Detection, Estimation, and Modulation Theory, Part I:
Detection, Estimation, and Linear Modulation Theory. Harry L. Van Trees

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6

Linear Estimation

In this chapter we shall study the linear estimation problem in detail. We recall from our work in Chapter 5 that in the linear modulation problem the received signal was given by

$$r(t) = c(t)\,a(t) + c_0(t) + n(t), \qquad T_i \le t \le T_f, \tag{1}$$

where $a(t)$ was the message, $c(t)$ was a deterministic carrier, $c_0(t)$ was a residual carrier component, and $n(t)$ was the additive noise. As we pointed out in Chapter 5, the effect of the residual carrier is not important in our logical development. Therefore we can assume that $c_0(t)$ equals zero for algebraic simplicity.

A more general form of linear modulation was obtained by passing $a(t)$ through a linear filter to obtain $x(t)$ and then modulating $c(t)$ with $x(t)$. In this case

$$r(t) = c(t)\,x(t) + n(t), \qquad T_i \le t \le T_f. \tag{2}$$

In Chapter 5 we defined linear modulation in terms of the derivative of the signal with respect to the message. An equivalent definition is the following:

Definition. The modulated signal is $s(t, a(t))$. Denote the component of $s(t, a(t))$ that does not depend on $a(t)$ as $c_0(t)$. If the signal $[s(t, a(t)) - c_0(t)]$ obeys superposition, then $s(t, a(t))$ is a linear modulation system.

We considered the problem of finding the maximum a posteriori probability (MAP) estimate of $a(t)$ over the interval $T_i \le t \le T_f$. In the case described by (1) the estimate $\hat{a}(t)$ was specified by two integral equations

$$\hat{a}(t) = \int_{T_i}^{T_f} K_a(t, u)\, c(u)[r_g(u) - g(u)]\, du, \qquad T_i \le t \le T_f, \tag{3}$$

and

$$r(t) - c(t)\,\hat{a}(t) = \int_{T_i}^{T_f} K_n(t, u)[r_g(u) - g(u)]\,du, \qquad T_i \leq t \leq T_f. \quad (4)$$

We shall study the solution of these equations and the properties of the resulting processors. Up to this point we have considered only *interval* estimation. In this chapter we also consider *point* estimators and show the relationship between the two types of estimators.

In Section 6.1 we develop some of the properties that result when we impose the linear modulation restriction. We shall explore the relation between the Gaussian assumption, the linear modulation assumption, the error criterion, and the structure of the resulting processor. In Section 6.2 we consider the special case in which the infinite past is available (i.e., $T_i = -\infty$), the processes of concern are stationary, and we want to make a minimum mean-square error point estimate. A constructive solution technique is obtained and its properties are discussed. In Section 6.3 we explore a different approach to point estimation. The result is a solution for the processor in terms of a feedback system. In Sections 6.2 and 6.3 we emphasize the case in which $c(t)$ is a constant. In Section 6.4 we look at conventional linear modulation systems such as amplitude modulation and single sideband. In the last two sections we summarize our results and comment on some related problems.

6.1 PROPERTIES OF OPTIMUM PROCESSORS

As suggested in the introduction, when we restrict our attention to linear modulation, certain simplifications are possible that were not possible in the general nonlinear modulation case.

The most important of these simplifications is contained in Property 1.

Property 1. The MAP interval estimate $\hat{a}(t)$ over the interval $T_i \leq t \leq T_f$, where

$$r(t) = c(t)\,a(t) + n(t), \qquad T_i \leq t \leq T_f, \quad (5)$$

is the received signal, can be obtained by using a *linear* processor.

Proof. A simple way to demonstrate that a linear processor can generate $\hat{a}(t)$ is to find an impulse response $h_o(t, u)$ such that

$$\hat{a}(t) = \int_{T_i}^{T_f} h_o(t, u)\,r(u)\,du, \qquad T_i \leq t \leq T_f. \quad (6)$$

First, we multiply (3) by $c(t)$ and add the result to (4), which gives

$$r(t) = \int_{T_i}^{T_f} [c(t)\,K_a(t, u)\,c(u) + K_n(t, u)][r_g(u) - g(u)]\,du, \qquad T_i \leq t \leq T_f. \quad (7)$$

We observe that the term in the bracket is $K_r(t, u)$. We rewrite (6) to indicate $K_r(t, u)$ explicitly. We also change t to x to avoid confusion of variables in the next step:

$$r(x) = \int_{T_i}^{T_f} K_r(x, u)[r_g(u) - g(u)]\, du, \qquad T_i \le x \le T_f. \tag{8}$$

Now multiply both sides of (8) by $h(t, x)$ and integrate with respect to x,

$$\int_{T_i}^{T_f} h(t, x)\, r(x)\, dx = \int_{T_i}^{T_f} [r_g(u) - g(u)]\, du \int_{T_i}^{T_f} h(t, x)\, K_r(x, u)\, dx. \tag{9}$$

We see that the left-hand side of (9) corresponds to passing the input $r(x)$, $T_i \le x \le T_f$ through a linear time-varying unrealizable filter. Comparing (3) and (9), we see that the output of the filter will equal $\hat{a}(t)$ if we require that the inner integral on the right-hand side of (9) equal $K_a(t, u)\, c(u)$ over the interval $T_i < u < T_f$, $T_i \le t \le T_f$. This gives the equation for the optimum impulse response.

$$K_a(t, u)\, c(u) = \int_{T_i}^{T_f} h_o(t, x)\, K_r(x, u)\, dx, \qquad T_i < u < T_f,\ T_i \le t \le T_f. \tag{10}$$

The subscript o denotes that $h_o(t, x)$ is the optimum processor. In (10) we have used a strict inequality on u. If $[r_g(u) - g(u)]$ does not contain impulses, either a strict or nonstrict equality is adequate. By choosing the inequality to be strict, however, we can find a continuous solution for $h_o(t, x)$. (See discussion in Chapter 3.) Whenever $r(t)$ contains a white noise component, this assumption is valid. As before, we define $h_o(t, x)$ at the end points by a continuity condition:

$$\begin{aligned} h_o(t, T_f) &\triangleq \lim_{x \to T_f^-} h_o(t, x), \\ h_o(t, T_i) &\triangleq \lim_{x \to T_i^+} h_o(t, x). \end{aligned} \tag{11}$$

It is frequently convenient to include the white noise component explicitly. Then we may write

$$K_r(x, u) = c(x)\, K_a(x, u)\, c(u) + K_c(x, u) + \frac{N_0}{2}\, \delta(x - u). \tag{12}$$

and (10) reduces to

$$K_a(t, u)\, c(u) = \frac{N_0}{2}\, h_o(t, u) + \int_{T_i}^{T_f} [c(x)\, K_a(x, u)\, c(u) + K_c(x, u)] h_o(t, x)\, dx,$$
$$T_i < u < T_f,\ T_i \le t \le T_f. \tag{13}$$

If $K_a(t, u)$, $K_c(t, u)$, and $c(t)$ are continuous square-integrable functions, our results in Chapter 4 guarantee that the integral equation specifying $h_o(t, x)$ will have a continuous square-integrable solution. Under these conditions (13) is also true for $u = T_f$ and $u = T_i$ because of our continuity assumption.

The importance of Property 1 is that it guarantees that the structure of the processor is linear and thus reduces the problem to one of finding the correct impulse response. A similar result follows easily for the case described by (2).

Property 1A. The MAP estimate $\hat{a}(t)$ of $a(t)$ over the interval $T_i \leq t \leq T_f$ using the received signal $r(t)$, where

$$r(t) = c(t)\, x(t) + n(t), \qquad T_i \leq t \leq T_f, \tag{14}$$

is obtained by using a linear processor.

The second property is one we have already proved in Chapter 5 (see p. 439). We include it here for completeness.

Property 2. The MAP estimate $\hat{a}(t)$ is also the minimum mean-square error interval estimate in the linear modulation case. (This results from the fact that the MAP estimate is efficient.)

Before solving (10) we shall discuss a related problem. Specifically, we shall look at the problem of estimating a waveform at a single point in time.

Point Estimation Model

Consider the typical estimation problem shown in Fig. 6.1. The signal available at the receiver for processing is $r(u)$. It is obtained by performing

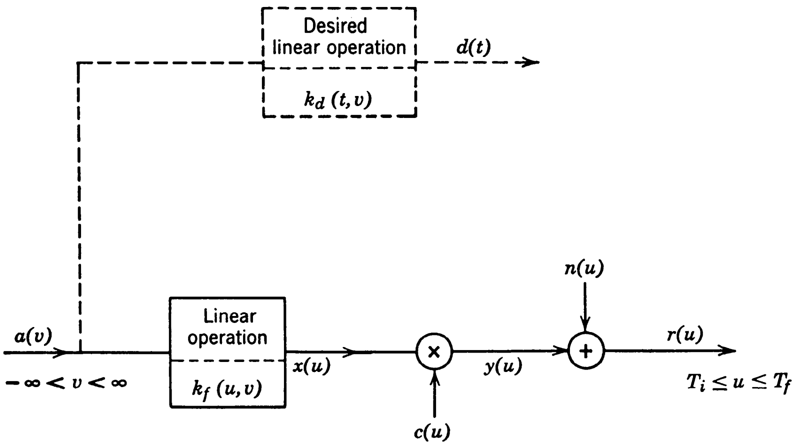


Fig. 6.1 Typical estimation problem.

a linear operation on $a(v)$ to obtain $x(u)$, which is then multiplied by a *known* modulation function. A noise $n(u)$ is added to the output $y(u)$ before it is observed. The dotted lines represent some linear operation (not necessarily time-invariant nor realizable) that we should like to perform on $a(v)$ *if* it were available (possibly for all time). The output is the desired signal $d(t)$ at some *particular* time t. The time t *may or may not* be included in the observation interval.

Common examples of desired signals are:

(i)
$$d(t) = a(t).$$

Here the output is simply the message. Clearly, *if* t were included in the observation interval, $x(u) = a(u)$, $n(u)$ were zero, and $c(u)$ were a constant, we could obtain the signal exactly. In general, this will not be the case.

(ii)
$$d(t) = a(t + \alpha).$$

Here, if α is a positive quantity, we wish to predict the value of $a(t)$ at some time in the future. Now, even in the absence of noise the estimation problem is nontrivial if $t + \alpha > T_f$. If α is a negative quantity, we shall want the value at some previous time.

(iii)
$$d(t) = \frac{d}{dt} a(t).$$

Here the desired signal is the derivative of the message. Other types of operations follow easily.

We shall assume that the linear operation is such that $d(t)$ is defined in the mean-square sense [i.e., if $d(t) = \dot{a}(t)$, as in (iii), we assume that $a(t)$ is a mean-square differentiable process]. Our discussion has been in the context of the linear modulation system in Fig. 6.1. We have not yet specified the statistics of the random processes. We describe the processes by the following assumption:

Gaussian Assumption. The message $a(t)$, the desired signal $d(t)$, and the received signal $r(t)$ are jointly Gaussian processes.

This assumption includes the linear modulation problem that we have discussed but avoids the necessity of describing the modulation system in detail. For algebraic simplicity we assume that the processes are zero-mean.

We now return to the optimum processing problem. We want to operate on $r(u)$, $T_i \le u \le T_f$ to obtain an estimate of $d(t)$. We denote this estimate as $\hat{d}(t)$ and choose our processor so that the quantity

$$\xi_P(t) \triangleq E\{[d(t) - \hat{d}(t)]^2\} = E[e^2(t)] \tag{15}$$

is minimized. First, observe that this is a *point* estimate (therefore the subscript P). Second, we observe that we are minimizing the mean-square error between the desired signal $d(t)$ and the estimate $\hat{d}(t)$.

We shall now find the optimum processor. The approach is as follows:

1. First, we shall find the optimum *linear* processor. Properties 3, 4, 5, and 6 deal with this problem. We shall see that the Gaussian assumption is not used in the derivation of the optimum linear processor.
2. Next, by including the Gaussian assumption, Property 7 shows that a linear processor is the best of *all possible* processors for the mean-square error criterion.
3. Property 8 demonstrates that under the Gaussian assumption the linear processor is optimum for a large class of error criteria.
4. Finally, Properties 9 and 10 show the relation between point estimators and interval estimators.

Property 3. The minimum mean-square *linear* estimate is the output of a linear processor whose impulse response is a solution to the integral equation

$$\boxed{K_{dr}(t, u) = \int_{T_i}^{T_f} h_o(t, \tau)\, K_r(\tau, u)\, d\tau, \qquad T_i < u < T_f.} \tag{16}$$

The proof of this property is analogous to the derivation in Section 3.4.5. The output of a linear processor can be written as

$$\hat{d}(t) = \int_{T_i}^{T_f} h(t, \tau)\, r(\tau)\, d\tau. \tag{17}$$

We assume that $h(t, \tau) = 0$, $\tau < T_i$, $\tau > T_f$. The mean-square error at time t is

$$\begin{aligned} \xi_P(t) &= \{E[d(t) - \hat{d}(t)]^2\} \\ &= E\left\{\left[d(t) - \int_{T_i}^{T_f} h(t, \tau)\, r(\tau)\, d\tau\right]^2\right\}. \end{aligned} \tag{18}$$

To minimize $\xi_P(t)$ we would go through the steps in Section 3.4.5 (pp. 198–204).

1. Let $h(t, \tau) = h_o(t, \tau) + \epsilon h_\epsilon(t, \tau)$.
2. Write $\xi_P(t)$ as the sum of the optimum error $\xi_{P_o}(t)$ and an incremental error $\Delta\xi(t, \epsilon)$.
3. Show that a necessary and sufficient condition for $\Delta\xi(t, \epsilon)$ to be greater than zero for $\epsilon \neq 0$ is the equation

$$E\left\{\left[d(t) - \int_{T_i}^{T_f} h_o(t, \tau)\, r(\tau)\, d\tau\right] r(u)\right\} = 0, \qquad T_i < u < T_f. \tag{19}$$

Bringing the expectation inside the integral, we obtain

$$K_{dr}(t, u) = \int_{T_i}^{T_f} h_o(t, \tau)\, K_r(\tau, u)\, d\tau, \qquad T_i < u < T_f, \tag{20}$$

which is the desired result. In Property 7A we shall show that the solution to (20) is unique iff $K_r(t, u)$ is positive-definite.

We observe that the only quantities needed to design the optimum linear processor for minimizing the mean-square error are the covariance function of the received signal $K_r(t, u)$ and the cross-covariance between the desired signal and the received signal, $K_{dr}(t, u)$. We emphasize that we have *not* used the Gaussian assumption.

Several special cases are important enough to be mentioned explicitly.

Property 3A. When $d(t) = a(t)$ and $T_f = t$, we have a realizable filtering problem, and (20) becomes

$$K_{ar}(t, u) = \int_{T_i}^{t} h_o(t, \tau)\, K_r(\tau, u)\, d\tau, \qquad T_i < u < t. \tag{21}$$

We use the term realizable because the filter indicated by (21) operates only on the past [i.e., $h_o(t, \tau) = 0$ for $\tau > t$].

Property 3B. Let $r(t) = c(t)\,x(t) + n(t) \triangleq y(t) + n(t)$. If the noise is white with spectral height $N_0/2$ and uncorrelated with $a(t)$, (20) becomes

$$K_{dy}(t, u) = \frac{N_0}{2}\, h_o(t, u) + \int_{T_i}^{T_f} h_o(t, \tau)\, K_y(\tau, u)\, d\tau, \qquad T_i \le u \le T_f. \tag{22}$$

Property 3C. When the assumptions of both 3A and 3B hold, and $x(t) = a(t)$, (20) becomes

$$K_a(t, u)\, c(u) = \frac{N_0}{2}\, h_o(t, u) + \int_{T_i}^{t} h_o(t, \tau)\, c(\tau)\, K_a(\tau, u)\, c(u)\, d\tau, \qquad T_i \le u \le t. \tag{23}$$

[The end point equalities were discussed after (13).]

Returning to the general case, we want to find an expression for the minimum mean-square error.

Property 4. The minimum mean-square error with the optimum linear processor is

$$\xi_{P_o}(t) \triangleq E[e_o^2(t)] = K_d(t, t) - \int_{T_i}^{T_f} h_o(t, \tau)\, K_{dr}(t, \tau)\, d\tau. \tag{24}$$

This follows by using (16) in (18). Hereafter we suppress the subscript o in the optimum error.

The error expressions for several special cases are also of interest. They all follow by straightforward substitution.

Property 4A. When $d(t) = a(t)$ and $T_f = t$, the minimum mean-square error is

$$\xi_P(t) = K_a(t, t) - \int_{T_i}^{t} h_o(t, \tau)\, K_{ar}(t, \tau)\, d\tau. \tag{25}$$

Property 4B. If the noise is white and uncorrelated with $a(t)$, the error is

$$\xi_P(t) = K_d(t, t) - \int_{T_i}^{T_f} h_o(t, \tau)\, K_{dy}(t, \tau)\, d\tau. \tag{26}$$

Property 4C. If the conditions of 4A and 4B hold and $x(t) = a(t)$, then

$$h_o(t, t) = \frac{2}{N_0}\, c(t)\, \xi_P(t). \tag{27}$$

If $c^{-1}(t)$ exists, (27) can be rewritten as

$$\xi_P(t) = \frac{N_0}{2}\, c^{-1}(t)\, h_o(t, t). \tag{28}$$

We may summarize the knowledge necessary to find the optimum linear processor in the following property:

Property 5. $K_r(t, u)$ and $K_{dr}(t, u)$ are the only quantities needed to find the MMSE point estimate when the processing is restricted to being linear. Any further statistical information about the processes cannot be used. All processes, Gaussian or non-Gaussian, with the same $K_r(t, u)$ and $K_{dr}(t, u)$ lead to the same processor and the same mean-square error if the processing is *required* to be linear.

Property 6. The error at time t using the optimum linear processor is uncorrelated with the input $r(u)$ at every point in the observation interval. This property follows directly from (19) by observing that the first term is the error using the optimum filter. Thus

$$E[e_o(t)\, r(u)] = 0, \qquad T_i < u < T_f. \tag{29}$$

We should observe that (29) can also be obtained by a simple heuristic geometric argument. In Fig. 6.2 we plot the desired signal $d(t)$ as a point in a vector space. The shaded plane area χ represents those points that can be achieved by a linear operation on the given input $r(u)$. We want to

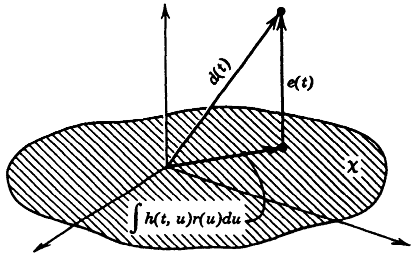


Fig. 6.2 Geometric interpretation of the optimum linear filter.

choose $\hat{d}(t)$ as the point in χ closest to $d(t)$. Intuitively, it is clear that we must choose the point directly under $d(t)$. Therefore the error vector is perpendicular to χ (or, equivalently, every vector in χ); that is, $e_0(t) \perp \int h(t, u)\, r(u)\, du$ for every $h(t, u)$.

The only difficulty is that the various functions are random. A suitable measure of the squared-magnitude of a vector is its mean-square value. The squared magnitude of the vector representing the error is $E[e^2(t)]$. Thus the condition of perpendicularity is expressed as an expectation:

$$E\left[e_o(t) \int_{T_i}^{T_f} h(t, u)\, r(u)\, du\right] = 0 \tag{30}$$

for every continuous $h(t, u)$; this implies

$$E[e_o(t)\, r(u)] = 0, \qquad T_i < u < T_f, \tag{31}$$

which is (29) [and, equivalently, (19)].†

Property 6A. If the processes of concern $d(t)$, $r(t)$, and $a(t)$ are jointly Gaussian, the error using the optimum linear processor is *statistically independent* of the input $r(u)$ at every point in the observation interval.

This property follows directly from the fact that uncorrelated Gaussian variables are statistically independent.

Property 7. When the Gaussian assumption holds, the optimum *linear* processor for minimizing the mean-square error is the best of *any* type. In other words, a nonlinear processor can not give an estimate with a smaller mean-square error.

Proof. Let $d_*(t)$ be an estimate generated by an arbitrary continuous processor operating on $r(u)$, $T_i \leq u \leq T_f$. We can denote it by

$$d_*(t) = f(t{:}r(u), T_i \leq u \leq T_f). \tag{32}$$

† Our discussion is obviously heuristic. It is easy to make it rigorous by introducing a few properties of linear vector spaces, but this is not necessary for our purposes.

Denote the mean-square error using this estimate as $\xi_*(t)$. We want to show that

$$\xi_*(t) \geq \xi_P(t), \tag{33}$$

with equality holding when the arbitrary processor is the optimum linear filter:

$$\begin{aligned}\xi_*(t) &= E\{[d_*(t) - d(t)]^2\} \\ &= E\{[d_*(t) - \hat{d}(t) + \hat{d}(t) - d(t)]^2\} \\ &= E\{[d_*(t) - \hat{d}(t)]^2\} + 2E\{[d_*(t) - \hat{d}(t)]e_o(t)\} + \xi_P(t).\end{aligned} \tag{34}$$

The first term is nonnegative. It remains to show that the second term is zero. Using (32) and (17), we can write the second term as

$$E\left\{\left[f(t{:}r(u), T_i \leq u \leq T_f) - \int_{T_i}^{T_f} h_o(t, u)\, r(u)\, du\right] e_o(t)\right\}. \tag{35}$$

This term is zero because $r(u)$ is *statistically independent* of $e_o(t)$ over the appropriate range, except for $u = T_f$ and $u = T_i$. (Because both processors are continuous, the expectation is also zero at the end point.) Therefore the optimum linear processor is as good as any other processor. The final question of interest is the uniqueness. To prove uniqueness we must show that the first term is strictly positive unless the two processors are equal. We discuss this issue in two parts.

Property 7A. First assume that $f(t{:}r(u), T_i \leq u \leq T_f)$ corresponds to a *linear* processor that is *not* equal to $h_o(t, u)$. Thus

$$f(t{:}r(u), T_i \leq u \leq T_f) = \int_{T_i}^{T_f} (h_o(t, u) + h_*(t, u))\, r(u)\, du, \tag{36}$$

where $h_*(t, u)$ represents the difference in the impulse responses.

Using (36) to evaluate the first term in (34), we have

$$E\{[d_*(t) - \hat{d}(t)]^2\} = \iint_{T_i}^{T_f} du\, dz\, h_*(t, u)\, K_r(u, z)\, h_*(t, z). \tag{37}$$

From (3.35) we know that if $K_r(u, z)$ is positive-definite the right-hand side will be positive for every $h_*(t, u)$ that is not identically zero. On the other hand, if $K_r(t, u)$ is only nonnegative definite, then from our discussion on p. 181 of Chapter 3 we know there exists an $h_*(t, u)$ such that

$$\int_{T_i}^{T_f} h_*(t, u)\, K_r(u, z)\, du = 0, \qquad T_i \leq z \leq T_f. \tag{38}$$

Because the eigenfunctions of $K_r(u, z)$ do not form a complete orthonormal set we can construct $h_*(t, u)$ out of functions that are orthogonal to $K_r(u, z)$.

Note that our discussion in 7A has not used the Gaussian assumption and that we have derived a necessary and sufficient condition for the uniqueness of the solution of (20). If $K_r(u, z)$ is not positive-definite, we can add an $h_*(t, u)$ satisfying (38) to any solution of (20) and still have a solution. Observe that the estimate $\hat{a}(t)$ is unique even if $K_r(u, z)$ is not positive-definite. This is because any $h_*(t, u)$ that we add to $h_o(t, u)$ must satisfy (38) and therefore cannot cause an output when the input is $r(t)$.

Property 7B. Now assume that $f(t:r(u), T_i \leq u \leq T_f)$ is a continuous nonlinear functional unequal to $\int h_o(t, u)\, r(u)\, du$. Thus

$$f(t:r(u), T_i \leq u \leq T_f) = \int_{T_i}^{T_f} h_o(t, u)\, r(u) du + f_*(t:r(u), T_i \leq u \leq T_f). \quad (39)$$

Then

$$E\{[d_*(t) - \hat{d}(t)]^2\} = E\,[f_*(t:r(u), T_i \leq u \leq T_f) f_*(t:r(z), T_i \leq z \leq T_f)]. \quad (40)$$

Because $r(u)$ is Gaussian and the higher moments factor, we can express the expectation on the right in terms of combinations of $K_r(u, z)$. Carrying out the tedious details gives the result that if $K_r(u, z)$ is positive-definite the expectation will be positive unless $f_*(t:r(z), T_i \leq z \leq T_f))$ is identically zero.

Property 7 is obviously quite important. It enables us to achieve two sets of results simultaneously by studying the linear processing problem.

1. If the Gaussian assumption holds, we are studying the best possible processor.
2. Even if the Gaussian assumption does not hold (or we cannot justify it), we shall have found the best possible linear processor.

In our discussion of waveform estimation we have considered only minimum mean-square error and MAP estimates. The next property generalizes the criterion.

Property 8A. Let $e(t)$ denote the error in estimating $d(t)$, using some estimate $\hat{d}(t)$.

$$e(t) = d(t) - \hat{d}(t). \quad (41)$$

The error is weighted with some cost function $C(e(t))$. The risk is the expected value of $C(e(t))$,

$$\mathcal{R}(\hat{d}(t), t) = E[C(e(t))] = E[C(d(t) - \hat{d}(t))]. \quad (42)$$

The Bayes point estimator is the estimate $\hat{d}_B(t)$ which minimizes the risk. If we assume that $C(e(t))$ is a symmetric convex upward function *and* the Gaussian assumption holds, the Bayes estimator is equal to the MMSE estimator.

$$\hat{d}_B(t) = \hat{d}_o(t). \tag{43}$$

Proof. The proof consists of three observations.

1. Under the Gaussian assumption the MMSE point estimator at any time (say t_1) is the conditional mean of the a posteriori density $p_{d_{t_1}|r(u)}[D_{t_1}|r(u):T_i \le u \le T_f]$. Observe that we are talking about a single random variable d_{t_1} so that this is a legitimate density. (See Problem 6.1.1.)
2. The a posteriori density is unimodal and symmetric about its conditional mean.
3. Property 1 on p. 60 of Chapter 2 is therefore applicable and gives the above conclusion.

Property 8B. If, in addition to the assumptions in Property 8A, we require the cost function to be *strictly* convex, then

$$\hat{d}_B(t) = \hat{d}_o(t) \tag{44}$$

is the unique Bayes point estimator.

This result follows from (2.158) in the derivation in Chapter 2.

Property 8C. If we replace the convexity requirement on the cost function with a requirement that it be a symmetric nondecreasing function such that

$$\lim_{X \to \infty} C(X) p_{d_{t_1}|r(u)}[X|r(u)\colon T_i \le u \le T_f] = 0 \tag{45}$$

for all t_1 and $r(t)$ of interest, then (44) is still valid.

These properties are important because they guarantee that the processors we are studying in this chapter are optimum for a large class of criteria when the Gaussian assumption holds.

Finally, we can relate our results with respect to point estimators and MMSE and MAP interval estimators.

Property 9. A minimum mean-square error interval estimator is just a collection of point estimators. Specifically, suppose we observe a waveform $r(u)$ over the interval $T_i \le u \le T_f$ and want a signal $d(t)$ over the interval $T_\alpha \le t \le T_\beta$ such that the mean-square error averaged over the interval is minimized.

$$\xi_I \triangleq E\left\{\int_{T_\alpha}^{T_\beta} [d(t) - \hat{d}(t)]^2 \, dt\right\}. \tag{46}$$

Clearly, if we can minimize the expectation of the bracket for each t then ξ_I will be minimized. This is precisely what a MMSE point estimator does. Observe that the point estimator uses $r(u)$ over the entire observation interval to generate $\hat{d}(t)$. (Note that Property 9 is true for nonlinear modulation also.)

Property 10. Under the Gaussian assumption the minimum mean-square error point estimate and MAP point estimate are identical. This is just a special case of Property 8C. Because the MAP interval estimate is a collection of MAP point estimates, the interval estimates also coincide.

These ten properties serve as background for our study of the linear modulation case. Property 7 enables us to concentrate our efforts in this chapter on the *optimum linear processing* problem. When the Gaussian assumption holds, our results will correspond to the best possible processor (for the class of criterion described above). For arbitrary processes the results will correspond to the best linear processor.

We observe that all results carry over to the vector case with obvious modifications. Some properties, however, are used in the sequel and therefore we state them explicitly. *A typical* vector problem is shown in Fig. 6.3.

The message $\mathbf{a}(t)$ is a p-dimensional vector. We operate on it with a matrix linear filter which has p inputs and n outputs.

$$\mathbf{x}(u) = \int_{-\infty}^{\infty} \mathbf{k}_f(u, v)\, \mathbf{a}(v)\, dv, \qquad T_i \le u \le T_f. \tag{47}$$

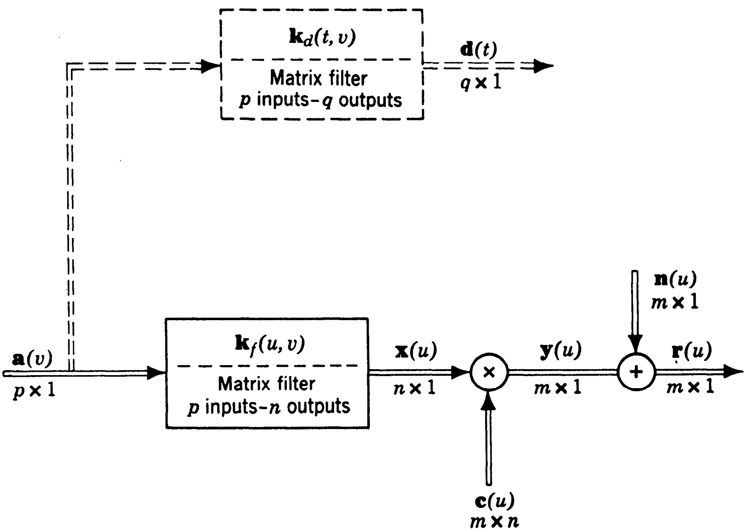


Fig. 6.3 Vector estimation problem.

The vector $\mathbf{x}(u)$ is multiplied by an $m \times n$ modulation matrix to give the m-dimensional vector $\mathbf{y}(t)$ which is transmitted over the channel. Observe that we have generated $\mathbf{y}(t)$ by a cascade of a linear operation with a memory and no-memory operation. The reason for this two-step procedure will become obvious later. The desired signal $\mathbf{d}(t)$ is a $q \times 1$-dimensional vector which is related to $\mathbf{a}(v)$ by a matrix filter with p inputs and q outputs Thus

$$\mathbf{d}(t) = \int_{-\infty}^{\infty} \mathbf{k}_d(t, v)\, \mathbf{a}(v)\, dv. \tag{48}$$

We shall encounter some typical vector problems later. Observe that p, q, m, and n, the dimensions of the various vectors, may all be different.

The desired signal $\mathbf{d}(t)$ has q components. Denote the estimate of the ith component as $\hat{d}_i(t)$. We want to minimize simultaneously

$$\xi_{P_i}(t) \triangleq E\{[d_i(t) - \hat{d}_i(t)]^2\}, \qquad i = 1, 2, \ldots, q. \tag{49}$$

The message $\mathbf{a}(v)$ is a zero-mean vector Gaussian process and the noise is an m-dimensional Gaussian random process. In general, we assume that it contains a white component $\mathbf{w}(t)$:

$$E[\mathbf{w}(t)\, \mathbf{w}^T(u)] \triangleq \mathbf{R}(t)\, \delta(t - u), \tag{50}$$

where $\mathbf{R}(t)$ is positive-definite. We assume also that the necessary covariance functions are known. We shall use the same property numbers as in the scalar case and add a V. We shall not restate the assumptions.

Property 3V.

$$\mathbf{K}_{\mathbf{dr}}(t, u) = \int_{T_i}^{T_f} \mathbf{h}_o(t, \tau)\mathbf{K}_{\mathbf{r}}(\tau, u)\, d\tau, \qquad T_i < u < T_f. \tag{51}$$

Proof. See Problem 6.1.2.

Property 3A-V.

$$\mathbf{K}_{\mathbf{ar}}(t, u) = \int_{T_i}^{t} \mathbf{h}_o(t, \tau)\, \mathbf{K}_{\mathbf{r}}(\tau, u)\, d\tau; \qquad T_i < u < t. \tag{52}$$

Property 4C-V.

$$\mathbf{h}_o(t, t) = \boldsymbol{\xi}_P(t)\, \mathbf{C}^T(t)\, \mathbf{R}^{-1}(t), \tag{53}$$

where $\xi_P(t)$ is the error covariance matrix whose elements are

$$\xi_{P_{ij}}(t) \triangleq E\{[a_i(t) - \hat{a}_i(t)][a_j(t) - \hat{a}_j(t)]\}. \tag{54}$$

(Because the errors are zero-mean, the correlation and covariance are identical.)

Proof. See Problem 6.1.3.

Other properties of the vector case follow by direct modification.

Summary

In this section we have explored properties that result when a linear modulation restriction is imposed. Although we have discussed the problem in the modulation context, it clearly has widespread applicability. We observe that if we let $c(t) = 1$ at certain instants of time and zero elsewhere, we will have the sampled observation model. This case and others of interest are illustrated in the problems (see Problems 6.1.4–6.1.9).

Up to this point we have restricted neither the processes nor the observation interval. In other words, the processes were stationary or nonstationary, the initial observation time T_i was arbitrary, and $T_f\,(\geq T_i)$ was arbitrary. Now we shall consider specific solution techniques. The easiest approach is by means of various special cases.

Throughout the rest of the chapter we shall be dealing with linear processors. In general, we do not specify explicitly that the Gaussian assumption holds. It is important to re-emphasize that in the absence of this assumption we are finding only the best *linear* processor (a nonlinear processor might be better). Corresponding to each problem we discuss there is another problem in which the processes are Gaussian, and for which the processor is the optimum of all processors for the given criterion.

It is also worthwhile to observe that the remainder of the chapter could have been studied directly after Chapter 1 if we had approached it as a "structured" problem and not used the Gaussian assumption. We feel that this places the emphasis incorrectly and that the linear processor should be viewed as a device that is generating the conditional mean. This viewpoint puts it into its proper place in the over-all statistical problem.

6.2 REALIZABLE LINEAR FILTERS: STATIONARY PROCESSES, INFINITE PAST: WIENER FILTERS

In this section we discuss an important case relating to (20). First, we assume that the final observation time corresponds to the time at which the estimate is desired. Thus $t = T_f$ and (20) becomes

$$K_{dr}(t, \sigma) = \int_{T_i}^{t} h_o(t, u)\, K_r(u, \sigma)\, du; \qquad T_i < \sigma < t. \tag{55}$$

Second, we assume that $T_i = -\infty$. This assumption means that we have the infinite past available to operate on to make our estimate. From a practical standpoint it simply means that the past is available beyond the significant memory time of our filter. In a later section, when we discuss finite T_i, we shall make some quantitative statements about how large $t - T_i$ must be in order to be considered infinite.

Third, we assume that the received signal is a sample function from a stationary process and that the desired signal and the received signal are jointly stationary. (In Fig. 6.1 we see that this implies that $c(t)$ is constant. Thus we say the process is unmodulated.†) Then we may write

$$K_{dr}(t - \sigma) = \int_{-\infty}^{t} h_o(t, u)\, K_r(u - \sigma)\, du, \qquad -\infty < \sigma < t. \tag{56}$$

Because the processes are stationary and the interval is infinite, let us try to find a solution to (56) which is time-invariant.

$$K_{dr}(t - \sigma) = \int_{-\infty}^{t} h_o(t - u)\, K_r(u - \sigma)\, du, \qquad -\infty < \sigma < t. \tag{57}$$

If we can find a solution to (57), it will also be a solution to (56). If $K_r(u - \sigma)$ is positive-definite, (56) has a unique solution. Thus, if (57) has a solution, it will be unique and will also be the only solution to (56). Letting $\tau = t - \sigma$ and $v = t - u$, we have

$$\boxed{K_{dr}(\tau) = \int_{0}^{\infty} h_o(v)\, K_r(\tau - v)\, dv, \qquad 0 < \tau < \infty,} \tag{58}$$

which is commonly referred to as the Wiener-Hopf equation. It was derived and solved by Wiener [1]. (The linear processing problem was studied independently by Kolmogoroff [2].)

6.2.1 Solution of Wiener-Hopf Equation

Our solution to the Wiener-Hopf equation is analogous to the approach by Bode and Shannon [3]. Although the amount of manipulation required is identical to that in Wiener's solution, the present procedure is more intuitive. We restrict our attention to the case in which the Fourier transform of $K_r(\tau)$, the input correlation function, is a rational function. This is not really a practical restriction because most spectra of interest can be approximated by a rational function. The general case is discussed by Wiener [1] but does not lead to a practical solution technique.

The first step in our solution is to observe that *if* $r(t)$ were white the solution to (58) would be trivial. If

$$K_r(\tau) = \delta(\tau), \tag{59}$$

then (58) becomes

$$K_{dr}(\tau) = \int_{0}^{\infty} h_o(v)\, \delta(\tau - v)\, dv, \qquad 0 < \tau < \infty, \tag{60}$$

† Our use of the term *modulated* is the opposite of the normal usage in which the message process *modulates* a carrier. The adjective *unmodulated* seems to be the best available.

Fig. 6.4 Whitening filter.

and

$$\begin{aligned} h_o(\tau) &= K_{dr}(\tau), \qquad & \tau \geq 0, \\ &= 0, & \tau < 0, \end{aligned} \tag{61}$$

where the value at $\tau = 0$ comes from our continuity restriction.

It is unlikely that (59) will be satisfied in many problems of interest. If, however, we could perform some preliminary operation on $r(t)$ to transform it into a white process, as shown in Fig. 6.4, the subsequent filtering problem in terms of the whitened process would be trivial. The idea of a whitening operation is familiar from Section 4.3 of Chapter 4. In that case the signal was deterministic and we whitened only the noise. In this case we whiten the entire input. In Section 4.3 we proved that *any* reversible operation could not degrade the over-all system performance. Now we also want the over-all processor to be a *realizable* linear filter. Therefore we show the following property:

Whitening Property. For all rational spectra there exists a realizable, time-invariant linear filter whose output $z(t)$ is a white process when the input is $r(t)$ *and* whose inverse is a realizable linear filter.

If we denote the impulse response of the whitening filter as $w(\tau)$ and the transfer function as $W(j\omega)$, then the property says:

(i) $$\iint_{-\infty}^{\infty} w(u)\, w(v)\, K_r(\tau - u - v)\, du\, dv = \delta(\tau), \qquad -\infty < \tau < \infty.$$

or

(ii) $$|W(j\omega)|^2 S_r(\omega) = 1.$$

If we denote the impulse response of the inverse filter as $w^{-1}(\tau)$, then

(iii) $$\int_{-\infty}^{\infty} w^{-1}(u - v)\, w(v)\, dv = \delta(u)$$

or

(iv) $$\mathcal{F}[w^{-1}(\tau)] = \frac{1}{W(j\omega)} = W^{-1}(j\omega)$$

and $w^{-1}(\tau)$ must be the impulse response of a realizable filter.

We derive this property by demonstrating a constructive technique for a simple example and then extending it to arbitrary rational spectra.

Example 1. Let

$$S_r(\omega) = \frac{2k}{\omega^2 + k^2}. \tag{62}$$

We want to choose the transfer function of the whitening filter so that it is realizable and the spectrum of its output $z(t)$ satisfies the equation

$$S_z(\omega) = S_r(\omega)|W(j\omega)|^2 = 1. \tag{63}$$

To accomplish this we divide $S_r(\omega)$ into two parts,

$$S_r(\omega) = \left(\frac{\sqrt{2k}}{j\omega + k}\right)\left(\frac{\sqrt{2k}}{-j\omega + k}\right) \triangleq [G^+(j\omega)][G^+(j\omega)]^*. \tag{64}$$

We denote the first term by $G^+(j\omega)$ because it is zero for negative time. The second term is its complex conjugate. Clearly, if we let

$$W(j\omega) = \frac{1}{G^+(j\omega)} = \frac{j\omega + k}{\sqrt{2k}}, \tag{65}$$

then (63) will be satisfied.

We observe that the whitening filter consists of a differentiator and a gain term in parallel. Because

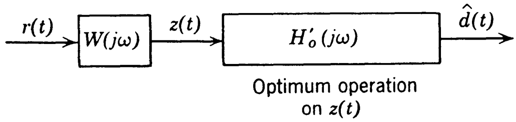
$$W^{-1}(j\omega) = G^+(j\omega) = \frac{\sqrt{2k}}{j\omega + k}, \tag{66}$$

it is clear that the inverse is a realizable linear filter and therefore $W(j\omega)$ is a legitimate reversible operation. Thus we could operate on $z(t)$ in either of the two ways shown in Fig. 6.5 and, as we proved in Section 4.3, if we choose $h_o'(\tau)$ in an optimum manner the output of both systems will be $\hat{d}(t)$.

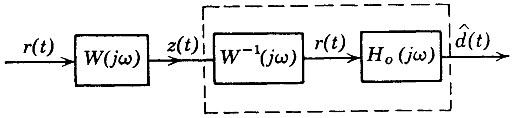
In this particular example the selection of $W(j\omega)$ was obvious. We now consider a more complicated example.

Example 2. Let

$$S_r(\omega) = \frac{c^2(j\omega + \alpha_1)(-j\omega + \alpha_1)}{(j\omega + \beta_1)(-j\omega + \beta_1)}. \tag{67}$$



(a)



(b)

Fig. 6.5 Optimum filter: (*a*) approach No. 1; (*b*) approach No. 2.

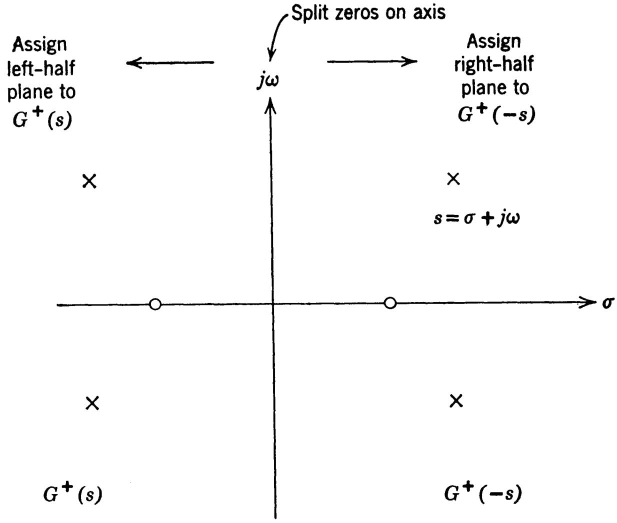


Fig. 6.6 A typical pole-zero plot.

We must choose $W(j\omega)$ so that

$$S_r(\omega) = |W^{-1}(j\omega)|^2 = |G^+(j\omega)|^2 \tag{68}$$

and both $W(j\omega)$ and $W^{-1}(j\omega)$ [or equivalently $G^+(j\omega)$ and $W(j\omega)$] are realizable. When discussing realizability, it is convenient to use the complex s-plane. We extend our functions to the entire complex plane by replacing $j\omega$ by s, where $s = \sigma + j\omega$. In order for $W(s)$ to be realizable, it cannot have any poles in the right half of the s-plane. Therefore we must assign the $(j\omega + \alpha_1)$ term to it. Similarly, for $W^{-1}(s)$ [or $G^+(s)$] to be realizable we assign to it the $(j\omega + \beta_1)$ term. The assignment of the constant is arbitrary because it adjusts only the white noise level. For simplicity we assume a unity level spectrum for $z(t)$ and divide the constant evenly. Therefore

$$G^+(j\omega) = c\,\frac{(j\omega + \alpha_1)}{(j\omega + \beta_1)}. \tag{69}$$

To study the general case we consider the pole-zero plot of the typical spectrum shown in Fig. 6.6. Assuming that this spectrum is typical, we then find that the procedure is clear. We factor $S_r(\omega)$ and assign all poles and zeros in the left half plane (and half of each pair of zeros on the axis) to $G^+(j\omega)$. The remaining poles and zeros will correspond exactly to the conjugate $[G^+(j\omega)]^*$. The fact that every rational spectrum can be divided in this manner follows directly from the fact that $S_r(\omega)$ is a real, even, nonnegative function of ω whose inverse transform is a correlation function. This implies the modes of behavior for the pole-zero plot shown in Fig. 6.7*a–c*:

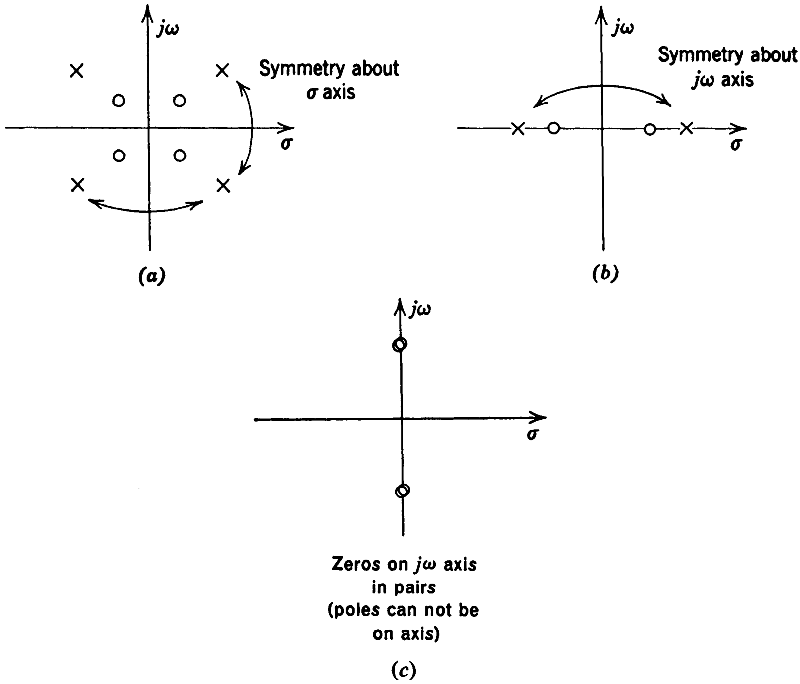


Fig. 6.7 Possible pole-zero plots in the s-plane.

1. Symmetry about the σ-axis. Otherwise $S_r(\omega)$ would not be real.
2. Symmetry about the $j\omega$-axis. Otherwise $S_r(\omega)$ would not also be even.
3. Any zeros on the $j\omega$-axis occur in pairs. Otherwise $S_r(\omega)$ would be negative for some value of ω.
4. No poles on the $j\omega$-axis. This would correspond to a $1/\omega^2$ term whose inverse is not the correlation function of a stationary process.

The verification of these properties is a straightforward exercise (see Problem 6.2.1).

We have now proved that we can always find a realizable, reversible whitening filter. The processing problem is now reduced to that shown in Fig. 6.8. We must design $H_o'(j\omega)$ so that it operates on $z(t)$ in such a way

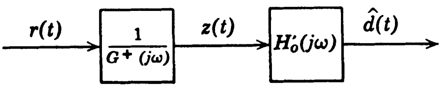


Fig. 6.8 Optimum filter.

that it produces the minimum mean-square error estimate of $d(t)$. Clearly, then, $h_o'(\tau)$ must satisfy (58) with r replaced by z,

$$K_{dz}(\tau) = \int_0^\infty h_o'(v)\, K_z(\tau - v)\, dv, \qquad 0 < \tau < \infty. \tag{70}$$

However, we have forced $z(t)$ to be white with unity spectral height. Therefore

$$h_o'(\tau) = K_{dz}(\tau), \qquad \tau \geq 0. \tag{71}$$

Thus, if we knew $K_{dz}(\tau)$, our solution would be complete. Because $z(t)$ is obtained from $r(t)$ by a linear operation, $K_{dz}(\tau)$ is easy to find,

$$\begin{aligned} K_{dz}(\tau) &\triangleq E\left[d(t)\int_{-\infty}^{\infty} w(v)\, r(t - \tau - v)\, dv\right] \\ &= \int_{-\infty}^{\infty} w(v)\, K_{dr}(\tau + v)\, dv = \int_{-\infty}^{\infty} w(-\beta)\, K_{dr}(\tau - \beta)\, d\beta. \end{aligned} \tag{72}$$

Transforming,

$$S_{dz}(j\omega) = W^*(j\omega)\, S_{dr}(j\omega) = \frac{S_{dr}(j\omega)}{[G^+(j\omega)]^*}. \tag{73}$$

We simply find the inverse transform of $S_{dz}(j\omega)$, $K_{dz}(\tau)$, and retain the part corresponding to $\tau \geq 0$. A typical $K_{dz}(\tau)$ is shown in Fig. 6.9*a*. The associated $h_o'(\tau)$ is shown in Fig. 6.9*b*.

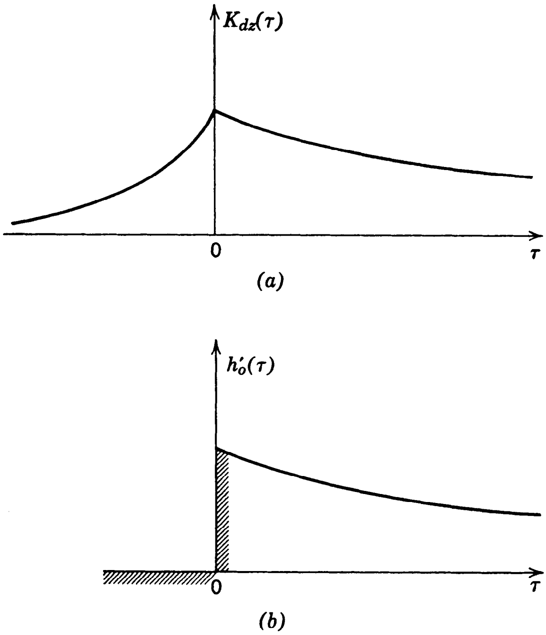


Fig. 6.9 Typical Functions: (*a*) a typical covariance function; (*b*) corresponding $h_o'(\tau)$.

We can denote the transform of $K_{dz}(\tau)$ for $\tau \geq 0$ by the symbol

$$[S_{dz}(j\omega)]_+ \triangleq \int_0^\infty K_{dz}(\tau)e^{-j\omega\tau}\,d\tau = \int_0^\infty h_o'(\tau)e^{-j\omega\tau}\,d\tau.\dagger \tag{74}$$

Similarly,

$$[S_{dz}(j\omega)]_- \triangleq \int_{-\infty}^{0^-} K_{dz}(\tau)e^{-j\omega\tau}\,d\tau. \tag{75}$$

Clearly,

$$S_{dz}(j\omega) = [S_{dz}(j\omega)]_+ + [S_{dz}(j\omega)]_-, \tag{76}$$

and we may write

$$H_o'(j\omega) = [S_{dz}(j\omega)]_+ = [W^*(j\omega)\,S_{dr}(j\omega)]_+ = \left[\frac{S_{dr}(j\omega)}{[G^+(j\omega)]^*}\right]_+. \tag{77}$$

Then the entire optimum filter is just a cascade of the whitening filter and $H_o'(j\omega)$,

$$\boxed{H_o(j\omega) = \left[\frac{1}{G^+(j\omega)}\right]\left[\frac{S_{dr}(j\omega)}{[G^+(j\omega)]^*}\right]_+.} \tag{78}$$

We see that by a series of routine, conceptually simple operations we have derived the desired filter. We summarize the steps briefly.

1. We factor the *input* spectrum into two parts. One term, $G^+(s)$, contains all the poles and zeros in the left half of the s-plane. The other factor is its mirror image about the $j\omega$-axis.
2. The cross-spectrum between $d(t)$ and $z(t)$ can be expressed in terms of the original cross-spectrum divided by $[G^+(j\omega)]^*$. This corresponds to a function that is nonzero for both positive and negative time. The realizable part of this function ($\tau \geq 0$) is $h_o'(\tau)$ and its transform is $H_o'(j\omega)$.
3. The transfer function of the optimum filter is a simple product of these two transfer functions. We shall see that the composite transfer function corresponds to a realizable system. Observe that we actually build the optimum linear filter as single system. The division into two parts is for conceptual purposes only.

Before we discuss the properties and implications of the solution, it will be worthwhile to consider a simple example to guarantee that we all agree on what (78) means.

Example 3. Assume that

$$r(t) = \sqrt{P}\,a(t) + n(t), \tag{79}$$

† In general, the symbol $[\sim]_+$ denotes the transform of the realizable part of the inverse transform of the expression inside the bracket.

where $a(t)$ and $n(t)$ are uncorrelated zero-mean stationary processes and

$$S_a(\omega) = \frac{2k}{\omega^2 + k^2}. \tag{80}$$

[We see that $a(t)$ has unity power so that P is the transmitted power.]

$$S_n(\omega) = \frac{N_0}{2}. \tag{81}$$

The desired signal is

$$d(t) = a(t + \alpha), \tag{82}$$

where α is a constant.

By choosing α to be positive we have the prediction problem, choosing α to be zero gives the conventional filtering problem, and choosing α to be negative gives the filtering-with-delay problem.

The solution is a simple application of the procedure outlined in the preceding section:

$$S_r(\omega) = \frac{2kP}{\omega^2 + k^2} + \frac{N_0}{2} = \frac{N_0}{2}\frac{\omega^2 + k^2(1 + 4P/kN_0)}{\omega^2 + k^2}. \tag{83}$$

It is convenient to define

$$\Lambda = \frac{4P}{kN_0}. \tag{84}$$

(This quantity has a physical significance we shall discuss later. For the moment, it can be regarded as a useful parameter.) First we factor the spectrum

$$S_r(\omega) = \frac{N_0}{2}\frac{\omega^2 + k^2(1 + \Lambda)}{\omega^2 + k^2} = G^+(j\omega)\,[G^+(j\omega)]^*. \tag{85}$$

so

$$G^+(j\omega) = \left(\frac{N_0}{2}\right)^{1/2}\left(\frac{j\omega + k\sqrt{1 + \Lambda}}{j\omega + k}\right). \tag{86}$$

Now

$$\begin{aligned} K_{dr}(\tau) &= E[d(t)\,r(t - \tau)] = E\{a(t + \alpha)[\sqrt{P}\,a(t - \tau) + n(t - \tau)]\} \\ &= \sqrt{P}\,E[a(t + \alpha)\,a(t - \tau)] = \sqrt{P}\,K_a(\tau + \alpha). \end{aligned} \tag{87}$$

Transforming,

$$S_{dr}(j\omega) = \sqrt{P}\,S_a(\omega)e^{+j\omega\alpha} = \frac{2k\sqrt{P}\,e^{+j\omega\alpha}}{\omega^2 + k^2} \tag{88}$$

and

$$S_{dz}(j\omega) = \frac{S_{dr}(j\omega)}{[G^+(j\omega)]^*} = \frac{2k\sqrt{P}\,e^{+j\omega\alpha}}{\omega^2 + k^2}\cdot\frac{(-j\omega + k)}{\sqrt{N_0/2}(-j\omega + k\sqrt{1 + \Lambda})}. \tag{89}$$

To find the realizable part, we take the inverse transform:

$$K_{dz}(\tau) = \mathcal{F}^{-1}\left\{\frac{S_{dr}(j\omega)}{[G^+(j\omega)]^*}\right\} = \mathcal{F}^{-1}\left[\frac{2k\sqrt{P}\,e^{+j\omega\alpha}}{(j\omega + k)\sqrt{N_0/2}(-j\omega + k\sqrt{1 + \Lambda})}\right]. \tag{90}$$

The inverse transform can be evaluated easily (either by residues or a partial fraction expansion and the shifting theorem). The result is

$$K_{dz}(\tau) = \begin{cases} \dfrac{2\sqrt{P}}{\sqrt{N_0/2}}\dfrac{1}{1 + \sqrt{1 + \Lambda}}\,e^{-k(\tau + \alpha)}, & \tau + \alpha \geq 0, \\[2ex] \dfrac{2\sqrt{P}}{\sqrt{N_0/2}}\dfrac{1}{1 + \sqrt{1 + \Lambda}}\,e^{+k\sqrt{1+\Lambda}(\tau + \alpha)}, & \tau + \alpha < 0. \end{cases} \tag{91}$$

The function is shown in Fig. 6.10.

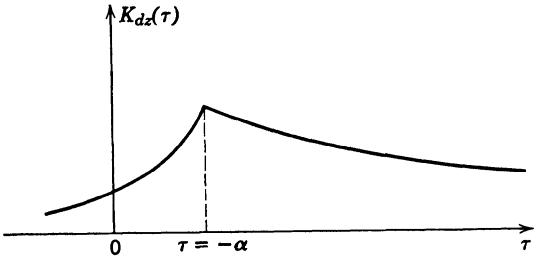


Fig. 6.10 Cross-covariance function.

Now $h_o'(\tau)$ depends on the value α. In other words, the amount of $K_{dz}(\tau)$ in the range $\tau \geq 0$ is a function of α. We consider three types of operations:

Case 1. $\alpha = 0$: filtering with zero delay. Letting $\alpha = 0$ in (91), we have

$$h_o'(\tau) = \frac{2\sqrt{P}}{\sqrt{N_0/2}} \frac{1}{1 + \sqrt{1 + \Lambda}} e^{-k\tau} u_{-1}(\tau), \tag{92}$$

or

$$H_o'(j\omega) = \frac{1}{1 + \sqrt{1 + \Lambda}} \frac{2\sqrt{P}}{\sqrt{N_0/2}} \frac{1}{j\omega + k}. \tag{93}$$

Then

$$H_0(j\omega) = \frac{H_o'(j\omega)}{G^+(j\omega)} = \frac{2\sqrt{P}}{(N_0/2)(1 + \sqrt{1 + \Lambda})} \frac{1}{j\omega + k\sqrt{1 + \Lambda}}. \tag{94}$$

We see that our result is intuitively logical. The amplitude of the filter response is shown in Fig. 6.11. The filter is a simple low-pass filter whose bandwidth varies as a function of k and Λ.

We now want to attach some physical significance to the parameter Λ. The bandwidth of the message process is directly proportional to k, as shown in Fig. 6.12*a*

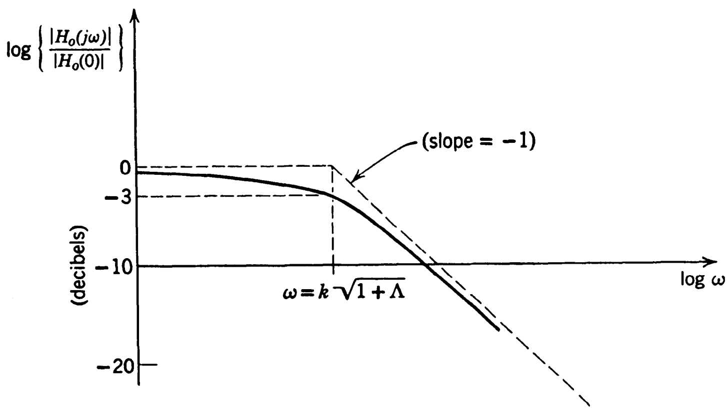


Fig. 6.11 Magnitude plot for optimum filter.

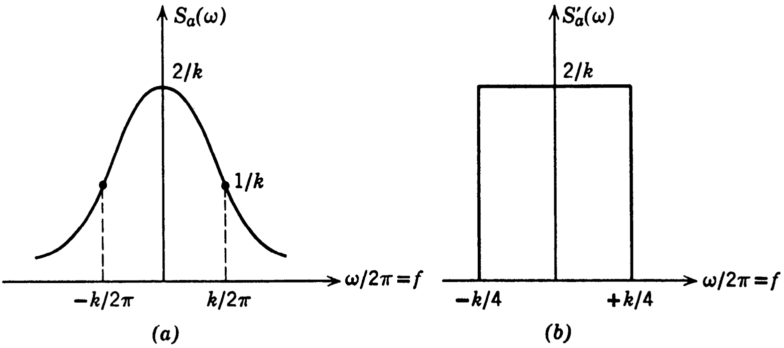


Fig. 6.12 Equivalent rectangular spectrum.

The 3-db bandwidth is k/π cps. Another common bandwidth measure is the equivalent rectangular bandwidth (ERB), which is the bandwidth of a rectangular spectrum with height $S_a(0)$ and the same total power of the actual message as shown in Fig. 6.12*b*. Physically, Λ is the signal-to-noise ratio in the message ERB. This ratio is most natural for most of our work. The relationship between Λ and the signal-to-noise ratio in the 3-db bandwidth depends on the particular spectrum. For *this particular case* $\Lambda_{3db} = (\pi/2)\Lambda$.

We see that for a fixed k the optimum filter bandwidth increases as Λ, the signal-to-noise ratio, increases. Thus, as $\Lambda \to \infty$, the filter magnitude approaches unity for all frequencies and it passes the message component without distortion. Because the noise is unimportant in this case, this is intuitively logical. On the other hand, as $\Lambda \to 0$, the filter 3-db point approaches k. The gain, however, approaches zero. Once again, this is intuitively logical. There is so much noise that, based on the mean square error criterion, the best filter output is zero (the mean value of the message).

Case 2. α is negative: filtering with delay. Here $h_o'(\tau)$ has the impulse response shown in Fig. 6.13. Transforming, we have

$$H_o'(j\omega) = \frac{2k\sqrt{P}}{\sqrt{N_0/2}}\left[\frac{e^{\alpha j\omega}}{(j\omega + k)(-j\omega + k\sqrt{1+\Lambda})} - \frac{-\,e^{\alpha k\sqrt{1+\Lambda}}}{k(1+\sqrt{1+\Lambda})(-j\omega + k\sqrt{1+\Lambda})}\right] \tag{95}$$

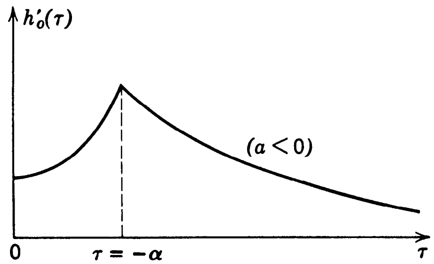


Fig. 6.13 Filtering with delay.

and

$$H_o(j\omega) = \frac{H_o'(j\omega)}{G^+(j\omega)} = \frac{2k\sqrt{P}}{N_0/2}\left\{\frac{e^{\alpha j\omega}}{[\omega^2 + k^2(1+\Lambda)]} - \frac{e^{\alpha k\sqrt{1+\Lambda}}(j\omega + k)}{k(1+\sqrt{1+\Lambda})[\omega^2 + k^2(1+\Lambda)]}\right\}. \tag{96a}$$

This can be rewritten as

$$H_o(j\omega) = \frac{2k\sqrt{P}\,e^{\alpha j\omega}}{(N_0/2)[\omega^2 + k^2(1+\Lambda)]}\left[1 - \frac{(j\omega + k)e^{\alpha(k\sqrt{1+\Lambda} - j\omega)}}{k(1+\sqrt{1+\Lambda})}\right]. \tag{96b}$$

We observe that the expression outside the bracket is just

$$\frac{S_{dr}(j\omega)}{S_r(\omega)}\,e^{\alpha j\omega}. \tag{97}$$

We see that when α is a large negative number the second term in the bracket is approximately zero. Thus $H_o(j\omega)$ approaches the expression in (97). This is just the ratio of the cross spectrum to the total input spectrum, with a delay to make the filter realizable.

We also observe that the impulse response in Fig. 6.13 is difficult to realize with conventional network synthesis techniques.

Case 3. α is positive: filtering with prediction. Here

$$h_o'(\tau) = \left(\frac{2\sqrt{P}}{\sqrt{N_0/2}}\frac{1}{1+\sqrt{1+\Lambda}}\,e^{-k\tau}\right)e^{-k\alpha}. \tag{98}$$

Comparing (98) with (92), we see that the optimum filter for prediction is just the optimum filter for estimating $a(t)$ multiplied by a gain $e^{-k\alpha}$, as shown in Fig. 6.14. The reason for this is that $a(t)$ is a first order wide-sense Markov process and the noise is white. We obtain a similar result for more general processes in Section 6.3.

Before concluding our discussion we amplify a point that was encountered in Case 1 of the example. One step of the solution is to find the realizable part of a function. Frequently it is unnecessary to find the time function and then retransform. Specifically, whenever $S_{dr}(j\omega)$ is a ratio of two polynomials in $j\omega$, we may write

$$\frac{S_{dr}(j\omega)}{[G^+(j\omega)]^*} = F(j\omega) + \sum_{i=1}^{N}\frac{c_i}{j\omega + p_i} + \sum_{j=1}^{N}\frac{d_j}{-j\omega + q_j}, \tag{99a}$$

where $F(j\omega)$ is a polynomial, the first sum contains all terms corresponding to poles in the left half of the s-plane (including the $j\omega$-axis), and the second sum contains all terms corresponding to poles in the right half of

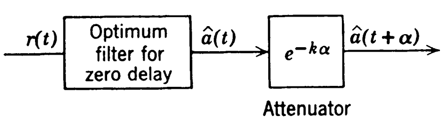


Fig. 6.14 Filtering with prediction.

the s-plane. In this expanded form the realizable part consists of the first two terms. Thus

$$\left[\frac{S_{dr}(j\omega)}{[G^+(j\omega)]^*}\right]_+ = F(j\omega) + \sum_{i=1}^{N} \frac{c_i}{j\omega + p_i}. \tag{99b}$$

The use of (99b) reduces the required manipulation.

In this section we have developed an algorithm for solving the Wiener-Hopf equation and presented a simple example to demonstrate the technique. Next we investigate the resulting mean-square error.

6.2.2 Errors in Optimum Systems

In order to evaluate the performance of the optimum linear filter we calculate the minimum mean-square error. The minimum mean-square error for the general case was given in (24) of Property 4. Because the processes are stationary and the filter is time-invariant, the mean-square error will not be a function of time. Thus (24) reduces to

$$\xi_P = K_d(0) - \int_0^\infty h_o(\tau)\, K_{dr}(\tau)\, d\tau. \tag{100}$$

Because $h_o(\tau) = 0$ for $\tau < 0$, we can equally well write (100) as

$$\xi_P = K_d(0) - \int_{-\infty}^\infty h_o(\tau)\, K_{dr}(\tau)\, d\tau. \tag{101}$$

Now

$$H_o(j\omega) = \frac{1}{G^+(j\omega)} \int_0^\infty K_{dz}(t) e^{-j\omega t}\, dt, \tag{102}$$

where

$$K_{dz}(t) = \mathcal{F}^{-1}\left\{\frac{S_{dr}(j\omega)}{[G^+(j\omega)]^*}\right\} = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{S_{dr}(j\omega)}{[G^+(j\omega)]^*} e^{j\omega t}\, d\omega. \tag{103}$$

Substituting the inverse transform of (102) into (101), we obtain,

$$\xi_P = K_d(0) - \int_{-\infty}^\infty K_{dr}(\tau)\, d\tau \left[\frac{1}{2\pi}\int_{-\infty}^\infty e^{j\omega\tau}\, d\omega \cdot \frac{1}{G^+(j\omega)} \int_0^\infty K_{dz}(t) e^{-j\omega t}\, dt\right]. \tag{104}$$

Changing orders of integration, we have

$$\xi_P = K_d(0) - \int_0^\infty K_{dz}(t)\, dt \left[\frac{1}{2\pi}\int_{-\infty}^\infty e^{-j\omega t}\, d\omega\, \frac{1}{G^+(j\omega)} \int_{-\infty}^\infty K_{dr}(\tau) e^{j\omega\tau}\, d\tau\right]. \tag{105}$$

The part of the integral inside the brackets is just $K_{dz}^*(t)$. Thus, since $K_{dz}(t)$ is real,

$$\xi_P = K_d(0) - \int_0^\infty K_{dz}^2(t)\, dt. \tag{106}$$

The result in (106) is a convenient expression for the mean-square error. Observe that we must factor the input spectrum and perform an inverse transform in order to evaluate it. (The same shortcuts discussed above are applicable.)

We can use (106) to study the effect of α on the mean-square error. Denote the desired signal when $\alpha = 0$ as $d_0(t)$ and the desired signal for arbitrary α as $d_\alpha(t) \triangleq d_0(t + \alpha)$. Then

$$E[d_0(t)\, z(t - \tau)] = K_{d_0 z}(\tau) \triangleq \phi(\tau), \tag{107a}$$

and

$$E[d_\alpha(t)\, z(t - \tau)] = E[d_0(t + \alpha)\, z(t - \tau)] = \phi(\tau + \alpha). \tag{107b}$$

We can now rewrite (106) in terms of $\phi(\tau)$. Letting

$$K_{dz}(t) = \phi(t + \alpha) \tag{108a}$$

in (106), we have

$$\xi_P{}^\alpha = K_d(0) - \int_0^\infty \phi^2(t + \alpha)\, dt = K_d(0) - \int_\alpha^\infty \phi^2(u)\, du. \tag{108b}$$

Note that $\phi(u)$ is not a function of α. We observe that because the integrand is a positive quantity the error is monotone increasing with increasing α. Thus the smallest error is achieved when $\alpha = -\infty$ (infinite delay) and increases monotonely to unity as $\alpha \to +\infty$. This result says that for *any* desired signal the minimum mean-square error will decrease if we allow delay in the processing. The mean-square error for infinite delay provides a lower bound on the mean-square error for any finite delay and is frequently called the *irreducible error*. A more interesting quantity in some cases is the normalized error. We define the normalized error as

$$\xi_{Pn}{}^\alpha \triangleq \frac{\xi_P{}^\alpha}{K_d(0)}, \tag{109a}$$

or

$$\xi_{Pn}{}^\alpha = 1 - \frac{1}{K_d(0)} \int_\alpha^\infty \phi^2(u)\, du. \tag{109b}$$

We may now apply our results to the preceding example.

Example 3 (continued). For our example

$$\xi_{Pn}{}^\alpha = \begin{cases} 1 - \dfrac{8P}{N_0} \dfrac{1}{(1 + \sqrt{1 + \Lambda})^2} \left(\displaystyle\int_\alpha^0 dt\, e^{+2k\sqrt{1+\Lambda}\,t} + \int_0^\infty e^{-2kt}\, dt \right), & \alpha \le 0, \\ 1 - \dfrac{8P}{N_0} \dfrac{1}{(1 + \sqrt{1 + \Lambda})^2} \displaystyle\int_\alpha^\infty e^{-2kt}\, dt, & \alpha \ge 0. \end{cases} \tag{110}$$

Evaluating the integrals, we have

$$\xi_{Pn}{}^{\alpha} = \frac{1}{\sqrt{1+\Lambda}} + \frac{\Lambda e^{+2k\sqrt{1+\Lambda}\,\alpha}}{(1+\sqrt{1+\Lambda})^2\sqrt{1+\Lambda}}, \qquad \alpha \le 0, \tag{111}$$

$$\xi_{Pn}{}^{0} = \frac{2}{1+\sqrt{1+\Lambda}}, \tag{112}$$

and

$$\xi_{Pn}{}^{\alpha} = \frac{2}{(1+\sqrt{1+\Lambda})} + \frac{\Lambda[1-e^{-2k\alpha}]}{(1+\sqrt{1+\Lambda})^2}, \qquad \alpha \ge 0. \tag{113}$$

The two limiting cases for (111) and (113) are $\alpha = -\infty$ and $\alpha = \infty$, respectively.

$$\xi_{Pn}{}^{-\infty} = \frac{1}{\sqrt{1+\Lambda}}. \tag{115}$$

$$\xi_{Pn}{}^{\infty} = 1, \tag{114}$$

A plot of $\xi_{Pn}{}^{\alpha}$ versus $(k\alpha)$ is shown in Fig. 6.15. Physically, the quantity $k\alpha$ is related to the reciprocal of the message bandwidth. If we define

$$\tau_c = \frac{1}{k}, \tag{116}$$

the units on the horizontal axis are α/τ_c, which corresponds to the delay measured in correlation times. We see that the error for a delay of one time constant is approximately the infinite delay error. Note that the error is not a symmetric function of α.

Before summarizing our discussion of realizable filters, we discuss the related problem of unrealizable filters.

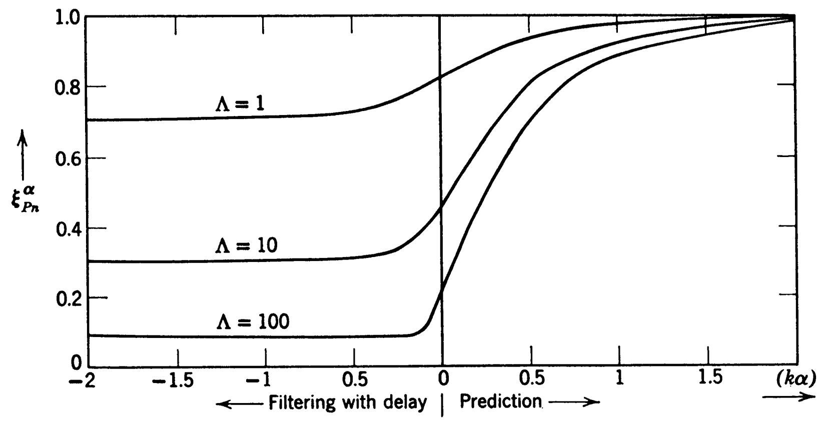


Fig. 6.15 Effect of time-shift on filtering error.

6.2.3 Unrealizable Filters

Instead of requiring the processor to be realizable, let us consider an optimum *unrealizable* system. This corresponds to letting $T_f > t$. In other words, we use the input $r(u)$ at times *later* than t to determine the estimate at t.

For the case in which $T_f > t$ we can modify (55) to obtain

$$K_{dr}(\tau) = \int_{t-T_f}^{\infty} h_o(t, t\text{-}v)\, K_r(\tau - v)\, dv, \qquad t - T_f < \tau < \infty. \quad (117)$$

In this case $h_o(t, t\text{-}v)$ is nonzero for all $v \geq t - T_f$. Because this includes values of v less than zero, the filter will be unrealizable. The case of most interest to us is the one in which $T_f = \infty$. Then (117) becomes

$$K_{dr}(\tau) = \int_{-\infty}^{\infty} h_{ou}(v)\, K_r(\tau - v)\, dv, \qquad -\infty < \tau < \infty. \quad (118)$$

We add the subscript u to emphasize that the filter is unrealizable. Because the equation is valid for all τ, we may solve by transforming

$$\boxed{H_{ou}(j\omega) = \frac{S_{dr}(j\omega)}{S_r(\omega)}.} \quad (119)$$

From Property 4, the mean-square error is

$$\xi_u = K_d(0) - \int_{-\infty}^{\infty} h_{ou}(\tau)\, K_{dr}(\tau)\, d\tau. \quad (120)$$

Note that ξ_u is a mean-square *point* estimation error. By Parseval's Theorem,

$$\xi_u = \frac{1}{2\pi} \int_{-\infty}^{\infty} [S_d(\omega) - H_{ou}(j\omega)\, S_{dr}^*(j\omega)]\, d\omega. \quad (121)$$

Substituting (119) into (121), we obtain

$$\boxed{\xi_u = \int_{-\infty}^{\infty} \frac{S_d(\omega)\, S_r(\omega) - |S_{dr}(j\omega)|^2}{S_r(\omega)} \frac{d\omega}{2\pi}.} \quad (122)$$

For the special case in which

$$\begin{aligned} d(t) &= a(t), \\ r(t) &= a(t) + n(t), \end{aligned} \quad (123)$$

and the message and noise are uncorrelated, (122) reduces to

$$\boxed{\xi_u = \int_{-\infty}^{\infty} \frac{S_n(\omega)\, S_a(\omega)}{S_a(\omega) + S_n(\omega)} \frac{d\omega}{2\pi}.} \quad (124)$$

In the example considered on p. 488, the noise is white. Therefore,

$$H_{ou}(j\omega) = \frac{\sqrt{P}\, S_a(\omega)}{S_r(\omega)}, \tag{125}$$

and

$$\xi_u = \frac{N_0}{2} \int_{-\infty}^{\infty} H_{ou}(j\omega) \frac{d\omega}{2\pi} = \frac{N_0}{2} \int_{-\infty}^{\infty} \frac{\sqrt{P}\, S_a(\omega)}{S_r(\omega)} \frac{d\omega}{2\pi}. \tag{126}$$

We now return to the general case. It is easy to demonstrate that the expression in (121) is also equal to

$$\xi_u = K_d(0) - \int_{-\infty}^{\infty} \phi^2(t)\, dt. \tag{127}$$

Comparing (127) with (107) we see that the effect of using an unrealizable filter is the same as allowing an infinite delay in the desired signal. This result is intuitively logical. In an unrealizable filter we allow ourselves (fictitiously, of course) to use the entire past and future of the input and produce the desired signal at the present time. A practical way to approximate this processing is to wait until more of the future input comes in and produce the desired output at a later time. In many, if not most, communications problems it is the unrealizable error that is a fundamental system limitation.

The essential points to remember when discussing unrealizable filters are the following:

1. The mean-square error using an unrealizable linear filter ($T_f = \infty$) provides a lower bound on the mean-square error for any realizable linear filter. It corresponds to the *irreducible* (or infinite delay) error that we encountered on p. 494. The computation of ξ_u (124) is usually easier than the computation of ξ_P (100) or (106). Therefore it is a logical preliminary calculation even if we are interested only in the realizable filtering problem.
2. We can build a realizable filter whose performance approaches the performance of the unrealizable filter by allowing delay in the output. We can obtain a mean-square error that is arbitrarily close to the irreducible error by increasing this delay. From the practical standpoint a delay of several times the reciprocal of the effective bandwidth of $[S_a(\omega) + S_n(\omega)]$ will usually result in a mean-square error close to the irreducible error.

We now return to the realizable filtering problem. In Sections 6.2.1 and 6.2.2 we devised an algorithm that gave us a constructive method for finding the optimum realizable filter and the resulting mean-square error. In other words, given the necessary information, we can always (conceptually, at least) proceed through a specified procedure and obtain the optimum filter and resulting performance. In practice, however, the algebraic complexity has caused most engineers studying optimum filters to use the one-pole spectrum as the canonic message spectrum. The lack of a closed-form mean-square error expression which did not require a spectrum factorization made it essentially impossible to study the effects of different message spectra.

In the next section we discuss a special class of linear estimation problems and develop a *closed-form* expression for the minimum mean-square error.

6.2.4 Closed-Form Error Expressions

In this section we shall derive some useful closed-form results for a special class of optimum linear filtering problems. The case of interest is when

$$r(u) = a(u) + n(u), \qquad -\infty < u \le t. \tag{128}$$

In other words, the received signal consists of the message plus additive noise. The desired signal $d(t)$ is the message $a(t)$. We assume that the noise and message are uncorrelated. The message spectrum is rational with a finite variance. Our goal is to find an expression for the error that does not require spectrum factorization. The major results in this section were obtained originally by Yovits and Jackson [4]. It is convenient to consider white and nonwhite noise separately.

Errors in the Presence of White Noise. We assume that $n(t)$ is white with spectral height $N_0/2$. Although the result was first obtained by Yovits and Jackson, appreciably simpler proofs have been given by Viterbi and Cahn [5], Snyder [6], and Helstrom [57]. We follow a combination of these proofs. From (128)

$$S_r(\omega) = S_a(\omega) + \frac{N_0}{2}, \tag{129}$$

and

$$G^+(j\omega) = \left[S_a(j\omega) + \frac{N_0}{2}\right]^+. \tag{130}$$

From (78)

$$H_o(j\omega) = \frac{1}{[S_a(\omega) + N_0/2]^+}\left\{\frac{S_a(\omega)}{[S_a(\omega) + N_0/2]^-}\right\}_+ \tag{131†}$$

†To avoid a double superscript we introduce the notation

$$G^-(j\omega) = [G^+(j\omega)]^*.$$

Recall that conjugation in the frequency domain corresponds to reversal in the time domain. The time function corresponding to $G^+(j\omega)$ is zero for negative time. Therefore the time function corresponding to $G^-(j\omega)$ is zero for positive time.

or

$$H_o(j\omega) = \frac{1}{[S_a(\omega) + N_0/2]^+} \left\{ \frac{S_a(\omega) + N_0/2}{[S_a(\omega) + N_0/2]^-} - \frac{N_0/2}{[S_a(\omega) + N_0/2]^-} \right\}_+. \tag{132}$$

Now, the first term in the bracket is just $[S_a(\omega) + N_0/2]^+$, which is realizable. Because the realizable part operator is linear, the first term comes out of the bracket without modification. Therefore

$$H_o(j\omega) = 1 - \frac{1}{[S_a(\omega) + N_0/2]^+} \left\{ \frac{N_0/2}{[S_a(\omega) + N_0/2]^-} \right\}_+, \tag{133}$$

We take $\sqrt{N_0/2}$ out of the brace and put the remaining $\sqrt{N_0/2}$ inside the $[\cdot]^-$. The operation $[\cdot]^-$ is a factoring operation so we obtain $N_0/2$ inside.

$$H_o(j\omega) = 1 - \frac{\sqrt{N_0/2}}{[S_a(\omega) + N_0/2]^+} \left\{ \frac{1}{\left[\dfrac{S_a(\omega) + N_0/2}{N_0/2}\right]^-} \right\}_+. \tag{134}$$

The next step is to prove that the realizable part of the term in the brace equals one.

Proof. Let $S_a(\omega)$ be a rational spectrum. Thus

$$S_a(\omega) = \frac{N(\omega^2)}{D(\omega^2)}, \tag{135}$$

where the denominator is a polynomial in ω^2 whose order is at least one higher than the numerator polynomial. Then

$$\frac{S_a(\omega) + N_0/2}{N_0/2} = \frac{N(\omega^2) + (N_0/2)\, D(\omega^2)}{(N_0/2)\, D(\omega^2)} \tag{136}$$

$$= \frac{D(\omega^2) + (2/N_0)\, N(\omega^2)}{D(\omega^2)} \tag{137}$$

$$= \prod_{i=1}^{n} \frac{\omega^2 + \alpha_i^2}{\omega^2 + \beta_i^2}. \tag{138}$$

Observe that there is no additional multiplier because the highest order term in the numerator and denominator are identical.

The α_i and β_i may always be chosen so that their real parts are positive. If any of the α_i or β_i are complex, the conjugate is also present. Inverting both sides of (138) and factoring the result, we have

$$\left\{ \left[\frac{S_a(\omega) + N_0/2}{N_0/2} \right]^- \right\}^{-1} = \prod_{i=1}^{n} \frac{(-j\omega + \beta_i)}{(-j\omega + \alpha_i)} \tag{139}$$

$$= \prod_{i=1}^{n} \left[1 + \frac{\beta_i - \alpha_i}{(-j\omega + \alpha_i)} \right]. \tag{140}$$

The transform of all terms in the product except the unity term will be zero for positive time (their poles are in the right-half s-plane). Multiplying the terms together corresponds to convolving their transforms. Convolving functions which are zero for positive time always gives functions that are zero for positive time. Therefore only the unity term remains when we take the realizable part of (140). This is the desired result. Therefore

$$\boxed{H_o(j\omega) = 1 - \frac{\sqrt{N_0/2}}{[S_a(j\omega) + N_0/2]^+}.} \tag{141}$$

The next step is to derive an expression for the error. From Property 4C (27–28) we know that

$$\xi_P = \frac{N_0}{2} \lim_{\tau \to t^-} h_o(t, \tau) = \frac{N_0}{2} \lim_{\epsilon \to 0^+} h_o(\epsilon) \triangleq \frac{N_0}{2} h_o(0^+) \tag{142}$$

for the time-invariant case. We also know that

$$\int_{-\infty}^{\infty} H_o(j\omega) \frac{d\omega}{2\pi} = \frac{h_o(0^+) + h_o(0^-)}{2} = \frac{h_o(0^+)}{2}, \tag{143}$$

because $h_0(\tau)$ is realizable. Combining (142) and (143), we obtain

$$\xi_P = N_0 \int_{-\infty}^{\infty} H_o(j\omega) \frac{d\omega}{2\pi}. \tag{144}$$

Using (141) in (144), we have

$$\xi_P = N_0 \int_{-\infty}^{\infty} \left(1 - \left\{\left[\frac{S_a(\omega) + N_0/2}{N_0/2}\right]^+\right\}^{-1}\right) \frac{d\omega}{2\pi}. \tag{145}$$

Using the conjugate of (139) in (145), we obtain

$$\xi_P = N_0 \int_{-\infty}^{\infty} \left[1 - \prod_{i=1}^{n} \frac{(j\omega + \beta_i)}{(j\omega + \alpha_i)}\right] \frac{d\omega}{2\pi}, \tag{146}$$

$$\xi_P = N_0 \int_{-\infty}^{\infty} \left[1 - \prod_{i=1}^{n} \left(1 + \frac{\beta_i - \alpha_i}{j\omega + \alpha_i}\right)\right] \frac{d\omega}{2\pi}. \tag{147}$$

Expanding the product, we have

$$\xi_P = N_0 \int_{-\infty}^{\infty} \left\{1 - \left[1 + \sum_{i=1}^{n} \frac{\beta_i - \alpha_i}{j\omega + \alpha_i} + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\gamma_{ij}}{(j\omega + \alpha_i)(j\omega + \alpha_j)} + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \cdots \right]\right\} \frac{d\omega}{2\pi}, \tag{148}$$

$$\xi_P = N_0 \int_{-\infty}^{\infty} \sum_{i=1}^{n} \frac{(\alpha_i - \beta_i)}{(j\omega + \alpha_i)} \frac{d\omega}{2\pi} - N_0 \int_{-\infty}^{\infty} \left[\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\gamma_{ij}}{(j\omega + \alpha_i)(j\omega + \alpha_j)} + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \cdots \right] \frac{d\omega}{2\pi}. \tag{149}$$

The integral in the first term is just one half the sum of the residues (this result can be verified easily). We now show that the second term is zero. Because the integrand in the second term is analytic in the right half of the s-plane, the integral $[-\infty, \infty]$ equals the integral around a semicircle with infinite radius. All terms in the brackets, however, are at least of order $|s|^{-2}$ for large $|s|$. Therefore the integral on the semicircle is zero, which implies that the second term is zero. Therefore

$$\boxed{\xi_P = \frac{N_0}{2} \sum_{i=1}^{n} (\alpha_i - \beta_i).} \tag{150}$$

The last step is to find a closed-form expression for the sum of the residues. This follows by observing that

$$\int_{-\infty}^{\infty} \ln\left(\frac{\omega^2 + \alpha_i^2}{\omega^2 + \beta_i^2}\right) \frac{d\omega}{2\pi} = (\alpha_i - \beta_i). \tag{151}$$

(To verify this equation integrate the left-hand side by parts with $u = \ln\,[(\omega^2 + \alpha_i^2)/(\omega^2 + \beta_i^2)]$ and $dv = d\omega/2\pi$.)

Comparing (150), (151), and (138), we have

$$\boxed{\xi_P = \frac{N_0}{2} \int_{-\infty}^{\infty} \ln\left[1 + \frac{S_a(\omega)}{N_0/2}\right] \frac{d\omega}{2\pi},} \tag{152}$$

which is the desired result. Both forms of the error expressions (150) and (152) are useful. The first form is often the most convenient way to actually evaluate the error. The second form is useful when we want to find the $S_a(\omega)$ that minimizes ξ_P subject to certain constraints.

It is worthwhile to emphasize the importance of (152). In conventional Wiener theory to investigate the effect of various message spectra we had to actually factor the input spectrum. The result in (152) enables us to explore the error behavior directly. In later chapters we shall find it essential to the solution for the optimum pre-emphasis problem in angle modulation and other similar problems.

We observe in passing that the integral on the right-hand side is equal to twice the average mutual information (as defined by Shannon) between $r(t)$ and $a(t)$.

Errors for Typical Message Spectra. In this section we consider two families of message spectra. For each family we use (152) to compute the error when the optimum realizable filter is used. To evaluate the improvement obtained by allowing delay we also use (124) to calculate the unrealizable error.

Case 1. Butterworth Family. The message processes in the first class have spectral densities that are inverse Butterworth polynomials of order $2n$. From (3.98),

$$S_a(\omega:n) = \frac{2nP}{k}\frac{\sin(\pi/2n)}{1 + (\omega/k)^{2n}} \triangleq \frac{c_n}{1 + (\omega/k)^{2n}}. \tag{153}$$

The numerator is just a gain adjusted so that the power in the message spectrum is P.

Some members of the family are shown in Fig. 6.16. For $n = 1$ we have the one-pole spectrum of Section 6.2.1. The break-point is at $\omega = k$ rad/sec and the magnitude decreases at 6 db/octave above this point. For higher n the break-point remains the same, but the magnitude decreases at $6n$ db/octave. For $n = \infty$ we have a rectangular bandlimited spectrum

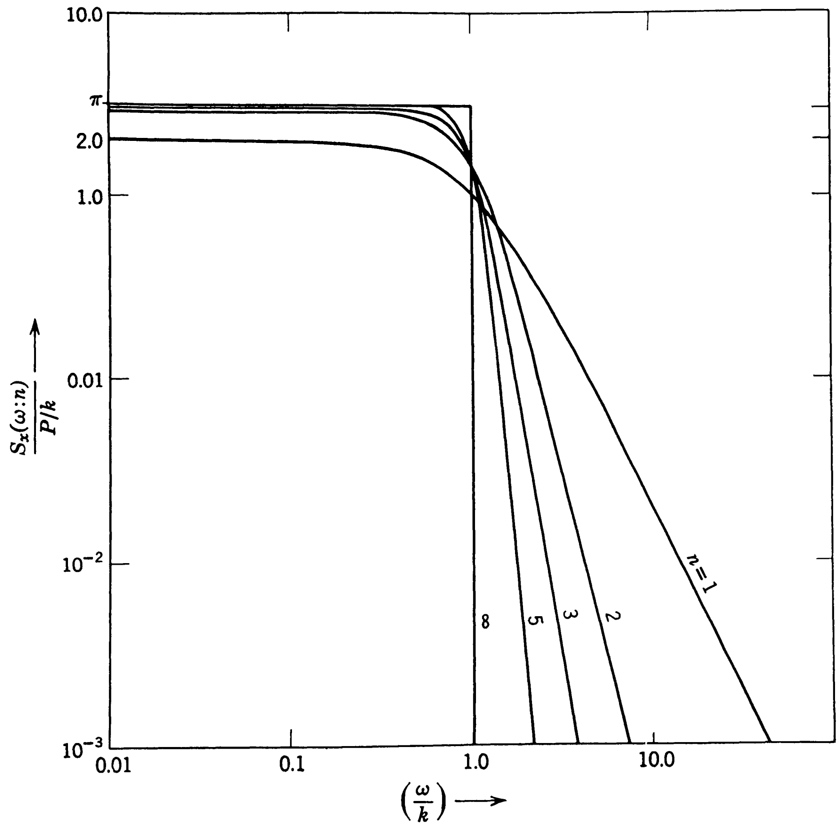


Fig. 6.16 Butterworth spectra.

[height $P\pi/k$; width k rad/sec]. We observe that if we use k/π cps as the reference bandwidth (double-sided) the signal-to-noise ratio will not be a function of n and will provide a useful reference:

$$\Lambda_B \triangleq \frac{P}{k/\pi \cdot N_0/2} = \frac{2\pi P}{kN_0}. \tag{154}$$

To find ξ_P we use (152):

$$\xi_P = \frac{N_0}{2}\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \ln\left[1 + \frac{2c_n/N_0}{1 + (\omega/k)^{2n}}\right]. \tag{155}$$

This can be integrated (see Problem 6.2.18) to give the following expression for the normalized error:

$$\xi_{Pn} = \frac{\pi}{\Lambda_B}\left(\sin\frac{\pi}{2n}\right)^{-1}\left[\left(1 + 2n\frac{\Lambda_B}{\pi}\sin\frac{\pi}{2n}\right)^{1/2n} - 1\right]. \tag{156}$$

Similarly, to find the unrealizable error we use (124):

$$\xi_u = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{c_n}{1 + (\omega/k)^{2n} + (2/N_0)c_n}. \tag{157}$$

This can be integrated (see Problem 6.2.19) to give

$$\xi_{un} = \left[1 + \frac{2n}{\pi}\Lambda_B \sin\frac{\pi}{2n}\right]^{(1/2n)-1}. \tag{158}$$

The reciprocal of the normalized error is plotted versus Λ_B in Fig. 6.17. We observe the vast difference in the error behavior as a function of n. The most difficult spectrum to filter is the one-pole spectrum. We see that asymptotically it behaves linearly whereas the bandlimited spectrum behaves exponentially. We also see that for $n = 3$ or 4 the performance is reasonably close to the bandlimited ($n = \infty$) case. Thus the one-pole message spectrum, which is commonly used as an example, has the worst error performance.

A second observation is the difference in improvement obtained by use of an unrealizable filter. For the one-pole spectrum

$$\xi_{un} > \tfrac{1}{2}\xi_{Pn}, \qquad (n = 1). \tag{159}$$

In other words, the maximum possible ratio is 2 (or 3 db). At the other extreme for $n = \infty$

$$\xi_{Pn} = \frac{1}{\Lambda_B}\ln(1 + \Lambda_B), \tag{160}$$

whereas

$$\xi_{un} = \frac{1}{1 + \Lambda_B}. \tag{161}$$