is used to transmit equally likely hypotheses over a Rayleigh channel. The received signal under the ith hypothesis is

$$r(t) = \sqrt{2}\, v\, f(t) \cos(\omega_i t + \theta) + w(t), \qquad 0 \le t \le T : H_i, \qquad i = 1, 2, \ldots, M,$$

where v is Rayleigh with variance $\bar{E}_r$, θ is uniform, $w(t)$ is white ($N_0/2$), and $f(t)$ is normalized.

1. Draw a block diagram of the minimum probability of error receiver.
2. Show that

$$\Pr(\epsilon) = \sum_{n=1}^{M-1} \binom{M-1}{n} \frac{(-1)^{n+1}}{n+1+n\beta},$$

where

$$\beta \triangleq \frac{\bar{E}_r}{N_0}.$$

Problem 4.4.25 [90]. In this problem we investigate the improvement obtained by using M orthogonal signals instead of two orthogonal signals to transmit information over a Rayleigh channel.

1. Show that

$$\Pr(\epsilon) = 1 - \frac{\Gamma[1/(\beta+1)+1]\Gamma(M)}{\Gamma[1/(\beta+1)+M]}.$$

Hint. Use the familiar expression

$$\frac{\Gamma(z)\,\Gamma(a+1)}{\Gamma(z+a)} = \sum_{n=0}^{a-1} (-1)^n \frac{a(a-1)\cdots(a-n)}{n!}\frac{1}{z+n}.$$

2. Consider the case in which $\beta \gg 1$. Use a Taylor series expansion and the properties of $\psi(x) \triangleq \dfrac{\Gamma'(x)}{\Gamma(x)}$ to obtain the approximate expression

$$\Pr(\epsilon) \simeq \frac{1}{\beta}\left(\ln M - \frac{1}{2M} + 0.577\right).$$

Recall that

$$\psi(1) = 0.577,$$
$$\psi(z) = \ln z - \frac{1}{2z} + o(z).$$

3. Now assume that the M hypotheses arise from the simple coding system in Fig. 4.24. Verify that the bit error probability is

$$P_B(\epsilon) = \frac{1}{2}\frac{M}{M-1}\Pr(\epsilon).$$

4. Find an expression for the ratio of the $\Pr_B(\epsilon)$ in a binary system to the $\Pr_B(\epsilon)$ in an M-ary system.
5. Show that $M \to \infty$, the power saving resulting from using M orthogonal signals, approaches $2/\ln 2 = 4.6$ db.

Problem 4.4.26. M Orthogonal Signals: Rician Channel. Consider the same system as in Problem 4.4.24, but assume that v is Rician.

1. Draw a block diagram of the minimum $\Pr(\epsilon)$ receiver.
2. Find the $\Pr(\epsilon)$.

Problem 4.4.27. Binary Orthogonal Signals: Square-Law Receiver [18]. Consider the problem of transmitting two equally likely bandpass orthogonal signals with energy E_t over the Rician channel defined in (416). Instead of using the optimum receiver shown in Fig. 4.74, we use the receiver for the Rayleigh channel (i.e., let $\alpha = 0$ in Fig. 4.74). Show that

$$\Pr(\epsilon) = \left[2\left(1 + \frac{\bar{E}_r}{2N_0}\right)\right]^{-1} \exp\left[\frac{-\alpha^2 E_t}{2N_0(1 + \bar{E}_r/2N_0)}\right]$$

Problem 4.4.28. Repeat Problem 4.4.27 for the case of M orthogonal signals.

Composite Signal Hypotheses

Problem 4.4.29. Detecting One of M Orthogonal Signals. Consider the following binary hypothesis testing problem. Under H_1 the signal is one of M orthogonal signals $\sqrt{E_1}\, s_1(t), \sqrt{E_2}\, s_2(t), \ldots, \sqrt{E_M}\, s_M(t)$:

$$\int_0^T s_i(t)\, s_j(t)\, dt = \delta_{ij}, \qquad i, j = 1, 2, \ldots, M.$$

Under H_1 the i^{th} signal occurs with probability p_i ($\sum_{i=1}^M p_i = 1$). Under H_0 there is no signal component. Under both hypotheses there is additive white Gaussian noise with spectral height $N_0/2$:

$$\begin{aligned} r(t) &= \sqrt{E_i}\, s_i(t) + w(t), \qquad 0 \le t \le T \text{ with probability } p_i : H_1, \\ r(t) &= w(t), \qquad 0 \le t \le T : H_0. \end{aligned}$$

1. Find the likelihood ratio test.
2. Draw a block diagram of the optimum receiver.

Problem 4.4.30 (continuation). Now assume that

$$p_i = \frac{1}{M}, \qquad i = 1, 2, \ldots, M$$

and

$$E_i = E.$$

One method of approximating the performance of the receiver was developed in Problem 2.2.14. Recall that we computed the variance of Λ (*not* $\ln \Lambda$) on H_0 and used the equation

$$d^2 = \ln(1 + \mathrm{Var}[\Lambda | H_0]). \tag{P.1}$$

We then used these values of d on the ROC of the known signal problem to find P_F and P_D.

1. Find $\mathrm{Var}[\Lambda | H_0]$.
2. Using (P.1), verify that

$$\frac{2E}{N_0} = \ln(1 - M + Me^{d^2}). \tag{P.2}$$

3. For $2E/N_0 \gtrsim 3$ verify that we may approximate (P.2) by

$$\frac{2E}{N_0} \simeq \ln M + \ln(e^{d^2} - 1). \tag{P.3}$$

The significance of (P.3) is that if we have a certain performance level (P_F, P_D) for a single known signal then to maintain the performance level when the signal is equally likely to be any one of M orthogonal signals requires an increase in the energy-to-noise ratio of $\ln M$. This can be considered as the cost of signal uncertainty.

4. Now remove the equal probability restriction. Show that (P.3) becomes

$$\frac{2E}{N_0} \simeq -\ln\left(\sum_{i=1}^{M} p_i^2\right) + \ln(e^{d^2} - 1).$$

What probability assignment maximizes the first term? Is this result intuitively logical?

Problem 4.4.31 (alternate continuation). Consider the special case of Problem 4.4.29 in which $M = 2$, $E_1 = E_2 = E$, and $p_1 = p_2 = \frac{1}{2}$. Define

$$l_i[r(t)] = \left[\frac{2\sqrt{E}}{N_0}\int_0^T dt\, r(t)\, s_i(t) - \frac{E}{N_0}\right], \qquad i = 1, 2. \tag{P.4}$$

1. Sketch the optimum decision boundary in l_1, l_2-plane for various values of η.
2. Verify that the decision boundary approaches the asymptotes $l_1 = 2\eta$ and $l_2 = 2\eta$.
3. Under what conditions would the following test be close to optimum.

Test. If either l_1 *or* $l_2 \geq 2\eta$, say H_1 is true. Otherwise say H_0 is true.

4. Find P_D and P_F for the suboptimum test in Part 3.

Problem 4.4.32 (continuation). Consider the special case of Problem 4.4.29 in which $E_i = E$, $i = 1, 2, \ldots, M$ and $p_i = 1/M$, $i = 1, 2, \ldots, M$. Extending the definition of $l_i[r(t)]$ in (P.4) to $i = 1, 2, \ldots, M$, we consider the suboptimum test.

Test. If one or more $l_i \geq \ln M\eta$, say H_1. Otherwise say H_0.

1. Define

$$\alpha = \Pr\,[l_1 > \ln M\eta \,|\, s_1(t) \text{ is not present}],$$
$$\beta = \Pr\,[l_1 < \ln M\eta \,|\, s_1(t) \text{ is present}].$$

Show

$$P_F = 1 - (1 - \alpha)^M$$

and

$$P_D = 1 - \beta(1 - \alpha)^{M-1}.$$

2. Verify that

$$P_F \leq M\alpha$$

and

$$P_D \leq \beta.$$

When are these bounds most accurate?

3. Find α and β.
4. Assume that $M = 1$ and E/N_0 gives a certain P_F, P_D performance. How must E/N_0 increase to maintain the same performance at M increases? (Assume that the relations in part 2 are exact.) Compare these results with those in Problem 4.4.30.

Problem 4.4.33. A similar problem is encountered when each of the M orthogonal signals has a random phase.

Under H_1:

$$r(t) = \sqrt{2E}\, f_i(t) \cos\,[\omega_c t + \phi_i(t) + \theta_i] + w(t), \qquad 0 \leq t \leq T \quad (\text{with probability } p_i).$$

Under H_0:

$$r(t) = w(t), \qquad 0 \leq t \leq T.$$

The signal components are orthogonal. The white noise has spectral height $N_0/2$. The probabilities, p_i, equal $1/M$, $i = 1, 2, \ldots, M$. The phase term in each signal θ_i is an independent, uniformly distributed random variable $(0, 2\pi)$.

1. Find the likelihood ratio test and draw a block diagram of the optimum receiver.
2. Find $\mathrm{Var}(\Lambda|H_0)$.
3. Using the same approximation techniques as in Problem 4.4.30, show that the correct value of d to use on the known signal ROC is

$$d \triangleq \ln [1 + \mathrm{Var}(\Lambda|H_0)] = \ln \left[1 - \frac{1}{M} + \frac{1}{M} I_0 \left(\frac{2E}{N_0}\right)\right].$$

Problem 4.4.34 (continuation). Use the same reasoning as in Problem 4.4.31 to derive a suboptimum test and find an expression for its performance.

Problem 4.4.35. Repeat Problem 4.4.33(1) and 4.4.34 for the case in which each of M orthogonal signals is received over a Rayleigh channel.

Problem 4.4.36. In Problem 4.4.30 we saw in the "one-of-M" orthogonal signal problem that to maintain the same performance we had to increase $2E/N_0$ by $\ln M$. Now suppose that under H_1 one of $N(N > M)$ equal-energy signals occurs with equal probability. The N signals, however, lie in an M-dimensional space. Thus, if we let $\phi_j(t)$, $j = 1, 2, \ldots, M$, be a set of orthonormal functions $(0, T)$, then

$$s_i(t) = \sum_{j=1}^{M} a_{ij}\phi_j(t), \qquad i = 1, 2, \ldots, N,$$

where

$$\sum_{j=1}^{M} a_{ij}^2 = 1, \qquad i = 1, 2, \ldots, N.$$

The other assumptions in Problem 4.4.29 remain the same.

1. Find the likelihood ratio test.
2. Discuss qualitatively (or quantitatively, if you wish) the cost of uncertainty in this problem.

Channel Measurement Receivers

Problem 4.4.37. Channel Measurement [18]. Consider the following approach to exploiting the phase stability in the channel. Use the first half of the signaling interval to transmit a channel measuring signal $\sqrt{2}\, s_m(t) \cos \omega_c t$ with energy E_m. Use the other half to send one of two equally likely signals $\pm \sqrt{2}\, s_d(t) \cos \omega_c t$ with energy E_d. Thus

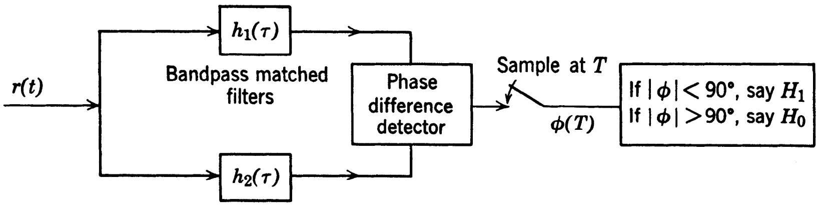
$$r(t) = [s_m(t) + s_d(t)] \sqrt{2} \cos (\omega_c t + \theta) + w(t) : H_1, \qquad 0 \le t \le T,$$

$$r(t) = [s_m(t) - s_d(t)] \sqrt{2} \cos (\omega_c t + \theta) + w(t) : H_0, \qquad 0 \le t \le T,$$

and

$$p_\theta(\theta) = \frac{1}{2\pi}, \qquad 0 \le \theta \le 2\pi.$$

1. Draw the optimum receiver and decision rule for the case in which $E_m = E_d$.
2. Find the optimum receiver and decision rule for the case in part 1.
3. Prove that the optimum receiver can also be implemented as shown in Fig. P4.8.
4. What is the $\Pr(\epsilon)$ of the optimum system?



$$h_1(\tau) = \sqrt{2}\, s_m(T - \tau) \cos \omega_c \tau$$
$$h_2(\tau) = \sqrt{2}\, s_d(T - \tau) \cos \omega_c \tau$$

Fig. P4.8

Problem 4.4.38 (continuation). Kineplex [81]. A clever way to take advantage of the result in Problem 4.4.37 is employed in the Kineplex system. The information is transmitted by the phase relationship between successive bauds. If $s_d(t)$ is transmitted in one interval, then to send H_1 in the next interval we transmit $+s_d(t)$; and to send H_0 we transmit $-s_d(t)$. A typical sequence is shown in Fig. P4.9.

Source		1	1	0	0	0	1	1
Trans. sequence	$+s_m(t)$	$+s_m(t)$	+	−	+	−	−	−
	(Initial reference)							

Fig. P4.9

1. Assuming that there is no phase change from baud-to-baud, adapt the receiver in Fig. P4.8 to this problem. Show that the resulting Pr (ϵ) is

$$\Pr(\epsilon) = \tfrac{1}{2} \exp\left(-\frac{E}{N_0}\right),$$

(where E is the energy per baud, $E = E_d = E_m$).

2. Compare the performance of this system with the optimum coherent system in the text for large E/N_0. Are decision errors in the Kineplex system independent from baud to baud?

3. Compare the performance of Kineplex to the partially coherent system performance shown in Figs. 4.62 and 4.63.

Problem 4.4.39 (continuation). Consider the signal system in Problem 4.4.37 and assume that $E_m \neq E_d$.

1. Is the phase-comparison receiver of Fig. P4.8 optimum?
2. Compute the Pr (ϵ) of the optimum receiver.

Comment. It is clear that the ideas of phase-comparison can be extended to M-ary systems. [72], [82], and [83] discuss systems of this type.

MISCELLANEOUS

Problem 4.4.40. Consider the communication system described below. A known signal $s(t)$ is transmitted. It arrives at the receiver through *one* of two possible channels. The output is corrupted by additive white Gaussian noise $w(t)$. *If* the signal passes through channel 1, the input to the receiver is

$$r(t) = a\, s(t) + w(t), \qquad 0 \le t \le T,$$

where a is constant over the interval. It is the value of a Gaussian random variable $N(0, \sigma_a)$. If the signal passes through channel 2, the input to the receiver is

$$r(t) = s(t) + w(t), \qquad 0 \le t \le T.$$

It is given that

$$\int_0^T s^2(t)\, dt = E.$$

The probability of passing through channel 1 is equal to the probability of passing through channel 2 (i.e., $P_1 = P_2 = \frac{1}{2}$).

1. Find a receiver that decides which channel the signal passed through with minimum probability of error.
2. Compute the Pr (ϵ).

Problem 4.4.41. A new engineering graduate is told to design an optimum detection system for the following problem:

$$H_1: r(t) = s(t) + w(t), \qquad T_i \le t \le T_f,$$
$$H_0: r(t) = n(t), \qquad T_i \le t \le T_f.$$

The signal $s(t)$ is known. To find a suitable covariance function $K_n(t, u)$ for the noise, he asks several engineers for an opinion.

Engineer A says

$$K_n(t, u) = \frac{N_0}{2}\, \delta(t - u).$$

Engineer B says

$$K_n(t, u) = \frac{N_0}{2}\, \delta(t - u) + K_c(t, u),$$

where $K_c(t, u)$ is a known, square-integrable, positive-definite function.

He must now reconcile these different opinions in order to design a signal detection system.

1. He decides to combine their opinions probabilistically. Specifically,

$$\Pr\,(\text{Engineer A is correct}) = P_A,$$
$$\Pr\,(\text{Engineer B is correct}) = P_B,$$

where $P_A + P_B = 1$.
   (a) Construct an optimum Bayes test (threshold η) to decide whether H_1 or H_0 is true.
   (b) Draw a block diagram of the receiver.
   (c) Check your answer for $P_A = 0$ and $P_B = 0$.
2. Discuss some other possible ways you might reconcile these different opinions.

Problem 4.4.42. Resolution. The following detection problem is a crude model of a simple radar resolution problem:

$$H_1: r(t) = b_d\, s_d(t) + b_I\, s_I(t) + w(t), \qquad T_i \le t \le T,$$
$$H_0: r(t) = b_I\, s_I(t) + w(t), \qquad T_i \le t \le T_f.$$

1. $\int_{T_i}^{T_f} s_d(t)\, s_I(t)\, dt = \rho.$
2. $s_d(t)$ and $s_I(t)$ are normalized to unit energy.
3. The multipliers b_d and b_I are *independent* zero-mean Gaussian variables with variances σ_d^2 and σ_I^2, respectively.
4. The noise $w(t)$ is white Gaussian with spectral height $N_0/2$ and is independent of the multipliers.

Find an explicit solution for the optimum likelihood ratio receiver. You do *not* need to specify the threshold.

Section P.4.5. Multiple Channels.

Mathematical Derivations

Problem 4.5.1. The definition of a matrix inverse kernel given in (4.434) is

$$\int_{T_i}^{T_f} \mathbf{K_n}(t, u)\, \mathbf{Q_n}(u, z)\, du = \mathbf{I}\, \delta(t - z).$$

1. Assume that

$$\mathbf{K_n}(t, u) = \frac{N_0}{2}\mathbf{I}\, \delta(t - u) + \mathbf{K}_c(t, u).$$

Show that we can write

$$\mathbf{Q_n}(t, u) = \frac{2}{N_0}\, [\mathbf{I}\, \delta(t - u) - \mathbf{h}_o(t, u)],$$

where $\mathbf{h}_o(t, u)$ is a square-integrable function. Find the matrix integral equation that $\mathbf{h}_o(t, u)$ must satisfy.

2. Consider the problem of a matrix linear filter operating on $\mathbf{n}(t)$.

$$\mathbf{d}(t) = \int_{T_i}^{T_f} \mathbf{h}(t, u)\, \mathbf{n}(u)\, du,$$

where

$$\mathbf{n}(t) = \mathbf{n}_c(t) + \mathbf{w}(t)$$

has the covariance function given in part 1. We want to choose $\mathbf{h}(t, u)$ so that

$$\xi_I \triangleq E\int_{T_i}^{T_f} [\mathbf{n}_c(t) - \mathbf{d}(t)]^T[\mathbf{n}_c(t) - \mathbf{d}(t)]\, dt$$

is minimized. Show that the linear matrix filter that does this is the $\mathbf{h}_o(t, u)$ found in part 1.

Problem 4.5.2 (continuation). In this problem we extend the derivation in Section 4.5 to include the case in which

$$\mathbf{K_n}(t, u) = \mathbf{N}\, \delta(t - u) + \mathbf{K}_c(t, u), \qquad T_i \le t, u \le T_f,$$

where $\mathbf{N}$ is a positive-definite matrix of numbers. We denote the eigenvalues of $\mathbf{N}$ as $\lambda_1, \lambda_2, \ldots, \lambda_M$ and define a diagonal matrix,

$$\mathbf{I}_\lambda{}^k = \begin{bmatrix} \lambda_1{}^k & & & 0 \\ & \lambda_2{}^k & & \\ & & \ddots & \\ 0 & & & \lambda_M{}^k \end{bmatrix}$$

To find the LRT we first perform two preliminary transformations on $\mathbf{r}$ as shown in Fig. P4.10.

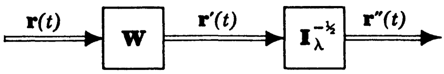


Fig. P4.10

The matrix $\mathbf{W}$ is an orthogonal matrix defined in (2.369) and has the properties

$$\mathbf{W}^T = \mathbf{W}^{-1},$$
$$\mathbf{N} = \mathbf{W}^{-1}\mathbf{I}_\lambda \mathbf{W}.$$

1. Verify that $\mathbf{r}''(t)$ has a covariance function matrix which satisfies (428).
2. Express l in terms of $\mathbf{r}''(t)$, $\mathbf{Q}_{\mathbf{n}}''(t, u)$, and $\mathbf{s}''(t)$.
3. Prove that

$$l = \iint_{T_i}^{T_f} \mathbf{r}^T(t)\, \mathbf{Q}_{\mathbf{n}}(t, u)\, \mathbf{s}(u)\, dt\, du,$$

where

$$\mathbf{Q}_{\mathbf{n}}(t, u) \triangleq \mathbf{N}^{-1}[\delta(t-u) - \mathbf{h}_o(t, u)],$$

and $\mathbf{h}_o(t, u)$ satisfies the equation

$$\mathbf{K}_c(t, u) = \mathbf{h}_o(t, u)\mathbf{N} + \int_{T_i}^{T_f} \mathbf{h}_o(t, z)\, \mathbf{K}_c(z, u)\, dz, \qquad T_i \le t, u \le T_f.$$

4. Repeat part (2) of Problem 4.5.1.

Problem 4.5.3. Consider the vector detection problem defined in (4.423). Assume that $\mathbf{K}_c(t, u) = 0$ and that $\mathbf{N}$ is not positive-definite. Find a signal vector $\mathbf{s}(t)$ with total energy E and a receiver that leads to perfect detectability.

Problem 4.5.4. Let

$$\mathbf{r}(t) = \mathbf{s}(t, A) + \mathbf{n}(t), \qquad T_i \le t \le T_f,$$

where the covariance of $\mathbf{n}(t)$ is given by (425) to (428) and A is a nonrandom parameter.

1. Find the equation the maximum-likelihood estimate of A must satisfy.
2. Find the Cramér-Rao inequality for an unbiased estimate $\hat{a}$.
3. Now assume that a is Gaussian, $N(0, \sigma_a)$. Find the MAP equation and the lower bound on the mean-square error.

Problem 4.5.5 (continuation). Let $\mathbf{L}$ denote a nonsingular linear transformation on a, where a is a zero-mean Gaussian random variable.

1. Show that an efficient estimate of A will exist if

$$\mathbf{s}(t, A) = \mathbf{L}A\,\mathbf{s}(t).$$

2. Find an explicit solution for $\hat{a}_{map}$ and an expression for the resulting mean-square error.

Problem 4.5.6. Let

$$r_i(t) = \sum_{j=1}^{K} a_{ij}\, s_{ij}(t) + w_i(t), \qquad i = 1, 2, \ldots, M{:}H_1,$$

$$r_i(t) = w_i(t), \qquad i = 1, 2, \ldots, M{:}H_0.$$

The noise in each channel is a sample function from a zero-mean white Gaussian random process

$$E[\mathbf{w}(t)\,\mathbf{w}^T(u)] = \frac{N_0}{2}\,\mathbf{I}\,\delta(t - u).$$

The a_{ij} are jointly Gaussian and zero-mean. The $s_{ij}(t)$ are orthogonal. Find an expression for the optimum Bayes receiver.

Problem 4.5.7. Consider the binary detection problem in which the received signal is an M-dimensional vector:

$$\begin{aligned}\mathbf{r}(t) &= \mathbf{s}(t) + \mathbf{n}_c(t) + \mathbf{w}(t), \qquad -\infty < t < \infty{:}H_1,\\ &= \mathbf{n}_c(t) + \mathbf{w}(t), \qquad -\infty < t < \infty{:}H_0.\end{aligned}$$

The total signal energy is ME:

$$\int_0^T \mathbf{s}^T(t)\,\mathbf{s}(t)\,dt = ME.$$

The signals are zero outside the interval $(0, T)$.

1. Draw a block diagram of the optimum receiver.
2. Verify that

$$d^2 = \int_{-\infty}^{\infty} \mathbf{S}^T(j\omega)\,\mathbf{S}_n^{-1}(\omega)\,\mathbf{S}(j\omega)\,\frac{d\omega}{2\pi}.$$

Problem 4.5.8. Maximal-Ratio Combiners. Let

$$\mathbf{r}(t) = \mathbf{s}(t) + \mathbf{w}(t), \qquad 0 \le t \le T.$$

The received signal $\mathbf{r}(t)$ is passed into a time-invariant matrix filter with M inputs and one output $y(t)$:

$$y(t) = \int_0^T \mathbf{h}(t - \tau)\,\mathbf{r}(\tau)\,d\tau.$$

The subscript s denotes the output due to the signal. The subscript n denotes the output due to the noise. Define

$$\left(\frac{S}{N}\right)_{\text{out}} \triangleq \frac{y_s^2(T)}{E[y_n^2(T)]}.$$

1. Assume that the covariance matrix of $\mathbf{w}(t)$ satisfies (439). Find the matrix filter $\mathbf{h}(\tau)$ that maximizes $(S/N)_{\text{out}}$. Compare your answer with (440).
2. Repeat part 1 for a noise vector with an arbitrary covariance matrix $\mathbf{K}_c(t, u)$.

RANDOM PHASE CHANNELS

Problem 4.5.9 [14]. Let

$$x = \sum_{i=1}^{M} a_i^2,$$

where each a_i is an independent random variable with the probability density

$$\begin{aligned} p_{a_i}(A) &= \frac{A}{\sigma^2} \exp\left(-\frac{A^2 + \alpha_i^2}{2\sigma^2}\right) I_0\left(\frac{\alpha_i A}{\sigma^2}\right), && 0 \le A < \infty, \\ &= 0, && \text{elsewhere.} \end{aligned}$$

Show that

$$\begin{aligned} p_x(X) &= \frac{1}{2\sigma^2}\left(\frac{X}{P}\right)^{\frac{M-1}{2}} \exp\left(-\frac{X + P}{2\sigma^2}\right) I_{M-1}\left(\frac{\sqrt{PX}}{\sigma^2}\right), && 0 \le X < \infty, \\ &= 0, && \text{elsewhere,} \end{aligned}$$

where

$$P = \sigma^2 \sum_{i=1}^{M} \alpha_i^2.$$

Problem 4.5.10. Generalized Q-Function.

The generalization of the Q-function to M channels is

$$Q_M(\alpha, \beta) = \int_{\beta}^{\infty} x\left(\frac{x}{\alpha}\right)^{M-1} \exp\left(-\frac{x^2 + \alpha^2}{2}\right) I_{M-1}(\alpha x)\, dx.$$

1. Verify the relation

$$Q_M(\alpha, \beta) = Q(\alpha, \beta) + \exp\left(-\frac{\alpha^2 + \beta^2}{2}\right) \sum_{k=1}^{M-1} \left(\frac{\beta}{\alpha}\right)^k I_k(\alpha\beta).$$

2. Find $Q_M(\alpha, 0)$.
3. Find $Q_M(0, \beta)$.

Problem 4.5.11. On-Off Signaling: N Incoherent Channels. Consider an on-off communication system that transmits over N fixed-amplitude random-phase channels. When H_1 is true, a bandpass signal is transmitted over each channel. When H_0 is true, no signal is transmitted. The received waveforms under the two hypotheses are

$$\begin{aligned} r_i(t) &= \sqrt{2E_i}\, f_i(t) \cos(\omega_i t + \phi_i(t) + \theta_i) + w(t), && 0 \le t \le T: H_1, \\ r_i(t) &= w(t), && 0 \le t \le T: H_0, \\ & && i = 1, 2, \dots, N. \end{aligned}$$

The carrier frequencies are separated enough so that the signals are in disjoint frequency bands. The $f_i(t)$ and $\phi_i(t)$ are known low-frequency functions. The amplitudes $\sqrt{E_i}$ are known. The θ_i are statistically independent phase angles with a uniform distribution. The additive noise $w(t)$ is a sample function from a white Gaussian random process ($N_0/2$) which is independent of the θ_i.

1. Show that the likelihood ratio test is

$$\Lambda = \prod_{i=1}^{N} \exp\left(-\frac{E_i}{N_0}\right) I_0\left[\frac{2E_i^{1/2}}{N_0}(L_{c_i}^2 + L_{s_i}^2)^{1/2}\right] \underset{H_0}{\overset{H_1}{\gtrless}} \eta,$$

where L_{c_i} and L_{s_i} are defined as in (361) and (362).

2. Draw a block diagram of the optimum receiver based on $\ln \Lambda$.
3. Using (371), find a good approximation to the optimum receiver for the case in which the argument of $I_0(\cdot)$ is small.
4. Repeat for the case in which the argument is large.
5. If the E_i are unknown nonrandom variables, does a UMP test exist?

Problem 4.5.12 (continuation). In this problem we analyze the performance of the suboptimum receiver developed in part 3 of the preceding problem. The test statistic is

$$l = \sum_{i=1}^{N} (L_{c_i}^2 + L_{s_i}^2) \underset{H0}{\overset{H1}{\gtrless}} \gamma.$$

1. Find $E[L_{c_i}|H_0]$, $E[L_{s_i}|H_0]$, $\text{Var}[L_{c_i}|H_0]$, $\text{Var}[L_{s_i}|H_0]$, $E[L_{c_i}|H_1, \theta]$, $E[L_{s_i}|H_1, \theta]$, $\text{Var}[L_{c_i}|H_1, \theta]$, $\text{Var}[L_{s_i}|H_1, \theta]$.
2. Use the result in Problem 2.6.4 to show that

$$M_{l|H_1}(jv) = (1 - jvN_0)^{-N} \exp\left(\frac{jv \sum_{i=1}^{N} E_i}{1 - jvN_0}\right)$$

and

$$M_{l|H_0}(jv) = (1 - jvN_0)^{-N}.$$

3. What is $p_{l|H_0}(X|H_0)$? Write an expression for P_F. The probability density of H_1 can be obtained from Fourier transform tables (e.g., [75], p. 197), It is

$$\begin{aligned} p_{l|H_1}(X|H_1) &= \frac{1}{N_0}\left(\frac{X}{E_T}\right)^{\frac{N-1}{2}} \exp\left(-\frac{X + E_T}{N_0}\right) I_{N-1}\left(\frac{2\sqrt{XE_T}}{N_0}\right), && X \geq 0, \\ &= 0, && \text{elsewhere,} \end{aligned}$$

where

$$E_T \triangleq \sum_{i=1}^{N} E_i.$$

4. Express P_D in terms of the generalized Q-function.

Comment. This problem was first studied by Marcum [46].

Problem 4.5.13 (continuation). Use the bounding and approximation techniques of Section 2.7 to evaluate the performance of the square-law receiver in Problem 4.5.11. Observe that the test statistic l is *not* equal to $\ln \Lambda$, so that the results in Section 2.7 must be modified.

Problem 4.5.14. N Pulse Radar: Nonfluctuating Target. In a conventional pulse radar the target is illuminated by a sequence of pulses, as shown in Fig. 4.5. If the target strength is constant during the period of illumination, the return signal will be

$$r(t) = \sqrt{2E} \sum_{k=1}^{M} f(t - \tau - kT_p) \cos(\omega_c t + \theta_i) + w(t), \qquad -\infty < t < \infty : H_1,$$

where τ is the round-trip time to the target, which is assumed known, and T_p is the interpulse time which is much larger than the pulse length $T\,[f(t) = 0 : t < 0, t > T]$. The phase angles of the received pulses are statistically independent random variables with uniform densities. The noise $w(t)$ is a sample function of a zero-mean white Gaussian process ($N_0/2$). Under H_0 no target is present and

$$r(t) = w(t), \qquad -\infty < t < \infty : H_0.$$

1. Show that the LRT for this problem is identical to that in Problem 4.5.11 (except for notation). This implies that the results of Problems 4.5.11 to 13 apply to this model also.
2. Draw a block diagram of the optimum receiver. Do not use more than one bandpass filter.

Problem 4.5.15. Orthogonal Signals: N Incoherent Channels. An alternate communication system to the one described in Problem 4.5.11 would transmit a signal on both hypotheses. Thus

$$r_i(t) = \sqrt{2E_{1i}}\, f_{1i}(t) \cos\left[\omega_i t + \phi_{1i}(t) + \theta_i\right] + w(t), \qquad 0 \le t \le T : H_1, \quad i = 1, 2, \ldots, N,$$

$$r_i(t) = \sqrt{2E_{0i}}\, f_{0i}(t) \cos\left[\omega_i t + \phi_{0i}(t) + \theta_i\right] + w(t), \qquad 0 \le t \le T : H_0, \quad i = 1, 2, \ldots, N.$$

All of the assumptions in 4.5.11 are valid. In addition, the signals on the two hypotheses are orthogonal.

1. Find the likelihood ratio test under the assumption of equally likely hypotheses and minimum Pr (ϵ) criterion.
2. Draw a block diagram of the suboptimum square-law receiver.
3. Assume that $E_i = E$. Find an expression for the probability of error in the square-law receiver.
4. Use the techniques of Section 2.7 to find a bound on the probability of error and an approximate expression for Pr (ϵ).

Problem 4.5.16 (continuation). N Partially Coherent Channels.

1. Consider the model in Problem 4.5.11. Now assume that the phase angles are independent random variables with probability density

$$p_{\theta_i}(\theta) = \frac{\exp\left(\Lambda_m \cos\theta\right)}{2\pi I_0(\Lambda_m)}, \qquad -\pi < \theta < \pi, \qquad i = 1, 2, \ldots, N.$$

Do parts 1, 2, and 3 of Problem 4.5.11, using this assumption.

2. Repeat part 1 for the model in Problem 4.5.15.

Random Amplitude and Phase Channels

Problem 4.5.17. Density of Rician Envelope and Phase. If a narrow-band signal is transmitted over a Rician channel, the output contains a specular component and a random component. Frequently it is convenient to use complex notation. Let

$$s_t(t) \triangleq \sqrt{2}\,\mathrm{Re}\left[f(t)e^{j\phi(t)}e^{j\omega_c t}\right]$$

denote the transmitted signal. Then, using (416), the received signal (without the additive noise) is

$$s_r(t) \triangleq \sqrt{2}\,\mathrm{Re}\left\{v' f(t) \exp\left[j\phi(t) + j\theta' + j\omega_c t\right]\right\},$$

where

$$v'e^{j\theta'} \triangleq \alpha e^{j\delta} + ve^{j\theta}$$

in order to agree with (416).

1. Show that

$$p_{v',\theta'}(V', \theta') = \begin{cases} \dfrac{V'}{2\pi\sigma^2} \exp\left(-\dfrac{V'^2 + \alpha^2 - 2V'\alpha\cos(\theta' - \delta)}{2\sigma^2}\right), & 0 \le V' < \infty, \quad 0 \le \theta' - \delta \le 2\pi, \\ 0, & \text{elsewhere.} \end{cases}$$

2. Show that

$$p_{v'}(V') = \begin{cases} \dfrac{V'}{\sigma^2} \exp\left(-\dfrac{V'^2 + \alpha^2}{2\sigma^2}\right) I_0\left(\dfrac{\alpha V'}{\sigma^2}\right), & 0 \le V' < \infty, \\ 0, & \text{elsewhere.} \end{cases}$$

3. Find $E(v')$ and $E(v'^2)$.
4. Find $p_{\theta'}(\theta')$, the probability density of θ'.

The probability densities in parts 2 and 4 are plotted in Fig. 4.73.

Problem 4.5.18. On-off Signaling: N Rayleigh Channels. In an on-off communication system a signal is transmitted over each of N Rayleigh channels when H_1 is true. The received signals are

$$H_1: r_i(t) = v_i\sqrt{2}\, f_i(t) \cos[\omega_i t + \phi_i(t) + \theta_i] + w_i(t), \qquad 0 \le t \le T, \quad i = 1, 2, \dots, N,$$

$$H_0: r_i(t) = w_i(t), \qquad 0 \le t \le T, \quad i = 1, 2, \dots, N,$$

where $f_i(t)$ and $\phi_i(t)$ are known waveforms, the v_i are statistically independent Rayleigh random variables with variance E_i, the θ_i are statistically independent random variables uniformly distributed $0 \le \theta \le 2\pi$, and $w_i(t)$ are independent white Gaussian noises $(N_0/2)$.

1. Find the LRT.
2. Draw a block diagram of the optimum receiver. Indicate both a bandpass filter realization and a filter-squarer realization.

Problem 4.5.19 (continuation). Optimum Diversity.
Now assume that $E_i = E$, $(i = 1, 2, \dots, N)$.

1. Verify that this problem is mathematically identical to Case 1A on p. 108 in Section 2.6. Find the relationships between the parameters in the two problems.
2. Use the identity in part 1 and the results in Example 2 on pp. 127–129 to find $\mu(s)$ and $\dot{\mu}(s)$ for this problem.
3. Assume that the hypotheses are equally likely and that minimum $\Pr(\epsilon)$ is the criterion. Find an upper bound on the $\Pr(\epsilon)$ and an approximate expression for the $\Pr(\epsilon)$.
4. Constrain $NE = E_T$. Use an approximate expression of the type given in (2.508) to find the optimum number of diversity channels.

Problem 4.5.20. N Pulse Radar: Fluctuating Target. Consider the pulsed model developed in Problem 4.5.14. If the target fluctuates, the amplitude of the reflected signal will change from pulse to pulse. A good model for this fluctuation is the Rayleigh model. Under H_1 the received signal is

$$r(t) = \sqrt{2} \sum_{i=1}^{N} v_i\, f(t - \tau - kT_p) \cos(\omega_c t + \theta_i) + w(t), \qquad -\infty < t < \infty,$$

where v_i, θ_i, and $w(t)$ are specified in Problem 4.5.18.

1. Verify that this problem is mathematically identical to Problem 4.5.18.
2. Draw a block diagram of the optimum receiver.
3. Verify that the results in Figs. 2.35 and 2.42 are immediately applicable to this problem.

4. If the required $P_F = 10^{-4}$ and the total average received energy is constrained $E[Nv_i^2] = 64$, what is the optimum number of pulses to transmit in order to maximize P_D?

Problem 4.5.21. Binary Orthogonal Signals: N Rayleigh Channels. Consider a binary communication system using orthogonal signals and operating over N Rayleigh channels. The hypotheses are equally likely and the criterion is minimum Pr (ϵ). The received waveforms are

$$\begin{aligned} r_i(t) &= \sqrt{2}\, v_i\, f_1(t) \cos [\omega_{1i}t + \phi_1(t) + \theta_i] + w_i(t), && 0 \le t \le T, \\ & && i = 1, 2, \dots, N{:}H_1 \\ &= \sqrt{2}\, v_i\, f_0(t) \cos [\omega_{0i}t + \phi_0(t) + \theta_i] + w_i(t), && 0 < t < T, \\ & && i = 1, 2, \dots, N{:}H_0. \end{aligned}$$

The signals are orthogonal. The quantities v_i, θ_i, and $w_i(t)$ are described in Problem 4.5.18. The system is an FSK system with diversity.

1. Draw a block diagram of the optimum receiver.
2. Assume $E_i = E$, $i = 1, 2, \dots, N$. Verify that this model is mathematically identical to Case 2A on p. 115. The resulting Pr (ϵ) is given in (2.434). Express this result in terms of E and N_0.

Problem 4.5.22 (continuation). Error Bounds: Optimal Diversity. Now assume the E_i may be different.

1. Compute $\mu(s)$. (Use the result in Example 3A on p. 130.)
2. Find the value of s which corresponds to the threshold $\gamma = \dot{\mu}(s)$ and evaluate $\mu(s)$ for this value.
3. Evaluate the upper bound on the Pr (ϵ) that is given by the Chernoff bound.
4. Express the result in terms of the probability of error in the individual channels:

$$P_i \triangleq \text{Pr} (\epsilon \text{ on the } i\text{th diversity channel})$$

$$P_i = \frac{1}{2}\left[\left(1 + \frac{E_i}{2N_0}\right)^{-1}\right].$$

5. Find an approximate expression for Pr(ϵ) using a Central Limit Theorem argument.
6. Now assume that $E_i = E$, $i = 1, 2, \dots, N$, and $NE = E_T$. Using an approximation of the type given in (2.473), find the optimum number of diversity channels.

Problem 4.5.23. M-ary Orthogonal Signals: N Rayleigh Channels. A generalization of the binary diversity system is an M-ary diversity system. The M hypotheses are equally likely. The received waveforms are

$$\begin{aligned} r_i(t) = \sqrt{2}\, v_i\, f_k(t) \cos [\omega_{ki}t + \phi_k(t) + \theta_i] + w_i(t), \quad & 0 \le t \le T{:}H_k, \\ & i = 1, 2, \dots, N, \\ & k = 1, 2, \dots, M. \end{aligned}$$

The signals are orthogonal. The quantities v_i, θ_i, and $w_i(t)$ are described in Problem 4.5.18. This type of system is usually referred to as multiple FSK (MFSK) with diversity.

1. Draw a block diagram of the optimum receiver.
2. Find an expression for the probability of error in deciding which hypothesis is true.

Comment. This problem is discussed in detail in Hahn [84] and results for various M and N are plotted.

Problem 4.5.24 (continuation). Bounds.

1. Combine the bounding techniques of Section 2.7 with the simple bounds in (4.63) through (4.65) to obtain a bound on the probability of error in the preceding problem.
2. Use a Central Limit Theorem argument to obtain an approximate expression.

Problem 4.5.25. M Orthogonal Signals: N Rician Channels. Consider the M-ary system in Problem 4.5.23. All the assumptions remain the same except now we assume that the channels are independent Rician instead of Rayleigh. (See Problem 4.5.17.) The amplitude and phase of the specular component are known.

1. Find the LRT and draw a block diagram of the optimum receiver.
2. What are some of the difficulties involved in implementing the optimum receiver?

Problem 4.5.26 (continuation). Frequently the phase of specular component is not accurately known. Consider the model in Problem 4.5.25 and assume that

$$p_{\delta_i}(X) = \frac{\exp(\Lambda_m \cos X)}{2\pi\, I_0(\Lambda_m)}, \qquad \pi \leq X \leq \pi,$$

and that the δ_i are independent of each other and all the other random quantities in the model.

1. Find the LRT and draw a block diagram of the optimum receiver.
2. Consider the special case where $\Lambda_m = 0$. Draw a block diagram of the optimum receiver.

Commentary. The preceding problems show the computational difficulties that are encountered in evaluating error probabilities for multiple-channel systems. There are two general approaches to the problem. The direct procedure is to set up the necessary integrals and attempt to express them in terms of Q-functions, confluent hypergeometric functions, Bessel functions, or some other tabulated function. Over the years a large number of results have been obtained. A summary of solved problems and an extensive list of references are given in [89]. A second approach is to try to find analytically tractable bounds to the error probability. The bounding technique derived in Section 2.7 is usually the most fruitful. The next two problems consider some useful examples.

Problem 4.5.27. Rician Channels: Optimum Diversity [86].

1. Using the approximation techniques of Section 2.7, find Pr (ϵ) expressions for binary orthogonal signals in N Rician channels.
2. Conduct the same type of analysis for a suboptimum receiver using square-law combining.
3. The question of optimum diversity is also appropriate in this case. Check your expressions in parts 1 and 2 with [86] and verify the optimum diversity results.

Problem 4.5.28. In part 3 of Problem 4.5.27 it was shown that if the ratio of the energy in the specular component to the energy in the random component exceeded a certain value, then infinite diversity was optimum. This result is not practical because it assumes perfect knowledge of the phase of the specular component. As N increases, the effect of small phase errors will become more important and should always lead to a finite optimum number of channels. Use the phase probability density in Problem 4.5.26 and investigate the effects of imperfect phase knowledge.

Section P4.6 Multiple Parameter Estimation

Problem 4.6.1. The received signal is

$$r(t) = s(t, \mathbf{A}) + w(t), \qquad 0 \le t \le T.$$

The parameter $\mathbf{a}$ is a Gaussian random vector with probability density

$$p_{\mathbf{a}}(\mathbf{A}) = [(2\pi)^{M/2}|\boldsymbol{\Lambda}_{\mathbf{a}}|^{1/2}]^{-1} \exp\left(-\tfrac{1}{2}\mathbf{A}^T\boldsymbol{\Lambda}_{\mathbf{a}}^{-1}\mathbf{A}\right).$$

1. Using the derivative matrix notation of Chapter 2 (p. 76), derive an integral equation for the MAP estimate of $\mathbf{a}$.
2. Use the property in (444) and the result in (447) to find the $\hat{\mathbf{a}}_{map}$.
3. Verify that the two results are identical.

Problem 4.6.2. Modify the result in Problem 4.6.1 to include the case in which $\boldsymbol{\Lambda}_{\mathbf{a}}$ is singular.

Problem 4.6.3. Modify the result in part 1 of Problem 4.6.1 to include the case in which $E(\mathbf{a}) = \mathbf{m}_{\mathbf{a}}$.

Problem 4.6.4. Consider the example on p. 372. Show that the actual mean-square errors approach the bound as E/N_0 increases.

Problem 4.6.5. Let

$$r(t) = s(t, a(t)) + n(t), \qquad 0 \le t \le T.$$

Assume that $a(t)$ is a zero-mean Gaussian random process with covariance function $K_a(t, u)$. Consider the function $a^*(t)$ obtained by sampling $a(t)$ every T/M seconds and reconstructing a waveform from the samples.

$$a^*(t) = \sum_{i=1}^{M} a(t_i) \frac{\sin[(\pi M/T)(t - t_i)]}{(\pi M/T)(t - t_i)}, \qquad t_i = 0, \frac{T}{M}, \frac{2T}{M}, \dots.$$

1. Define

$$\hat{a}^*(t) = \sum_{i=1}^{M} \hat{a}(t_i) \frac{\sin[(\pi M/T)(t - t_i)]}{(\pi M/T)(t - t_i)}.$$

Find an equation for $\hat{a}^*(t)$.

2. Proceeding formally, show that as $M \to \infty$ the equation for the MAP estimate of $a(t)$ is

$$\hat{a}(t) = \frac{2}{N_0}\int_0^T [r(u) - s(u, \hat{a}(u))] \frac{\partial s(u, \hat{a}(u))}{\partial \hat{a}(u)} K_a(t, u)\, du, \qquad 0 \le t \le T.$$

Problem 4.6.6. Let

$$r(t) = s(t, \mathbf{A}) + n(t), \qquad 0 \le t \le T,$$

where $\mathbf{a}$ is a zero-mean Gaussian vector with a diagonal covariance matrix and $n(t)$ is a sample function from a zero-mean Gaussian random process with covariance function $K_n(t, u)$. Find the MAP estimate of $\mathbf{a}$.

Problem 4.6.7. The multiple channel estimation problem is

$$\mathbf{r}(t) = \mathbf{s}(t, \mathbf{A}) + \mathbf{n}(t), \qquad 0 \le t \le T,$$

where $\mathbf{r}(t)$ is an N-dimensional vector and $\mathbf{a}$ is an M-dimensional parameter. Assume that $\mathbf{a}$ is a zero-mean Gaussian vector with a diagonal covariance matrix. Let

$$E[\mathbf{n}(t)\,\mathbf{n}^T(u)] = \mathbf{K}_{\mathbf{n}}(t, u).$$

Find an equation that specifies the MAP estimate of $\mathbf{a}$.

Problem 4.6.8. Let

$$r(t) = \sqrt{2}\, v\, f(t, \mathrm{A}) \cos\left[\omega_c t + \phi(t, \mathrm{A}) + \theta\right] + w(t), \qquad 0 \le t \le T,$$

where v is a Rayleigh variable and θ is a uniform variable. The additive noise $w(t)$ is a sample function from a white Gaussian process with spectral height $N_0/2$. The parameter $\mathbf{a}$ is a zero-mean Gaussian vector with a diagonal covariance matrix; $\mathbf{a}$, v, θ, and $w(t)$ are statistically independent. Find the likelihood function as a function of $\mathbf{a}$.

Problem 4.6.9. Let

$$r(t) = \sqrt{2}\, v\, f(t - \tau) \cos\left[\omega_c t + \phi(t - \tau) + \omega t + \theta\right] + w(t), \qquad -\infty < t < \infty,$$

where $w(t)$ is a sample function from a zero-mean white Gaussian noise process with spectral height $N_0/2$. The functions $f(t)$ and $\phi(t)$ are deterministic functions that are low-pass compared with ω_c. The random variable v is Rayleigh and the random variable θ is uniform. The parameters τ and ω are nonrandom.

1. Find the likelihood function as a function of τ and ω.
2. Draw the block diagram of a receiver that provides an approximate implementation of the maximum-likelihood estimator.

Problem 4.6.10. A sequence of amplitude modulated signals is transmitted. The signal transmitted in the kth interval is

$$s_k(t, A) = A_k\, s(t), \qquad (k - 1)T \le t \le kT, \qquad k = 1, 2, \ldots.$$

The sequence of random variables is zero-mean Gaussian; the variables are related in the following manner:

$$\begin{aligned} &a_1 \text{ is } N(0, \sigma_a) \\ &a_2 = \Phi a_1 + u_1 \\ &\vdots \\ &a_k = \Phi a_{k-1} + u_{k-1}. \end{aligned}$$

The multiplier Φ is fixed. The u_i are independent, zero-mean Gaussian random variables, $N(0, \sigma_u)$. The received signal in the kth interval is

$$r(t) = s_k(t, A) + w(t), \qquad (k - 1)T \le t \le kT, \qquad k = 1, 2, \ldots.$$

Find the MAP estimate of a_k, $k = 1, 2, \ldots$. (Note the similarity to Problem 2.6.15.)

REFERENCES

[1] G. E. Valley and H. Wallman, *Vacuum Tube Amplifiers*, McGraw-Hill, New York, 1948.

[2] W. B. Davenport and W. L. Root, *An Introduction to the Theory of Random Signals and Noise*, McGraw-Hill, New York, 1958.

[3] C. Helstrom, "Solution of the Detection Integral Equation for Stationary Filtered White Noise," *IEEE Trans. Inform. Theory*, **IT-11**, 335–339 (July 1965).

[4] H. Nyquist, "Certain Factors Affecting Telegraph Speed," *Bell System Tech. J.*, **3**, No. 2, 324–346 (1924).

[5] U. Grenander, "Stochastic Processes and Statistical Inference," *Arkiv Mat.*, **1**, 295 (1950).

[6] J. L. Lawson and G. E. Uhlenbeck, *Threshold Signals*, McGraw-Hill, New York, 1950.