In both cases (318) assumes $E[\mathbf{x}(T_i)] = \mathbf{0}$. The modification for other initial conditions is straightforward (see Problem 6.3.20). Differentiating (318), we have

$$\frac{d\hat{\mathbf{x}}(t)}{dt} = \mathbf{h}_o(t, t)\,\mathbf{r}(t) + \int_{T_i}^{t} \frac{\partial \mathbf{h}_o(t, \tau)}{\partial t}\,\mathbf{r}(\tau)\,d\tau. \tag{319}$$

Substituting (317) into the second term on the right-hand side of (319) and using (318), we obtain

$$\boxed{\frac{d\hat{\mathbf{x}}(t)}{dt} = \mathbf{F}(t)\,\hat{\mathbf{x}}(t) + \mathbf{h}_o(t, t)[\mathbf{r}(t) - \mathbf{C}(t)\,\hat{\mathbf{x}}(t)].} \tag{320}$$

It is convenient to introduce a new symbol for $\mathbf{h}_o(t, t)$ to indicate that it is only a function of one variable

$$\mathbf{z}(t) \triangleq \mathbf{h}_o(t, t). \tag{321}$$

The operations in (322) can be represented by the matrix block diagram of Fig. 6.36. We see that all the coefficients are known except $\mathbf{z}(t)$, but Property 4C–V (53) expresses $\mathbf{h}_o(t, t)$ in terms of the error matrix,

$$\boxed{\mathbf{z}(t) = \mathbf{h}_o(t, t) = \boldsymbol{\xi}_P(t)\mathbf{C}^T(t)\,\mathbf{R}^{-1}(t).} \tag{322}$$

Thus (320) will be completely determined if we can find an expression for $\boldsymbol{\xi}_P(t)$, the error covariance matrix for the optimum realizable point estimator.

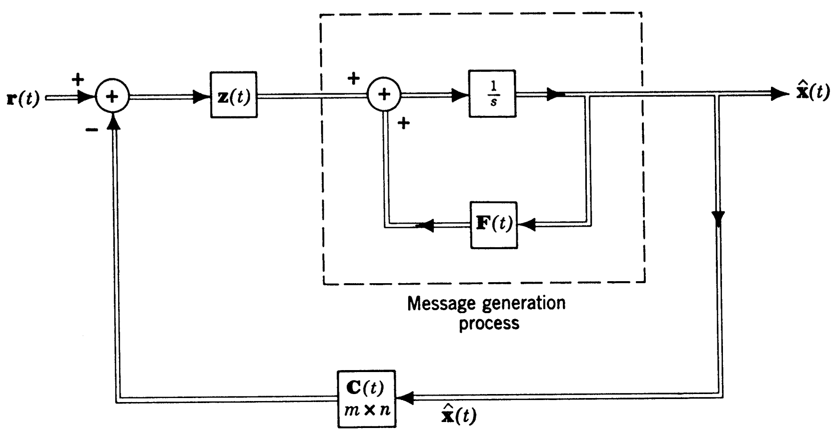


Fig. 6.36 Feedback estimator structure.

Step 3. We first find a differential equation for the error $\mathbf{x}_\epsilon(t)$, where

$$\mathbf{x}_\epsilon(t) \triangleq \mathbf{x}(t) - \hat{\mathbf{x}}(t). \tag{323}$$

Differentiating, we have

$$\frac{d\mathbf{x}_\epsilon(t)}{dt} = \frac{d\mathbf{x}(t)}{dt} - \frac{d\hat{\mathbf{x}}(t)}{dt}. \tag{324}$$

Substituting (302) for the first term on the right-hand side of (324), substituting (320) for the second term, and using (305), we obtain the desired equation

$$\frac{d\mathbf{x}_\epsilon(t)}{dt} = [\mathbf{F}(t) - \mathbf{z}(t)\,\mathbf{C}(t)]\mathbf{x}_\epsilon(t) - \mathbf{z}(t)\,\mathbf{w}(t) + \mathbf{G}(t)\,\mathbf{u}(t). \tag{325}$$

The last step is to derive a differential equation for $\boldsymbol{\xi}_P(t)$.

Step 4. Differentiating

$$\boldsymbol{\xi}_P(t) \triangleq E[\mathbf{x}_\epsilon(t)\,\mathbf{x}_\epsilon^T(t)], \tag{326}$$

we have

$$\frac{d\boldsymbol{\xi}_P(t)}{dt} = E\left[\frac{d\mathbf{x}_\epsilon(t)}{dt}\,\mathbf{x}_\epsilon^T(t)\right] + E\left[\mathbf{x}_\epsilon(t)\,\frac{d\mathbf{x}_\epsilon^T(t)}{dt}\right]. \tag{327}$$

Substituting (325) into the first term of (327), we have

$$\begin{aligned} E\left[\frac{d\mathbf{x}_\epsilon(t)}{dt}\,\mathbf{x}_\epsilon^T(t)\right] = {} & E\{[\mathbf{F}(t) - \mathbf{z}(t)\,\mathbf{C}(t)]\,\mathbf{x}_\epsilon(t)\,\mathbf{x}_\epsilon^T(t) \\ & - \mathbf{z}(t)\,\mathbf{w}(t)\,\mathbf{x}_\epsilon^T(t) + \mathbf{G}(t)\,\mathbf{u}(t)\,\mathbf{x}_\epsilon^T(t)\}. \end{aligned} \tag{328}$$

Looking at (325), we see $\mathbf{x}_\epsilon(t)$ is the state vector for a system driven by the weighted sum of two independent white noises $\mathbf{w}(t)$ and $\mathbf{u}(t)$. Therefore the expectations in the second and third terms are precisely the same type as we evaluated in Property 13 (second line of 266).

$$\begin{aligned} E\left[\frac{d\mathbf{x}_\epsilon(t)}{dt}\,\mathbf{x}_\epsilon^T(t)\right] = {} & \mathbf{F}(t)\,\boldsymbol{\xi}_P(t) - \mathbf{z}(t)\,\mathbf{C}(t)\,\boldsymbol{\xi}_P(t) \\ & + \tfrac{1}{2}\mathbf{z}(t)\,\mathbf{R}(t)\,\mathbf{z}^T(t) + \tfrac{1}{2}\,\mathbf{G}(t)\mathbf{Q}\,\mathbf{G}^T(t). \end{aligned} \tag{329}$$

Adding the transpose and replacing $\mathbf{z}(t)$ with the right-hand side of (322), we have

$$\boxed{\begin{aligned} \frac{d\boldsymbol{\xi}_P(t)}{dt} = {} & \mathbf{F}(t)\,\boldsymbol{\xi}_P(t) + \boldsymbol{\xi}_P(t)\,\mathbf{F}^T(t) - \boldsymbol{\xi}_P(t)\,\mathbf{C}^T(t)\,\mathbf{R}^{-1}(t)\,\mathbf{C}(t)\,\boldsymbol{\xi}_P(t) \\ & + \mathbf{G}(t)\mathbf{Q}\,\mathbf{G}^T(t), \end{aligned}} \tag{330}$$

which is called the *variance equation*. This equation and the initial condition

$$\boldsymbol{\xi}_P(T_i) = E[\mathbf{x}_\epsilon(T_i)\,\mathbf{x}_\epsilon^T(T_i)] \tag{331}$$

determine $\boldsymbol{\xi}_P(t)$ uniquely. Using (322) we obtain $\mathbf{z}(t)$, the gain in the optimal filter.

Observe that the variance equation does not contain the received signal. Therefore it may be solved before any data is received and used to determine the gains in the optimum filter. The variance equation is a matrix equation equivalent to n^2 scalar differential equations. However, because $\boldsymbol{\xi}_P(t)$ is a symmetric matrix, we have $\frac{1}{2}n(n+1)$ scalar nonlinear coupled differential equations to solve. In the general case it is impossible to obtain an explicit analytic solution, but this is unimportant because the equation is in a form which may be integrated using either an analog or digital computer.

The variance equation is a matrix Riccati equation whose properties have been studied extensively in other contexts (e.g., McLachlan [31]; Levin [32]; Reid [33], [34]; or Coles [35]). To study its behavior adequately requires more background than we have developed. Two properties are of interest. The first deals with the infinite memory, stationary process case (the Wiener problem) and the second deals with analytic solutions.

Property 15. Assume that T_i is fixed and that the matrices **F**, **G**, **C**, **R**, and **Q** are constant. Under certain conditions, as t increases there will be an initial transient period after which the filter gains will approach constant values. Looking at (322) and (330), we see that as $\dot{\boldsymbol{\xi}}_P(t)$ approaches zero the error covariance matrix and gain matrix will approach constants. We refer to the problem when the condition $\dot{\boldsymbol{\xi}}_P(t) = \mathbf{0}$ is true as the *steady-state estimation problem*.

The left-hand side of (330) is then zero and the variance equation becomes a set of $\frac{1}{2}n(n+1)$ quadratic algebraic equations. The nonnegative definite solution is $\boldsymbol{\xi}_P$.

Some comments regarding this statement are necessary.

1. How do we tell if the steady-state problem is meaningful? To give the *best* general answer requires notions that we have not developed [23]. A *sufficient* condition is that the message correspond to a stationary random process.

2. For small n it is feasible to calculate the various solutions and select the correct one. For even moderate n (e.g. $n = 2$) it is more practical to solve (330) numerically. We may start with some arbitrary nonnegative definite $\boldsymbol{\xi}_P(T_i)$ and let the solution converge to the steady-state result (once again see [23], Theorem 4, p. 8, for a precise statement).

3. Once again we observe that we can generate $\boldsymbol{\xi}_P(t)$ before the data is received or in real time. As a simple example of generating the variance using an analog computer, consider the equation:

$$\frac{d\xi_P(t)}{dt} = -2k\,\xi_P(t) - \frac{2}{N_0}\,\xi_P{}^2(t) + 2kP. \tag{332}$$

(This will appear in Example 1.) A simple analog method of generation is shown in Fig. 6.37. The initial condition is $\xi_P(T_i) = P$ (see discussion in the next paragraph).

4. To solve (or mechanize) the variance equation we must specify $\boldsymbol{\xi}_P(T_i)$. There are several possibilities.
   (a) The process may begin at T_i with a known value (i.e., zero variance) or with a random value having a known variance.
   (b) The process may have started at some time t_0 which is much earlier than T_i and reached a statistical steady state. In Property 14 on p. 533 we derived a differential equation that $\mathbf{\Lambda}_\mathbf{x}(t)$ satisfied. If it has reached a statistical steady state, $\dot{\mathbf{\Lambda}}_\mathbf{x}(t) = \mathbf{0}$ and (273) reduces to

$$\mathbf{0} = \mathbf{F}\mathbf{\Lambda}_\mathbf{x} + \mathbf{\Lambda}_\mathbf{x}\mathbf{F}^T + \mathbf{G}\mathbf{Q}\mathbf{G}^T. \tag{333a}$$

This is an algebraic equation whose solution is $\mathbf{\Lambda}_\mathbf{x}$. Then

$$\boldsymbol{\xi}_P(T_i) = \mathbf{\Lambda}_\mathbf{x} \tag{333b}$$

if the process has reached steady state before T_i. In order for the unobserved process to reach statistical steady state ($\dot{\mathbf{\Lambda}}_\mathbf{x}(t) = \mathbf{0}$), it is necessary and sufficient that the eigenvalues of $\mathbf{F}$ have negative real parts. This condition guarantees that the solution to (333) is nonnegative definite.

In many cases the basic characterization of an unobserved stationary process is in terms of its spectrum $S_y(\omega)$. The elements

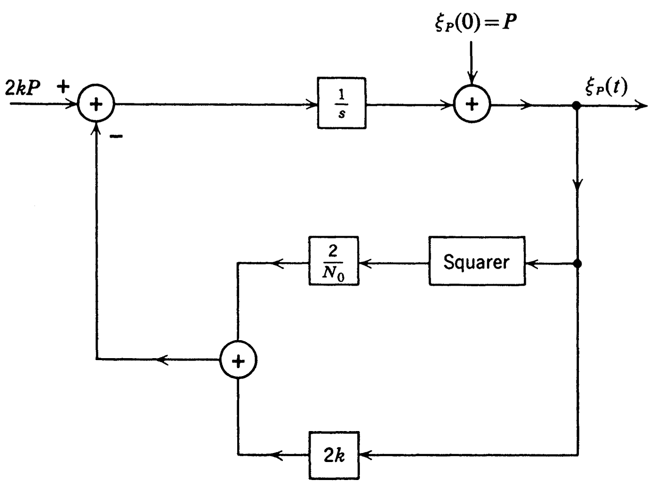


Fig. 6.37 Analog solution to variance equation.

in $\mathbf{\Lambda_x}$ follow easily from $S_y(\omega)$. As an example consider the state vector in (191),

$$x_i = \frac{d^{(i-1)}y(t)}{dt^{(i-1)}}, \qquad i = 1, 2, \ldots, n. \tag{334a}$$

If $y(t)$ is stationary, then

$$\Lambda_{\mathbf{x},11} = \int_{-\infty}^{\infty} S_y(\omega) \frac{d\omega}{2\pi} \tag{334b}$$

or, more generally,

$$\Lambda_{\mathbf{x},ik} = \int_{-\infty}^{\infty} (j\omega)^{i+k} S_y(\omega) \frac{d\omega}{2\pi}, \qquad \begin{matrix} i = 1, 2, \ldots, n, \\ k = 1, 2, \ldots, n. \end{matrix} \tag{334c}$$

Note that, for *this particular state vector,*

$$\Lambda_{\mathbf{x},ik} = 0 \quad \text{when} \quad i + k \text{ is odd}, \tag{334d}$$

because $S_y(\omega)$ is an even function.

A second property of the variance equation enables us to obtain analytic solutions in some cases (principally, the constant matrix, finite-time problem). We do not use the details in the text but some of the problems exploit them.

Property 16. The variance equation can be related to two simultaneous linear equations,

$$\begin{aligned} \frac{d\mathbf{v}_1(t)}{dt} &= \mathbf{F}(t)\,\mathbf{v}_1(t) + \mathbf{G}(t)\,\mathbf{Q}\mathbf{G}^T(t)\,\mathbf{v}_2(t), \\ \frac{d\mathbf{v}_2(t)}{dt} &= \mathbf{C}^T(t)\,\mathbf{R}^{-1}(t)\,\mathbf{C}(t)\,\mathbf{v}_1(t) - \mathbf{F}^T(t)\,\mathbf{v}_2(t), \end{aligned} \tag{335}$$

or, equivalently,

$$\frac{d}{dt}\left[\begin{array}{c} \mathbf{v}_1(t) \\ \hline \mathbf{v}_2(t) \end{array}\right] = \left[\begin{array}{c|c} \mathbf{F}(t) & \mathbf{G}(t)\,\mathbf{Q}\mathbf{G}^T(t) \\ \hline \mathbf{C}^T(t)\,\mathbf{R}^{-1}(t)\,\mathbf{C}(t) & -\mathbf{F}^T(t) \end{array}\right] \left[\begin{array}{c} \mathbf{v}_1(t) \\ \hline \mathbf{v}_2(t) \end{array}\right]. \tag{336}$$

Denote the transition matrix of (336) by

$$\mathbf{T}(t, T_i) = \left[\begin{array}{c|c} \mathbf{T}_{11}(t, T_i) & \mathbf{T}_{12}(t, T_i) \\ \hline \mathbf{T}_{21}(t, T_i) & \mathbf{T}_{22}(t, T_i) \end{array}\right]. \tag{337}$$

Then we can show [32],

$$\boldsymbol{\xi}_P(t) = [\mathbf{T}_{11}(t, T_i)\,\boldsymbol{\xi}_P(T_i) + \mathbf{T}_{12}(t, T_i)][\mathbf{T}_{21}(t, T_i)\,\boldsymbol{\xi}_P(T_i) + \mathbf{T}_{22}(t, T_i)]^{-1}. \tag{338}$$

When the matrices of concern are constant, we can always find the transition matrix $\mathbf{T}$ (see Problem 6.3.21 for an example in which we find $\mathbf{T}$ by using Laplace transform techniques. As discussed in that problem, we must take the contour to the right of all the poles in order to include all the eigenvalues) of the coefficient matrix in (336).

In this section we have transformed the optimum linear filtering problem into a state variable formulation. All the quantities of interest are expressed as outputs of dynamic systems. The three equations that describe these dynamic systems are our principal results.

The Estimator Equation.

$$\boxed{\frac{d\hat{\mathbf{x}}(t)}{dt} = \mathbf{F}(t)\,\hat{\mathbf{x}}(t) + \mathbf{z}(t)[\mathbf{r}(t) - \mathbf{C}(t)\,\hat{\mathbf{x}}(t)].} \tag{339}$$

The Gain Equation.

$$\boxed{\mathbf{z}(t) = \boldsymbol{\xi}_P(t)\,\mathbf{C}^T(t)\,\mathbf{R}^{-1}(t).} \tag{340}$$

The Variance Equation.

$$\boxed{\begin{aligned}\frac{d\boldsymbol{\xi}_P(t)}{dt} = \mathbf{F}(t)\,\boldsymbol{\xi}_P(t) + \boldsymbol{\xi}_P(t)\,\mathbf{F}^T(t) - \boldsymbol{\xi}_P(t)\,\mathbf{C}^T(t)\,\mathbf{R}^{-1}(t)\,\mathbf{C}(t)\,\boldsymbol{\xi}_P(t)& \\ + \mathbf{G}(t)\,\mathbf{Q}\mathbf{G}^T(t).&\end{aligned}} \tag{341}$$

To illustrate their application we consider some simple examples, chosen for one of three purposes:

1. To show an alternate approach to a problem that could be solved by conventional Wiener theory.
2. To illustrate a problem that could not be solved by conventional Wiener theory.
3. To develop a specific result that will be useful in the sequel.

6.3.3 Applications

In this section we consider some examples to illustrate the application of the results derived in Section 6.3.2.

Example 1. Consider the first-order message spectrum

$$S_a(\omega) = \frac{2kP}{\omega^2 + k^2}. \tag{342}$$

In this case $\mathbf{x}(t)$ is a scalar; $x(t) = a(t)$. If we assume that the message is not modulated and the measurement noise is white, then

$$r(t) = x(t) + w(t). \tag{343}$$

The necessary quantities follow by inspection:

$$\begin{aligned} \mathbf{F}(t) &= -k, \\ \mathbf{G}(t) &= 1, \\ \mathbf{C}(t) &= 1, \\ \mathbf{Q} &= 2kP, \\ \mathbf{R}(t) &= \frac{N_0}{2}. \end{aligned} \tag{344}$$

Substituting these quantities into (339), we obtain the differential equation for the optimum estimate:

$$\frac{d\hat{x}(t)}{dt} = -k\hat{x}(t) + z(t)[r(t) - \hat{x}(t)]. \tag{345}$$

The resulting filter is shown in Fig. 6.38. The value of the gain $z(t)$ is determined by solving the variance equation.

First, we assume that the estimator has reached steady state. Then the steady-state solution to the variance equation can be obtained easily. Setting the left-hand side of (341) equal to zero, we obtain

$$0 = -2k\xi_{P\infty} - \xi_{P\infty}^2 \frac{2}{N_0} + 2kP. \tag{346}$$

where $\xi_{P\infty}$ denotes the steady-state variance.

$$\xi_{P\infty} \triangleq \lim_{t\to\infty} \xi_P(t).$$

There are two solutions to (346); one is positive and one is negative. Because $\xi_{P\infty}$ is a mean-square error it must be positive. Therefore

$$\xi_{P\infty} = k\frac{N_0}{2}(-1 + \sqrt{1 + \Lambda}) \tag{347}$$

(recall that $\Lambda = 4P/kN_0$). From (340)

$$z(\infty) \triangleq z_\infty = \xi_{P\infty}R^{-1} = k(-1 + \sqrt{1 + \Lambda}). \tag{348}$$

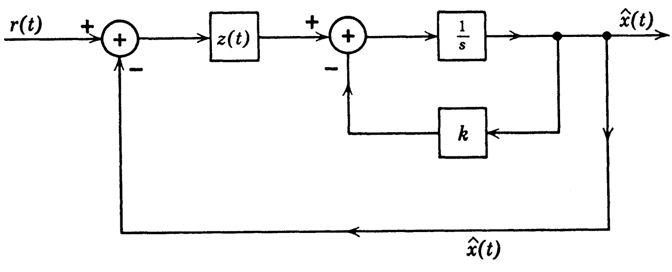


Fig. 6.38 Optimum filter: example 1.

Clearly, the filter must be equivalent to the one obtained in the example in Section 6.2. The closed loop transfer function is

$$H_o(j\omega) = \frac{z_\infty}{j\omega + k + z_\infty}. \tag{349}$$

Substituting (348) in (349), we have

$$H_o(j\omega) = \frac{k(\sqrt{1 + \Lambda} - 1)}{j\omega + k\sqrt{1 + \Lambda}}, \tag{350}$$

which is the same as (94).

The transient problem can be solved analytically or numerically. The details of the analytic solution are carried out in Problem 6.3.21 by using Property 16 on p. 545. The transition matrix is

$$\mathbf{T}(T_i + \tau, T_i) = \left[\begin{array}{c|c} \cosh(\gamma\tau) - \dfrac{k}{\gamma}\sinh(\gamma\tau) & \dfrac{2kP}{\gamma}\sinh(\gamma\tau) \\ \hline \dfrac{2}{N_0\gamma}\sinh(\gamma\tau) & \cosh(\gamma\tau) + \dfrac{k}{\gamma}\sinh(\gamma\tau) \end{array}\right], \tag{351}$$

where

$$\gamma \triangleq k\sqrt{1 + \Lambda}. \tag{352}$$

If we assume that the unobserved message is in a statistical steady state then $\hat{x}(T_i) = 0$ and $\xi_P(T_i) = P$. [(342) implies $a(t)$ is zero-mean.] Using these assumptions and (351) in (338), we obtain

$$\xi_P(t + T_i) = 2kP\left[\frac{(\gamma + k)e^{+\gamma t} + (\gamma + k)e^{-\gamma t}}{(\gamma + k)^2 e^{+\gamma t} - (\gamma - k)^2 e^{-\gamma t}}\right] = \frac{2kP}{\gamma + k}\left(\frac{1 + \left(\dfrac{\gamma - k}{\gamma + k}\right)e^{-2\gamma t}}{1 - \left(\dfrac{\gamma - k}{\gamma + k}\right)^2 e^{-2\gamma t}}\right) \tag{353}$$

As $t \to \infty$, we have

$$\lim_{t\to\infty} \xi_P(t + T_i) = \frac{2kP}{\gamma + k} = \frac{N_0}{2}[\gamma - k] = \xi_{P\infty}, \tag{354}$$

which agrees with (347). In Fig. 6.39 we show the behavior of the normalized error as a function of time for various values of Λ. The number on the right end of each curve is $\xi_P(1.2) - \xi_{P\infty}$. This is a measure of how close the error is to its steady-state value.

Example 2. A logical generalization of the one-pole case is the Butterworth family defined in (153):

$$S_a(\omega: n) = \frac{2nP}{k}\frac{\sin(\pi/2n)}{(1 + (\omega/k)^{2n})}. \tag{355}$$

To formulate this equation in state-variable terms we need the differential equation of the message generation process.

$$a^{(n)}(t) + p_{n-1}\, a^{(n-1)}(t) + \cdots + p_0\, a(t) = u(t). \tag{356}$$

The coefficients are tabulated for various n in circuit theory texts (e.g., Guillemin [37] or Weinberg [38]). The values for $k = 1$ are shown in Fig. 6.40a. The pole locations for various n are shown in Fig. 6.40b. If we are interested only in the

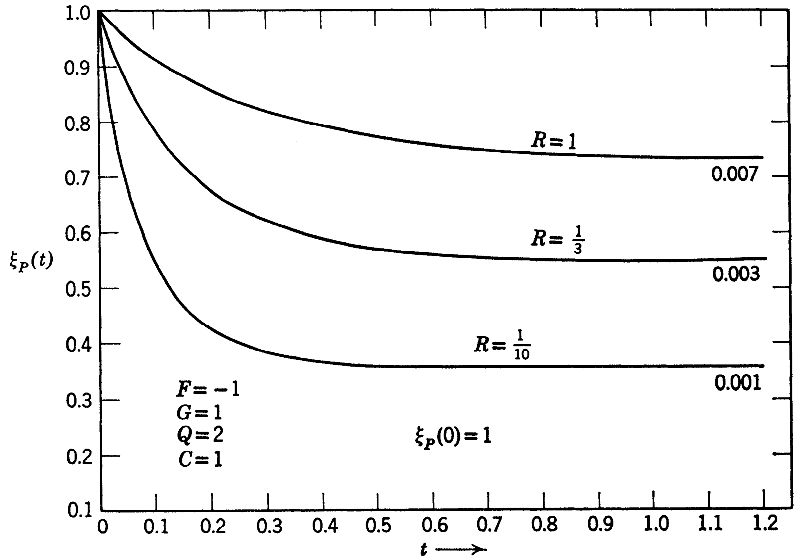
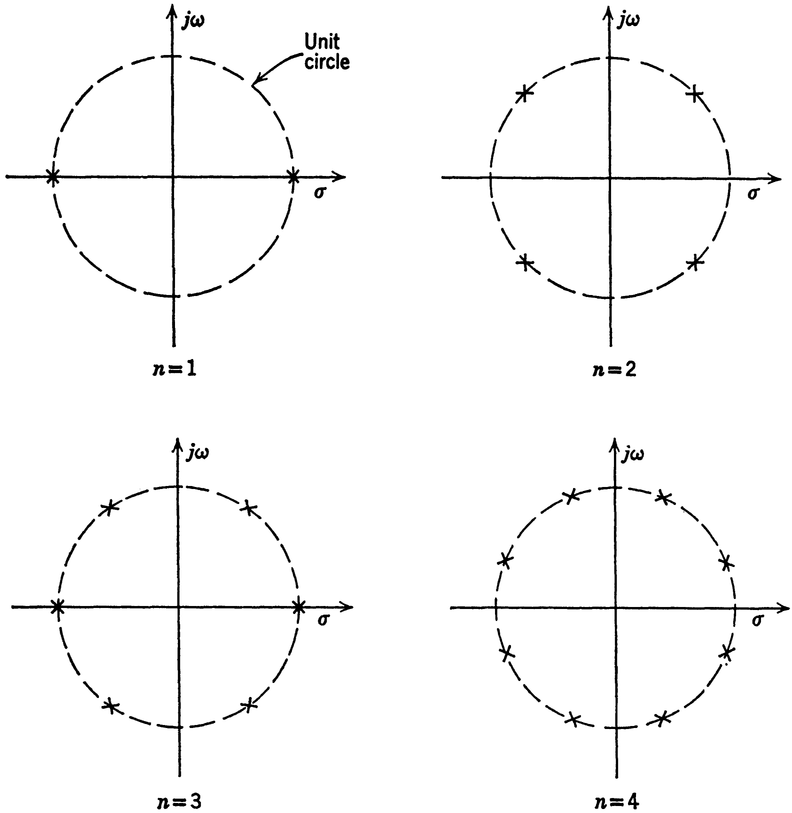


Fig. 6.39 Mean-square error, one-pole spectrum.

n	p_{n-1}	p_{n-2}	p_{n-3}	p_{n-4}	p_{n-5}	p_{n-6}	p_{n-7}
2	1.414	1.000					
3	2.000	2.000	1.000				
4	2.613	3.414	2.613	1.000			
5	3.236	5.236	5.236	3.236	1.000		
6	3.864	7.464	9.141	7.464	3.864	1.000	
7	4.494	10.103	14.606	14.606	10.103	4.494	1.000

$$a^{(n)}(t) + p_{n-1}a^{(n-1)}(t) + \cdots + p_0 a(t) = u(t)$$

Fig. 6.40 (*a*) Coefficients in the differential equation describing the Butterworth spectra [38].



Poles are at $s = \exp\{j\pi(2m+n-1)\} : m = 1, 2, \ldots, 2n$

Fig. 6.40 (*b*) pole plots, Butterworth spectra.

message process, we can choose any convenient state representation. An example is defined by (191),

$$
\begin{aligned}
x_1(t) &= a(t) \\
x_2(t) &= \dot{a}(t) = \dot{x}_1(t) \\
x_3(t) &= \ddot{a}(t) = \dot{x}_2(t) \\
&\vdots \\
x_n(t) &= a^{(n-1)}(t) = \dot{x}_{n-1}(t) \\
\dot{x}_n(t) &= -\sum_{k=1}^{n} p_{k-1}\, a^{(k-1)}(t) + u(t) \\
&= -\sum_{k=1}^{n} p_{k-1}\, x_k(t) + u(t)
\end{aligned}
\qquad (357)
$$

The resulting **F** matrix for any n is given by using (356) and (193). The other quantities needed are

$$\mathbf{G}(t) = \begin{bmatrix} 0 \\ \cdot\cdot \\ 0 \\ \cdot\cdot \\ \vdots \\ 1 \end{bmatrix} \tag{358}$$

$$\mathbf{C}(t) = [1 \mid 0 \mid \cdots \mid 0] \tag{359}$$

$$\mathbf{Q} = 2nPk^{2n-1} \sin\left(\frac{\pi}{2n}\right) \tag{360}$$

$$\mathbf{R}(t) = \frac{N_0}{2}. \tag{361}$$

From (340) we observe that $\mathbf{z}(t)$ is an $n \times 1$ matrix,

$$\mathbf{z}(t) = \frac{2}{N_0} \begin{bmatrix} \xi_{11}(t) \\ \xi_{12}(t) \\ \vdots \\ \xi_{1n}(t) \end{bmatrix}, \tag{362}$$

$$\begin{aligned} \dot{\hat{x}}_1(t) &= \hat{x}_2(t) + \frac{2}{N_0}\, \xi_{11}(t)[r(t) - \hat{x}_1(t)] \\ \dot{\hat{x}}_2(t) &= \hat{x}_3(t) + \frac{2}{N_0}\, \xi_{12}(t)[r(t) - \hat{x}_1(t)] \\ &\vdots \\ \dot{\hat{x}}_n(t) &= -p_0\hat{x}_1(t) - p_1\hat{x}_2(t) - \cdots - p_{n-1}\, \hat{x}_n(t) + \frac{2}{N_0}\, \xi_{1n}(t)[r(t) - \hat{x}_1(t)]. \end{aligned} \tag{363}$$

The block diagram is shown in Fig. 6.41.

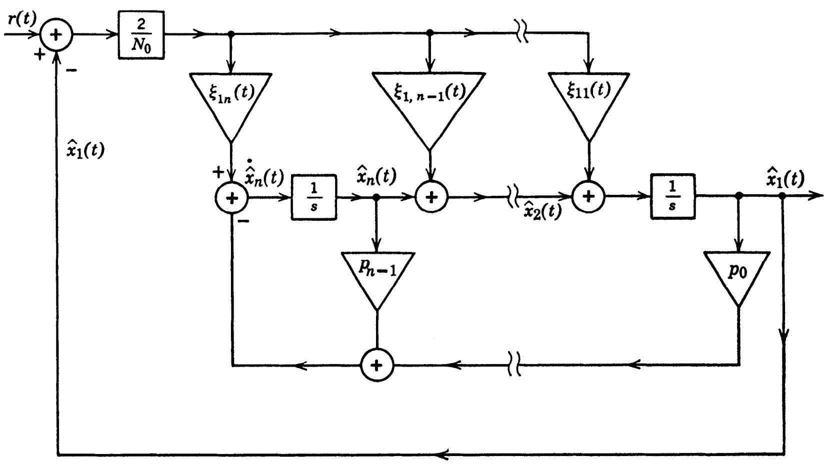
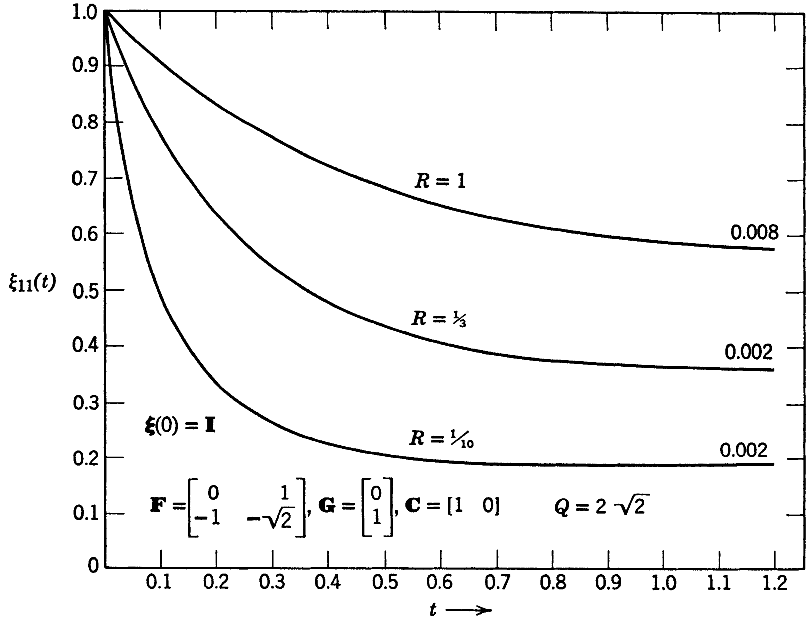


Fig. 6.41 Optimum filter: *n*th-order Butterworth message.



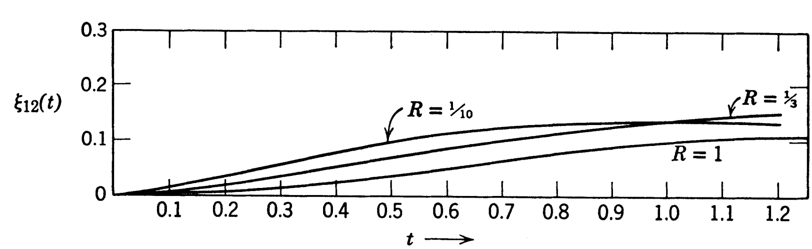


Fig. 6.42 **(*a*) Mean-square error, second-order Butterworth; (*b*) filter gains, second-order Butterworth.**

To find the values of $\xi_{11}(t), \ldots, \xi_{1n}(t)$, we solve the variance equation. This could be done analytically by using Property 16 on p. 545, but a numerical solution is much more practical. We assume that $T_i = 0$ and that the unobserved process is in a statistical steady state. We use (334) to find $\xi_P(0)$. Note that our choice of state variables causes (334*d*) to be satisfied. This is reflected by the zeros in $\boldsymbol{\xi}_P(0)$ as shown in the figures. In Fig. 6.42*a* we show the error as a function of time for the two-pole case. Once again the number on the right end of each curve is $\xi_P(1.2) - \xi_{P\infty}$. We see that for $t = 1$ the error has essentially reached its steady-state value. In Fig. 6.42*b* we show the term $\xi_{12}(t)$. Similar results are shown for the three-pole and four-pole cases in Figs. 6.43 and 6.44, respectively.† In all cases steady state is essentially reached by $t = 1$. (Observe that $k = 1$ so our time scale is normalized.) This means

† The numerical results in Figs. 6.39 and 6.41 through 6.44 are due to Baggeroer [36].

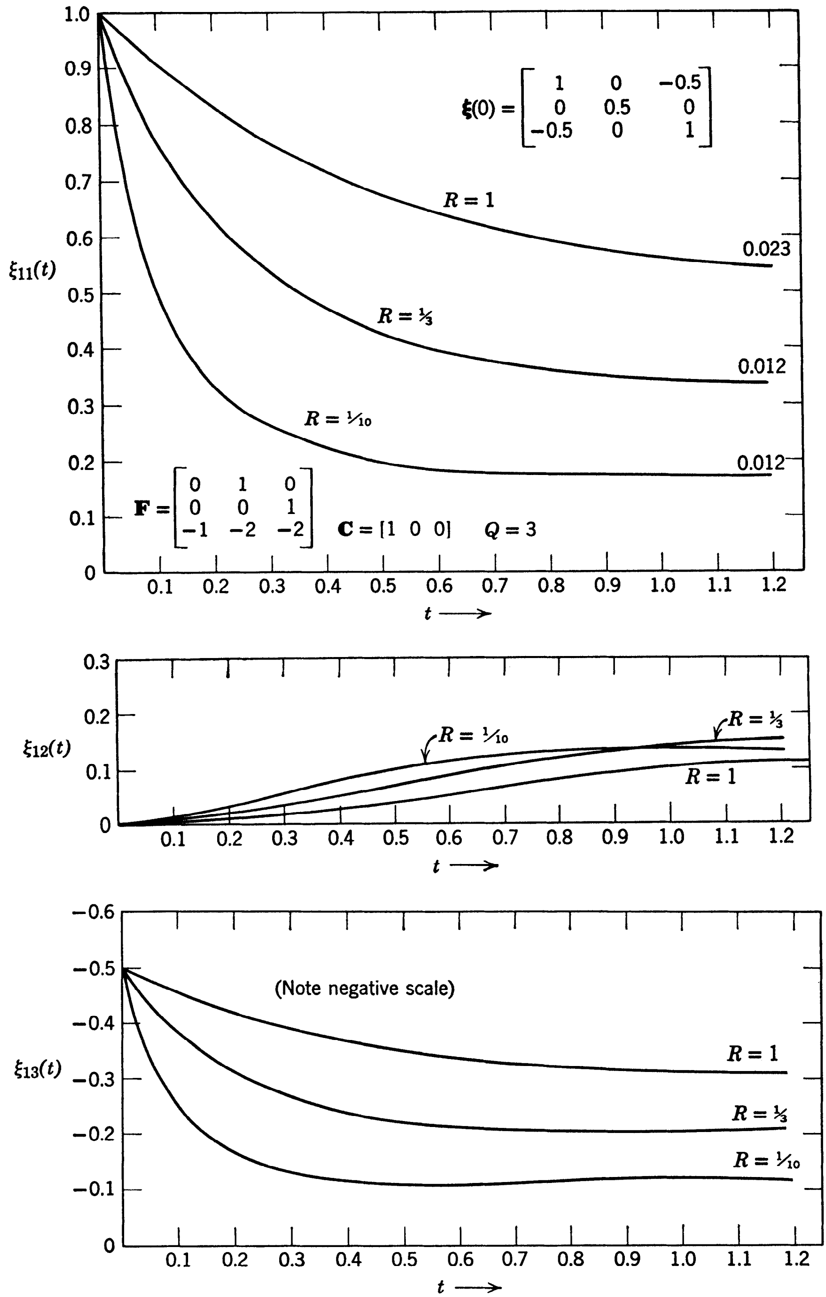


Fig. 6.43 (*a*) Mean-square error, third-order Butterworth; (*b*) filter gain, third-order Butterworth; (*c*) filter gain, third-order Butterworth.

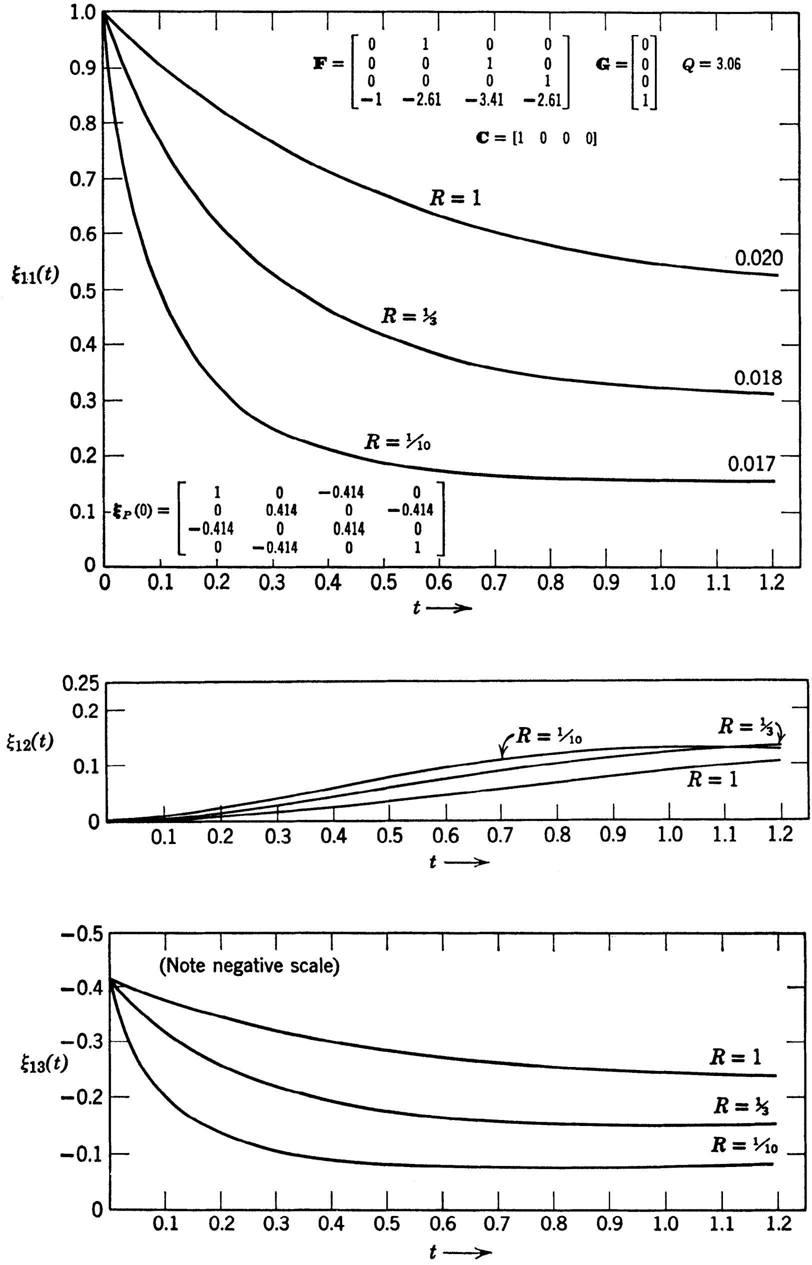


Fig. 6.44 (*a*) Mean-square error, fourth-order Butterworth; (*b*) filter gains, fourth-order Butterworth; (*c*) filter gains, fourth-order Butterworth.

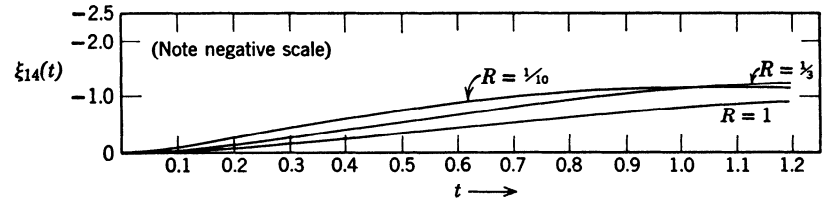


Fig. 6.44 (*d*) filter gains, fourth-order Butterworth.

that after $t = 1$ the filters are essentially time-invariant. (This does not imply that *all* terms in $\xi_P(t)$ have reached steady state.) A related question, which we leave as an exercise, is, "If we use a time-invariant filter designed for the steady-state gains, how will its performance during the initial transient compare with the optimum time-varying filter?" (See Problems 6.3.23 and 6.3.26.)

Example 3. The two preceding examples dealt with stationary processes. A simple nonstationary process is the Wiener process. It can be represented in differential equation form as

$$\begin{aligned} \dot{x}(t) &= G\,u(t), \\ x(0) &= 0. \end{aligned} \tag{364}$$

Observe that even though the coefficients in the differential equation are constant the process is nonstationary. If we assume that

$$r(t) = x(t) + w(t), \tag{365}$$

the estimator follows easily

$$\dot{\hat{x}}(t) = z(t)[r(t) - \hat{x}(t)], \tag{366}$$

where

$$z(t) = \frac{2}{N_0}\,\xi_P(t) \tag{367}$$

and

$$\dot{\xi}_P(t) = -\frac{2}{N_0}\,\xi_P{}^2(t) + G^2 Q. \tag{368}$$

The transient problem can be solved easily by using Property 16 on p. 545 (see Problem 6.3.25). The result is

$$\xi_P(t) = \left(\frac{N_0}{2}\,G^2 Q\right)^{1/2}\left(\frac{e^{\gamma t} - e^{-\gamma t}}{e^{\gamma t} + e^{-\gamma t}}\right) = \left(\frac{N_0 G^2 Q}{2}\right)^{1/2}\left(\frac{1 - e^{-2\gamma t}}{1 + e^{-2\gamma t}}\right), \tag{369}$$

where $\gamma = [2G^2Q/N_0]^{1/2}$. [Observe that (369) is not a limiting case of (353) because the initial condition is different.] As $t \to \infty$, the error approaches steady state.

$$\xi_{P\infty} = \left(\frac{N_0}{2}\,G^2 Q\right)^{1/2} \tag{370}$$

[(370) can also be obtained directly from (368) by letting $\dot{\xi}_P(t) = 0$.]

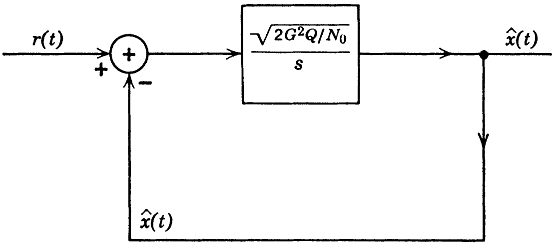


Fig. 6.45 Steady-state filter: Example 3.

The steady-state filter is shown in Fig. 6.45. It is interesting to observe that this problem is not included in the Wiener-Hopf model in Section 6.2. A heuristic way to include it is to write

$$S_x(\omega) = \frac{G^2Q}{\omega^2 + \epsilon^2}, \tag{371}$$

solve the problem by using spectral factorization techniques, and then let $\epsilon \to 0$. It is easy to verify that this approach gives the system shown in Fig. 6.45.

Example 4. In this example we derive a canonic receiver model for the following problem:

1. The message has a rational spectrum in which the order of the numerator as a function of ω^2 is at least one smaller than the order of the denominator. We use the state variable model described in Fig. 6.30. The **F** and **G** matrices are given by (212) and (213), respectively (Canonic Realization No. 2).
2. The received signal is scalar function.
3. The modulation matrix has unity in its first column and zero elsewhere. In other words, only the unmodulated message would be observed in the absence of measurement noise,

$$\mathbf{C}(t) = [1 \mid 0 \cdots 0]. \tag{372}$$

The equation describing the estimator is obtained from (339),

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{F}\hat{\mathbf{x}}(t) + \mathbf{z}(t)[r(t) - \hat{x}_1(t)] \tag{373}$$

and

$$\mathbf{z}(t) = \frac{2}{N_0}\, \boldsymbol{\xi}_P(t)\, \mathbf{C}^T(t). \tag{374}$$

As in Example 2, the gains are simply $2/N_0$ times the first row of the error matrix. The resulting filter structure is shown in Fig. 6.46. As $t \to \infty$, the gains become constant.

For the constant-gain case, by comparing the system inside the block to the two diagrams in Fig. 6.30 and 31*a*, we obtain the equivalent filter structure shown in Fig. 6.47.

Writing the loop filter in terms of its transfer function, we have

$$G_{lo}(s) = \frac{2}{N_0}\frac{\xi_{11}s^{n-1} + \cdots + \xi_{1n}}{s^n + p_{n-1}s^{n-1} + \cdots + p_0}. \tag{375}$$

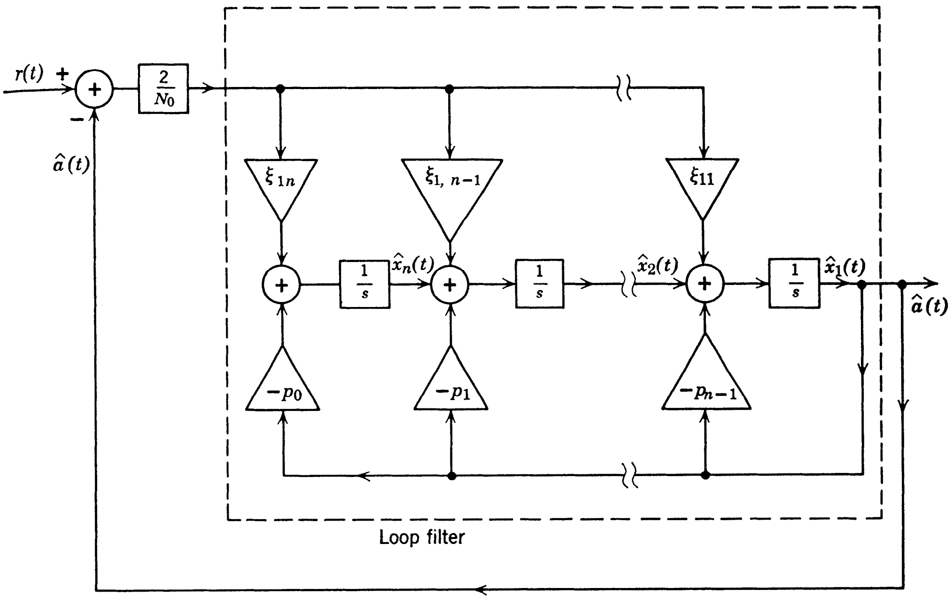


Fig. 6.46 Canonic estimator: stationary messages, statistical steady-state.

Thus the coefficients of the numerator of the loop-filter transfer function correspond to the first column in the error matrix. The poles (as we have seen before) are identical to those of the message spectrum.

Observe that we still have to solve the variance equation to obtain the numerator coefficients.

Example 5A [*23*]. As a simple example of the general case just discussed, consider the message process shown in Fig. 6.48*a*. If we want to use the canonic receiver structure we have just derived, we can redraw the message generator process as shown in Fig. 6.48*b*.

We see that

$$p_1 = k, \qquad p_0 = 0, \qquad b_1 = 0, \qquad b_0 = 1. \tag{376}$$

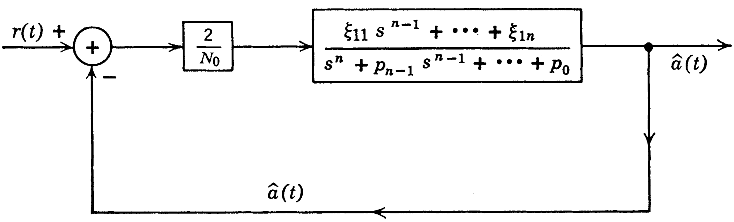


Fig. 6.47 Canonic estimator: stationary messages, statistical steady-state.

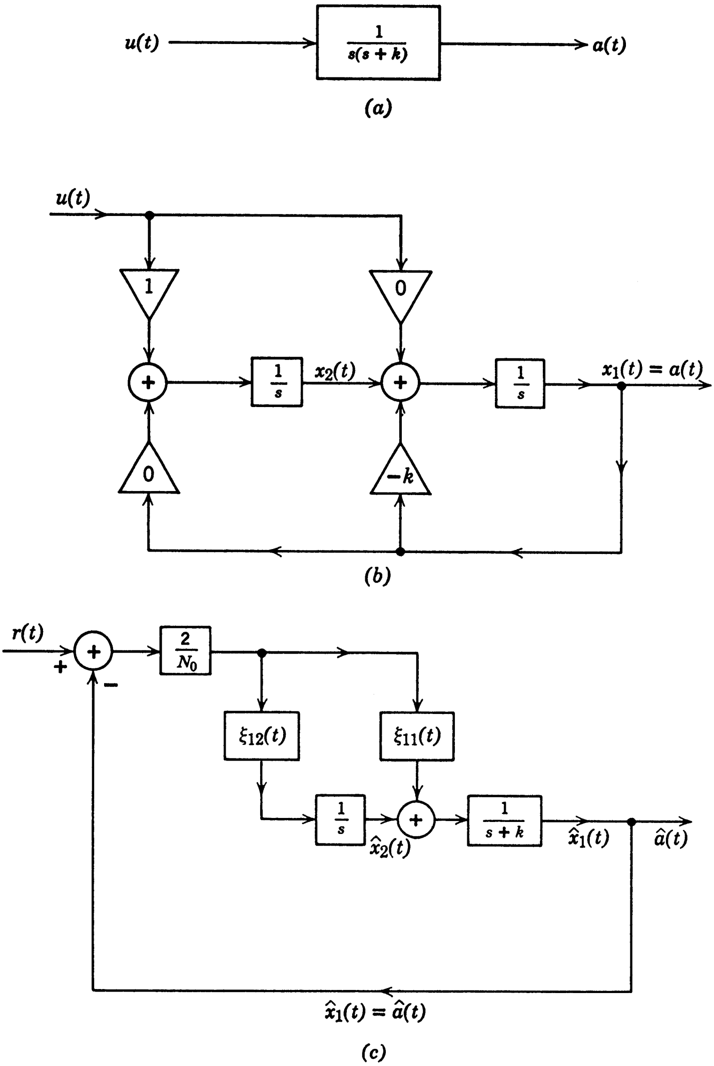


Fig. 6.48 **Systems for example 5A: (*a*) message generator; (*b*) analog representation; (*c*) optimum estimator.**

Then, using (212) and (213), we have

$$\mathbf{F} = \begin{bmatrix} -k & 1 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{G} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \mathbf{C} = [1 \quad 0] \tag{377}$$

$\mathbf{Q} = q$, $\mathbf{R} = N_0/2$.

The resulting filter is just a special case of Fig. 6.47 as shown in Fig. 6.48.

The variance equation in the steady state is

$$
\begin{aligned}
2(-k\xi_{11} + \xi_{12}) - \frac{2}{N_0}\,\xi_{11}^2 &= 0, \\
-k\xi_{12} + \xi_{22} - \frac{2}{N_0}\,\xi_{11}\xi_{12} &= 0, \\
\frac{2}{N_0}\,\xi_{12}^2 &= q.
\end{aligned} \tag{378}
$$

Thus the steady-state errors are

$$
\begin{aligned}
\xi_{12} &= \frac{N_0}{2}\left(\frac{2q}{N_0}\right)^{1/2} \\
\xi_{11} &= \frac{N_0}{2}\left\{-k + \left[k^2 + 2\left(\frac{2q}{N_0}\right)^{1/2}\right]^{1/2}\right\}.
\end{aligned} \tag{379}
$$

We have taken the positive definite solution. The loop filter for the steady-state case is

$$
G_{lo}(s) = \frac{\left\{-k + \left[k^2 + 2\left(\frac{2q}{N_0}\right)^{1/2}\right]^{1/2}\right\} s + \left(\frac{2q}{N_0}\right)^{1/2}}{s(s + k)}. \tag{380}
$$

Example 5B. An interesting example related to the preceding one is shown in Fig. 6.49. We now add A and B subscripts to denote quantities in Examples 5A and 5B, respectively. We see that, except for some constants, the output is the same as in Example 5A. The intermediate variable $x_{2_B}(t)$, however, did not appear in that realization.

We assume that the message of interest is $x_{2_B}(t)$. In Chapter II.2 we shall see the model in Fig. 6.49 and the resulting estimator play an important role in the FM problem. This is just a particular example of the general problem in which the message is subjected to a linear operation *before* transmission. There are two easy ways to solve this problem. One way is to observe that because we have already solved Example 5A we can use that result to obtain the answer. To use it we must express $x_{2_B}(t)$ as a linear transformation of $x_{1_A}(t)$ and $x_{2_A}(t)$, the state variables in Example 5A.

$$
\beta\, x_{2_B}(t) = \dot{x}_{1_B}(t) \tag{381}
$$

and

$$
x_{1_B}(t) = x_{1_A}(t), \tag{382}
$$

if we require

$$
q_A = \beta^2 q_B. \tag{383}
$$

Observing that

$$
x_{2_A}(t) = k\, x_{1_A}(t) + \dot{x}_{1_A}(t), \tag{384}
$$

we obtain

$$
\beta\, x_{2_B}(t) = \dot{x}_{1_A}(t) = -k x_{1_A}(t) + x_{2_A}(t). \tag{385}
$$

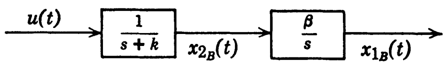


Fig. 6.49 Message generation, example 5B.

Now minimum mean-square error filtering commutes over *linear transformations.* (The proof is identical to the proof of 2.237.) Therefore

$$\hat{x}_{2_B}(t) = \frac{-1}{\beta}[k\,\hat{x}_{1_A}(t) + \hat{x}_{2_A}(t)]. \tag{386}$$

Observe that this is *not* equivalent to letting $\beta\hat{x}_{2_B}(t)$ equal the derivative of $\hat{x}_{1_A}(t)$. Thus

$$\hat{x}_{2_B}(t) \neq \frac{1}{\beta}\,\hat{x}_{1_A}(t). \tag{387}$$

With these observations we may draw the optimum filter by modifying Fig. 6.48. The result is shown in Fig. 6.50. The error variance follows easily:

$$\beta^2\,\xi_{22_B}(t) = k^2\,\xi_{11_A}(t) - 2k\,\xi_{12_A}(t) + \xi_{22_A}(t). \tag{388}$$

Alternatively, if we had not solved Example 5A, we would approach the problem directly. We identify the message as one of the state variables. The appropriate matrices are

$$\mathbf{F} = \begin{bmatrix} 0 & \beta \\ 0 & -k \end{bmatrix}, \qquad \mathbf{G} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \mathbf{C} = [1 \quad 0] \tag{389}$$

and

$$\mathbf{Q} = q_B. \tag{390}$$

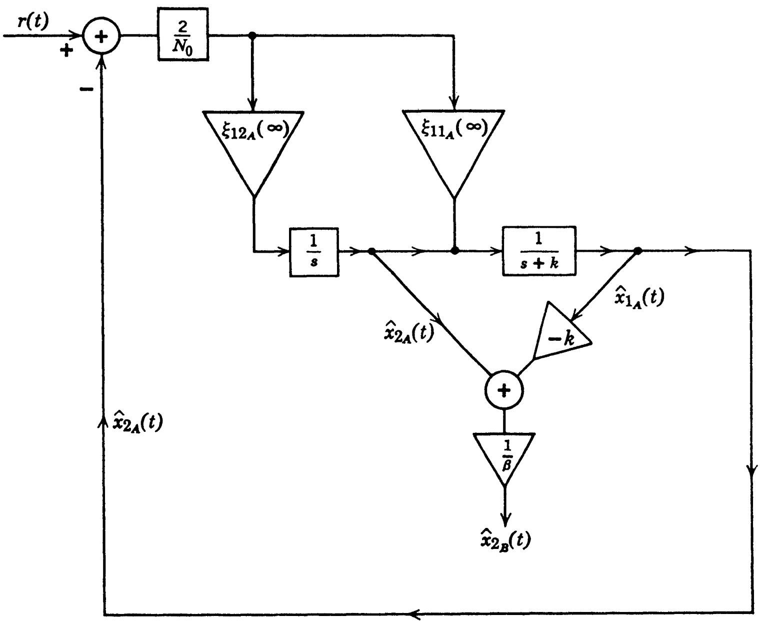


Fig. 6.50 Estimator for example 5B.

The structure of the estimator is shown in Fig. 6.51. The variance equation is

$$
\begin{aligned}
\dot{\xi}_{11}(t) &= 2\beta\xi_{12}(t) - \frac{2}{N_0}\,\xi_{11}^2(t),\\
\dot{\xi}_{12}(t) &= \beta\xi_{22}(t) - k\xi_{12}(t) - \frac{2}{N_0}\,\xi_{11}(t)\xi_{12}(t),\\
\dot{\xi}_{22}(t) &= -2k\xi_{22}(t) - \frac{2}{N_0}\,\xi_{12}^2(t) + q_B.
\end{aligned}
\tag{391}
$$

Even in the steady state, these equations appear difficult to solve analytically. In this particular case we are helped by having just solved Example 5A. Clearly, $\xi_{11}(t)$ must be the same in both cases, if we let $q_A = \beta^2 q_B$. From (379)

$$
\begin{aligned}
\xi_{11,\infty} &= \frac{kN_0}{2}\left\{-1 + \left[1 + \frac{2}{k^2}\left(\frac{2q_B\beta^2}{N_0}\right)^{1/2}\right]^{1/2}\right\}\\
&\triangleq \frac{kN_0\kappa}{2}.
\end{aligned}
\tag{392}
$$

The other gain $\xi_{12,\infty}$ now follows easily

$$
\xi_{12,\infty} = \frac{k^2N_0\kappa^2}{4\beta}. \tag{393}
$$

Because we have assumed that the message of interest is $x_2(t)$, we can also easily calculate its error variance:

$$
\xi_{22,\infty} = \frac{1}{2k}\left\{q_B - \frac{k^4\kappa^4N_0}{8\beta^2}\right\}. \tag{394}
$$

It is straightforward to verify that (388) and (394) give the same result and that the block diagrams in Figs. 6.50 and 6.51 have identical responses between $r(t)$ and $\hat{x}_2(t)$. The internal difference in the two systems developed from the two different state representations we chose.

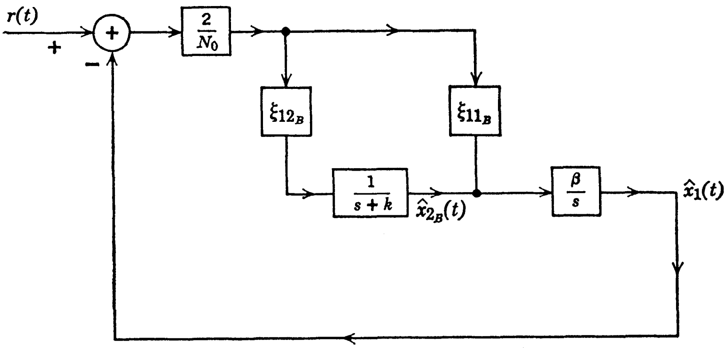


Fig. 6.51 Optimum estimator: example 5B (form #2).

Example 6. Now consider the same message process as in Example 1 but assume that the noise consists of the sum of a white noise and an uncorrelated colored noise,

$$n(t) = n_c(t) + w(t) \tag{395}$$

and

$$S_c(\omega) = \frac{2k_c P_c}{\omega^2 + k_c{}^2}. \tag{396}$$

As already discussed, we simply include $n_c(t)$ as a component in the state vector. Thus,

$$\mathbf{x}(t) \triangleq \begin{bmatrix} a(t) \\ n_c(t) \end{bmatrix}, \tag{397}$$

and

$$\mathbf{F}(t) = \begin{bmatrix} -k & 0 \\ 0 & -k_c \end{bmatrix}, \tag{398}$$

$$\mathbf{G}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{399}$$

$$\mathbf{C}(t) = [1 \quad 1], \tag{400}$$

$$\mathbf{Q}(t) = \begin{bmatrix} 2kP & 0 \\ 0 & 2k_cP_c \end{bmatrix}, \tag{401}$$

and

$$\mathbf{R}(t) = \frac{N_0}{2}. \tag{402}$$

The gain matrix $\mathbf{z}(t)$ becomes

$$\mathbf{z}(t) = \frac{2}{N_0}\, \boldsymbol{\xi}_P(t)\, \mathbf{C}^T(t), \tag{403}$$

or

$$z_{11}(t) = \frac{2}{N_0}\,[\xi_{11}(t) + \xi_{12}(t)], \tag{404}$$

$$z_{21}(t) = \frac{2}{N_0}\,[\xi_{12}(t) + \xi_{22}(t)]. \tag{405}$$

The equation specifying the estimator is

$$\frac{d\hat{\mathbf{x}}(t)}{dt} = \mathbf{F}(t)\,\hat{\mathbf{x}}(t) + \mathbf{z}(t)[r(t) - \mathbf{C}(t)\,\hat{\mathbf{x}}(t)], \tag{406}$$

or in terms of the components

$$\frac{d\hat{x}_1(t)}{dt} = -k\hat{x}_1(t) + z_{11}(t)[r(t) - \hat{x}_1(t) - \hat{x}_2(t)], \tag{407}$$

$$\frac{d\hat{x}_2(t)}{dt} = -k_c\hat{x}_2(t) + z_{21}(t)[r(t) - \hat{x}_1(t) - \hat{x}_2(t)]. \tag{408}$$

The structure is shown in Fig. 6.52*a*. This form of the structure exhibits the symmetry of the estimation process. Observe that the estimates are coupled through the feedback path.

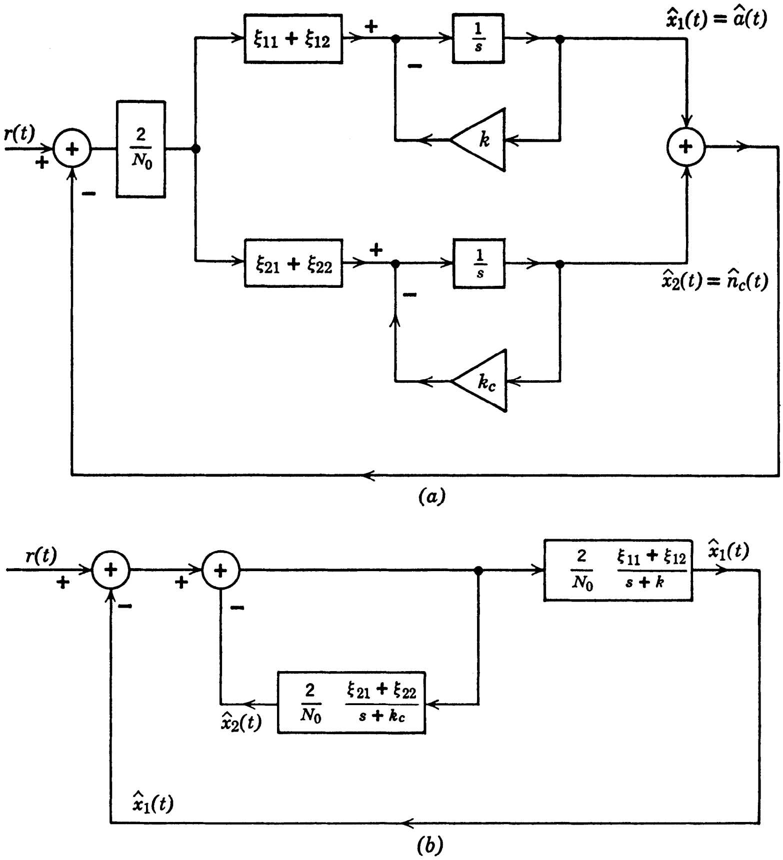


Fig. 6.52 Optimum estimator: colored and white noise (statistical steady-state).

An equivalent asymmetrical structure which shows the effect of the colored noise on the message is shown in Fig. 6.52*b*. To find the gains we must solve the variance equation. Substituting into (341), we find

$$\dot{\xi}_{11}(t) = -2k\xi_{11}(t) - \frac{2}{N_0}(\xi_{11}(t) + \xi_{12}(t))^2 + 2kP, \tag{409}$$

$$\dot{\xi}_{12}(t) = -(k + k_c)\xi_{12}(t) - \frac{2}{N_0}(\xi_{11}(t) + \xi_{12}(t))(\xi_{12}(t) + \xi_{22}(t)), \tag{410}$$

$$\dot{\xi}_{22}(t) = -2k_c\xi_{22}(t) - \frac{2}{N_0}(\xi_{12}(t) + \xi_{22}(t))^2 + 2k_cP_c. \tag{411}$$

Two comments are in order.

1. The system in Fig. 6.52*a* exhibits all of the essential features of the canonic structure for estimating a set of independent random processes whose sum is observed

in the presence of white noise (see Problem 6.3.31 for a derivation of the general canonical structure). In the general case, the coupling arises in exactly the same manner.

2. We are tempted to approach the case when there is no white noise ($N_0 = 0$) with a limiting operation. The difficulty is that the variance equation degenerates. A derivation of the receiver structure for the pure colored noise case is discussed in Section 6.3.4.

Example 7. The most common example of a multiple observation case in communications is a diversity system. A simplified version is given here. Assume that the message is transmitted over m channels with known gains as shown in Fig. 6.53. Each channel is corrupted by white noise. The modulation matrix is $m \times n$, but only the first column is nonzero:

$$\mathbf{C}(t) = \left[\begin{array}{c|c} c_2 & \\ c_2 & \mathbf{0} \\ \vdots & \\ c_m & \end{array}\right] \triangleq [\mathbf{c} \mid \mathbf{0}]. \tag{412}$$

For simplicity we assume first that the channel noises are uncorrelated. Therefore $\mathbf{R}(t)$ is diagonal:

$$\mathbf{R}(t) = \begin{bmatrix} \frac{N_1}{2} & & 0 \\ & \frac{N_2}{2} & \\ & & \ddots \\ 0 & & \frac{N_m}{2} \end{bmatrix}. \tag{413}$$

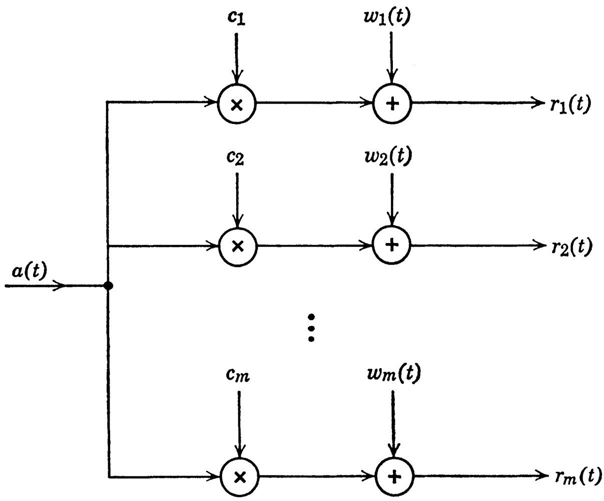


Fig. 6.53 Diversity system.

The gain matrix $\mathbf{z}(t)$ is an $n \times m$ matrix whose ijth element is

$$z_{ij}(t) = \frac{2c_j}{N_j}\,\xi_{i1}(t). \tag{414}$$

The general receiver structure is shown in Fig. 6.54*a*. We denote the input to the inner loop as $\mathbf{v}(t)$,

$$\mathbf{v}(t) = \mathbf{z}(t)[\mathbf{r}(t) - \mathbf{C}(t)\,\hat{\mathbf{x}}(t)]. \tag{415a}$$

Using (412)–(414), we have

$$v_i(t) = \xi_{i1}(t)\left[\sum_{j=1}^{m} \frac{2c_j}{N_j}\,r_j(t) - \left(\sum_{j=1}^{m} \frac{2c_j^2}{N_j}\right)\hat{x}_1(t)\right]. \tag{415b}$$

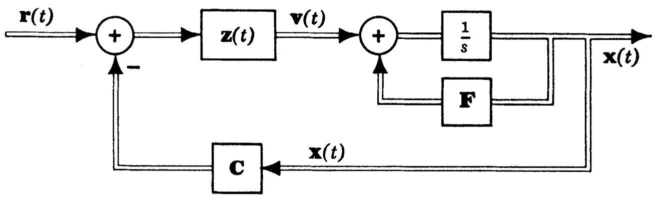
We see that the first term in the bracket represents a no-memory combiner of the received waveforms.

Denote the output of this combiner by $r_c(t)$,

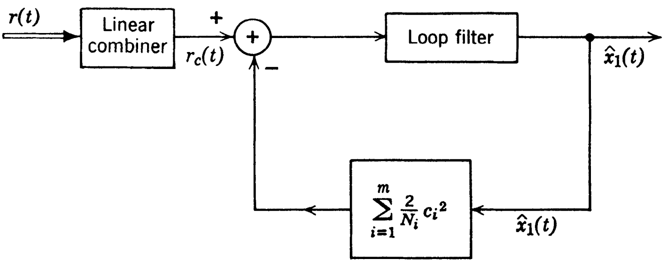
$$r_c(t) = \sum_{j=1}^{m} \frac{2c_j}{N_j}\,r_j(t). \tag{416}$$

We see that it is precisely the maximal-ratio-combiner that we have already encountered in Chapter 4. The optimum filter may be redrawn as shown in Fig. 6.54*b*. We see that the problem is reduced to a single channel problem with the received waveform $r_c(t)$,

$$r_c(t) = \left(\sum_{j=1}^{m} \frac{2c_j^2}{N_j}\right) a(t) + \sum_{j=1}^{m} \frac{2c_j}{N_j}\,n_j(t). \tag{417a}$$



(*a*)



(*b*)

Fig. 6.54 Diversity receiver.

The modulation matrix is a scalar

$$c = \left(\sum_{j=1}^{m} \frac{2c_j^2}{N_j}\right). \tag{417b}$$

and the noise is

$$n_c(t) \triangleq \sum_{j=1}^{m} \frac{2c_j}{N_j} n_j(t). \tag{417c}$$

If the message $a(t)$ has unity variance then we have an effective power of

$$P_{ef} = \left(\sum_{j=1}^{m} \frac{2}{N_j} c_j^2\right)^2 \tag{418}$$

and an effective noise level of

$$\frac{N_{ef}}{2} = \sum_{j=1}^{m} \frac{2}{N_j} c_j^2. \tag{419}$$

Therefore all of the results for the single channel can be used with a simple scale change; for example, for the one-pole message spectrum in Example 1 we would have

$$\xi_{P\infty} = k\frac{N_{ef}}{2}(-1 + \sqrt{1 + \Lambda_{ef}}), \tag{420}$$

where

$$\Lambda_{ef} = \frac{2}{k}\sum_{j=1}^{m} \frac{2c_j^2}{N_j} = \frac{2}{k}\sum_{j=1}^{m} \frac{2P_j}{N_j}. \tag{421}$$

Similar results hold when $\mathbf{R}(t)$ is not diagonal and when the message is nonstationary.

These seven examples illustrate the problems encountered most frequently in the communications area. Other examples are included in the problems.

6.3.4 Generalizations

Several generalizations are necessary in order to include other problems of interest. We discuss them briefly in this section.

Prediction. In this case $\mathbf{d}(t) = \mathbf{x}(t + \alpha)$, where α is positive. We can show easily that

$$\hat{\mathbf{d}}(t) = \boldsymbol{\phi}(t + \alpha, t)\, \hat{\mathbf{x}}(t), \qquad \alpha > 0, \tag{422}$$

where $\boldsymbol{\phi}(t, \tau)$ is the transition matrix of the system,

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\, \mathbf{x}(t) + \mathbf{G}(t)\, \mathbf{u}(t) \tag{423}$$

(see Problem 6.3.37).

When we deal with a constant parameter system,

$$\boldsymbol{\phi}(t + \alpha, t) = e^{\mathbf{F}\alpha}, \tag{424}$$

and (422) becomes

$$\hat{\mathbf{d}}(t) = e^{\mathbf{F}\alpha}\, \hat{\mathbf{x}}(t). \tag{425}$$

Filtering with Delay. In this case $\mathbf{d}(t) = \mathbf{x}(t + \alpha)$, but α is negative. From our discussions we know that considerable improvement is available and we would like to include it. The modification is not so straightforward as the prediction case. It turns out that the canonic receiver first finds the realizable estimate and then uses it to obtain the desired estimate. A good reference for this type of problem is Baggeroer [40]. The problem of estimating $\mathbf{x}(t_1)$, where t_1 is a point interior to a fixed observation interval, is also discussed in this reference. These problems are the state-variable counterparts to the unrealizable filters discussed in Section 6.2.3 (see Problem 6.6.4).

Linear Transformations on the State Vector. If $\mathbf{d}(t)$ is a linear transformation of the state variables $\mathbf{x}(t)$, that is,

$$\mathbf{d}(t) = \mathbf{k}_d(t)\,\mathbf{x}(t), \tag{426}$$

then

$$\hat{\mathbf{d}}(t) = \mathbf{k}_d(t)\,\hat{\mathbf{x}}(t). \tag{427}$$

Observe that $\mathbf{k}_d(t)$ is *not* a linear filter. It is a linear transformation of the state variables. This is simply a statement of the fact that minimum mean-square estimation and linear transformation commute. The error matrix follows easily,

$$\boldsymbol{\xi}_d(t) \triangleq E[(\mathbf{d}(t) - \hat{\mathbf{d}}(t))(\mathbf{d}^T(t) - \hat{\mathbf{d}}^T(t))] = \mathbf{k}_d(t)\,\boldsymbol{\xi}_P(t)\,\mathbf{k}_d^T(t). \tag{428}$$

In Example 5B we used this technique.

Desired Linear Operations. In many cases the desired signal is obtained by passing $\mathbf{x}(t)$ or $\mathbf{y}(t)$ through a linear filter. This is shown in Fig. 6.55.

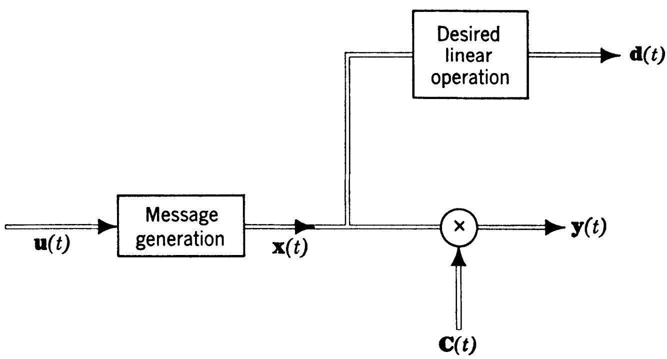


Fig. 6.55 Desired linear operations.

Three types of linear operations are of interest.

1. *Operations such as differentiation*; for example,

$$d(t) = \frac{d}{dt}\,x_1(t). \tag{429}$$

This expression is meaningful only if $x_1(t)$ is a sample function from a mean-square differentiable process.† If this is true, we can always choose $d(t)$ as one of the components of the message state vector and the results in Section 6.3.2 are immediately applicable. Observe that

$$\hat{d}(t) \neq \frac{d}{dt}\,\hat{x}_1(t). \tag{430}$$

In other words, linear filtering and realizable estimation do *not* commute. The result in (430) is obvious if we look at the estimator structure in Fig. 6.51.

2. *Improper operations*; for example, assume $y(t)$ is a scalar function and

$$d(t) = \int_0^\infty k_d(\tau)\,y(t-\tau)\,d\tau, \tag{431a}$$

where

$$K_d(j\omega) = \frac{j\omega+\alpha}{j\omega+\beta} = 1 + \frac{\alpha-\beta}{j\omega+\beta}. \tag{431b}$$

In this case the desired signal is the sum of two terms. The first term is $y(t)$. The second term is the output of a convolution operation on the past of $y(t)$. In general, an improper operation consists of a weighted sum of $y(t)$ and its derivatives plus an operation with memory. To get a state representation we must modify our results slightly. We denote the state vector of the dynamic system whose impulse response is $k_d(\tau)$ as $\mathbf{x}_d(t)$. (Here it is a scalar.) Then we have

$$\dot{x}_d(t) = -\beta x_d(t) + y(t) \tag{432}$$

and

$$d(t) = (\alpha-\beta)\,x_d(t) + y(t). \tag{433}$$

Thus the output equation contains an extra term. In general,

$$\dot{\mathbf{x}}_d(t) = \mathbf{F}_d(t)\,\mathbf{x}_d(t) + \mathbf{G}_d(t)\,\mathbf{x}(t), \tag{434}$$

$$\mathbf{d}(t) = \mathbf{C}_d(t)\,\mathbf{x}_d(t) + \mathbf{B}_d(t)\,\mathbf{x}(t). \tag{435}$$

Looking at (435) we see that if we *augment* the state-vector so that it contains both $\mathbf{x}_d(t)$ and $\mathbf{x}(t)$ then (427) will be valid. We define an augmented state vector

$$\mathbf{x}_a(t) \triangleq \begin{bmatrix}\mathbf{x}(t)\\ \hdashline \mathbf{x}_d(t)\end{bmatrix}. \tag{436}$$

† Note that we have worked with $\dot{\mathbf{x}}(t)$ when one of its components was not differentiable. However the output of the system always existed in the mean-square sense.

The equation for the augmented process is

$$\dot{\mathbf{x}}_a(t) = \left[\begin{array}{c|c} \mathbf{F}(t) & \mathbf{0} \\ \hline \mathbf{G}_d(t) & \mathbf{F}_d(t) \end{array}\right] \mathbf{x}_a(t) + \left[\begin{array}{c} \mathbf{G}(t) \\ \hline \mathbf{0} \end{array}\right] \mathbf{u}(t). \tag{437}$$

The observation equation is unchanged:

$$\mathbf{r}(t) = \mathbf{C}(t)\,\mathbf{x}(t) + \mathbf{w}(t); \tag{438}$$

However we must rewrite this in terms of the augmented state vector

$$\mathbf{r}(t) = \mathbf{C}_a(t)\,\mathbf{x}_a(t) + \mathbf{w}(t), \tag{439}$$

where

$$\mathbf{C}_a(t) \triangleq [\mathbf{C}(t) \mid \mathbf{0}]. \tag{440}$$

Now $\mathbf{d}(t)$ is obtained by a *linear* transformation on the augmented state vector. Thus

$$\hat{\mathbf{d}}(t) = \mathbf{C}_d(t)\,\hat{\mathbf{x}}_d(t) + \mathbf{B}_d(t)\,\hat{\mathbf{x}}(t) = [\mathbf{B}_d(t) \mid \mathbf{C}_d(t)][\hat{\mathbf{x}}_a(t)]. \tag{441}$$

(See Problem 6.3.41 for a simple example.)

3. *Proper operation*: In this case the impulse response of $k_d(\tau)$ does not contain any impulses or derivatives of impulses. The comments for the improper case apply directly by letting $\mathbf{B}_d(t) = \mathbf{0}$.

Linear Filtering Before Transmission. Here the message is passed through a linear filter before transmission, as shown in Fig. 6.56. All comments for the preceding case apply with obvious modification; for example, if the linear filter is an improper operation, we can write

$$\dot{\mathbf{x}}_f(t) = \mathbf{F}_f(t)\,\mathbf{x}_f(t) + \mathbf{G}_f(t)\,\mathbf{x}(t), \tag{442}$$

$$\mathbf{y}_f(t) = \mathbf{C}_f(t)\,\mathbf{x}_f(t) + \mathbf{B}_f(t)\,\mathbf{x}(t). \tag{443}$$

Then the augmented state vector is

$$\mathbf{x}_a(t) = \left[\begin{array}{c} \mathbf{x}(t) \\ \hline \mathbf{x}_f(t) \end{array}\right]. \tag{444}$$

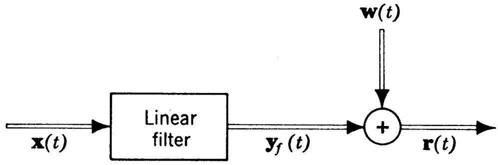


Fig. 6.56 Linear filtering before transmission.

The equation for the augmented process is

$$\dot{\mathbf{x}}_a(t) = \left[\begin{array}{c:c} \mathbf{F}(t) & \mathbf{0} \\ \hdashline \mathbf{G}_f(t) & \mathbf{F}_f(t) \end{array}\right] \mathbf{x}_a(t) + \left[\begin{array}{c} \mathbf{G}(t) \\ \hdashline \mathbf{0} \end{array}\right] \mathbf{u}(t). \tag{445}$$

The observation equation is modified to give

$$\mathbf{r}(t) = \mathbf{y}_f(t) + \mathbf{w}(t) = [\mathbf{B}_f(t) \mid \mathbf{C}_f(t)] \left[\begin{array}{c} \mathbf{x}(t) \\ \hdashline \mathbf{x}_f(t) \end{array}\right] + \mathbf{w}(t). \tag{446}$$

For the last two cases the key to the solution is the augmented state vector.

Correlation Between u(t) and w(t). We encounter cases in practice in which the vector white noise process $\mathbf{u}(t)$ that generates the message is correlated with the vector observation noise $\mathbf{w}(t)$. The modification in the derivation of the optimum estimator is straightforward.† Looking at the original derivation, we find that the results for the first two steps are unchanged. Thus

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{F}(t)\,\hat{\mathbf{x}}(t) + \mathbf{h}_o(t, t)[\mathbf{r}(t) - \mathbf{C}(t)\,\hat{\mathbf{x}}(t)]. \tag{447}$$

In the case of correlated $\mathbf{u}(t)$ and $\mathbf{w}(t)$ the expression for

$$\mathbf{z}(t) \triangleq \mathbf{h}_o(t, t) \triangleq \lim_{u \to t^-} \mathbf{h}_o(t, u) \tag{448}$$

must be modified. From Property 3A–V and the definition of $\boldsymbol{\xi}_P(t)$ we have

$$\boldsymbol{\xi}_P(t) = \lim_{u \to t^-} \left[\mathbf{K}_{\mathbf{x}}(t, u) - \int_{T_i}^{t} \mathbf{h}_o(t, \tau)\,\mathbf{K}_{\mathbf{rx}}(\tau, u)\,d\tau\right]. \tag{449}$$

Multiplying by $\mathbf{C}^T(t)$,

$$\begin{aligned} \boldsymbol{\xi}_P(t)\,\mathbf{C}^T(t) &= \lim_{u \to t^-} \left[\mathbf{K}_{\mathbf{x}}(t, u)\,\mathbf{C}^T(t) - \int_{T_i}^{t} \mathbf{h}_o(t, \tau)\,\mathbf{K}_{\mathbf{rx}}(\tau, u)\,\mathbf{C}^T(t) d\tau\right] \\ &= \lim_{u \to t^-} \left[\mathbf{K}_{\mathbf{xr}}(t, u) - \mathbf{K}_{\mathbf{xw}}(t, u) - \int_{T_i}^{t} \mathbf{h}_o(t, \tau)\,\mathbf{K}_{\mathbf{rx}}(\tau, u)\,\mathbf{C}^T(u)\,d\tau.\right] \end{aligned} \tag{450}$$

Now, the vector Wiener-Hopf equation implies that

$$\begin{aligned} \lim_{u \to t^-} \mathbf{K}_{\mathbf{xr}}(t, u) &= \lim_{u \to t^-} \left[\int_{T_i}^{t} \mathbf{h}_o(t, \tau)\,\mathbf{K}_{\mathbf{r}}(\tau, u)\,d\tau\right] \\ &= \lim_{u \to t^-} \Big[\int_{T_i}^{t} \mathbf{h}_o(t, \tau)[\mathbf{K}_{\mathbf{rx}}(\tau, u)\,\mathbf{C}^T(u) + \mathbf{R}(\tau)\,\delta(\tau - u) \\ &\qquad + \mathbf{C}(\tau)\mathbf{K}_{\mathbf{xw}}(\tau, u)]\,d\tau\Big]. \end{aligned} \tag{451}$$

† This particular case was first considered in [41]. Our derivation follows Collins [42].

Using (451) in (450), we obtain

$$\boldsymbol{\xi}_P(t)\,\mathbf{C}^T(t) = \lim_{u\to t^-}\left[\mathbf{h}_o(t,u)\mathbf{R}(u) + \int_{T_i}^{t} \mathbf{h}_o(t,\tau)\,\mathbf{C}(\tau)\,\mathbf{K}_{\mathbf{xw}}(\tau,u)\,d\tau - \mathbf{K}_{\mathbf{xw}}(t,u)\right]. \tag{452}$$

The first term is continuous. The second term is zero in the limit because $\mathbf{K}_{\mathbf{xw}}(\tau, u)$ is zero except when $u = t$. Due to the continuity of $\mathbf{h}_o(t, \tau)$ the integral is zero. The third term represents the effect of the correlation. Thus

$$\boldsymbol{\xi}_P(t)\mathbf{C}(t) = \mathbf{h}_o(t, t)\,\mathbf{R}(t) - \lim_{u\to t^-} \mathbf{K}_{\mathbf{xw}}(t, u). \tag{453}$$

Using Property 13 on p. 532, we have

$$\lim_{u\to t^-} \mathbf{K}_{\mathbf{xw}}(t, u) = \mathbf{G}(t)\,\mathbf{P}(t), \tag{454}$$

where

$$E[\mathbf{u}(t)\,\mathbf{w}^T(\tau)] \triangleq \delta(t-\tau)\,\mathbf{P}(t). \tag{455}$$

Then

$$\boxed{\mathbf{z}(t) \triangleq \mathbf{h}_o(t, t) = [\boldsymbol{\xi}_P(t)\,\mathbf{C}^T(t) + \mathbf{G}(t)\,\mathbf{P}(t)]\mathbf{R}^{-1}(t).} \tag{456}$$

The final step is to modify the variance equation. Looking at (328), we see that we have to evaluate the expectation

$$E\{[-\mathbf{z}(t)\,\mathbf{w}(t) + \mathbf{G}(t)\,\mathbf{u}(t)]\mathbf{x}_\epsilon{}^T(t)\}. \tag{457}$$

To do this we define a new white-noise driving function

$$\mathbf{v}(t) \triangleq -\mathbf{z}(t)\,\mathbf{w}(t) + \mathbf{G}(t)\,\mathbf{u}(t). \tag{458}$$

Then

$$\begin{aligned} E[\mathbf{v}(t)\mathbf{v}^T(\tau)] &= [\mathbf{z}(t)\mathbf{R}(t)\mathbf{z}^T(t) + \mathbf{G}(t)\mathbf{Q}\mathbf{G}^T(t)]\delta(t-\tau) \\ &\quad - [\mathbf{z}(t)E[\mathbf{w}(t)\mathbf{u}^T(t)]\mathbf{G}^T(t) + \mathbf{G}(t)E[\mathbf{u}(t)\mathbf{w}^T(t)]\mathbf{z}^T(t)] \end{aligned} \tag{459}$$

or

$$\begin{aligned} E[\mathbf{v}(t)\,\mathbf{v}^T(\tau)] &= [\mathbf{z}(t)\,\mathbf{R}(t)\,\mathbf{z}^T(t) - \mathbf{z}(t)\,\mathbf{P}^T(t)\,\mathbf{G}^T(t) - \mathbf{G}(t)\,\mathbf{P}(t)\mathbf{z}^T(t) \\ &\quad + \mathbf{G}(t)\,\mathbf{Q}\,\mathbf{G}^T(t)]\,\delta(t-\tau) \triangleq \mathbf{M}(t)\,\delta(t-\tau), \end{aligned} \tag{460}$$

and, using Property 13 on p. 532, we have

$$E[(-\mathbf{z}(t)\,\mathbf{w}(t) + \mathbf{G}(t)\,\mathbf{u}(t))\mathbf{x}_\epsilon{}^T(t)] = \tfrac{1}{2}\mathbf{M}(t). \tag{461}$$

Substituting into the variance equation (327), we have

$$\begin{aligned} \dot{\boldsymbol{\xi}}_P(t) &= \{\mathbf{F}(t)\boldsymbol{\xi}_P(t) - \mathbf{z}(t)\,\mathbf{C}(t)\boldsymbol{\xi}_P(t)\} + \{\sim\}^T \\ &\quad + \mathbf{z}(t)\,\mathbf{R}(t)\,\mathbf{z}^T(t) - \mathbf{z}(t)\,\mathbf{P}^T(t)\,\mathbf{G}^T(t) \\ &\quad - \mathbf{G}(t)\,\mathbf{P}(t)\,\mathbf{z}^T(t) + \mathbf{G}^T(t)\,\mathbf{Q}\,\mathbf{G}^T(t). \end{aligned} \tag{462}$$

Using the expression in (456) for $\mathbf{z}(t)$, (462) reduces to

$$\boxed{\begin{aligned}\dot{\boldsymbol{\xi}}_P(t) = {} & [\mathbf{F}(t) - \mathbf{G}(t)\,\mathbf{P}(t)\,\mathbf{R}^{-1}(t)\,\mathbf{C}(t)]\,\boldsymbol{\xi}_P(t) \\ & + \boldsymbol{\xi}_P(t)[\mathbf{F}^T(t) - \mathbf{C}^T(t)\,\mathbf{R}^{-1}(t)\mathbf{P}^T(t)\mathbf{G}(t)] \\ & - \boldsymbol{\xi}_P(t)\,\mathbf{C}^T(t)\,\mathbf{R}^{-1}(t)\,\mathbf{C}(t)\boldsymbol{\xi}_P(t) \\ & + \mathbf{G}(t)[\mathbf{Q} - \mathbf{P}(t)\,\mathbf{R}^{-1}(t)\mathbf{P}^T(t)]\,\mathbf{G}^T(t),\end{aligned}} \tag{463}$$

which is the desired result. Comparing (463) with the conventional variance equation (341), we see that we have exactly the same structure. The correlated noise has the same effect as changing $\mathbf{F}(t)$ and $\mathbf{Q}$ in (330). If we define

$$\mathbf{F}_{ef}(t) \triangleq \mathbf{F}(t) - \mathbf{G}(t)\,\mathbf{P}(t)\,\mathbf{R}^{-1}(t)\,\mathbf{C}(t) \tag{464}$$

and

$$\mathbf{Q}_{ef}(t) \triangleq \mathbf{Q} - \mathbf{P}(t)\,\mathbf{R}^{-1}(t)\,\mathbf{P}^T(t), \tag{465}$$

we can use (341) directly. Observe that the filter structure is identical to the case without correlation; only the time-varying gain $\mathbf{z}(t)$ is changed. This is the first time we have encountered a time-varying $\mathbf{Q}(t)$. The results in (339)–(341) are all valid for this case. Some interesting cases in which this correlation occurs are included in the problems.

Colored Noise Only. Throughout our discussion we have assumed that a nonzero white noise component is present. In the detection problem we encountered cases in which the removal of this assumption led to singular tests. Thus, even though the assumption is justified on physical grounds, it is worthwhile to investigate the case in which there is no white noise component. We begin our discussion with a simple example.

Example. The message process generation is described by the differential equation

$$\dot{x}_1(t) = F_1(t)\,x_1(t) + G_1(t)\,u_1(t). \tag{466}$$

The colored-noise generation is described by the differential equation

$$\dot{x}_2(t) = F_2(t)\,x_2(t) + G_2(t)\,u_2(t). \tag{467}$$

The observation process is the sum of these two processes:

$$y(t) = x_1(t) + x_2(t). \tag{468}$$

Observe that there is no white noise present. Our previous work with whitening filters suggests that in one procedure we could pass $y(t)$ through a filter designed so that the output due to $x_2(t)$ would be white. (Note that we whiten only the colored noise, not the entire input.) Looking at (468), we see that the desired output is $\dot{y}(t) - F_2(t)y(t)$. Denoting this new output as $r'(t)$, we have

$$\begin{aligned} r'(t) &\triangleq \dot{y}(t) - F_2(t)\,y(t) \\ &= \dot{x}_1(t) - F_2(t)\,x_1(t) + G_2(t)\,u_2(t), \end{aligned} \tag{469}$$

$$r'(t) = [F_1(t) - F_2(t)]x_1(t) + w'(t), \tag{470}$$

where

$$w'(t) \triangleq G_1(t)\, u_1(t) + G_2(t)\, u_2(t). \tag{471}$$

We now have the problem in a familiar format:

$$r'(t) = \mathbf{C}(t)\,\mathbf{x}(t) + w'(t), \tag{472}$$

where

$$\mathbf{C}(t) = [F_1(t) - F_2(t) \mid 0]. \tag{473}$$

The state equation is

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} F_1(t) & 0 \\ 0 & F_2(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} G_1(t) & 0 \\ 0 & G_2(t) \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}. \tag{474}$$

We observe that the observation noise $w'(t)$ is correlated with $\mathbf{u}(t)$.

$$E[\mathbf{u}(t)\, w'(\tau)] = \delta(t - \tau)\, \mathbf{P}(t), \tag{475}$$

so that

$$\mathbf{P}(t) = \begin{bmatrix} G_1(t) Q_1 \\ G_2(t) Q_2 \end{bmatrix}. \tag{476}$$

The optimal filter follows easily by using the gain equation (456) and the variance equation (463) derived in the last section. The general case† is somewhat more complicated but the basic ideas carry through (e.g., [43], [41], or Problem 6.3.45).

Sensitivity. In all of our discussion we have assumed that the matrices $\mathbf{F}(t)$, $\mathbf{G}(t)$, $\mathbf{C}(t)$, $\mathbf{Q}$, and $\mathbf{R}(t)$ were known exactly. In practice, the actual matrices may be different from those assumed. The sensitivity problem is to find the increase in error when the actual matrices are different. We assume the following model:

$$\dot{\mathbf{x}}_{\text{mo}}(t) = \mathbf{F}_{\text{mo}}(t)\, \mathbf{x}_{\text{mo}}(t) + \mathbf{G}_{\text{mo}}(t)\, \mathbf{u}_{\text{mo}}(t), \tag{477}$$

$$\mathbf{r}_{\text{mo}}(t) = \mathbf{C}_{\text{mo}}(t)\, \mathbf{x}_{\text{mo}}(t) + \mathbf{w}_{\text{mo}}(t). \tag{478}$$

The correlation matrices are $\mathbf{Q}_{\text{mo}}$ and $\mathbf{R}_{\text{mo}}(t)$. We denote the error matrix under the model assumptions as $\boldsymbol{\xi}_{\text{mo}}(t)$. (The subscript "mo" denotes model.) Now assume that the actual situation is described by the equations,

$$\dot{\mathbf{x}}_{\text{ac}}(t) = \mathbf{F}_{\text{ac}}(t)\, \mathbf{x}_{\text{ac}}(t) + \mathbf{G}_{\text{ac}}(t)\, \mathbf{u}_{\text{ac}}(t), \tag{479}$$

$$\mathbf{r}_{\text{ac}}(t) = \mathbf{C}_{\text{ac}}(t)\, \mathbf{x}_{\text{ac}}(t) + \mathbf{w}_{\text{ac}}(t), \tag{480}$$

† We have glossed over two important issues because the colored-noise-only problem does not occur frequently enough to warrant a lengthy discussion. The first issue is the minimal dimensionality of the problem. In this example, by a suitable choice of state variables, we can end up with a scalar variance equation instead of a 2×2 equation. We then retrieve $\hat{\mathbf{x}}(t)$ by a linear transformation on our estimate of the state variable for the minimal dimensional system and the received signal. The second issue is that of initial conditions in the variance equation. $\xi_P(0^-)$ may not equal $\xi_P(0^+)$. The interested reader should consult the two references listed for a more detailed discussion.

with correlation matrices $\mathbf{Q}_{ac}$ and $\mathbf{R}_{ac}(t)$. We want to find the actual error covariance matrix $\boldsymbol{\xi}_{ac}(t)$ for a system which is optimum under the model assumptions when the input is $\mathbf{r}_{ac}(t)$. (The subscript "ac" denotes actual.) The derivation is carried out in detail in [44]. The results are given in (484)–(487). We define the quantities:

$$\boldsymbol{\xi}_{ac}(t) \triangleq E[\mathbf{x}_{\epsilon_{ac}}(t)\, \mathbf{x}_{\epsilon_{ac}}^T(t)], \tag{481}$$

$$\boldsymbol{\xi}_{a\epsilon}(t) \triangleq E[\mathbf{x}_{ac}(t)\, \mathbf{x}_{\epsilon_{ac}}^T(t)], \tag{482}$$

$$\mathbf{F}_{\epsilon}(t) \triangleq \mathbf{F}_{ac}(t) - \mathbf{F}_{mo}(t). \tag{483a}$$

$$\mathbf{C}_{\epsilon}(t) \triangleq \mathbf{C}_{ac}(t) - \mathbf{C}_{mo}(t). \tag{483b}$$

The actual error covariance matrix is specified by three matrix equations:

$$\begin{aligned}\dot{\boldsymbol{\xi}}_{ac}(t) = {} & \{[\mathbf{F}_{mo}(t) - \boldsymbol{\xi}_{mo}(t)\, \mathbf{C}_{mo}^T(t)\, \mathbf{R}_{mo}^{-1}(t)\, \mathbf{C}_{mo}(t)]\, \boldsymbol{\xi}_{ac}(t) \\ & - [\mathbf{F}_{\epsilon}(t) - \boldsymbol{\xi}_{mo}(t)\, \mathbf{C}_{mo}^T(t)\, \mathbf{R}_{mo}^{-1}(t)\, \mathbf{C}_{\epsilon}(t)]\, \boldsymbol{\xi}_{a\epsilon}(t)\} \\ & + \{\sim\}^T + \mathbf{G}_{ac}(t)\, \mathbf{Q}_{ac}\, \mathbf{G}_{ac}^T(t) \\ & + \boldsymbol{\xi}_{mo}(t)\, \mathbf{C}_{mo}^T(t)\, \mathbf{R}_{mo}^{-1}(t)\, \mathbf{R}_{ac}(t)\, \mathbf{R}_{mo}^{-1}(t)\, \mathbf{C}_{mo}(t)\, \boldsymbol{\xi}_{mo}(t),\end{aligned} \tag{484}$$

$$\begin{aligned}\dot{\boldsymbol{\xi}}_{a\epsilon}(t) = {} & \mathbf{F}_{ac}(t)\, \boldsymbol{\xi}_{a\epsilon}(t) + \boldsymbol{\xi}_{a\epsilon}(t)\, \mathbf{F}_{mo}^T(t) \\ & - \boldsymbol{\xi}_{a\epsilon}(t)\, \mathbf{C}_{mo}^T(t)\, \mathbf{R}_{mo}^{-1}(t)\, \mathbf{C}_{mo}(t)\, \boldsymbol{\xi}_{mo}(t) \\ & - \boldsymbol{\Lambda}_{ac}(t)\, \mathbf{F}_{\epsilon}^T(t) + \boldsymbol{\Lambda}_{ac}(t)\, \mathbf{C}_{\epsilon}^T(t)\, \mathbf{R}_{mo}^{-1}(t)\, \mathbf{C}_{mo}(t)\, \boldsymbol{\xi}_{mo}(t) \\ & + \mathbf{G}_{ac}(t)\, \mathbf{Q}_{ac}\, \mathbf{G}_{ac}^T(t),\end{aligned} \tag{485}$$

where

$$\boldsymbol{\Lambda}_{ac}(t) \triangleq E[\mathbf{x}_{ac}(t)\, \mathbf{x}_{ac}^T(t)] \tag{486}$$

satisfies the familiar linear equation

$$\dot{\boldsymbol{\Lambda}}_{ac}(t) = \mathbf{F}_{ac}(t)\, \boldsymbol{\Lambda}_{ac}(t) + \boldsymbol{\Lambda}_{ac}(t)\, \mathbf{F}_{ac}^T(t) + \mathbf{G}_{ac}(t)\, \mathbf{Q}_{ac}\, \mathbf{G}_{ac}^T(t). \tag{487}$$

We observe that we can solve (487) then (485) and (484). In other words the equations are coupled in only one direction. Solving in this manner and assuming that the variance equation for $\boldsymbol{\xi}_{mo}(t)$ has already been solved, we see that the equations are linear and time-varying. Some typical examples are discussed in [44] and the problems.

Summary

With the inclusion of these generalizations, the feedback filter formulation can accommodate all the problems that we can solve by using conventional Wiener theory. (A possible exception is a stationary process with a nonrational spectrum. In theory, the spectral factorization techniques will work for nonrational spectra if they satisfy the Paley-Wiener criterion, but the actual solution is not practical to carry out in most cases of interest).

We summarize some of the advantages of the state-variable formulation.

1. Because it is a time-domain formulation, nonstationary processes and finite time intervals are easily included.

2. The form of the solution is such that it can be implemented on a computer. This advantage should not be underestimated. Frequently, when a problem is simple enough to solve analytically, our intuition is good enough so that the optimum processor will turn out to be only slightly better than a logically designed, but *ad hoc*, processor. However, as the complexity of the model increases, our intuition will start to break down, and the optimum scheme is frequently essential as a guide to design. If we cannot get quantitative answers for the optimum processor in an easy manner, the advantage is lost.

3. A third advantage is not evident from our discussion. The original work [23] recognizes and exploits heavily the duality between the estimation and control problem. This enables us to prove many of the desired results rigorously by using techniques from the control area.

4. Another advantage of the state-variable approach which we shall not exploit fully is its use in nonlinear system problems. In Chapter II.2 we indicate some of the results that can be derived with this approach.

Clearly, there are disadvantages also. Some of the more important ones are the following:

1. It appears difficult to obtain closed-form expressions for the error such as (152).

2. Several cases, such as unrealizable filters, are more difficult to solve with this formulation.

Our discussion in this section has served as an introduction to the role of the state-variable formulation. Since the original work of Kalman and Bucy a great deal of research has been done in the area. Various facets of the problem and related areas are discussed in many papers and books. In Chapters II.2, II.3, and II.4, we shall once again encounter interesting problems in which the state-variable approach is useful.

We now turn to the problem of amplitude modulation to see how the results of Sections 6.1 through 6.3 may be applied.

6.4 LINEAR MODULATION: COMMUNICATIONS CONTEXT

The general discussion in Section 6.1 was applicable to arbitrary linear modulation problems. In Section 6.2 we discussed solution techniques which were applicable to unmodulated messages. In Section 6.3 the general results were for linear modulations but the examples dealt primarily with unmodulated signals.

We now want to discuss a particular category of linear modulation that occurs frequently in communication problems. These are characterized by the property that the carrier is a waveform whose frequency is high compared with the message bandwidth. Common examples are

$$s(t, a(t)) = \sqrt{2P}\,[1 + ma(t)] \cos \omega_c t. \tag{488}$$

This is conventional AM with a residual carrier.

$$s(t, a(t)) = \sqrt{2P}\, a(t) \cos \omega_c t. \tag{489}$$

This is double-sideband suppressed-carrier amplitude modulation.

$$s(t, a(t)) = \sqrt{P}\,[a(t) \cos \omega_c t - \tilde{a}(t) \sin \omega_c t], \tag{490}$$

where $\tilde{a}(t)$ is related to the message $a(t)$ by a particular linear transformation (we discuss this in more detail on p. 581). By choosing the transformation properly we can obtain a single-sideband signal.

All of these systems are characterized by the property that $c(t)$ is essentially disjoint in frequency from the message process. We shall see that this leads to simplification of the estimator structure (or demodulator).

We consider several interesting cases and discuss both the realizable and unrealizable problems. Because the approach is fresh in our minds, let us look at a realizable point-estimation problem first.

6.4.1 DSB-AM: Realizable Demodulation

As a first example consider a double-sideband suppressed-carrier amplitude modulation system

$$s(t, a(t)) = [\sqrt{2P} \cos \omega_c t]\, a(t) = y(t). \tag{491}$$

We write this equation in the general linear modulation form by letting

$$c(t) \triangleq \sqrt{2P} \cos \omega_c t. \tag{492}$$

First, consider the problem of *realizable* demodulation in the presence of white noise. We denote the message state vector as $\mathbf{x}(t)$ and assume that the message $a(t)$ is its first component. The optimum estimate is specified by the equations,

$$\frac{d\hat{\mathbf{x}}(t)}{dt} = \mathbf{F}(t)\, \hat{\mathbf{x}}(t) + \mathbf{z}(t)[r(t) - \mathbf{C}(t)\, \hat{\mathbf{x}}(t)], \tag{493}$$

where

$$\mathbf{C}(t) = [c(t) \,\vdots\, 0 \,\vdots\, 0 \cdots 0] \tag{494}$$

and

$$\mathbf{z}(t) = \boldsymbol{\xi}_P(t)\, \mathbf{C}^T(t) \frac{2}{N_0}. \tag{495}$$

The block diagram of the receiver is shown in Fig. 6.57.

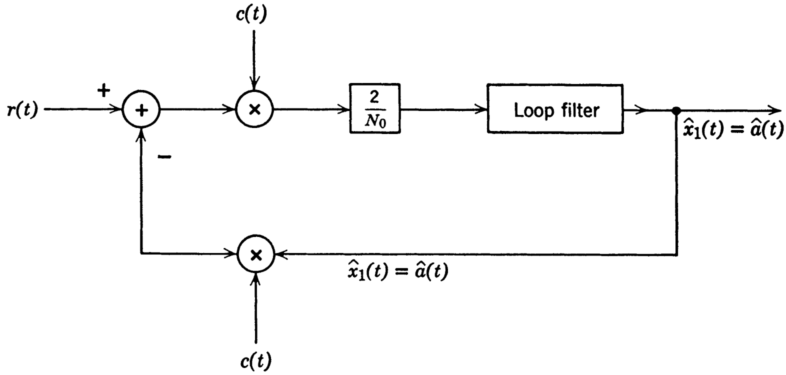


Fig. 6.57 DSB-AM receiver: form 1.

To simplify this structure we must examine the character of the loop filter and $c(t)$. First, let us *conjecture* that the loop filter will be low-pass with respect to ω_c, the carrier frequency. This is a logical conjecture because in an AM system the message is low-pass with respect to ω_c.

Now, let us look at what happens to $\hat{a}(t)$ in the feedback path when it is multiplied twice by $c(t)$. The result is

$$c^2(t)\,\hat{a}(t) = P(1 + \cos 2\omega_c t)\,\hat{a}(t). \tag{496}$$

From our original assumption, $\hat{a}(t)$ has no frequency components near $2\omega_c$. Thus, if the loop filter is low-pass, the term near $2\omega_c$ will not pass through and we could redraw the loop as shown in Fig. 6.58 to obtain the same output. We see that the loop is now operating at low-pass frequencies.

It remains to be shown that the resulting filter is just the conventional low-pass optimum filter. This follows easily by determining how $c(t)$ enters into the variance equation (see Problem 6.4.1).

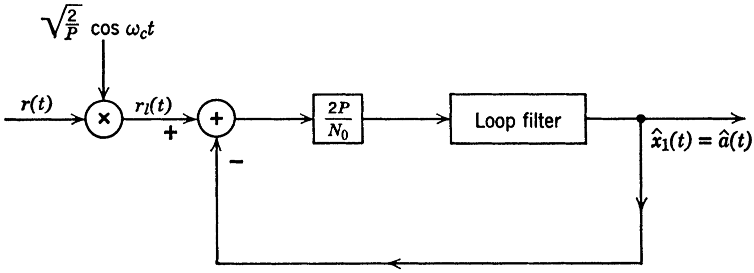


Fig. 6.58 DSB-AM receiver: final form.

We find that the resulting low-pass filter is identical to the unmodulated case. Because the modulator simply shifted the message to a higher frequency, we would expect the error variance to be the same. The error variance follows easily.

Looking at the input to the system and recalling that

$$r(t) = [\sqrt{2P} \cos \omega_c t]\, a(t) + n(t), \tag{497}$$

we see that the input to the loop is

$$r_l(t) = [\sqrt{P}\, a(t) + n_s(t)] + \text{double frequency terms}, \tag{498}$$

where $n_s(t)$ is the original noise $n(t)$ multiplied by $\sqrt{2/P} \cos \omega_c t$.

$$S_{n_s}(\omega) = \frac{N_0}{2P} \tag{499}$$

We see that this input is identical to that in Section 6.2. Thus the error expression in (152) carries over directly.

$$\xi_{Pn} = \frac{N_0}{2P} \int_{-\infty}^{\infty} \ln \left[1 + \frac{S_a(\omega)}{N_0/2P}\right] \frac{d\omega}{2\pi} \tag{500}$$

for DSB-SC amplitude modulation. The curves in Figs. 6.17 and 6.19 for the Butterworth and Gaussian spectra are directly applicable.

We should observe that the noise actually has the spectrum shown in Fig. 6.59 because there are elements operating at bandpass that the received waveform passes through before being available for processing. Because the filter in Fig. 6.58 is low-pass, the white noise approximation will be valid as long as the spectrum is flat over the effective filter bandwidth.

6.4.2 DSB-AM: Demodulation with Delay

Now consider the same problem for the case in which unrealizable filtering (or filtering with delay) is allowed. As a further simplification we

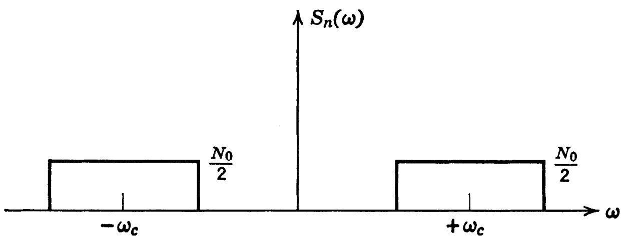


Fig. 6.59 Bandlimited noise with flat spectrum.

assume that the message process is stationary. Thus $T_f = \infty$ and $K_a(t, u) = K_a(t - u)$. In this case, the easiest approach is to assume the Gaussian model is valid and use the MAP estimation procedure developed in Chapter 5. The MAP estimator equation is obtained from (6.3) and (6.4),

$$\hat{a}_u(t) = \int_{-\infty}^{\infty} \frac{2}{N_0} K_a(t - u)\, c(u)[r(u) - c(u)\, \hat{a}(u)]\, du, \qquad -\infty < t < \infty. \tag{501}$$

The subscript u emphasizes the estimator is unrealizable. We see that the operation inside the integral is a convolution, which suggests the block

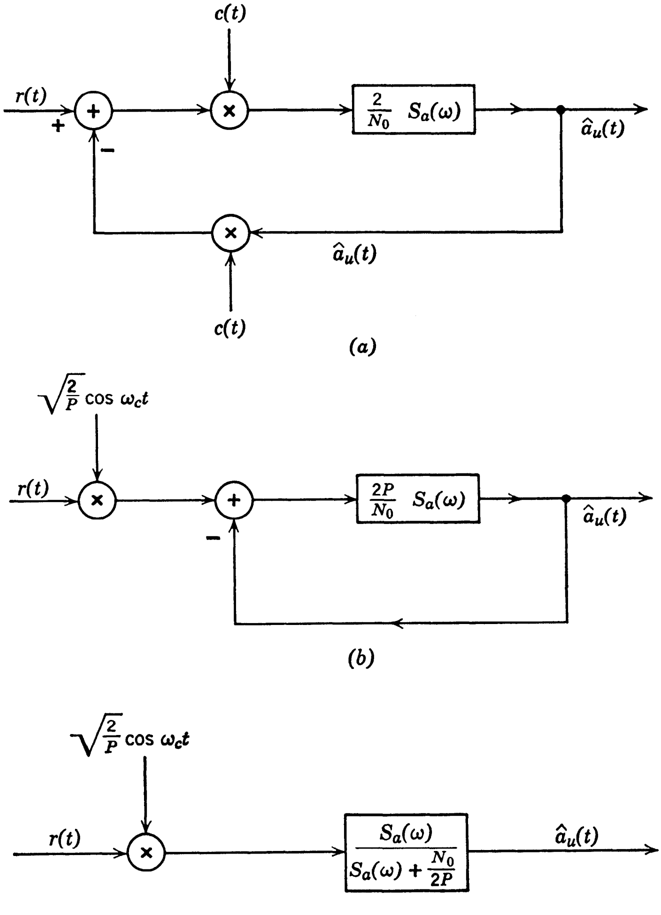


Fig. 6.60 Demodulation with delay.

diagram shown in Fig. 6.60*a*. Using an argument identical to that in Section 6.4.1, we obtain the block diagram in Fig. 6.60*b* and finally in Fig. 6.60*c*.

The optimum demodulator is simply a multiplication followed by an optimum unrealizable filter.† The error expression follows easily:

$$\xi_u = \int_{-\infty}^{\infty} \frac{S_a(\omega)(N_0/2P)}{S_a(\omega) + N_0/2P} \frac{d\omega}{2\pi}. \tag{502}$$

This result, of course, is identical to that of the unmodulated process case. As we have pointed out, this is because linear modulation is just a simple spectrum shifting operation.

6.4.3 Amplitude Modulation: Generalized Carriers

In most conventional communication systems the carrier is a sine wave. Many situations, however, develop when it is desirable to use a more general waveform as a carrier. Some simple examples are the following:

1. Depending on the spectrum of the noise, we shall see that different carriers will result in different demodulation errors. Thus in many cases a sine wave carrier is inefficient. A particular case is that in which additive noise is intentional jamming.
2. In communication nets with a large number of infrequent users we may want to assign many systems in the same frequency band. An example is a random access satellite communication system. By using different wideband orthogonal carriers, this assignment can be accomplished.
3. We shall see in Part II that a wideband carrier will enable us to combat randomly time-varying channels.

Many other cases arise in which it is useful to depart from sinusoidal carriers. The modification of the work in Section 6.4.1 is obvious. Let

$$s(t, a(t)) = c(t)\, a(t). \tag{503}$$

Looking at Fig. 6.60, we see that if

$$c^2(t)\, a(t) = k[a(t) + \text{a high frequency term}] \tag{504}$$

the succeeding steps are identical. If this is not true, the problem must be re-examined.

As a second example of linear modulation we consider single-sideband amplitude modulation.

† This particular result was first obtained in [46] (see also [45]).

6.4.4 Amplitude Modulation: Single-Sideband Suppressed-Carrier†

In a single-sideband communication system we modulate the carrier $\cos \omega_c t$ with the message $a(t)$. In addition, we modulate a carrier $\sin \omega_c t$ with a time function $\tilde{a}(t)$, which is linearly related to $a(t)$. Thus the transmitted signal is

$$s(t, a(t)) = \sqrt{P}\,[a(t) \cos \omega_c t - \tilde{a}(t) \sin \omega_c t]. \tag{505}$$

We have removed the $\sqrt{2}$ so that the transmitted power is the same as the DSB-AM case. Once again, the carrier is suppressed.

The function $\tilde{a}(t)$ is the Hilbert transform of $a(t)$. It corresponds to the output of a linear filter $h(\tau)$ when the input is $a(t)$. The transfer function of the filter is

$$H(j\omega) = \begin{cases} -j, & \omega > 0, \\ 0, & \omega = 0, \\ +j, & \omega < 0. \end{cases} \tag{506}$$

We can show (Problem 6.4.2) that the resulting signal has a spectrum that is entirely above the carrier (Fig. 6.61).

To find the structure of the optimum demodulator and the resulting performance we consider the case $T_f = \infty$ because it is somewhat easier. From (505) we observe that SSB is a mixture of a no-memory term and a memory term.

There are several easy ways to derive the estimator equation. We can return to the derivation in Chapter 5 (5.25), modify the expression for $\partial s(t, a(t))/\partial A_r$, and proceed from that point (see Problem 6.4.3). Alternatively, we can view it as a vector problem and jointly estimate $a(t)$ and $\tilde{a}(t)$ (see Problem 6.4.4). This equivalence is present because the transmitted signal contains $a(t)$ and $\tilde{a}(t)$ in a linear manner. Note that a state-

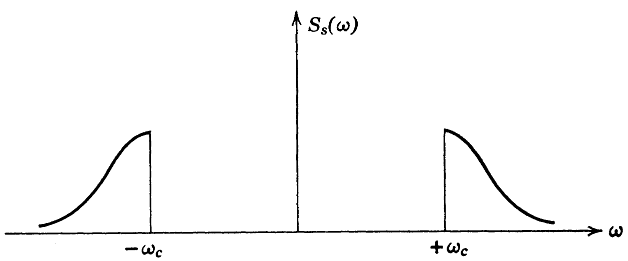


Fig. 6.61 SSB spectrum.

† We assume that the reader has heard of SSB. Suitable references are [47] to [49].