variable approach is not useful because the filter that generates the Hilbert transform (506) is not a finite-dimensional dynamic system.

We leave the derivation as an exercise and simply state the results. Assuming that the noise is white with spectral height $N_0/2$ and using (5.160), we obtain the estimator equations

$$\hat{a}(t) = \frac{2}{N_0}\int_{-\infty}^{\infty} \sqrt{P}\left\{\cos \omega_c z\, R_a(t, z) - \sin \omega_c z\left[\int_{-\infty}^{\infty} h(z - y)R_a(t,y)\, dy\right]\right\}$$
$$\{r(z) - \sqrt{P}\,[\hat{a}(z)\cos \omega_c z - \hat{\check{a}}(z)\sin \omega_c z]\}\, dz \qquad (507)$$

and

$$\hat{\check{a}}(t) = \frac{2}{N_0}\int_{-\infty}^{\infty} \sqrt{P}\left\{\cos \omega_c z\left[\int_{-\infty}^{\infty} h(t - y)R_a(y, z)\, dy\right] - \sin \omega_c z\, R_a(t, y)\right\}$$
$$\{r(z) - \sqrt{P}\,[\hat{a}(z)\cos \omega_c z - \hat{\check{a}}(z)\sin \omega_c z]\}\, dz. \qquad (508)$$

These equations look complicated. However, drawing the block diagram and using the definition of $H(j\omega)$ (506), we are led to the simple receiver in Fig. 6.62 (see Problem 6.4.5 for details).

We can easily verify that $n_s(t)$ has a spectral height of $N_0/2$. A comparison of Figs. 6.60 and 6.62 makes it clear that the mean-square performance of SSB-SC and DSB-SC are identical. Thus we may use other considerations such as bandwidth occupancy when selecting a system for a particular application.

These two examples demonstrate the basic ideas involved in the estimation of messages in the linear modulation systems used in conventional communication systems.

Two other systems of interest are double-sideband and single-sideband in which the carrier is *not* suppressed. The transmitted signal for the first case was given in (488). The resulting receivers follow in a similar manner.

From the standpoint of estimation accuracy we would expect that because part of the available power is devoted to transmitting a residual carrier the estimation error would increase. This qualitative increase is easy to demonstrate (see Problems 6.4.6, 6.4.7). We might ask why we

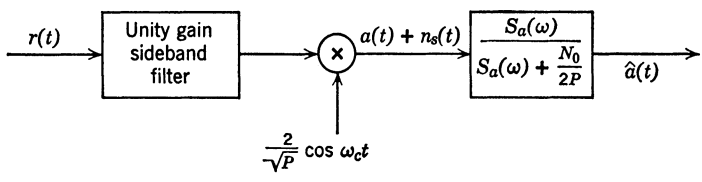


Fig. 6.62 SSB receiver.

would ever use a residual carrier. The answer, of course, lies in our model of the communication link.

We have assumed that $c(t)$, the modulation function (or carrier), is *exactly* known at the receiver. In other words, we assume that the oscillator at the receiver is *synchronized* in phase with the transmitter oscillator. For this reason, the optimum receivers in Figs. 6.58 and 6.60 are frequently referred to as *synchronous demodulators*. To implement such a demodulator in actual practice the receiver must be supplied with the carrier in some manner. In a simple method a pilot tune uniquely related to the carrier is transmitted. The receiver uses the pilot tone to construct a replica of $c(t)$. As soon as we consider systems in which the carrier is reconstructed from a signal sent from the transmitter, we encounter the question of an imperfect copy of $c(t)$. The imperfection occurs because there is noise in the channel in which we transmit the pilot tone. Although the details of reconstructing the carrier will develop more logically in Chapter II.2, we can illustrate the effect of a phase error in an AM system with a simple example.

Example. Let

$$r(t) = \sqrt{2P}\, a(t) \cos \omega_c t + n(t). \tag{509}$$

Assume that we are using the detector of Fig. 6.58 or 6.60. Instead of multiplying by exactly $c(t)$, however, we multiply by $\sqrt{2P} \cos (\omega_c t + \phi)$, where ϕ is phase angle that is a random variable governed by some probability density $p_\phi(\phi)$. We assume that ϕ is independent of $n(t)$.

It follows directly that for a given value of ϕ the effect of an imperfect phase reference is equivalent to a signal power reduction:

$$P_{ef} = P \cos^2 \phi. \tag{510}$$

We can then find an expression for the mean-square error (either realizable or nonrealizable) for the reduced power signal and average the result over $p_\phi(\phi)$. The calculations are conceptually straightforward but tedious (see Problems 6.4.8 and 6.4.9).

We can see intuitively that if ϕ is almost always small (say $|\phi| < 15°$) the effect will be negligible. In this case our model which assumes $c(t)$ is known exactly is a good approximation to the actual physical situation, and the results obtained from this model will accurately predict the performance of the actual system. A number of related questions arise:

1. Can we reconstruct the carrier without devoting any power to a pilot tone? This question is discussed by Costas [50]. We discuss it in the problem section of Chapter II.2.
2. If there is a random error in estimating $c(t)$, is the receiver structure of Fig. 6.58 or 6.60 optimum? The answer in general is "no". Fortunately, it is not too far from optimum in many practical cases.
3. Can we construct an optimum estimation theory that leads to practical receivers for the general case of a *random* modulation matrix, that is,

$$\mathbf{r}(t) = \mathbf{C}(t)\, \mathbf{x}(t) + \mathbf{n}(t), \tag{511}$$

where $C(t)$ is random? We find that the practicality depends on the statistics of $C(t)$. (It turns out that the easiest place to answer this question is in the problem section of Chapter II.3.)

4. If synchronous detection is optimum, why is it not used more often? Here, the answer is complexity. In Problem 6.4.10 we compute the performance of a DSB residual-carrier system when a simple detector is used. For high-input SNR the degradation is minor. Thus, whenever we have a single transmitter and many receivers (e.g., commercial broadcasting), it is far easier to increase transmitter power than receiver complexity. In military and space applications, however, it is frequently easier to increase receiver complexity than transmitter power.

This completes our discussion of linear modulation systems. We now comment briefly on some of the results obtained in this chapter.

6.5 THE FUNDAMENTAL ROLE OF THE OPTIMUM LINEAR FILTER

Because we have already summarized the results of the various sections in detail it is not necessary to repeat the comments. Instead, we shall discuss briefly three distinct areas in which the techniques developed in this chapter are important.

Linear Systems. We introduced the topic by demonstrating that for linear modulation systems the MAP interval estimate of the message was obtained by processing $r(t)$ with a linear system. To consider point estimates we resorted to an approach that we had not used before. We required that the processor be a linear system and found the best possible linear system. We saw that if we constrained the structure to be linear then only the second moments of the processes were relevant. This is an example of the type mentioned in Chapter 1 in which a partial characterization is adequate because we employed a structure-oriented approach. We then completed our development by showing that a linear system was the best possible processor whenever the Gaussian assumption was valid. Thus all of our results in this chapter play a double role. They are the best processors under the Gaussian assumptions for the classes of criteria assumed and they are the best *linear* processors for any random process.

The techniques of this chapter play a fundamental role in two other areas.

Nonlinear Systems. In Chapter II.2 we shall develop optimum receivers for nonlinear modulation systems. As we would expect, these receivers are nonlinear systems. We shall find that the optimum linear filters we have derived in this chapter appear as components of the over-all nonlinear system. We shall also see that the model of the system with respect to its effect on the message is linear in many cases. In these cases the results in this chapter will be directly applicable. Finally, as we showed in Chapter 5 the demodulation error in a nonlinear system can be bounded by the error in some related linear system.

Detection of Random Processes. In Chapter II.3 were turn to the detection and estimation problem in the context of a more general model. We shall find that the linear filters we have discussed are components of the optimum detector (or estimator).

We shall demonstrate why the presence of an optimum linear filter should be expected in these two areas. When our study is completed the fundamental importance of optimum linear filters in many diverse contexts will be clear.

6.6 COMMENTS

It is worthwhile to comment on some related issues.

1. In Section 6.2.4 we saw that for stationary processes in white noise the realizable mean-square error was related to Shannon's mutual information. For the nonstationary, finite-interval case a similar relation may also be derived

$$\xi_P = 2\,\frac{N_0}{2}\,\frac{\partial I(T\!:r(t), a(t))}{\partial T}. \tag{512}$$

2. The discussion with respect to state-variable filters considered only the continuous-time case. We can easily modify the approach to include discrete-time systems. (The discrete system results were derived in Problem 2.6.15 of Chapter 2 by using a sequential estimation approach.)

3. Occasionally a problem is presented in which the input has a transient nonrandom component and a stationary random component. We may want to minimize the mean-square error caused by the random input while constraining the squared error due to transient component. This is a straightforward modification of the techniques discussed (e.g., [51]).

4. In Chapter 3 we discussed the eigenfunctions and eigenvalues of the integral equation,

$$\lambda\,\phi(t) = \int_{T_i}^{T_f} K_y(t, u)\,\phi(u)\,du, \qquad T_i \le t \le T_f. \tag{513}$$

For rational spectra we obtained solutions by finding the associated differential equation, solving it, and using the integral equation to evaluate the boundary conditions. From our discussion in Section 6.3 we anticipate that a computationally more efficient method could be found by using state-variable techniques. These techniques are developed in [52] and [54] (see also Problems 6.6.1–6.6.4) The specific results developed are:

(*a*) A solution technique for homogeneous Fredholm equations using state-variable methods. This technique enables us to find the eigenvalues and eigenfunctions of scalar and vector random processes in an efficient manner.

(*b*) A solution technique for nonhomogeneous Fredholm equations using state-variable methods. This technique enables us to find the function $g(t)$ that appears in optimum detector for the colored noise problem. It is also the key to the optimum signal design problem.

(*c*) A solution of the optimum unrealizable filter problem using state-variable techniques. This enables us to achieve the best possible performance using a given amount of input data.

The importance of these results should not be underestimated because they lead to solutions that can be evaluated easily with numerical technniques. We develop these techniques in greater detail in Part II and use them to solve various problems.

5. In Chapter 4 we discussed whitening filters for the problem of detecting signals in colored noise. In the initial discussion we did not require realizability. When we examined the infinite interval stationary process case (p. 312), we determined that a realizable filter could be found and one component interpreted as an optimum realizable estimate of the colored noise. A similar result can be derived for the finite interval nonstationary case (see Problem 6.6.5). This enables us to use state-variable techniques to find the whitening filter. This result will also be valuable in Chapter II.3.

6.7 PROBLEMS

P6.1 Properties of Linear Processors

Problem 6.1.1. Let

$$r(t) = a(t) + n(t), \qquad T_i \leq t \leq T_f,$$

where $a(t)$ and $n(t)$ are uncorrelated Gaussian zero-mean processes with covariance functions $K_a(t, u)$ and $K_n(t, u)$, respectively. Find $p_{a(t_1)|r(t):T_i \leq t \leq T_f}(A|r(t):T_i \leq t \leq T_f)$.

Problem 6.1.2. Consider the model in Fig. 6.3.

1. Derive Property 3V (51).
2. Specialize (51) to the case in which $\mathbf{d}(t) = \mathbf{x}(t)$.

Problem 6.1.3.

Consider the vector model in Fig. 6.3.
Prove that

$$\mathbf{h}_o(t, t)\,\mathbf{R}(t) = \boldsymbol{\xi}_P(t)\,\mathbf{C}^T(t).$$

Comment. Problems 6.1.4 to 6.1.9 illustrate cases in which the observation is a finite set of random variables. In addition, the observation noise is zero. They illustrate the simplicity that (29) leads to in linear estimation problems.

Problem 6.1.4. Consider a simple prediction problem. We observe $a(t)$ at a *single time*. The desired signal is

$$d(t) = a(t + \alpha),$$

where α is a positive constant. Assume that

$$E[a(t)] = 0,$$

$$E[a(t)\, a(u)] = K_a(t - u) \triangleq K_a(\tau).$$

1. Find the best linear MMSE estimate of $d(t)$.
2. What is the mean-square error?
3. Specialize to the case $K_a(\tau) = e^{-k|\tau|}$.
4. Show that, for the correlation function in part 3, the MMSE estimate would not change if the entire past were available.
5. Is this true for any other correlation function? Justify your answer.

Problem 6.1.5. Consider the following interpolation problem. You are given the values $a(0)$ and $a(T)$:

$$E[a(t)] = 0, \qquad -\infty < t < \infty,$$

$$E[a(t)a(u)] = K_a(t - u), \qquad -\infty < t, u < \infty.$$

1. Find the MMSE estimate of $a(t)$.
2. What is the resulting mean-square error?
3. Evaluate for $t = T/2$.
4. Consider the special case, $K_a(\tau) = e^{-k|\tau|}$, and evaluate the processor constants

Problem 6.1.6 [55]. We observe $a(t)$ and $\dot{a}(t)$. Let $d(t) = a(t + \alpha)$, where α is a positive constant.

1. Find the MMSE linear estimate of $d(t)$.
2. State the conditions on $K_a(\tau)$ for your answer to be meaningful.
3. Check for small α.

Problem 6.1.7 [55]. We observe $a(0)$ and $a(t)$. Let

$$d(t) = \int_0^t a(u)\, du.$$

1. Find the MMSE linear estimate of $d(t)$.
2. Check your result for $t \ll 1$.

Problem 6.1.8. Generalize the preceding model to $n + 1$ observations; $a(0)$, $a(t)$. $a(2t) \cdots a(nt)$.

$$d(t) = \int_0^{nt} a(u)\, du.$$

1. Find the equations which specify the optimum linear processor.
2. Find an explicit solution for $nt \ll 1$.

Problem 6.1.9. [55]. We want to reconstruct $a(t)$ from an infinite number of samples; $a(nT)$, $n = \cdots -1, 0, +1, \ldots$, using a MMSE linear estimate:

$$\hat{a}(t) = \sum_{n=-\infty}^{\infty} c_n(t)\, a(nT).$$

1. Find an expression that the coefficients $c_n(t)$ must satisfy.
2. Consider the special case in which

$$S_a(\omega) = 0 \qquad |\omega| > \frac{\pi}{T}.$$

Evaluate the coefficients.

3. Prove that the resulting mean-square error is zero. (Observe that this proves the sampling theorem for random processes.)

Problem 6.1.10. In (29) we saw that

$$E[e_o(t)\, r(u)] = 0, \qquad T_i < u < T_f.$$

1. In our derivation we assumed $h_o(t, u)$ was continuous and defined $h_o(t, T_i)$ and $h_o(t, T_f)$ by the continuity requirement. Assume $r(u)$ contains a white noise component. Prove

$$E[e_o(t)r(T_i)] \neq 0,$$

$$E[e_o(t)r(T_f)] \neq 0.$$

2. Now remove the continuity assumption on $h_o(t, u)$ and assume $r(u)$ contains a white noise component. Find an equation specifying an $h_o(t, u)$, such that

$$E[e_o(t)r(u)] = 0, \qquad T_i \leq u \leq T_f.$$

Are the mean-square errors for the filters in parts 1 and 2 the same? Why?

3. Discuss the implications of removing the white noise component from $r(u)$. Will $h_o(t, u)$ be continuous? Do we use strict or nonstrict inequalities in the integral equation?

P6.2 Stationary Processes, Infinite Past, (Wiener Filters)

Realizable and Unrealizable Filtering

Problem 6.2.1. We have restricted our attention to rational spectra. We write the spectrum as

$$S_r(\omega) = c\,\frac{(\omega - n_1)(\omega - n_2)\cdots(\omega - n_N)}{(\omega - d_1)(\omega - d_2)\cdots(\omega - d_M)}, \qquad n_i \neq d_j,$$

where N and M are even. We assume that $S_r(\omega)$ is integrable on the real line. Prove the following statements:

1. $S_r(\omega) = S_r^*(\omega)$.
2. c is real.
3. All n_i's and d_i's with nonzero imaginary parts occur in conjugate pairs.
4. $S_r(\omega) \geq 0$.
5. Any real roots of numerator occur with even multiplicity.
6. No root of the denominator can be real.
7. $N < M$.

Verify that these results imply all the properties indicated in Fig. 6.7.

Problem 6.2.2. Let

$$r(u) = a(u) + n(u), \qquad -\infty < u \le t.$$

The waveforms $a(u)$ and $n(u)$ are sample functions from uncorrelated zero-mean processes with spectra

$$S_a(\omega) = \frac{2k\sigma_a^2}{\omega^2 + k^2}$$

and

$$S_n(\omega) = N_2\omega^2,$$

respectively.

1. The desired signal is $a(t)$. Find the realizable linear filter which minimizes the mean-square error.
2. What is the resulting mean-square error?
3. Repeat parts 1 and 2 for the case in which the filter may be unrealizable and compare the resulting mean-square errors.

Problem 6.2.3. Consider the model in Problem 6.2.2. Assume that

$$S_n(\omega) = N_0 + N_2\omega^2.$$

1. Repeat Problem 6.2.2.
2. Verify that your answers reduce to those in Problem 6.2.2 when $N_0 = 0$ and to those in the text when $N_2 = 0$.

Problem 6.2.4. Let

$$r(u) = a(u) + n(u), \qquad -\infty < u \le t.$$

The functions $a(u)$ and $n(u)$ are sample functions from independent zero-mean Gaussian random processes.

$$S_a(\omega) = \frac{2k\sigma_a^2}{\omega^2 + k^2},$$

$$S_n(\omega) = \frac{2c\sigma_n^2}{\omega^2 + c^2}.$$

We want to find the MMSE point estimate of $a(t)$.

1. Set up an expression for the optimum processor.
2. Find an explicit expression for the special case

$$\sigma_n^2 = \sigma_a^2,$$
$$c = 2k.$$

3. Look at your answer in (2) and check to see if it is intuitively correct.

Problem 6.2.5. Consider the model in Problem 6.2.4. Now let

$$S_n(\omega) = \frac{N_0}{2} + \frac{2c\sigma_n^2}{\omega^2 + c^2}.$$

1. Find the optimum realizable linear filter (MMSE).
2. Find an expression for ξ_{Pn}.
3. Verify that the result in (1) reduces to the result in Problem 6.2.4 when $N_0 = 0$ and to the result in the text when $\sigma_n^2 = 0$.

Problem 6.2.6. Let

$$r(u) = a(u) + w(u), \qquad -\infty < u \le t.$$

The processes are uncorrelated with spectra

$$S_a(\omega) = \frac{2\sqrt{2}\,P/k}{1 + (\omega^2/k^2)^2}$$

and

$$S_w(\omega) = \frac{N_0}{2}.$$

The desired signal is $a(t)$. Find the optimum realizable linear filter (MMSE).

Problem 6.2.7. The message $a(t)$ is passed through a linear network before transmission as shown in Fig. P6.1. The output $y(t)$ is corrupted by uncorrelated white noise $(N_0/2)$. The message spectrum is $S_a(\omega)$.

$$S_a(\omega) = \frac{2k\sigma_a^2}{\omega^2 + k^2}.$$

1. A minimum mean-square error realizable estimate of $a(t)$ is desired. Find the optimum linear filter.
2. Find ξ_{Pn} as a function of α and $\Lambda \triangleq 4\sigma_a^2/kN_0$.
3. Find the value of α that minimizes ξ_{Pn}.
4. How do the results change if the zero in the prefilter is at $+k$ instead of $-k$.

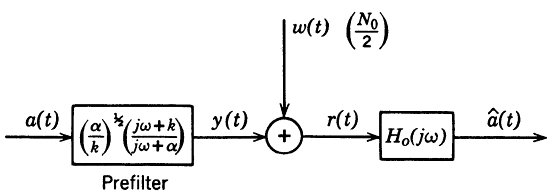


Fig. P6.1

Pure Prediction. The next four problems deal with pure prediction. The model is

$$r(u) = a(u), \qquad -\infty < u \le t,$$

and

$$d(t) = a(t + \alpha),$$

where $\alpha \ge 0$. We see that there is no noise in the received waveform. The object is to *predict* $a(t)$.

Problem 6.2.8. Let

$$S_a(\omega) = \frac{2k}{\omega^2 + k^2}.$$

1. Find the optimum (MMSE) realizable filter.
2. Find the normalized prediction error ξ_{Pn}^{α}.

Problem 6.2.9. Let

$$S_a(\omega) = \frac{1}{(1 + \omega^2)^2}.$$

Repeat Problem 6.2.8.

Problem 6.2.10. Let

$$S_a(\omega) = \frac{1 + \omega^2}{1 + \omega^4}.$$

Repeat Problem 6.2.8.

Problem 6.2.11.

1. The received signal is $a(u)$, $-\infty < u \le t$. The desired signal is

$$d(t) = a(t + \alpha), \qquad \alpha > 0.$$

Find $H_o(j\omega)$ to minimize the mean-square error

$$E\{[\hat{d}(t) - d(t)]^2\},$$

where

$$d(t) = \int_{-\infty}^{t} h_o(t - u)\, a(u)\, du.$$

The spectrum of $a(t)$ is

$$S_a(\omega) = \prod_{i=1}^{n} \frac{A^2}{(\omega^2 + k_i^2)},$$

where $k_i \ne k_j$; $i \ne j$ for $i = 1, \dots, n$, $j = 1, \dots, n$.

2. Now assume that the received signal is $a(u)$, $T_i \le u \le t$, where T_i is a *finite* number. Find $h_o(t, \tau)$ to minimize the mean-square error.

$$\hat{d}(t) = \int_{T_i}^{t} h_o(t, u)\, a(u)\, du.$$

3. Do your answers to parts 1 and 2 enable you to make any general statements about pure prediction problems in which the message spectrum has no zeros?

Problem 6.2.12. The message is generated as shown in Fig. P6.2, where $u(t)$ is a white noise process (unity spectral height) and α_i, $i = 1, 2$, and λ_i, $i = 1, 2$, are known positive constants. The additive white noise $w(t)(N_0/2)$ is uncorrelated with $u(t)$.

1. Find an expression for the linear filters whose outputs are the MMSE realizable estimates of $x_i(t)$, $i = 1, 2$.
2. Prove that

$$\hat{a}(t) = \sum_{i=1}^{2} \hat{x}_i(t).$$

3. Assume that

$$d(t) = \sum_{i=1}^{2} d_i\, x_i(t).$$

Prove that

$$\hat{d}(t) = \sum_{i=1}^{2} d_i\, \hat{x}_i(t).$$

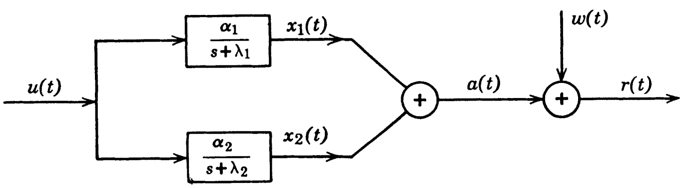


Fig. P6.2

Problem 6.2.13. Let

$$r(u) = a(u) + n(u), \qquad -\infty < u \le t,$$

where $a(u)$ and $n(u)$ are uncorrelated random processes with spectra

$$S_a(\omega) = \frac{\omega^2}{\omega^4 + 1},$$

$$S_n(\omega) = \frac{1}{\omega^2 + \epsilon^2}.$$

The desired signal is $a(t)$. Find the optimum (MMSE) linear filter and the resulting error for the limiting case in which $\epsilon \to 0$. Sketch the magnitude and phase of $H_o(j\omega)$.

Problem 6.2.14. The received waveform $r(u)$ is

$$r(u) = a(u) + w(u), \qquad -\infty < u \le t,$$

where $a(u)$ and $w(u)$ are uncorrelated random processes with spectra

$$S_a(\omega) = \frac{2k\sigma_a^2}{\omega^2 + k^2},$$

$$S_n(\omega) = \frac{N_0}{2}.$$

Let

$$d(t) \triangleq \int_t^{t+\alpha} a(u)\,du, \qquad \alpha > 0.$$

1. Find the optimum (MMSE) linear filter for estimating $d(t)$.
2. Find ξ_P^α.

Problem 6.2.15 (continuation). Consider the same model as Problem 6.2.14. Repeat that problem for the following desired signals:

1. $d(t) = \dfrac{1}{\alpha} \displaystyle\int_{t-\alpha}^{t} a(u)\,du, \qquad \alpha > 0.$

2. $d(t) = \dfrac{1}{\beta - \alpha} \displaystyle\int_{t+\alpha}^{t+\beta} a(u)\,du, \qquad \alpha > 0, \beta > 0, \beta \ge \alpha.$

What happens as $(\beta - \alpha) \to 0$?

3. $d(t) = \displaystyle\sum_{n=-1}^{+1} k_n\, a(t - n\alpha), \qquad \alpha > 0.$

Problem 6.2.16. Consider the model in Fig. P6.3. The function $u(t)$ is a sample function from a white process (unity spectral height). Find the MMSE realizable linear estimates, $\hat{x}_1(t)$ and $\hat{x}_2(t)$. Compute the mean-square errors and the cross correlation between the errors ($T_i = -\infty$).

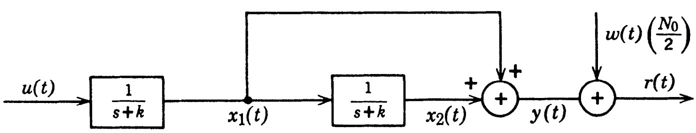


Fig. P6.3

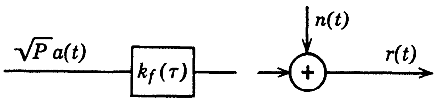


Fig. P6.4

Problem 6.2.17. Consider the communication problem in Fig. P6.4. The message $a(t)$ is a sample function from a stationary, zero-mean Gaussian process with unity variance. The channel $k_f(\tau)$ is a linear, time-invariant, not necessarily realizable system. The additive noise $n(t)$ is a sample function from a zero-mean white Gaussian process $(N_0/2)$.

1. We process $r(t)$ with the optimum unrealizable linear filter to find $\hat{a}(t)$. Assuming $\int_{-\infty}^{\infty} |K_f(j\omega)|^2 (d\omega/2\pi) = 1$, find the $k_f(\tau)$ that minimizes the minimum mean-square error.
2. Sketch for

$$S_a(\omega) = \frac{2k}{\omega^2 + k^2}.$$

CLOSED FORM ERROR EXPRESSIONS

Problem 6.2.18. We want to integrate

$$\xi_P = \frac{N_0}{2}\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \ln\left[1 + \frac{2c_n/N_0}{1 + (\omega/k)^{2n}}\right].$$

1. Do this by letting $y = 2c_n/N_0$. Differentiate with respect to y and then integrate with respect to ω. Integrate the result from 0 to y.
2. Discuss the conditions under which this technique is valid.

Problem 6.2.19. Evaluate

$$\xi_u = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{c_n}{1 + (\omega/k)^{2n} + (2/N_0)c_n}.$$

Comment. In the next seven problems we develop closed-form error expressions for some interesting cases. In most of these problems the solutions are difficult. In all problems

$$r(u) = a(u) + n(u), \qquad -\infty < u \le t,$$

where $a(u)$ and $n(u)$ are uncorrelated. The desired signal is $a(t)$ and optimum (MMSE) linear filtering is used. The optimum realizable linear filter is $H_o(j\omega)$ and

$$G_o(j\omega) \triangleq 1 - H_o(j\omega).$$

Most of the results were obtained in [4].

Problem 6.2.20. Let

$$S_n(\omega) = \frac{N_0 a^2}{\omega^2 + a^2}.$$

Show that

$$H_o(j\omega) = 1 - k\,\frac{[S_n(\omega)]^+}{[S_a(\omega) + S_n(\omega)]^+},$$

where

$$k = \exp\left[\frac{2}{N_0 a}\int_0^\infty S_n(\omega)\ln\frac{S_n(\omega)}{S_a(\omega) + S_n(\omega)}\frac{d\omega}{2\pi}\right].$$

Problem 6.2.21. Show that if $\lim_{\omega\to\infty} S_n(\omega) \to 0$ then

$$\xi_P = 2\int_0^\infty \{S_n(\omega) - |G_o(j\omega)|^2[S_a(\omega) + S_n(\omega)]\}\frac{d\omega}{2\pi}.$$

Use this result and that of the preceding problem to show that for one-pole noise

$$\xi_P = \frac{N_0 a}{2}(1 - k^2).$$

Problem 6.2.22. Consider the case

$$S_n(\omega) = N_0 + N_2\omega^2 + N_4\omega^4.$$

Show that

$$|G_o(j\omega)|^2 = \frac{S_n(\omega) + K}{S_n(\omega) + S_a(\omega)},$$

where

$$\int_0^\infty \ln\left[\frac{S_n(\omega) + K}{S_n(\omega) + S_a(\omega)}\right] d\omega = 0$$

determines K.

Problem 6.2.23. Show that when $S_n(\omega)$ is a polynomial

$$\xi_P = -\frac{1}{\pi}\int_0^\infty d\omega\{S_n(\omega) - |G_o(j\omega)|^2[S_a(\omega) + S_n(\omega)] + S_n(\omega)\ln|G_o(j\omega)|^2\}.$$

Problem 6.2.24. As pointed out in the text, we can double the size of the class of problems for which these results apply by a simple observation. Figure P6.5*a* represents a typical system in which the message is filtered before transmission.

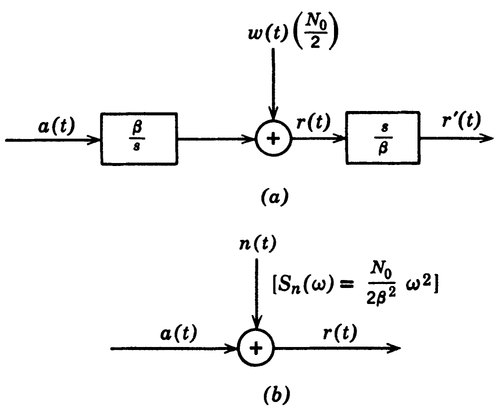


Fig. P6.5

Clearly the mean-square error in this system is identical to the error in the system in Fig. P6.5*b*. Using Problem 6.2.23, verify that

$$\xi_P = -\frac{1}{\pi}\int_0^\infty \left[\frac{N_0}{2\beta^2}\,\omega^2 - \left[\frac{N_0}{2\beta^2}\,\omega^2 + K\right] + \frac{N_0\omega^2}{2\beta^2}\ln|G_o(j\omega)|^2\right] d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \left[K - \frac{N_0\omega^2}{2\beta^2}\ln\left(\frac{S_n(\omega) + K}{S_a(\omega) + S_n(\omega)}\right)\right] d\omega.$$

Problem 6.2.25 (continuation) [39]. Using the model of Problem 6.2.24, show that

$$\xi_P = \frac{N_0}{6}\,[f(0)]^3 + F(0),$$

where

$$f(0) = \int_{-\infty}^{\infty} \ln\left[1 + \frac{2\beta^2 S_a(\omega)}{\omega^2 N_0}\right]\frac{d\omega}{2\pi}$$

and

$$F(0) = \int_{-\infty}^{\infty} \omega^2\,\frac{N_0}{2\beta^2}\ln\left[1 + \frac{2\beta^2 S_a(\omega)}{\omega^2 N_0}\right]\frac{d\omega}{2\pi}.$$

Problem 6.2.26 [20]. Extend the results in Problem 6.2.20 to the case

$$S_n(\omega) = \frac{N_0}{2} + \frac{N_1 a^2}{\omega^2 + a^2}$$

to find $|G_o(j\omega)|^2$ and ξ_P.

Feedback Realizations

Problem 6.2.27. Verify that the optimum loop filter is of the form indicated in Fig. 6.21*b*. Denote the numerator by $F(s)$.

1. Show that

$$F(s) = \left[\frac{2q}{N_0}\,B(s)\,B(-s) + P(s)\,P(-s)\right]^+ - P(s),$$

where $B(s)$ and $P(s)$ are defined in Fig. 6.21*a*.

2. Show that $F(s)$ is exactly one degree less than $P(s)$.

Problem 6.2.28. Prove

$$\xi_P = \frac{N_0}{2}\lim_{s\to\infty} s\,G_{lo}(s) = \frac{N_0}{2}\,f_{n-1}.$$

where $G_{lo}(s)$ is the optimum loop filter and f_{n-1} is defined in Fig. 6.21*b*.

Problem 6.2.29. In this problem we construct a realizable whitening filter. In Chapter 4 we saw that a conceptual unrealizable whitening filter may readily be obtained in terms of a Karhunen-Loève expansion. Let

$$r(u) = n_c(u) + w(u), \qquad -\infty < u \le t,$$

where $n_c(u)$ has a rational spectrum and $w(u)$ is an uncorrelated white noise process. Denote the optimum (MMSE) realizable linear filter for estimating $n_c(t)$ as $H_o(j\omega)$.

1. Prove that $1 - H_o(j\omega)$ is a realizable whitening filter. Draw a feedback realization of the whitening filter.

Hint. Recall the feedback structure of the optimum filter (173).

2. Find the inverse filter for $1 - H_o(j\omega)$. Draw a feedback realization of the inverse filter.

Problem 6.2.30 (continuation). What are the necessary and sufficient conditions for the inverse of the whitening filter found in Problem 6.2.29 to be stable?

Generalizations

Problem 6.2.31. Consider the simple unrealizable filter problem in which

$$r(u) = a(u) + n(u), \qquad -\infty < u < \infty$$

and

$$d(t) = a(t).$$

Assume that we design the optimum unrealizable filter $H_{ou}(j\omega)$ using the spectrum $S_a(\omega)$ and $S_n(\omega)$. In practice the noise spectrum is

$$S_{np}(\omega) = S_{nd}(\omega) + S_{ne}(\omega).$$

1. Show that the mean-square error using $H_{ou}(j\omega)$ is

$$\xi_{up} = \xi_{uo} + \int_{-\infty}^{\infty} |H_{ou}(j\omega)|^2 \, S_{ne}(\omega) \frac{d\omega}{2\pi},$$

where *up* denotes unrealizable mean-square error in practice and *uo* denotes unrealizable mean-square error in the optimum filter when the design assumptions are exact.

2. Show that the change in error is

$$\Delta\xi_u = \int_{-\infty}^{\infty} \left[\frac{S_a(\omega)}{S_a(\omega) + S_{nd}(\omega)}\right]^2 S_{ne}(\omega) \frac{d\omega}{2\pi}.$$

3. Consider the case

$$S_{nd}(\omega) = \frac{N_0}{2},$$

$$S_{ne}(\omega) = \epsilon \frac{N_0}{2}.$$

The message spectrum is flat and bandlimited. Show that

$$\Delta\xi_u = \frac{\epsilon\Lambda}{(1 + \Lambda)^2},$$

where Λ is the signal-to-noise ratio in the message bandwidth.

Problem 6.2.32. Derive an expression for the change in the mean-square error in an optimum unrealizable filter when the actual message spectrum is different from the design message spectrum.

Problem 6.2.33. Repeat Problem 6.2.32 for an optimum realizable filter and white noise.

Problem 6.2.34. Prove that the system in Figs. 6.23*b* is the optimum realizable filter for estimating $a(t)$.

Problem 6.2.35. Derive (181) and (183) for arbitrary $K_d(j\omega)$.

Problem 6.2.36. The mean-square error using an optimum unrealizable filter is given by (183):

$$\xi_{uo} = \int_{-\infty}^{\infty} \frac{S_a(\omega)\, S_n(\omega)}{S_a(\omega)\, S_f(\omega) + S_n(\omega)} \frac{d\omega}{2\pi},$$

where $S_f(\omega) \triangleq |K_f(j\omega)|^2$.

1. Consider the following problem. Constrain the transmitted power

$$P = \int_{-\infty}^{\infty} S_a(\omega)\, S_f(\omega) \frac{d\omega}{2\pi}.$$

Find an expression for $S_f(\omega)$ that minimizes the mean-square error.
2. Evaluate the resulting mean-square error.

Problem 6.2.37. Let

$$r(u) = a(u) + n(u), \qquad -\infty < u \le t,$$

where $a(u)$ and $n(u)$ are uncorrelated. Let

$$S_a(\omega) = \frac{1}{1 + \omega^2}, \qquad S_n(\omega) = \epsilon^2.$$

The desired signal is $d(t) = (d/dt)\, a(t)$.

1. Find $H_o(j\omega)$.
2. Discuss the behavior of $H_o(j\omega)$ and ξ_P as $\epsilon \to 0$. Why is the answer misleading?

Problem 6.2.38. Repeat Problem 6.2.37 for the case

$$S_a(\omega) = \frac{1}{1 + \omega^4}, \qquad S_n(\omega) = \epsilon^4.$$

What is the important difference between the message random processes in the two problems? Verify that differentiation and optimum realizable filtering do not commute.

Problem 6.2.39. Let

$$a(u) = \cos(2\pi f u + \phi), \qquad -\infty < u \le t,$$

where ϕ and f are independent variables:

$$p_\phi(\phi) = \frac{1}{2\pi}, \qquad 0 \le \phi \le 2\pi$$

and

$$p_f(X) = 0, \qquad X \le 0.$$

1. Describe the resulting ensemble.
2. Prove that $S_a(f) = p_f(|f|)/4$.
3. Choose a $p_f(X)$ to make $a(t)$ a deterministic process (see p. 512). Demonstrate a linear predictor whose mean-square error is zero.
4. Choose a $p_f(X)$ to make $a(t)$ a nondeterministic process. Show that you can predict $a(t)$ with zero mean-square error by using a nonlinear predictor.

Problem 6.2.40. Let

$$g^+(\tau) = \mathcal{F}^{-1}[G^+(j\omega)].$$

Prove that the MMSE error for *pure* prediction is

$$\xi_P^\alpha = \int_0^\alpha [g^+(\tau)]^2\, d\tau.$$

Problem 6.2.41 [1]. Consider the message spectrum

$$S_a(\omega) = \left[\left(1 + \frac{\omega^2}{n}\right)^n\right]^{-1}.$$

1. Show that

$$g^+(\tau) = \frac{\tau^{n-1}\exp(-\tau\sqrt{n})}{n^{-n/2}(n-1)!}.$$

2. Show that (for large n)

$$\xi_P^\alpha \simeq \int_0^\alpha \frac{1}{2\pi} \exp\left[-2\left(t - \frac{n-1}{\sqrt{n}}\right)^2\right] dt.$$

3. Use part 2 to show that for any ϵ^2 and α we can make

$$\xi_P^\alpha < \epsilon^2$$

by increasing n sufficiently. Explain why this result is true.

Problem 6.2.42. The message $a(t)$ is a zero-mean process observed in the absence of noise. The desired signal $d(t) = a(t + \alpha)$, $\alpha > 0$.

1. Assume

$$K_a(\tau) = \frac{1}{\tau^2 + k^2}.$$

Find $\hat{d}(t)$ by using $a(t)$ and its derivatives. What is the mean-square error for $\alpha < k$?

2. Assume

$$K_a(\tau) = e^{-k\tau^2}.$$

Show that

$$\hat{a}(t + \alpha) = \sum_{n=0}^{\infty} \left[\frac{d^n}{dt^n} a(t)\right] \frac{\alpha^n}{n!},$$

and that the mean-square error is zero for all α.

Problem 6.2.43. Consider a simple diversity system,

$$r_1(t) = a(t) + n_1(t),$$

$$r_2(t) = a(t) + n_2(t),$$

where $a(t)$, $n_1(t)$, and $n_2(t)$ are independent zero-mean, stationary Gaussian processes with finite variances. We wish to process $r_1(t)$ and $r_2(t)$, as shown in Fig. P6.6. The spectra $S_{n_1}(\omega)$ and $S_{n_2}(\omega)$ are known; $S_a(\omega)$, however, is *unknown*. We require that the message $a(t)$ be undistorted. In other words, if $n_1(t)$ and $n_2(t)$ are zero, the output will be exactly $a(t)$.

1. What condition does this impose on $H_1(j\omega)$ and $H_2(j\omega)$?
2. We want to choose $H_1(j\omega)$ to minimize $E[n_c^2(t)]$, subject to the constraint that

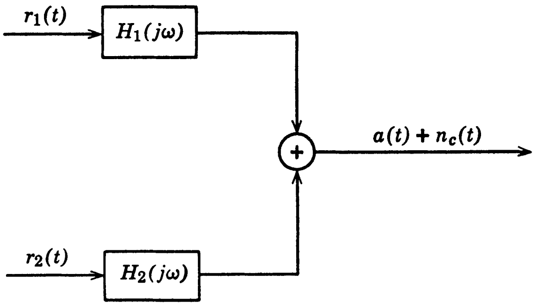


Fig. P6.6

$a(t)$ be reproduced exactly in the absence of input noise. The filters must be realizable and may operate on the infinite past. Find an expression for $H_{1_0}(j\omega)$ and $H_{2_0}(j\omega)$ in terms of the given quantities.

3. Prove that the $\hat{a}(t)$ obtained in part 2 is an unbiased, efficient estimate of the sample function $a(t)$. [Therefore $\hat{a}(t) = \hat{a}_{ml}(t)$.]

Problem 6.2.44. Generalize the result in Problem 6.2.43 to the n-input problem. Prove that any n-dimensional distortionless filter problem may be recast as an $(n - 1)$-dimensional Wiener filter problem.

P6.3 Finite-time, Nonstationary Processes (Kalman-Bucy filters)

STATE-VARIABLE REPRESENTATIONS

Problem 6.3.1. Consider the differential equation

$$y^{(n)}(t) + p_{n-1}y^{(n-1)}(t) + \cdots + p_0 y(t) = b_{n-1}u^{(n-1)}(t) + \cdots + b_0 u(t).$$

Extend Canonical Realization 1 on p. 522 to include this case. The desired $\mathbf{F}$ is

$$\mathbf{F} = \left[\begin{array}{c|ccc} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ \hline -p_0 & -p_1 & \cdots & -p_{n-1} \end{array}\right]$$

Draw an analog computer realization and find the $\mathbf{G}$ matrix.

Problem 6.3.2. Consider the differential equation in Problem 6.3.1. Derive Canonical Realization 3 (see p. 526) for the case of repeated roots.

Problem 6.3.3 [27]. Consider the differential equation

$$y^{(n)}(t) + p_{n-1}y^{(n-1)}(t) + \cdots + p_0 y(t) = b_{n-1}u^{(n-1)}(t) + \cdots + b_0 u(t).$$

1. Show that the system in Fig. P6.7 is a correct analog computer realization.
2. Write the vector differential equation that describes the system.

Problem 6.3.4. Draw an analog computer realization for the following systems:

$$\begin{aligned} &1. \quad \ddot{y}(t) + 3\dot{y}(t) + 4y(t) = \dot{u}(t) + u(t), \\ &2. \quad \ddot{y}_1(t) + 3\dot{y}_1(t) + 2y_2(t) = u_1(t) + 2\dot{u}_2(t) + 2u_2(t), \\ &\quad\;\; \ddot{y}_2(t) + 4\dot{y}_1(t) + 3y_2(t) = 3u_2(t) + u_1(t). \end{aligned}$$

Write the associated vector differential equation.

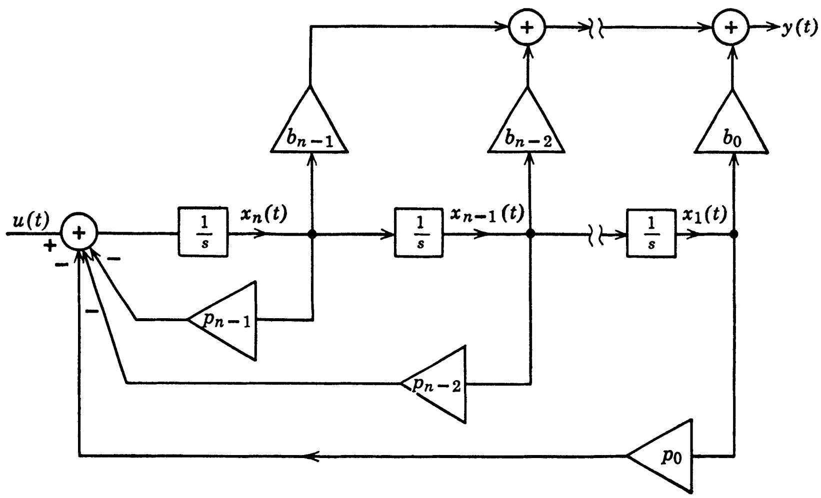


Fig. P6.7

Problem 6.3.5 [27]. Find the transfer function matrix and draw the transfer function diagram for the systems described below. Comment on the number of integrators required.

1. $\ddot{y}_1(t) + 3\dot{y}_1(t) + 2y_1(t) = \dot{u}_1(t) + 2u_1(t) + \dot{u}_2(t) + u_2(t),$
 $\dot{y}_2(t) + 2y_2(t) = -\dot{u}_1(t) - 2u_1(t) + u_2(t).$
2. $\dot{y}_1(t) + y_1(t) = u_1(t) + 2u_2(t),$
 $\ddot{y}_2(t) + 3\dot{y}_2(t) + 2y_2(t) = \dot{u}_2(t) + u_2(t) - u_1(t).$
3. $\ddot{y}_1(t) + 2\dot{y}_2(t) + y_1(t) = \dot{u}_1(t) + u_1(t) + u_2(t),$
 $\ddot{y}_2(t) + \dot{y}_1(t) + y_2(t) = u_2(t) + u_1(t).$
4. $\ddot{y}_1(t) + 3\dot{y}_1(t) + 2y_1(t) = 3\dot{u}_1(t) + 4\dot{u}_2(t) + 8u_2(t),$
 $\dot{y}_2(t) + 3y_2(t) - 4y_1(t) - \dot{y}_1(t) = \dot{u}_1(t) + 2\dot{u}_2(t) + 2u_2(t).$

Problem 6.3.6 [27]. Find the vector differential equations for the following systems, using the partial fraction technique.

1. $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = u(t),$
2. $\dddot{y}(t) + 4\ddot{y}(t) + 5\dot{y}(t) + 2y(t) = u(t),$
3. $\dddot{y}(t) + 4\ddot{y}(t) + 6\dot{y}(t) + 4y(t) = u(t),$
4. $\ddot{y}_1(t) - 10\dot{y}_2(t) + y_1(t) = u_1(t),$
 $\dot{y}_2(t) + 6y_2(t) = u_2(t).$

Problem 6.3.7. Compute $e^{\mathbf{F}t}$ for the following matrices:

1. $$\mathbf{F} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$
2. $$\mathbf{F} = \begin{bmatrix} 3 & 2 \\ -1 & 6 \end{bmatrix}.$$
3. $$\mathbf{F} = \begin{bmatrix} -2 & 5 \\ -4 & -3 \end{bmatrix}.$$

Problem 6.3.8. Compute $e^{\mathbf{F}t}$ for the following matrices:

1. $$\mathbf{F} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

2. $$\mathbf{F} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

3. $$\mathbf{F} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}.$$

Problem 6.3.9. Given the system with state representation as follows,

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{F}\,\mathbf{x}(t) + \mathbf{G}\,u(t), \\ y(t) &= \mathbf{C}\,\mathbf{x}(t), \\ \mathbf{x}(0) &= \mathbf{0}. \end{aligned}$$

Let $U(s)$ and $Y(s)$ denote the Laplace transform of $u(t)$ and $y(t)$, respectively. We found that the transfer function was

$$\begin{aligned} H(s) = \frac{Y(s)}{U(s)} &= \mathbf{C}\,\mathbf{\Phi}(s)\mathbf{G} \\ &= \mathbf{C}(s\mathbf{I} - \mathbf{F})^{-1}\mathbf{G}. \end{aligned}$$

Show that the poles of $H(s)$ are the eigenvalues of the matrix $\mathbf{F}$.

Problem 6.3.10. Consider the circuit shown in Fig. P6.8. The source is turned on at $t = 0$. The current $i(0^-)$ and the voltage across the capacitor $v_c(0^-)$ are both zero. The observed quantity is the voltage across R.

1. Write the vector differential equations that describe the system and an equation that describes the observation process.
2. Draw an analog computer realization of the circuit.

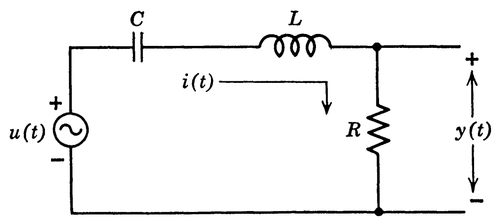


Fig. P6.8

Problem 6.3.11. Consider the control system shown in Fig. P6.9. The output of the system is $a(t)$. The two inputs, $b(t)$ and $n(t)$, are sample functions from zero-mean, uncorrelated, stationary random processes. Their spectra are

$$S_b(\omega) = \frac{2\sigma_b^2 k}{\omega^2 + k^2}$$

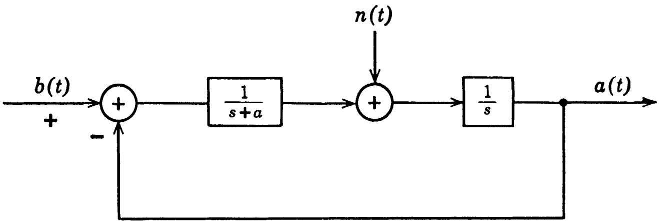


Fig. P6.9

and

$$S_n(\omega) = \frac{N_0}{2}.$$

Write the vector differential equation that describes a mathematically equivalent system whose input is a vector white noise $\mathbf{u}(t)$ and whose output is $a(t)$.

Problem 6.3.12. Consider the discrete multipath model shown in Fig. P6.10. The time delays are assumed known. The channel multipliers are independent, zero-mean processes with spectra

$$S_{b_j}(\omega) = \frac{2k_j\sigma_j^2}{\omega^2 + k_j^2}, \qquad \text{for } j = 1,2,3.$$

The additive white noise is uncorrelated and has spectral height $N_0/2$. The input signal $s(t)$ is a known waveform.

1. Write the state and observation equations for the process.
2. Indicate how this would be modified if the channel gains were correlated.

Problem 6.3.13. In the text we considered in detail state representations for time invariant systems.

Consider the time varying system

$$\ddot{y}(t) + p_1(t)\,\dot{y}(t) + p_0(t)\,y(t) = b_1(t)\,\dot{u}(t) + b_0(t)\,u(t).$$

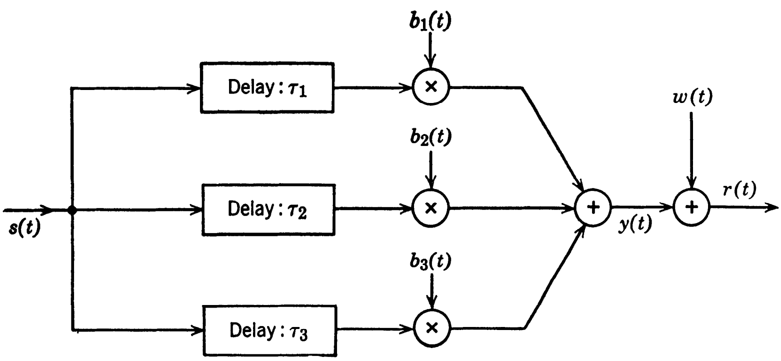


Fig. P6.10

Show that this system has a state representation of the same form as that in Example 2.

$$\frac{d}{dt}\mathbf{x}(t) = \begin{bmatrix} 0 & 1 \\ -p_0(t) & -p_1(t) \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix} u(t),$$

$$y(t) = [1 \quad 0]\,\mathbf{x}(t) = x_1(t),$$

where $h_1(t)$ and $h_2(t)$ are functions that you must find.

Problem 6.3.14 [27]. Given the system defined by the time-varying differential equation

$$y^{(n)}(t) + \sum_{k=0}^{n-1} p_{n-k}(t)\, y^{(k)}(t) = \sum_{k=0}^{n} b_{n-k}(t)\, u^{(k)}(t),$$

Show that this system has the state equations

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & 1 \\ -p_n & -p_{n-1} & \cdot & \cdots & -p_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{bmatrix} + \begin{bmatrix} g_1(t) \\ g_2(t) \\ \cdot \\ \cdot \\ \cdot \\ g_n(t) \end{bmatrix} u(t)$$

$$y(t) = x_1(t) + g_0(t)\, u(t),$$

where

$$g_0(t) = b_0(t),$$

$$g_i(t) = b_i(t) - \sum_{r=0}^{i-1} \sum_{m=0}^{i-r} \binom{n+m-i}{n-i} p_{i-r-m}(t)\, g_r^{(m)}(t).$$

Problem 6.3.15. Demonstrate that the following is a solution to (273).

$$\mathbf{\Lambda_x}(t) = \mathbf{\Phi}(t, t_0)\left[\mathbf{\Lambda_x}(t_0) + \int_{t_0}^{t} \mathbf{\Phi}(t_0, \tau)\mathbf{G}(\tau)\,\mathbf{Q}(\tau)\,\mathbf{G}^T(\tau)\mathbf{\Phi}^T(t_0, \tau)\, d\tau\right]\mathbf{\Phi}^T(t, t_0),$$

where $\mathbf{\Phi}(t, t_0)$ is the fundamental transition matrix; that is,

$$\frac{d}{dt}\mathbf{\Phi}(t, t_0) = \mathbf{F}(t)\,\mathbf{\Phi}(t, t_0),$$

$$\mathbf{\Phi}(t_0, t_0) = \mathbf{I}.$$

Demonstrate that this solution is unique.

Problem 6.3.16. Evaluate $\mathbf{K_y}(t, \tau)$ in terms of $\mathbf{K_x}(t, t)$ and $\mathbf{\Phi}(t, \tau)$, where

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{F}(t)\,\mathbf{x}(t) + \mathbf{G}(t)\,\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}(t)\,\mathbf{x}(t), \\ E[\mathbf{u}(t)\,\mathbf{u}^T(\tau)] &= \mathbf{Q}\,\delta(t - \tau). \end{aligned}$$

Problem 6.3.17. Consider the first-order system defined by

$$\frac{dx(t)}{dt} = -k(t)\,x(t) + g(t)\,u(t).$$

$$y(t) = x(t).$$

1. Determine a general expression for the transition matrix for this system.
2. What is $h(t, \tau)$ for this system?
3. Evaluate $h(t, \tau)$ for

$$k(t) = k(1 + m \sin(\omega_0 t)),$$
$$g(t) = 1.$$

4. Does this technique generalize to vector equations?

Problem 6.3.18. Show that for constant parameter systems the steady-state variance of the unobserved process is given by

$$\lim_{t \to \infty} \mathbf{K_x}(t, t) = \int_0^{\infty} e^{+\mathbf{F}\tau}\mathbf{GQG}^T e^{\mathbf{F}^T\tau}\, d\tau,$$

where

$$\dot{\mathbf{x}}(t) = \mathbf{Fx}(t) + \mathbf{Gu}(t),$$
$$E[\mathbf{u}(t)\, \mathbf{u}^T(\tau)] = \mathbf{Q}\, \delta(t - \tau),$$

or, equivalently,

$$\lim_{t \to \infty} \mathbf{K_x}(t, t) = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} [s\mathbf{I} - \mathbf{F}]^{-1}\mathbf{GQG}^T[-s\mathbf{I} - \mathbf{F}^T]^{-1}\, ds.$$

Problem 6.3.19. Prove that the condition in (317) is necessary when $\mathbf{R}(t)$ is positive-definite.

Problem 6.3.20. In this problem we incorporate the effect of nonzero means into our estimation procedure. The equations describing the model are (302)–(306).

1. Assume that $\mathbf{x}(T_i)$ is a Gaussian random vector

$$E[\mathbf{x}(T_i)] \triangleq \mathbf{m}(T_i) \neq \mathbf{0},$$

and

$$E\{[\mathbf{x}(T_i) - \mathbf{m}(T_i)][\mathbf{x}^T(T_i) - \mathbf{m}^T(T_i)]\} = \mathbf{K_x}(T_i, T_i).$$

It is statistically independent of $\mathbf{u}(t)$ and $\mathbf{w}(t)$. Find the vector differential equations that specify the MMSE estimate $\hat{\mathbf{x}}(t)$, $t \geq T_i$.

2. Assume that $\mathbf{m}(T_i) = \mathbf{0}$. Remove the zero-mean assumption on $\mathbf{u}(t)$,

$$E[\mathbf{u}(t)] = \mathbf{m_u}(t),$$

and

$$E\{[\mathbf{u}(t) - \mathbf{m_u}(t)][\mathbf{u}^T(\tau) - \mathbf{m_u}^T(\tau)]\} = \mathbf{Q}(t)\delta(t - \tau).$$

Find the vector differential equations that specify $\hat{\mathbf{x}}(t)$.

Problem 6.3.21. Consider Example 1 on p. 546. Use Property 16 to derive (351). Remember that when using the Laplace transform technique the contour must be taken to the right of all the poles.

Problem 6.3.22. Consider the second-order system illustrated in Fig. P6.11, where

$$E[u(t)\, u(\tau)] = 2Pab(a + b)\, \delta(t - \tau),$$

$$E[w(t)\, w(\tau)] = \frac{N_0}{2}\, \delta(t - \tau).$$

(a, b are possibly complex conjugates.) The state variables are

$$x_1(t) = y(t),$$

$$x_2(t) = \dot{y}(t).$$

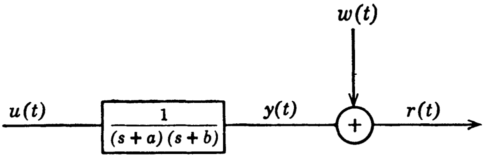


Fig. P6.11

1. Write the state equation and the output equation for the system.
2. For this state representation determine the steady state variance matrix $\mathbf{\Lambda_x}$ of the unobserved process. In other words, find

$$\mathbf{\Lambda_x} = \lim_{t\to\infty} E[\mathbf{x}(t)\,\mathbf{x}^T(t)],$$

where $\mathbf{x}(t)$ is the state vector of the system.
3. Find the transition matrix $\mathbf{T}(t, T_i)$ for the equation,

$$\frac{d\mathbf{T}(t, T_i)}{dt} = \begin{bmatrix} \mathbf{F} & \mathbf{GQG}^T \\ \mathbf{C}^T\mathbf{R}^{-1}\mathbf{C} & -\mathbf{F}^T \end{bmatrix} \mathbf{T}(t, T_i), \qquad \text{[text (336)]}$$

by using Laplace transform techniques. (Depending on the values of a, b, q, and $N_0/2$, the exponentials involved will be real, complex, or both.)
4. Find $\xi_P(t)$ when the initial condition is

$$\xi_P(T_i) = \mathbf{\Lambda_x}.$$

Comment. Although we have an analytic means of determining $\xi_P(t)$ for a system of any order, this problem illustrates that numerical means are more appropriate.

Problem 6.3.23. Because of its time-invariant nature, the optimal linear filter, as determined by Wiener spectral factorization techniques, will lead to a nonoptimal estimate when a finite observation interval is involved. The purpose of this problem is to determine how much we degrade our estimate by using a Wiener filter when the observation interval is finite. Consider the first order system.

$$\dot{x}(t) = -kx(t) + u(t),$$

where

$$\begin{aligned} r(t) &= x(t) + w(t), \\ E[u(t)\,u(\tau)] &= 2kP\,\delta(t-\tau), \\ E[w(t)\,w(\tau)] &= \frac{N_0}{2}\,\delta(t-\tau), \\ E[x(0)] &= 0, \\ E[x^2(0)] &= P_0. \\ T_i &= 0. \end{aligned}$$

1. What is the variance of error obtained by using Kalman-Bucy filtering?
2. Show that the steady-state filter (i.e., the Wiener filter) is given by

$$H_0(j\omega) = \frac{4kP/N_0}{(k+\gamma)(j\omega+\gamma)},$$

where $\gamma = k(1 + 4P/kN_0)^{1/2}$. Denote the output of the Wiener filter as $\hat{x}_{w_0}(t)$.

3. Show that a state representation for the Wiener filter is

$$\dot{\hat{x}}_{w_o}(t) = -\gamma\,\hat{x}_{w_o}(t) + \frac{4Pk}{N_0(k+\gamma)}\,r(t),$$

where

$$\hat{x}_{w_o}(0) = 0.$$

4. Show that the error for this system is

$$\dot{\epsilon}_{w_o}(t) = -\gamma\,\epsilon_{w_o}(t) - u(t) + \frac{4Pk}{N_0(k+\gamma)}\,w(t),$$

$$\epsilon_{w_o}(0) = -x(0).$$

5. Define

$$\xi_{w_o}(t) = E[\epsilon_{w_o}^{\,2}(t)].$$

Show that

$$\dot{\xi}_{w_o}(t) = -2\gamma\xi_{w_o}(t) + \frac{4kP\gamma}{\gamma + k}$$

$$\xi_{w_o}(0) = P_0$$

and verify that

$$\xi_{w_o}(t) = \xi_{P\infty}(1 - e^{-2\gamma t}) + P_o e^{-2\gamma t}.$$

6. Plot the ratio of the mean-square error using the Kalman-Bucy filter to the mean-square error using the Wiener filter. (Note that both errors are a function of time.)

$$\beta(t) = \frac{\xi_P(t)}{\xi_{w_o}(t)} \quad \text{for} \quad \gamma = 1.5k,\ 2k, \text{ and } 3k.$$

$$P_0 = 0,\ 0.5P, \text{ and } P.$$

Note that the expression for $\xi_P(t)$ in (353) is only valid for $P_0 = P$. Is your result intuitively correct?

Problem 6.3.24.

Consider the following system:

$$\dot{\mathbf{x}}(t) = \mathbf{F}\mathbf{x}(t) + \mathbf{G}u(t)$$

$$y(t) = \mathbf{C}\,\mathbf{x}(t)$$

where

$$\mathbf{F} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -p_0 & -p_1 & -p_2 & -p_3 \end{bmatrix}, \qquad \mathbf{G} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{C} = [1\,0\,0\,0].$$

$$E[u(t)\,u(\tau)] = Q\,\delta(t - \tau).$$

Find the steady-state covariance matrix, that is,

$$\lim_{t \to \infty} \mathbf{K}_{\mathbf{x}}(t, t),$$

for a fourth-order Butterworth process using the above representation.

$$S_a(\omega) = \frac{8 \sin(\pi/16)}{1 + \omega^8}$$

Hint. Use the results on p. 545.

Problem 6.3.25. Consider Example 3 on p. 555. Use Property 16 to solve (368).

Problem 6.3.26 (continuation). Assume that the steady-state filter shown in Fig. 6.45 is used. Compute the transient behavior of the error variance for this filter. Compare it with the optimum error variance given in (369).

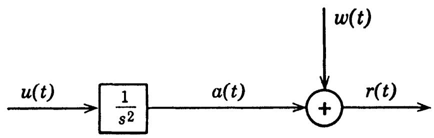
Problem 6.3.27. Consider the system shown in Fig. P6.12*a* where

$$E[u(t)\, u(\tau)] = \sigma^2\, \delta(t - \tau),$$

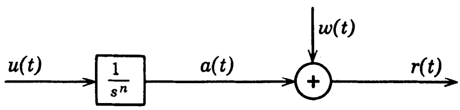
$$E[w(t)\, w(\tau)] = \frac{N_0}{2}\, \delta(t - \tau).$$

$$a(T_i) = \dot{a}(T_i) = 0.$$

1. Find the optimum linear filter.
2. Solve the steady-state variance equation.
3. Verify that the "pole-splitting" technique of conventional Wiener theory gives the correct answer.



(*a*)



(*b*)

Fig. P6.12

Problem 6.3.28 (continuation). A generalization of Problem 6.3.27 is shown in Fig. P6.12*b*. Repeat Problem 6.3.27.

Hint. Use the tabulated characteristics of Butterworth polynomials given in Fig. 6.40.

Problem 6.3.29. Consider the model in Problem 6.3.27. Define the state-vector as

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \triangleq \begin{bmatrix} a(t) \\ \dot{a}(t) \end{bmatrix}, \qquad \mathbf{x}(T_i) = \mathbf{0}.$$

1. Determine $\mathbf{K}_{\mathbf{x}}(t, u) = E[\mathbf{x}(t)\, \mathbf{x}^T(u)]$.
2. Determine the optimum realizable filter for estimating $\mathbf{x}(t)$ (calculate the gains analytically).
3. Verify that your answer reduces to the answer in Problem 6.3.27 as $t \to \infty$.

Problem 6.3.30. Consider Example 5*A* on p. 557. Write the variance equation for arbitrary time ($t \geq 0$) and solve it.

Problem 6.3.31. Let

$$r(t) + \sum_{i=1}^{n} k_i a_i(t) + w(t),$$

where the $a_i(t)$ are statistically independent messages with state representations,

$$\dot{\mathbf{x}}_i(t) = \mathbf{F}_i(t)\, \mathbf{x}_i(t) + \mathbf{G}_i(t)\, \mathbf{u}_i(t),$$
$$a_i(t) = \mathbf{C}_i(t)\, \mathbf{x}_i(t),$$

and $w(t)$ is white noise $(N_0/2)$. Generalize the optimum filter in Fig. 6.52*a* to include this case.

Problem 6.3.32. Assume that a particle leaves the origin at $t = 0$ and travels at a constant but unknown velocity. The observation is corrupted by additive white Gaussian noise of spectral height $N_0/2$. Thus

$$r(t) = vt + w(t), \qquad t \geq 0.$$

Assume that

$$E(v) = 0,$$
$$E(v^2) = \sigma^2,$$

and that v is a Gaussian random variable.

1. Find the equation specifying the MAP estimate of vt.
2. Find the equation specifying the MMSE estimate of vt.

Use the techniques of Chapter 4 to solve this problem.

Problem 6.3.33. Consider the model in Problem 6.3.32. Use the techniques of Section 6.3 to solve this problem.

1. Find the minimum mean-square error linear estimate of the message

$$a(t) \triangleq vt.$$

2. Find the resulting mean-square error.
3. Show that for large t

$$\xi_P(t) \simeq \left(\frac{3N_0}{2t}\right)^{1/2}.$$

Problem 6.3.34 (continuation).

1. Verify that the answers to Problems 6.3.32 and 6.3.33 are the same.
2. Modify your estimation procedure in Problem 6.3.32 to obtain a maximum likelihood estimate (assume that v is an unknown nonrandom variable).
3. Discuss qualitatively when the a priori knowledge is useful.

Problem 6.3.35 (continuation).

1. Generalize the model of Problem 6.3.32 to include an arbitrary polynomial message:

$$a(t) = \sum_{i=1}^{K} v_i t^i,$$

where

$$E(v_i) = 0,$$
$$E(v_i v_j) = \sigma_i^2\, \delta_{ij}.$$

2. Solve for $k = 0$, 1, and 2.

Problem 6.3.36. Consider the second-order system shown in Fig. P6.13.

$$E[u(t)\, u(\tau)] = Q\, \delta(t - \tau),$$

$$E[w(t)\, w(\tau)] = \frac{N_0}{2}\, \delta(t - \tau),$$

$$x_1(t) = y(t),$$
$$x_2(t) = \dot{y}(t).$$

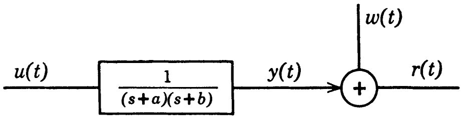


Fig. P6.13

1. Write the state equation and determine the steady state solution to the covariance equation by setting $\dot{\boldsymbol{\xi}}_P(t) = \mathbf{0}$.
2. Do the values of a, b, Q, N_0 influence the roots we select in order that the covariance matrix will be positive definite?
3. In general there are eight possible roots. In the a,b-plane determine which root is selected for any particular point for fixed Q and N_0.

Problem 6.3.37. Consider the prediction problem discussed on p. 566.

1. Derive the result stated in (422). Recall $\mathbf{d}(t) = \mathbf{x}(t+\alpha)$; $\alpha > 0$.
2. Define the prediction covariance matrix as

$$\boldsymbol{\xi}_P{}^{\alpha} \triangleq E\{[\hat{\mathbf{d}}(t) - \mathbf{d}(t)][\hat{\mathbf{d}}^T(t) - \mathbf{d}^T(t)]\}$$

Find an expression for $\boldsymbol{\xi}_P{}^{\alpha}$. Verify that your answer has the correct behavior for $\alpha \to \infty$.

Problem 6.3.38 (continuation). Apply the result in part 2 to the message and noise model in Example 3 on p. 494. Verify that the result is identical to (113).

Problem 6.3.39 (continuation). Let

$$r(u) = a(u) + w(u), \qquad -\infty < u \le t$$

and

$$d(t) = a(t + \alpha).$$

The processes $a(u)$ and $w(u)$ are uncorrelated with spectra

$$S_a(\omega) = \frac{2\sqrt{2}\,P/k}{1 + (\omega^2/k^2)^2},$$

$$S_n(\omega) = \frac{N_0}{2}.$$

Use the result of Problem 6.3.37 to find $E[(\hat{d}(t) - d(t))^2]$ as a function of α.

Compare your result with the result in Problem 6.3.38. Would you expect that the prediction error is a monotone function of n, the order of the Butterworth spectra?

Problem 6.3.40. Consider the following optimum realizable filtering problem:

$$r(u) = a(u) + w(u), \qquad 0 \le u \le t$$

$$S_a(\omega) = \frac{1}{(\omega^2 + k^2)^2},$$

$$S_w(\omega) = \frac{N_0}{2},$$

$$S_{aw}(\omega) = 0.$$

The desired signal $d(t)$ is

$$d(t) = \frac{da(t)}{dt}.$$

We want to find the optimum linear filter by using state-variable techniques.

1. Set the problem up. Define explicitly the state variables you are using and *all* matrices.
2. Draw an explicit block diagram of the optimum receiver. (Do not use matrix notation here.)
3. Write the variance equation as a set of scalar equations. *Comment* on how you would solve it.
4. Find the steady-state solution by letting $\dot{\boldsymbol{\xi}}_P(t) = \mathbf{0}$.

Problem 6.3.41. Let

$$r(u) = a(u) + w(u), \qquad 0 \leq u \leq t,$$

where $a(u)$ and $n(u)$ are uncorrelated processes with spectra

$$S_a(\omega) = \frac{2k\sigma_a^2}{\omega^2 + k^2}$$

and

$$S_n(\omega) = \frac{N_0}{2}.$$

The desired signal is obtained by passing $a(t)$ through a linear system whose transfer function is

$$K_d(j\omega) = \frac{-j\omega + k}{j\omega + \beta}$$

1. Find the optimum linear filter to estimate $d(t)$ and the variance equation.
2. Solve the variance equation for the steady-state case.

Problem 6.3.42. Consider the model in Problem 6.3.41. Let

$$K_d(j\omega) = \left(\frac{1}{j\omega + \beta}\right).$$

Repeat Problem 6.3.41.

Problem 6.3.43. Consider the model in Problem 6.3.41. Let

$$d(t) = \frac{1}{\beta - \alpha}\int_{t+\alpha}^{t+\beta} a(u)\,du, \qquad \alpha > 0,\ \beta > 0,\ \beta > \alpha.$$

1. Does this problem fit into any of the cases discussed in Section 6.3.4 of the text?
2. Demonstrate that you can solve it by using state variable techniques.
3. What is the basic reason that the solution in part 2 is possible?

Problem 6.3.44. Consider the following pre-emphasis problem shown in Fig. P6.14.

$$S_a(\omega) = \frac{2p_1\sigma_a^2}{\omega^2 + p_1^2},$$

$$S_w(\omega) = \frac{N_0}{2},$$

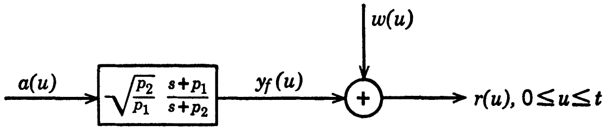


Fig. P6.14

and the processes are uncorrelated.

$$d(t) = a(t).$$

1. Find the optimum realizable linear filter by using a state-variable formulation.
2. Solve the variance equation for $t \to \infty$. (You may assume that a statistical steady state exists.) Observe that $a(u)$, not $y_f(t)$, is the message of interest.

Problem 6.3.45. Estimation in Colored Noise. In this problem we consider a simple example of state-variable estimation in colored noise. Our approach is simpler than that in the text because we are required to estimate only one state variable. Consider the following system.

$$\begin{aligned}
\dot{x}_1(t) &= -k_1\, x_1(t) + u_1(t),\\
\dot{x}_2(t) &= -k_2\, x_2(t) + u_2(t).\\
E[u_1(t)\, u_2(\tau)] &= 0,\\
E[u_1(t)\, u_1(\tau)] &= 2k_1 P_1\, \delta(t-\tau),\\
E[u_2(t)\, u_2(\tau)] &= 2k_1 P_2\, \delta(t-\tau),\\
E[x_1^2(0)] &= P_1,\\
E[x_2^2(0)] &= P_2,\\
E[x_1(0)\, x_2(0)] &= 0.
\end{aligned}$$

We observe

$$r(t) = x_1(t) + x_2(t);$$

that is, no white noise is present in the observation. We want to apply whitening concepts to estimating $x_1(t)$ and $x_2(t)$. First we generate a signal $r'(t)$ which has a white component.

1. Define a linear transformation of the state variables by

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x_1(t) - x_2(t) \\ r(t) \end{bmatrix}.$$

Note that one of our new state variables, $y_2(t)$, is $\frac{1}{2}r(t)$; therefore, it is known at the receiver.

Find the state equations for $y(t)$.

2. Show that the new state equations may be written as

$$\begin{aligned}
\dot{y}_1(t) &= -k'y_1(t) + [u'(t) + m_r(t)]\\
r'(t) &= \dot{y}_2(t) + k'y_2(t) = C'y_1(t) + w'(t),
\end{aligned}$$

where

$$k' = \frac{(k_1 + k_2)}{2},$$

$$C' = \frac{-(k_1 - k_2)}{2},$$

$$u'(t) = \tfrac{1}{2}[u_1(t) - u_2(t)],$$

$$w'(t) = \tfrac{1}{2}[u_1(t) + u_2(t)],$$

$$m_r(t) = C'y_2(t).$$

Notice that $m_r(t)$ is a known function at the receiver; therefore, its effect upon $y_1(t)$ is known. Also notice that our new observation signal $r'(t)$ consists of a linear modulation of $y_1(t)$ plus a white noise component.

3. Apply the estimation with correlated noise results to derive the Kalman-Bucy realizable filter equations.

4. Find equations which relate the error variance of $\hat{x}_1(t)$ to

$$E[(\hat{y}_1(t) - y_1(t))^2] \triangleq P_y(t).$$

5. Specialize your results to the case in which $k_1 = k_2$. What does the variance equation become in the limit as $t \to \infty$? Is this intuitively satisfying?

6. Comment on how this technique generalizes for higher dimension systems when there is no white noise component present in the observation. In the case of multiple observations can we ever have a singular problem: i.e., perfect estimation?

Comment. The solution to the unrealizable filter problem was done in a different manner in [56].

Problem 6.3.46. Let

$$\begin{aligned} \dot{a}(\tau) &= -k_{mo}a(\tau) + u(\tau), \qquad & 0 \le \tau \le t, \\ r(\tau) &= a(\tau) + w(\tau), & 0 \le \tau \le t, \end{aligned}$$

where

$$E[a(0)] = 0,\ E[a^2(0)] = \sigma^2/2k_{mo},$$

$$K_u(t, \tau) = \sigma^2\, \delta(t - \tau),$$

$$K_w(t, \tau) = \frac{N_0}{2}\, \delta(t - \tau).$$

Assume that we are processing $r(\tau)$ by using a realizable filter which is optimum for the above message model. The actual message process is

$$\dot{a}(\tau) = -k_{ac}\, a(\tau) + u(\tau), \qquad 0 \le \tau \le t.$$

Find the equations which specify $\xi_{ao}(t)$, the actual error variance.

P6.4 Linear Modulation, Communications Context

Problem 6.4.1. Write the variance equation for the DSB-AM example discussed on p. 576. Draw a block diagram of the system to generate it and verify that the high-frequency terms can be ignored.

Problem 6.4.2. Let

$$s(t, a(t)) = \sqrt{P}\,[a(t)\cos(\omega_c t + \theta) - \tilde{a}(t)\sin(\omega_c t + \theta)],$$

where

$$\tilde{A}(j\omega) = H(j\omega)\,A(j\omega).$$

$H(j\omega)$ is specified by (506) and θ is independent of $a(t)$ and uniformly distributed $(0, 2\pi)$. Find the power-density spectrum of $s(t, a(t))$ in terms of $S_a(\omega)$.

Problem 6.4.3. In this problem we derive the integral equation that specifies the optimum estimate of a SSB signal [see (505, 506)]. Start the derivation with (5.25) and obtain (507).

Problem 6.4.4. Consider the model in Problem 6.4.3. Define

$$\mathbf{a}(t) = \begin{bmatrix} a(t) \\ \tilde{a}(t) \end{bmatrix}.$$

Use the vector process estimation results of Section 5.4 to derive (507).

Problem 6.4.5.

1. Draw the block diagram corresponding to (508).
2. Use block diagram manipulation and the properties of $H(j\omega)$ given in (506) to obtain Fig. 6.62.

Problem 6.4.6. Let

$$s(t, a(t)) = \left(\frac{P}{1 + m^2}\right)^{1/2} [1 + ma(t)] \cos \omega_c t,$$

where

$$S_a(\omega) = \frac{2k}{\omega^2 + k^2}.$$

The received waveform is

$$r(t) = s(t, a(t)) + w(t), \qquad -\infty < t < \infty$$

where $w(t)$ is white $(N_0/2)$. Find the optimum unrealizable demodulator and plot the mean-square error as a function of m.

Problem 6.4.7 (continuation). Consider the model in Problem 6.4.6. Let

$$S_a(\omega) = \begin{cases} \dfrac{1}{2W}, & |\omega| \le 2\pi W, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the optimum unrealizable demodulator and plot the mean-square error as a function of m.

Problem 6.4.8. Consider the example on p. 583. Assume that

$$p_\phi(\theta) = \frac{e^{\Lambda_m \cos\theta}}{2\pi I_0(\Lambda_m)}, \qquad -\pi \le \theta \le \pi,$$

and

$$S_a(\omega) = \frac{2k}{\omega^2 + k^2}.$$

1. Find an expression for the mean-square error using an unrealizable demodulator designed to be optimum for the known-phase case.
2. Approximate the integral in part 1 for the case in which $\Lambda_m \gg 1$.

Problem 6.4.9 (continuation). Consider the model in Problem 6.4.8. Let

$$S_a(\omega) = \begin{cases} \dfrac{1}{2W}, & |\omega| \le 2\pi W, \\ 0, & \text{elsewhere.} \end{cases}$$

Repeat Problem 6.4.8.

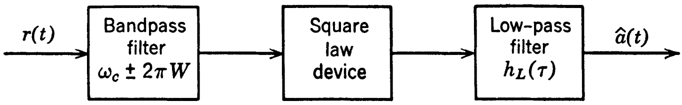


Fig. P6.15

Problem 6.4.10. Consider the model in Problem 6.4.7. The demodulator is shown in Fig. P6.15. Assume $m \ll 1$ and

$$S_a(\omega) = \begin{cases} \dfrac{1}{2W}, & 2\pi W_1 \le |\omega| \le 2\pi(W_1 + W), \\ 0, & \text{elsewhere,} \end{cases}$$

where $2\pi(W_1 + W) \ll \omega_c$.

Choose $h_L(\tau)$ to minimize the mean-square error. Calculate the resulting error ξ_P.

P6.6 Related Issues

In Problems 6.6.1 through 6.6.4, we show how the state-variable techniques we have developed can be used in several important applications. The first problem develops a necessary preliminary result. The second and third problems develop a solution technique for homogeneous and nonhomogeneous Fredholm equations (either vector or scalar). The fourth problem develops the optimum unrealizable filter. A complete development is given in [54]. The model for the four problems is

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{F}(t)\mathbf{x}(t) + \mathbf{G}(t)\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}(t)\mathbf{x}(t) \\ E[\mathbf{u}(t)\mathbf{u}^T(\tau)] &= \mathbf{Q}\delta(t - \tau) \end{aligned}$$

We use a function $\boldsymbol{\xi}(t)$ to agree with the notation of [54]. It is not related to the variance matrix $\boldsymbol{\xi}_P(t)$.

Problem 6.6.1. Define the linear functional

$$\boldsymbol{\xi}(t) = \int_{T_i}^{T_f} \mathbf{K_x}(t, \tau)\, \mathbf{s}(\tau)\, d\tau,$$

where $\mathbf{s}(t)$ is a bounded vector function.

We want to show that when $\mathbf{K_x}(t, \tau)$ is the covariance matrix for a state-variable random process $\mathbf{x}(t)$ we can represent this functional as the solution to the differential equations

$$\frac{d\boldsymbol{\xi}(t)}{dt} = \mathbf{F}(t)\,\boldsymbol{\xi}(t) + \mathbf{G}(t)\,\mathbf{Q}\mathbf{G}^T(t)\,\boldsymbol{\eta}(t)$$

and

$$\frac{d\boldsymbol{\eta}(t)}{dt} = -\mathbf{F}^T(t)\,\boldsymbol{\eta}(t) - \mathbf{s}(t),$$

with the boundary conditions

$$\boldsymbol{\eta}(T_f) = \mathbf{0},$$

and

$$\boldsymbol{\xi}(T_i) = \mathbf{P}_0\boldsymbol{\eta}(T_i),$$

where

$$\mathbf{P}_0 = \mathbf{K_x}(T_i, T_i).$$

1. Show that we can write the above integral as

$$\boldsymbol{\xi}(t) = \int_{T_i}^{t} \boldsymbol{\Phi}(t, \tau)\, \mathbf{K_x}(\tau, \tau)\, \mathbf{s}(\tau)\, d\tau + \int_{t}^{T_f} \mathbf{K_x}(t, t)\, \boldsymbol{\Phi}^T(\tau, t)\, \mathbf{s}(\tau)\, d\tau.$$

Hint. See Problem 6.3.16.

2. By using Leibnitz's rule, show that

$$\frac{d\boldsymbol{\xi}(t)}{dt} = \mathbf{F}(t)\, \boldsymbol{\xi}(t) + \mathbf{G}(t)\mathbf{Q}\mathbf{G}^T(t) \int_{t}^{T_f} \boldsymbol{\Phi}^T(\tau, t)\, \mathbf{s}(\tau)\, d\tau.$$

Hint. Note that $\mathbf{K_x}(t, t)$ satisfies the differential equation

$$\frac{d\mathbf{K_x}(t, t)}{dt} = \mathbf{F}(t)\, \mathbf{K_x}(t, t) + \mathbf{K_x}(t, t)\mathbf{F}^T(t) + \mathbf{G}(t)\mathbf{Q}\mathbf{G}^T(t), \qquad \text{(text 273)}$$

with $\mathbf{K_x}(T_i, T_i) = \mathbf{P}_0$, and $\boldsymbol{\Phi}^T(\tau, t)$ satisfies the adjoint equation; that is,

$$\frac{d\boldsymbol{\Phi}^T(\tau, t)}{dt} = -\mathbf{F}^T(t)\, \boldsymbol{\Phi}^T(\tau, t),$$

with $\boldsymbol{\Phi}(T_i, T_i) = \mathbf{I}$.

3. Define a second functional $\boldsymbol{\eta}(t)$ by

$$\boldsymbol{\eta}(t) = \int_{t}^{T_f} \boldsymbol{\Phi}^T(\tau, t)\, \mathbf{s}(\tau)\, d\tau.$$

Show that it satisfies the differential equation

$$\frac{d\boldsymbol{\eta}(t)}{dt} = -\mathbf{F}^T(t)\, \boldsymbol{\eta}(t) - \mathbf{s}(t).$$

4. Show that the differential equations must satisfy the two independent boundary conditions

$$\boldsymbol{\eta}(T_f) = \mathbf{0},$$
$$\boldsymbol{\xi}(T_i) = \mathbf{P}_0 \boldsymbol{\eta}(T_i).$$

5. By combining the results in parts 2, 3, and 4, show the desired result.

Problem 6.6.2. Homogeneous Fredholm Equation. In this problem we derive a set of differential equations to determine the eigenfunctions for the homogeneous Fredholm equation. The equation is given by

$$\int_{T_i}^{T_f} \mathbf{K_y}(t, \tau)\boldsymbol{\phi}(\tau)\, d\tau = \lambda \boldsymbol{\phi}(t), \qquad T_i \le t \le T_f,$$

or

$$\boldsymbol{\phi}(t) = \frac{1}{\lambda}\mathbf{C}(t) \int_{T_i}^{T_f} \mathbf{K_x}(t, \tau)\, \mathbf{C}^T(\tau)\boldsymbol{\phi}(\tau)\, d\tau, \qquad \text{for } \lambda > 0.$$

Define

$$\boldsymbol{\xi}(t) = \int_{T_i}^{T_f} \mathbf{K_x}(t, \tau)\mathbf{C}^T(\tau)\boldsymbol{\phi}(\tau)\, d\tau,$$

so that

$$\boldsymbol{\phi}(t) = \frac{1}{\lambda}\mathbf{C}(t)\, \boldsymbol{\xi}(t).$$

1. Show that $\xi(t)$ satisfies the differential equations

$$\frac{d}{dt}\begin{bmatrix}\xi(t)\\ \eta(t)\end{bmatrix} = \left[\begin{array}{c|c} \mathbf{F}(t) & \mathbf{G}(t)\mathbf{Q}\mathbf{G}^T(t) \\ \hline -\dfrac{\mathbf{C}^T(t)\,\mathbf{C}(t)}{\lambda} & -\mathbf{F}^T(t)\end{array}\right]\begin{bmatrix}\xi(t)\\ \eta(t)\end{bmatrix},$$

with

$$\begin{aligned}\xi(T_i) &= \mathbf{P}_0\eta(T_i),\\ \eta(T_f) &= \mathbf{0}.\end{aligned}$$

(Use the results of Problem 6.6.1.)

2. Show that to have a nontrivial solution which satisfies the boundary conditions we need

$$\det\left[\mathbf{\Psi}_{\eta\xi}(T_f, T_i:\lambda)\mathbf{P}_0 + \mathbf{\Psi}_{\eta\eta}(T_f, T_i:\lambda)\right] = 0,$$

where $\mathbf{\Psi}(t, T_i:\lambda)$ is given by

$$\frac{d}{dt}\left[\begin{array}{c|c}\mathbf{\Psi}_{\xi\xi}(t, T_i:\lambda) & \mathbf{\Psi}_{\xi\eta}(t, T_i:\lambda)\\ \hline \mathbf{\Psi}_{\eta\xi}(t, T_i:\lambda) & \mathbf{\Psi}_{\eta\eta}(t, T_i:\lambda)\end{array}\right]$$

$$= \left[\begin{array}{c|c} \mathbf{F}(t) & \mathbf{G}(t)\,\mathbf{Q}\mathbf{G}^T(t) \\ \hline -\dfrac{\mathbf{C}^T(t)\,\mathbf{C}(t)}{\lambda} & -\mathbf{F}^T(t)\end{array}\right]\left[\begin{array}{c|c}\mathbf{\Psi}_{\xi\xi}(t, T_i:\lambda) & \mathbf{\Psi}_{\xi\eta}(t, T_i:\lambda)\\ \hline \mathbf{\Psi}_{\eta\xi}(t, T_i:\lambda) & \mathbf{\Psi}_{\eta\eta}(t, T_i:\lambda)\end{array}\right],$$

and $\mathbf{\Psi}(T_i, T_i:\lambda) = \mathbf{I}$. The values of λ which satisfy this equation are the eigenvalues.

3. Show that the eigenfunctions are given by

$$\boldsymbol{\phi}(t, T_i:\lambda) = \frac{\mathbf{C}(t)}{\lambda}\left[\mathbf{\Psi}_{\xi\xi}(t, T_i:\lambda)\,\mathbf{P}_0 + \mathbf{\Psi}_{\xi\eta}(t, T_i:\lambda)\right]\eta(T_i)$$

where $\eta(T_i)$ satisfies the orthogonality relationship

$$\left[\mathbf{\Psi}_{\eta\xi}(T_f, T_i:\lambda)\,\mathbf{P}_0 + \mathbf{\Psi}_{\eta\eta}(T_f, T_i:\lambda)\right]\eta(T_i) = \mathbf{0}.$$

Problem 6.6.3. Nonhomogeneous Fredholm Equation. In this problem we derive a set of differential equations to determine the solution to the nonhomogeneous Fredholm equation. This equation is given by

$$\int_{T_i}^{T_f}\mathbf{K}_{\mathbf{y}}(t, \tau)\,\mathbf{g}(\tau)\,d\tau + \sigma\,\mathbf{g}(t) = \mathbf{s}(t), \qquad T_i \le t \le T_f,\ \sigma > 0.$$

1. If we define $\xi(t)$ as

$$\xi(t) = \int_{T_i}^{T_f}\mathbf{K}_{\mathbf{x}}(t, \tau)\,\mathbf{C}^T(\tau)\,\mathbf{g}(\tau)\,d\tau,$$

show that we may write the nonhomogeneous equation as

$$\mathbf{g}(t) = \frac{1}{\sigma}\left[\mathbf{s}(t) - \mathbf{C}(t)\,\xi(t)\right].$$

2. Using Problem 6.6.1, show that $\xi(t)$ satisfies the differential equations

$$\frac{d}{dt}\begin{bmatrix}\xi(t)\\ \eta(t)\end{bmatrix} = \left[\begin{array}{c|c} \mathbf{F}(t) & \mathbf{G}(t)\,\mathbf{Q}\mathbf{G}^T(t) \\ \hline \dfrac{\mathbf{C}^T(t)\,\mathbf{C}(t)}{\sigma} & -\mathbf{F}^T(t)\end{array}\right]\begin{bmatrix}\xi(t)\\ \eta(t)\end{bmatrix} - \left[\begin{array}{c}\mathbf{0}\\ \hline \dfrac{\mathbf{C}^T(t)\,\mathbf{s}(t)}{\sigma}\end{array}\right],$$

with

$$\boldsymbol{\xi}(T_i) = \mathbf{P}_0\, \boldsymbol{\eta}(T_i),$$
$$\boldsymbol{\eta}(T_f) = \mathbf{0}.$$

Comment. In general we can replace σ by an arbitrary positive-definite time-varying matrix ($\mathbf{R}(t)$) and the derivation is valid with obvious modifications.

Problem No. 6.6.4. ***Unrealizable Filters.*** In this problem we show how the nonhomogeneous Fredholm equation may be used to determine the optimal unrealizable filter structure. For algebraic simplicity we assume $\mathbf{r}(t)$ is a scalar.

$$\mathbf{R}(t) = N_0/2.$$

1. Show that the integral equation specifying the optimum unrealizable filter for estimating $\mathbf{x}(t)$ at any point t in the interval $[T_i, T_f]$ is

$$\mathbf{K_x}(t, \tau)\, \mathbf{C}^T(\tau) = \int_{T_i}^{T_f} \mathbf{h}_o(t, \sigma)\, K_r(\sigma, \tau)\, d\sigma, \qquad T_i \le t \le T_f, \qquad T_i < \tau < T_f. \tag{1}$$

2. Using the inverse kernel of $K_r(t, \tau)$ [Chapter 4, (4.161)] show that

$$\hat{\mathbf{x}}(t) = \int_{T_i}^{T_f} \mathbf{K_x}(t, \tau)\mathbf{C}^T(\tau)\left(\int_{T_i}^{T_f} Q_r(\tau, \sigma)\, r(\sigma)\, d\sigma\right), \qquad T_i \le t \le T_f. \tag{2}$$

3. As in Chapter 5, define the term in parentheses as $r_g(\tau)$. We note that $r_g(\tau)$ solves the nonhomogeneous Fredholm equation when the input is $r(t)$. Using Problem No. 6.6.3, show that $r_g(t)$ is given by

$$r_g(t) = \frac{2}{N_0}\,(r(t) - \mathbf{C}(t)\boldsymbol{\xi}(t)), \tag{3}$$

where

$$\frac{d\boldsymbol{\xi}(t)}{dt} = \mathbf{F}(t)\boldsymbol{\xi}(t) + \mathbf{G}(t)\mathbf{Q}\mathbf{G}^T(t)\boldsymbol{\eta}_1(t), \tag{4a}$$

$$\frac{d\boldsymbol{\eta}_1(t)}{dt} = \frac{2}{N_0}\,\mathbf{C}^T(t)\mathbf{C}(t)\boldsymbol{\xi}(t) - \mathbf{F}^T(t)\boldsymbol{\eta}_1(t) - \frac{2}{N_0}\,\mathbf{C}^T(t)r(t), \tag{4b}$$

and

$$\boldsymbol{\xi}(T_i) = \mathbf{K_x}(T_i, T_i)\boldsymbol{\eta}_1(T_i), \tag{5a}$$

$$\boldsymbol{\eta}_1(T_f) = \mathbf{0}. \tag{5b}$$

4. Using the results of Problem No. 6.6.1, show that $\hat{\mathbf{x}}(t)$ satisfies the differential equation,

$$\frac{d\hat{\mathbf{x}}(t)}{dt} = \mathbf{F}(t)\hat{\mathbf{x}}(t) + \mathbf{G}(t)\mathbf{Q}\mathbf{G}^T(t)\boldsymbol{\eta}_2(t), \tag{6a}$$

$$\frac{d\boldsymbol{\eta}_2(t)}{dt} = -\mathbf{C}^T(t)r_g(t) - \mathbf{F}^T(t)\boldsymbol{\eta}_2(t), \tag{6b}$$

where

$$\hat{\mathbf{x}}(T_i) = \mathbf{K_x}(T_i, T_i)\boldsymbol{\eta}_2(T_i), \tag{7a}$$

$$\boldsymbol{\eta}_2(T_f) = \mathbf{0}. \tag{7b}$$

5. Substitute (3) into (6*b*). Show that

$$\boldsymbol{\eta}_1(t) = \boldsymbol{\eta}_2(t), \tag{8a}$$

and

$$\hat{\mathbf{x}}(t) = \boldsymbol{\xi}(t). \tag{8b}$$

6. Show that the differential equation structure for the optimum unrealizable estimate of $\hat{\mathbf{x}}(t)$ is

$$\frac{d\hat{\mathbf{x}}(t)}{dt} = \mathbf{F}(t)\hat{\mathbf{x}}(t) + \mathbf{G}(t)\mathbf{Q}\mathbf{G}^T(t)\boldsymbol{\eta}(t), \tag{9a}$$

$$\frac{d\boldsymbol{\eta}(t)}{dt} = \mathbf{C}^T(t)\frac{2}{N_0}\mathbf{C}(t)\hat{\mathbf{x}}(t) - \mathbf{F}^T(t)\boldsymbol{\eta}(t) - \mathbf{C}^T(t)\frac{2}{N_0}r(t), \tag{9b}$$

where

$$\hat{\mathbf{x}}(T_i) = \mathbf{K}_{\mathbf{x}}(T_i, T_i)\boldsymbol{\eta}(T_i), \tag{10a}$$

$$\boldsymbol{\eta}(T_f) = \mathbf{0}. \tag{10b}$$

Comments

1. We have two n-dimensional linear vector differential equations to solve.
2. The performance given by the unrealizable error covariance matrix is not part of the filter structure.
3. By letting T_f be a variable we can determine a differential equation structure for $\hat{\mathbf{x}}(T_f)$ as a function of T_f. These equations are just the Kalman-Bucy equations for the optimum realizable filter.

Problem 6.6.5. In Problems 6.2.29 and 6.2.30 we discussed a realizable whitening filter for infinite intervals and stationary processes. In this problem we verify that these results generalize to finite intervals and nonstationary processes.

Let

$$r(\tau) = n_c(\tau) + w(\tau), \qquad T_i \le \tau \le t,$$

where $n_c(\tau)$ can be generated as the output of a dynamic system,

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{F}(t)\mathbf{x}(t) + \mathbf{G}(t)u(t)\\ n_c(t) &= \mathbf{C}(t)\mathbf{x}(t),\end{aligned}$$

driven by white noise $u(t)$.

Show that the process

$$\begin{aligned}r'(t) &= r(t) - \hat{n}_c(t),\\ &= r(t) - \mathbf{C}(t)\hat{\mathbf{x}}(t),\end{aligned}$$

is white.

Problem 6.6.6. Sequential Estimation. Consider the convolutional encoder in Fig. P6.16.

1. What is the "state" of this system?
2. Show that an appropriate set of state equations is

$$\mathbf{x}_{n+1} = \begin{bmatrix} x_{1,n+1} \\ x_{2,n+1} \\ x_{3,n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_{1,n} \\ x_{2,n} \\ x_{3,n} \end{bmatrix} \oplus \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_n$$

$$\mathbf{y}_{n+1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_{1,n+1} \\ x_{2,n+1} \\ x_{3,n+1} \end{bmatrix}$$

(all additions are modulo 2).

Assume that the process u is composed of a sequence of independent binary random variables with

$$\begin{aligned}E(u_n = 0) &= P_u,\\ E(u_n = 1) &= 1 - P_u.\end{aligned}$$

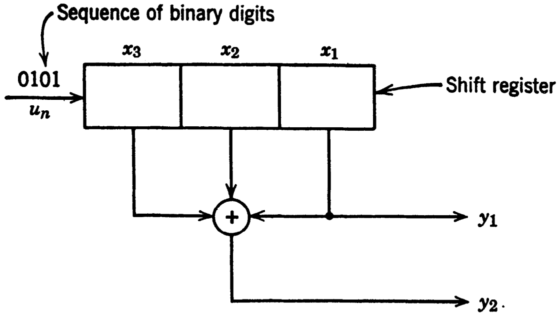


Fig. P6.16

In addition, assume that the components of y are sent over two independent identical binary symmetric channels such that

$$\mathbf{r}_n = \mathbf{y} \oplus \mathbf{w},$$

where

$$E(w_{n,1} = 0) = P_w,$$
$$E(w_{n,1} = 1) = 1 - P_w.$$

Finally, let the sequence of measurements

$$\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n, \text{ be denoted by } \mathbf{z}_n.$$

3. Show that the a posteriori probability density $p_{\mathbf{x}_{n+1}|\mathbf{z}_{n+1}}(\mathbf{X}_{n+1}|\mathbf{Z}_{n+1})$ satisfies the following recursive relationship:

$$p_{\mathbf{x}_{n+1}|\mathbf{z}_{n+1}}(\mathbf{X}_{n+1}|\mathbf{Z}_{n+1})$$
$$= \frac{p_{\mathbf{r}_{n+1}|\mathbf{z}_{n+1}}(\mathbf{R}_{n+1}|\mathbf{X}_{n+1}) \sum_{\mathbf{x}_n} p_{\mathbf{x}_{n+1}|\mathbf{x}_n}(\mathbf{X}_{n+1}|\mathbf{X}_n) p_{\mathbf{x}_n|\mathbf{z}_n}(\mathbf{X}_n|\mathbf{Z}_n)}{\sum_{\mathbf{x}_{n+1}} p_{\mathbf{r}_{n+1}|\mathbf{x}_{n+1}}(\mathbf{R}_{n+1}|\mathbf{X}_{n+1}) \sum_{\mathbf{x}_n} p_{\mathbf{x}_{n+1}|\mathbf{x}_n}(\mathbf{X}_{n+1}|\mathbf{X}_n) p_{\mathbf{x}_n|\mathbf{z}_n}(\mathbf{X}_n|\mathbf{Z}_n)}$$

where $\sum_{\mathbf{x}_n}$ denotes the sum over all possible states.

4. How would you design the MAP receiver that estimates $\mathbf{x}_{n+1}$? What must be computed for each receiver estimate?

5. How does this estimation procedure compare with the discrete Kalman filter?

6. How does the complexity of this procedure increase as the length of the convolutional encoder increases?

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