

2

Detection of Gaussian Signals in White Gaussian Noise

In this chapter we consider the problem of detecting a sample function from a Gaussian random process in the presence of additive white Gaussian noise. This problem is a special case of the general Gaussian problem described in Chapter 1. It is characterized by the property that on both hypotheses, the received waveform contains an additive noise component $w(t)$, which is a sample function from a zero-mean white Gaussian process with spectral height $N_0/2$. When H_1 is true, the received waveform also contains a signal $s(t)$, which is a sample function from a Gaussian random process whose mean and covariance function are known. Thus,

$$r(t) = s(t) + w(t), \quad T_i \leq t \leq T_f; H_1 \quad (1)$$

and

$$r(t) = w(t), \quad T_i \leq t \leq T_f; H_0. \quad (2)$$

The signal process has a mean value function $m(t)$,

$$E[s(t)] = m(t), \quad T_i \leq t \leq T_f, \quad (3)$$

and a covariance function $K_s(t, u)$,

$$E[(s(t) - m(t))(s(u) - m(u))] \triangleq K_s(t, u), \quad T_i \leq t, u \leq T_f. \quad (4)$$

Both $m(t)$ and $K_s(t, u)$ are known. We assume that the signal process has a finite mean-square value and is statistically independent of the additive noise. Thus, the covariance function of $r(t)$ on H_1 is

$$E[(r(t) - m(t))(r(u) - m(u)) \mid H_1] \triangleq K_1(t, u) = K_s(t, u) + \frac{N_0}{2} \delta(t - u),$$

$$T_i \leq t, u \leq T_f. \quad (5)$$

We refer to $r(t)$ as a *conditionally Gaussian* random process. The term “conditionally Gaussian” is used because $r(t)$, given H_1 is true, and $r(t)$, given H_0 is true, are the two Gaussian processes in the model.

We observe that the mean value function can be viewed as a deterministic component in the input. When we want to emphasize this we write

$$\begin{aligned} r(t) &= m(t) + [s(t) - m(t)] + w(t) \\ &= m(t) + s_R(t) + w(t), \quad T_i \leq t \leq T_f; H_1. \end{aligned} \quad (6)$$

(The subscript R denotes the random component of the signal process.) Now the waveform on H_1 consists of a known signal corrupted by two independent zero-mean Gaussian processes. If $K_s(t, u)$ is identically zero, the problem degenerates into the known signal in white noise problem of Chapter I-4. As we proceed, we shall find that all of the results in Chapter I-4 except for the random phase case in Section I-4.4.1 can be viewed as special cases of various problems in Chapters 2 and 3.

In Section 2.1, we derive the optimum receiver and discuss various procedures for implementing it. In Section 2.2, we analyze the performance of the optimum receiver. Finally, in Section 2.3, we summarize our results.

Most of the original work on the detection of Gaussian signals is due to Price [1]–[4] and Middleton [17]–[20]. Other references are cited at various points in the Chapter.

2.1 OPTIMUM RECEIVERS

Our approach to designing the optimum receiver is analogous to the approach in the deterministic signal case (see pages I-250–I-253). The essential steps are the following:

1. We expand $r(t)$ in a series, using the eigenfunctions of the signal process as coordinate functions. The noise term $w(t)$ is white, and so the coefficients of the expansion will be conditionally uncorrelated on both hypotheses. Because the input $r(t)$ is Gaussian on both hypotheses, the coefficients are conditionally statistically independent.

2. We truncate the expansion at the K th term and denote the first K coefficients by the vector \mathbf{r} . The waveform corresponding to the sum of the first K terms in the series is $r_K(t)$.

3. We then construct the likelihood ratio,

$$\Lambda(r_K(t)) = \Lambda(\mathbf{R}) = \frac{p_{r|H_1}(\mathbf{R} | H_1)}{p_{r|H_0}(\mathbf{R} | H_0)}, \quad (7)$$

and manipulate it into a form so that we can let $K \rightarrow \infty$.

4. We denote the limit of $\Lambda(r_K(t))$ as $\Lambda(r(t))$. The test consists of comparing the likelihood ratio with a threshold η ,

$$\Lambda[r(t)] \underset{H_0}{\overset{H_1}{\gtrless}} \eta. \quad (8)$$

As before, the threshold η is determined by the costs and a-priori probabilities in a Bayes test and the desired P_F in a Neyman-Pearson test.

We now carry out these steps in detail and then investigate the properties of the resulting tests.

The orthonormal functions for the series expansion are the eigenfunctions of the integral equation†

$$\hat{\lambda}_i^s \phi_i(t) = \int_{T_i}^{T_f} K_s(t, u) \phi_i(u) du, \quad T_i \leq t \leq T_f. \quad (9)$$

We shall assume that the orthonormal functions form a complete set. This will occur naturally if $K_s(t, u)$ is positive-definite. If $K_s(t, u)$ is only non-negative-definite, we augment the set to make it complete.

The coefficients in the series expansion are

$$r_i \triangleq \int_{T_i}^{T_f} r(t) \phi_i(t) dt. \quad (10)$$

The K -term approximation is

$$r_K(t) = \sum_{i=1}^K r_i \phi_i(t), \quad T_i \leq t \leq T_f \quad (11)$$

and

$$r(t) = \text{l.i.m.}_{K \rightarrow \infty} r_K(t), \quad T_i \leq t \leq T_f. \quad (12)$$

The statistical properties of the coefficients on the two hypotheses follow easily.

$$E[r_i | H_0] = E \left[\int_{T_i}^{T_f} w(t) \phi_i(t) dt \right] = 0. \quad (13)$$

$$E[r_i r_j | H_0] = \frac{N_0}{2} \delta_{ij}. \quad (14)$$

$$\begin{aligned} E[r_i | H_1] &= E \left[\int_{T_i}^{T_f} s(t) \phi_i(t) dt + \int_{T_i}^{T_f} w(t) \phi_i(t) dt \right] \\ &= \int_{T_i}^{T_f} m(t) \phi_i(t) dt \triangleq m_i. \end{aligned} \quad (15)$$

† Series expansions were developed in detail in Chapter I-3.

Notice that (15) implies that the m_i are the coefficients of an orthogonal expansion of the mean-value function; that is,

$$m(t) = \sum_{i=1}^{\infty} m_i \phi_i(t), \quad T_i \leq t \leq T_f. \quad (16)$$

The covariance between coefficients is

$$E[(r_i - m_i)(r_j - m_j) | H_1] = \left(\lambda_i^s + \frac{N_0}{2} \right) \delta_{ij}, \quad (17)$$

where λ_i^s is the i th eigenvalue of (9). The superscript s emphasizes that it is an eigenvalue of the signal process, $s(t)$.

Under both hypotheses, the coefficients r_i are statistically independent Gaussian random variables. The probability density of \mathbf{r} is just the product of the densities of the coefficients. Thus,

$$\begin{aligned} \Lambda(\mathbf{R}) &\triangleq \frac{p_{r|H_1}(\mathbf{R} | H_1)}{p_{r|H_0}(\mathbf{R} | H_0)} \\ &= \frac{\left(\prod_{i=1}^K \frac{1}{[2\pi(N_0/2 + \lambda_i^s)]^{1/2}} \right) \exp\left(-\frac{1}{2} \sum_{i=1}^K \frac{(R_i - m_i)^2}{\lambda_i^s + (N_0/2)}\right)}{\left(\prod_{i=1}^K \frac{1}{[2\pi(N_0/2)]^{1/2}} \right) \exp\left(-\frac{1}{2} \sum_{i=1}^K \frac{R_i^2}{N_0/2}\right)}. \end{aligned} \quad (18)$$

Multiplying out each term in the exponent, canceling common factors, taking the logarithm, and rearranging the results, we have

$$\begin{aligned} \ln \Lambda(\mathbf{R}) &= \frac{1}{N_0} \sum_{i=1}^K \left(\frac{\lambda_i^s}{\lambda_i^s + N_0/2} \right) R_i^2 + \sum_{i=1}^K \left(\frac{1}{\lambda_i^s + N_0/2} \right) m_i R_i \\ &\quad - \frac{1}{2} \sum_{i=1}^K \left(\frac{1}{\lambda_i^s + N_0/2} \right) m_i^2 - \frac{1}{2} \sum_{i=1}^K \ln \left(1 + \frac{2\lambda_i^s}{N_0} \right). \end{aligned} \quad (19)$$

The final step is to obtain closed form expressions for the various terms when $K \rightarrow \infty$. To do this, we need the inverse kernel that was first introduced in Chapter I-4 [see (I-4.152)]. The covariance function of the *entire* input $r(t)$ on H_1 is $K_1(t, u)$. The corresponding inverse kernel is defined by the relation

$$\int_{T_i}^{T_f} K_1(t, u) Q_1(u, z) du = \delta(t - z), \quad T_i < t, z < T_f. \quad (20)$$

In terms of eigenfunctions and eigenvalues,

$$Q_1(t, u) = \sum_{i=1}^{\infty} \frac{1}{\lambda_i^s + N_0/2} \phi_i(t) \phi_i(u), \quad T_i < t, z < T_f. \quad (21)$$

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We also saw in Chapter I-4 (I-4.162) that we could write $Q_1(t, u)$ as a sum of an impulse component and a well-behaved function,

$$Q_1(t, u) = \frac{2}{N_0} (\delta(t - u) - h_1(t, u)), \quad T_i < t, u < T_f, \quad (22)$$

where the function $h_1(t, u)$ satisfies the integral equation

$$\frac{N_0}{2} h_1(t, u) + \int_{T_i}^{T_f} h_1(t, z) K_s(z, u) dz = K_s(t, u), \quad T_i \leq t, u \leq T_f.$$

(23)

The endpoint values of $h_1(t, u)$ are defined as a limit of the open-interval values because we assume that $h_1(t, u)$ is continuous. (Recall the discussion on page I-296.) We also recall that we could write the solution to (23) in terms of eigenfunctions and eigenvalues.

$$h_1(t, u) = \sum_{i=1}^{\infty} \frac{\lambda_i^s}{\lambda_i^s + N_0/2} \phi_i(t) \phi_i(u), \quad T_i \leq t, u \leq T_f. \quad (24)$$

We now rewrite the first three terms in (19) by using (10) and (15) to obtain

$$\begin{aligned} \ln \Lambda(r_K(t)) &= \frac{1}{N_0} \iint_{T_i}^{T_f} r(t) \left[\sum_{i=1}^K \left(\frac{\lambda_i^s}{\lambda_i^s + N_0/2} \right) \phi_i(t) \phi_i(u) \right] r(u) dt du \\ &\quad + \iint_{T_i}^{T_f} m(t) \left[\sum_{i=1}^K \left(\frac{1}{\lambda_i^s + N_0/2} \right) \phi_i(t) \phi_i(u) \right] r(u) dt du \\ &\quad - \frac{1}{2} \iint_{T_i}^{T_f} m(t) \left[\sum_{i=1}^K \left(\frac{1}{\lambda_i^s + N_0/2} \right) \phi_i(t) \phi_i(u) \right] m(u) dt du \\ &\quad - \frac{1}{2} \sum_{i=1}^K \ln \left(1 + \frac{2\lambda_i^s}{N_0} \right) \end{aligned} \quad (25)$$

We now let $K \rightarrow \infty$ in (25) and use (21) and (24) to evaluate the first three terms in (25). The result is

$$\begin{aligned} \ln \Lambda(r(t)) &= \frac{1}{N_0} \iint_{T_i}^{T_f} r(t) h_1(t, u) r(u) dt du + \iint_{T_i}^{T_f} m(t) Q_1(t, u) r(u) dt du \\ &\quad - \frac{1}{2} \iint_{T_i}^{T_f} m(t) Q_1(t, u) m(u) dt du - \frac{1}{2} \sum_{i=1}^{\infty} \ln \left(1 + \frac{2\lambda_i^s}{N_0} \right). \end{aligned} \quad (26)$$

We can further simplify the second and third terms on the right side of (26) by recalling the definition of $g(u)$ in (I-4.168),

$$g_1(u) \triangleq \int_{T_i}^{T_f} m(t) Q_1(t, u) dt, \quad T_i < u < T_f. \quad (27)$$

Notice that $m(t)$ plays the role of the known signal [which was denoted by $s(t)$ in Chapter I-4]. We also observe that the third and fourth term are *not* functions of $r(t)$ and may be absorbed in the threshold. Thus, the likelihood ratio test (LRT) is,

$$\frac{1}{N_0} \int_{T_i}^{T_f} r(t) h_1(t, u) r(u) dt du + \int_{T_i}^{T_f} g_1(u) r(u) du \underset{H_0}{\overset{H_1}{\gtrless}} \gamma_*, \quad (28)$$

where

$$\gamma_* \triangleq \ln \eta + \frac{1}{2} \int_{T_i}^{T_f} g_1(u) m(u) du + \frac{1}{2} \sum_{i=1}^{\infty} \ln \left(1 + \frac{2\lambda_i^s}{N_0} \right). \quad (29)$$

If we are using a Bayes test, we must evaluate the infinite sum on the right side in order to set the threshold. On page 22 we develop a convenient closed-form expression for this sum. For the Neyman-Pearson test we adjust γ_* directly to obtain the desired P_F so that the exact value of the sum is not needed as long as we know the sum converges. The convergence follows easily.

$$\sum_{i=1}^{\infty} \ln \left(1 + \frac{2\lambda_i^s}{N_0} \right) \leq \sum_{i=1}^{\infty} \frac{2\lambda_i^s}{N_0} = \frac{2}{N_0} \int_{T_i}^{T_f} K_s(t, t) dt. \quad (30)$$

The integral is just the expected value of the energy in the process, which was assumed to be finite.

The first term on the left side of (28) is a quadratic operation on $r(t)$ and arises because the signal is random. If $K_s(t, u)$ is zero (i.e., the signal is deterministic), this term disappears. We denote the first term by l_R . (The subscript R denotes random.) The second term on the left side is a linear operation on $r(t)$ and arises because of the mean value $m(t)$. Whenever the signal is a zero-mean process, this term disappears. We denote the second term by l_D . (The subscript D denotes deterministic.) It is also convenient to denote the last two terms on the right side of (29) as

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$(-l_B^{[2]})$ and $(-l_B^{[1]})$. Thus, we have the definitions

$$l_R \triangleq \frac{1}{N_0} \int_{T_i}^{T_f} r(t) h_1(t, u) r(u) dt du, \tag{31}$$

$$l_D \triangleq \int_{T_i}^{T_f} g_1(u) r(u) du, \tag{32}$$

$$l_B^{[1]} \triangleq -\frac{1}{2} \sum_{i=1}^{\infty} \ln \left(1 + \frac{2\lambda_i^s}{N_0} \right), \tag{33}$$

$$l_B^{[2]} \triangleq -\frac{1}{2} \int_{T_i}^{T_f} g_1(u) m(u) du. \tag{34}$$

In this notation, the LRT is

$$l_R + l_D \underset{H_0}{\overset{H_1}{\geq}} \ln \eta - l_B^{[1]} - l_B^{[2]} = \gamma - l_B^{[1]} - l_B^{[2]} \triangleq \gamma_*. \tag{35}$$

The second term on the left side of (35) is generated physically by either a cross-correlation or a matched filter operation, as shown in Fig. 2.1. The impulse response of the matched filter in Fig. 2.1b is

$$h(\tau) = \begin{cases} g_1(T_f - \tau), & 0 \leq \tau \leq T_f - T_i, \\ 0, & \text{elsewhere.} \end{cases} \tag{36}$$

We previously encountered these operations in the colored noise detection problem discussed in Section I-4.3. Thus, the only new component in the optimum receiver is a device to generate l_R . In the next several paragraphs we develop a number of methods of generating l_R .

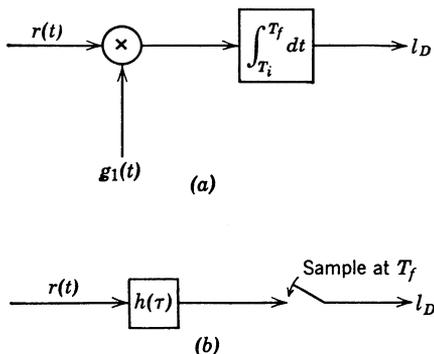


Fig. 2.1 Generation of l_D .

2.1.1 Canonical Realization No. 1: Estimator-Correlator

We want to generate l_R , where

$$l_R = \frac{1}{N_0} \int_{T_i}^{T_f} r(t) h_1(t, u) r(u) dt du, \tag{37}$$

and $h_1(t, u)$ satisfies (23). An obvious realization is shown in Fig. 2.2a. Notice that $h_1(t, u)$ is an unrealizable filter. Therefore, in order actually to build it, we would have to allow a delay in the filter in the system in Fig. 2.2a. This is done by defining a new filter whose output is a delayed version of the output of $h_1(t, u)$,

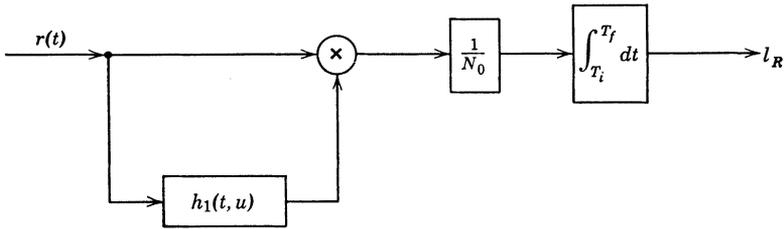
$$h_{1d}(t, u) = \begin{cases} h_1(t - T, u), & T_i + T \leq t \leq T_f + T, T_i \leq u \leq T_f, \\ 0, & \text{elsewhere,} \end{cases} \tag{38}$$

where

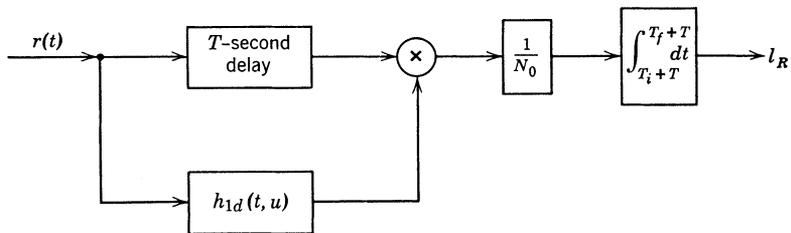
$$T \triangleq T_f - T_i \tag{39}$$

is the length of the observation interval. Adding a corresponding delay in the upper path and the integrator gives the system in Fig. 2.2b.

This realization has an interesting interpretation. We first assume that $m(t)$ is zero and then recall that we have previously encountered (23) in the



(a) Unrealizable filter



(b) Realization with delay

Fig. 2.2 Generation of l_R .

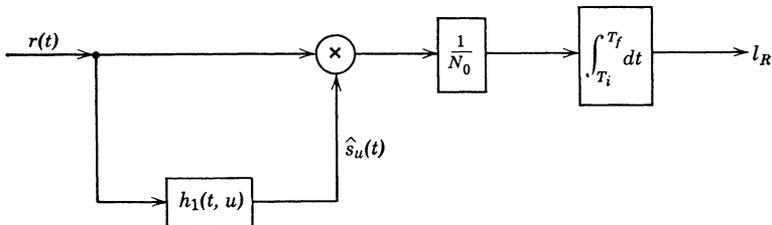


Fig. 2.3 Estimator-correlator (zero-mean case).

linear filter context. Specifically, if we had available a waveform

$$r(t) = s(t) + w(t), \quad T_i \leq t \leq T_f \tag{40}$$

and wanted to estimate $s(t)$ using a minimum mean-square error (MMSE) or maximum a-posteriori probability (MAP) criterion, then, from (I-6.16), we know that the resulting estimate $\hat{s}_u(t)$ would be obtained by passing $r(t)$ through $h_1(t, u)$.

$$\hat{s}_u(t) = \int_{T_i}^{T_f} h_1(t, u)r(u) du, \quad T_i \leq t \leq T_f, \tag{41}$$

where $h_1(t, u)$ satisfies (23) and the subscript u emphasizes that the estimate is unrealizable. Looking at Fig. 2.3, we see that the receiver is correlating $r(t)$ with the MMSE estimate of $s(t)$. For this reason, the realization in Fig. 2.3 is frequently referred to as an estimator-correlator receiver. This is an intuitively pleasing interpretation. (This result is due to Price [1]–[4].)

Notice that the interpretation of the left side of (41) as the MMSE estimate is only valid when $r(t)$ is zero-mean. However, the output of the receiver in Fig. 2.3 is l_R for either the zero-mean or the non-zero-mean case. We also obtain an estimator-correlator interpretation in the non-zero-mean case by a straightforward modification of the above discussion (see Problem 2.1.1).

Up to this point all of the filters except the one in Fig. 2.2b are unrealizable and are obtained by solving (23). The next configuration eliminates the unrealizability problem.

2.1.2 Canonical Realization No. 2: Filter-Correlator Receiver

The realization follows directly from (37). We see that because of the symmetry of the kernel $h_1(t, u)$, (37) can be rewritten as

$$l_R = \frac{2}{N_0} \int_{T_i}^{T_f} r(t) \left[\int_{T_i}^t h_1(t, u)r(u) du \right] dt. \tag{42}$$

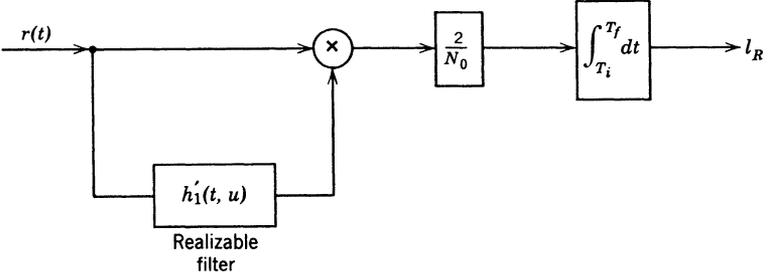


Fig. 2.4 Filter-correlator receiver.

In this form, the inner integral represents a *realizable* operation. Thus, we can build the receiver using a realizable filter,

$$h'_1(t, u) = \begin{cases} h_1(t, u), & t \geq u, \\ 0, & t < u, \end{cases} \quad (43)$$

This realization is shown in Fig. 2.4. Observe that the output of the realizable filter $h'_1(t, u)$ is *not* the realizable MMSE estimate of $s(t)$. The impulse response of the optimum realizable linear filter for estimating $s(t)$ is $h_{or}(t, u)$ and it satisfies the equation

$$\frac{N_0}{2} h_{or}(t, u) + \int_{T_i}^t h_{or}(t, z) K_s(z, u) dz = K_s(t, u), \quad T_i \leq u \leq t, \quad (44)$$

which is not the same filter specified by (23) plus (43). (This canonical realization is also due to Price [1].) The receiver in Fig. 2.4 is referred to as a filter-correlator receiver. We have included it for completeness. It is used infrequently in practice and we shall not use it in any subsequent discussions.

2.1.3 Canonical Realization No. 3: Filter-Squarer-Integrator (FSI) Receiver

A third canonical form can be derived by factoring $h_1(t, u)$. We define $h_f(z, t)$ by the relation

$$h_1(t, u) = \int_{T_i}^{T_f} h_f(z, t) h_f(z, u) dz, \quad T_i \leq t, u \leq T_f. \quad (45)$$

If we do not require that $h_f(z, t)$ be realizable, we can find an infinite number of solutions to (45). From (24), we recall that

$$h_1(t, u) = \sum_{i=1}^{\infty} h_i \phi_i(t) \phi_i(u), \quad T_i \leq t, u \leq T_f, \quad (46)$$

where

$$h_i = \frac{\lambda_i^s}{\lambda_i^s + N_0/2} \tag{47}$$

We see that

$$h_{fu}(z, t) = \sum_{i=1}^{\infty} \pm \sqrt{h_i} \phi_i(z) \phi_i(t), \quad T_i \leq z, t \leq T_f \tag{48}$$

is a solution to (45) for any assignment of plus and minus signs in the series.

Using (45) in (37), l_R becomes

$$l_R = \frac{1}{N_0} \int_{T_i}^{T_f} dz \left[\int_{T_i}^{T_f} h_{fu}(z, t) r(t) dt \right]^2 \tag{49}$$

This can be realized by a cascade of an unrealizable filter, a square-law device, and an integrator as shown in Fig. 2.5.

Alternatively, we can require that $h_1(t, u)$ be factored using realizable filters. In other words, we must find a solution $h_{fr}(z, t)$ to (45) that is zero for $t > z$. Then,

$$l_R = \frac{1}{N_0} \int_{T_i}^{T_f} dz \left[\int_{T_i}^z h_{fr}(z, t) r(t) dt \right]^2, \tag{50}$$

and the resulting receiver is shown in Fig. 2.6. If the time interval is finite, a realizable solution to (45) is difficult to find for arbitrary signal processes. Later we shall encounter several special situations that lead to simple solutions.

The integral equation (45) is a functional relationship somewhat analogous to the square-root relation. Thus, we refer to $h_f(z, t)$ as the *functional square root* of $h_1(t, u)$. We shall only define functional square roots for symmetric two-variable functions that can be expanded as in (46) with non-negative coefficients. We frequently use the notation

$$h_1(t, u) = \int_{T_i}^{T_f} h_1^{[1/2]}(z, t) h_1^{[1/2]}(z, u) dz. \tag{51}$$

Any solution to (51) is called a functional square root. Notice that the solutions are not necessarily symmetric.

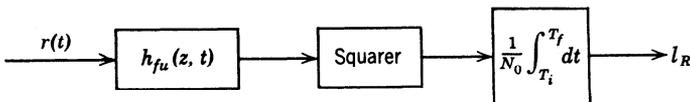


Fig. 2.5 Filter-squarer receiver (unrealizable).

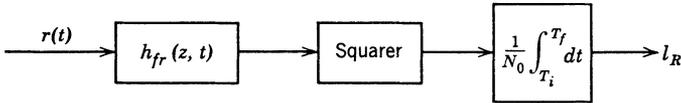


Fig. 2.6 Filter-squarer receiver (realizable).

The difficulty with all of the configurations that we have derived up to this point is that to actually implement them we must solve (23). From our experience in Chapter I-4 we know that we can do this for certain classes of kernels and certain conditions on T_i and T_f . We explore problems of this type in Chapter 4. On the other hand, in Section I-6.3 we saw that whenever the processes could be generated by exciting a linear finite-dimensional dynamic system with white noise, we had an effective procedure for solving (44). Fortunately, many of the processes (both nonstationary and stationary) that we encounter in practice have a finite-dimensional state representation.

In order to exploit the effective computation procedures that we have developed, we now modify our results to obtain an expression for l_R in which the optimum realizable linear filter specified by (44) is the only filter that we must find.

2.1.4 Canonical Realization No. 4: Optimum Realizable Filter Receiver

The basic concept involved in this realization is that of generating the likelihood ratio in real time as the output of a nonlinear dynamic system.† The derivation is of interest because the basic technique is applicable to many problems. For notational simplicity, we let $T_i = 0$ and $T_f = T$ in this section. Initially we shall assume that $m(t) = 0$ and consider only l_R .

Clearly, l_R is a function of the length of the observation interval T . To emphasize this, we can write

$$l_R(T | r(u), 0 \leq u \leq T) \triangleq l_R(T). \quad (52)$$

More generally, we could define a likelihood function for any value of time t .

$$l_R(t | r(u), 0 \leq u \leq t) \triangleq l_R(t), \quad (53)$$

where $l_R(0) = 0$. We can write $l_R(T)$ as

$$l_R(T) = \int_0^T \frac{dl_R(t)}{dt} dt = \int_0^T \dot{l}_R(t) dt. \quad (54)$$

† The original derivation of (66) was done by Schwegge [5]. The technique is a modification of the linear filter derivation in [6].

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Now we want to find an easy method for generating $\dot{i}_R(t)$. Replacing T by t in (31), we have

$$I_R(t) = \frac{1}{N_0} \int_0^t d\tau r(\tau) \int_0^t du h_1(\tau, u:t)r(u), \quad (55)$$

where $h_1(\tau, u:t)$ satisfies the integral equation

$$\frac{N_0}{2} h_1(\tau, u:t) + \int_0^t h_1(\tau, z:t)K_s(z, u) dz = K_s(\tau, u), \quad 0 \leq \tau, u \leq t. \quad (56)$$

[Observe that the solution to (56) depends on t . We emphasize this with the notation $h_1(\cdot, \cdot:t)$.] Differentiating (55), we obtain

$$\begin{aligned} \dot{i}_R(t) = \frac{1}{N_0} \left[r(t) \int_0^t du h_1(t, u:t)r(u) \right. \\ \left. + \int_0^t d\tau r(\tau) \left(h_1(\tau, t:t)r(t) + \int_0^t \frac{\partial h_1(\tau, u:t)}{\partial t} r(u) du \right) \right]. \quad (57) \end{aligned}$$

We see that the first two terms in (57) depend on $h_1(t, u:t)$. For this case, (56) reduces to

$$\frac{N_0}{2} h_1(t, u:t) + \int_0^t h_1(t, z:t)K_s(z, u) dz = K_s(t, u), \quad 0 \leq u \leq t. \quad (58)^\dagger$$

We know from our previous work in Chapter I-6 that

$$\hat{s}_r(t) = \int_0^t h_1(t, u:t)r(u) du \quad (59)$$

or

$$\hat{s}_r(t) = \int_0^t h_1(u, t:t)r(u) du. \quad (60)$$

[The subscript r means that the operation in (59) can be implemented with a realizable filter.] The result in (60) follows from the symmetry of the solution to (56). Using (59) and (60) in (57) gives

$$\dot{i}_R(t) = \frac{1}{N_0} \left[2r(t)\hat{s}_r(t) + \int_0^t d\tau \int_0^t du r(\tau) \frac{\partial h_1(\tau, u:t)}{\partial t} r(u) \right]. \quad (61)$$

In Problem I-4.3.3, we proved that

$$\frac{\partial h_1(\tau, u:t)}{\partial t} = -h_1(\tau, t:t)h_1(t, u:t), \quad 0 \leq \tau, u \leq t. \quad (62)$$

Because the result is the key step, we include the proof (from [7]).

\dagger Notice that $h_1(t, u:t) = h_{or}(t, u)$ [compare (44) and (58)].

Proof of (62). Differentiating (56) gives

$$\frac{N_0}{2} \frac{\partial h_1(\tau, u:t)}{\partial t} + \int_0^t \frac{\partial h_1(\tau, z:t)}{\partial t} K_s(z, u) dz + h_1(\tau, t:t)K_s(t, u) = 0, \quad 0 \leq \tau, u \leq t. \quad (63)$$

Now replace $K_s(t, u)$ with the left side of (58) and rearrange terms. This gives

$$-\frac{N_0}{2} \left\{ \frac{\partial h_1(\tau, u:t)}{\partial t} + h_1(\tau, t:t)h_1(t, u:t) \right\} = \int_0^t \left\{ \frac{\partial h_1(\tau, z:t)}{\partial t} + h_1(\tau, t:t)h_1(t, z:t) \right\} \times K_s(z, u) dz, \quad 0 \leq \tau, u \leq t. \quad (64)$$

We see that the terms in braces play the role of an eigenfunction with an eigenvalue of $(-N_0/2)$. However, $K_s(z, u)$ is non-negative definite, and so it cannot have a negative eigenvalue. Thus, the term in braces must be identically zero in order for (64) to hold. This is the desired result.

Substituting (62) into (61) and using (58), we obtain the desired result,

$$\dot{l}_R(t) = \frac{1}{N_0} [2r(t)\hat{s}_r(t) - \hat{s}_r^2(t)]. \quad (65)$$

Then

$$l_R = l_R(T) = \frac{1}{N_0} \int_0^T [2r(t)\hat{s}_r(t) - \hat{s}_r^2(t)] dt. \quad (66)^\dagger$$

Before looking at the optimum receiver configuration and some examples, it is appropriate to digress briefly and demonstrate an algorithm for computing the infinite sum $\sum_{i=1}^{\infty} \ln(1 + 2\lambda_i^s/N_0)$ that is needed to evaluate the bias in the Bayes test. We do this now because the derivation is analogous to the one we just completed. Two notational comments are necessary:

1. The eigenvalues in the sum depend on the length of the interval. We emphasize this with the notation $\lambda_i^s(T)$.
2. The eigenfunctions also depend on the length of the interval, and so we use the notation $\phi_i(t; T)$.

This notation was used previously in Chapter I-3 (page I-204).

† A result equivalent to that in (66) was derived independently by Stratonovich and Sosulin [21]–[24]. The integral in (66) is a stochastic integral, and some care must be used when one is dealing with arbitrary (not necessarily Gaussian) random processes. For Gaussian processes it can be interpreted as a Stratonovich integral and used rigorously [25]. For arbitrary processes an Itô integral formulation is preferable [26]–[28]. Interested readers should consult these references or [29]–[30]. For our purposes, it is adequate to treat (66) as an ordinary integral and manipulate it using the normal rules of calculus

We write

$$\sum_{i=1}^{\infty} \ln \left(1 + \frac{2}{N_0} \lambda_i^s(T) \right) = \int_0^T dt \left[\frac{d}{dt} \sum_{i=1}^{\infty} \ln \left(1 + \frac{2}{N_0} \lambda_i^s(t) \right) \right]. \quad (67)$$

Performing the indicated differentiation, we have

$$\frac{d}{dt} \sum_{i=1}^{\infty} \ln \left(1 + \frac{2}{N_0} \lambda_i^s(t) \right) = \frac{2}{N_0} \sum_{i=1}^{\infty} \frac{[d\lambda_i^s(t)]/dt}{1 + (2/N_0)\lambda_i^s(t)}. \quad (68)$$

In Chapter I-3 (page I-3.163), we proved that

$$\frac{d\lambda_i^s(t)}{dt} = \lambda_i^s(t)\phi_i^2(t:t), \quad (69)$$

and we showed that (I-3.154),

$$h_1(t, t:t) = \sum_{i=1}^{\infty} \frac{\lambda_i^s(t)}{\lambda_i^s(t) + N_0/2} \phi_i^2(t:t), \quad (70)$$

where $h_1(t, t:t)$ is the optimum MMSE realizable linear filter specified by (58). From (I-3.155), (44), and (58), the minimum mean-square realizable estimation error $\xi_{P^s}(t)$ is

$$\xi_{P^s}(t) = \frac{N_0}{2} h_1(t, t:t) \triangleq \frac{N_0}{2} h_{or}(t, t). \quad (71)$$

Thus

$$\sum_{i=1}^{\infty} \ln \left(1 + \frac{2\lambda_i^s(T)}{N_0} \right) = \int_0^T h_{or}(t, t) dt = \frac{2}{N_0} \int_0^T \xi_{P^s}(t) dt. \quad (72)$$

From (33),

$$\boxed{l_B^{[1]} = -\frac{1}{2} \sum_{i=1}^{\infty} \ln \left(1 + \frac{2\lambda_i^s}{N_0} \right) = -\frac{1}{N_0} \int_0^T \xi_{P^s}(t) dt.} \quad (73)$$

We see that whenever we use Canonical Realization No. 4, we obtain the first bias term needed for the Bayes test as a by-product. The second bias term [see (34)] is due to the mean, and its computation will be discussed shortly. A block diagram of Realization No. 4 for generating l_R and $l_B^{[1]}$ is shown in Fig. 2.7.

Before leaving our discussion of the bias term, some additional comments are in order. The infinite sum of the left side of (72) will appear in several different contexts, so that an efficient procedure for evaluating it is important. It can also be written as the logarithm of the Fredholm

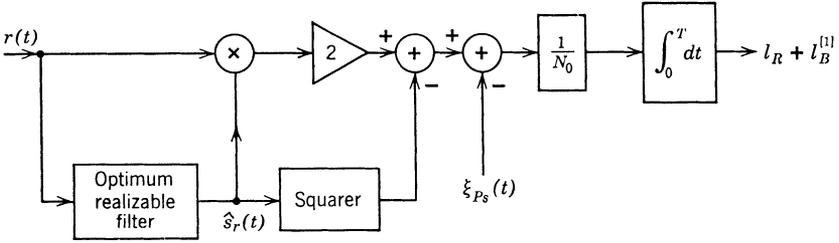


Fig. 2.7 Optimum realizable filter realization (Canonical Realization No. 4).

determinant [8],

$$\sum_{i=1}^{\infty} \ln \left(1 + \frac{2\lambda_i^s}{N_0} \right) = \left[\ln \left\{ \prod_{i=1}^{\infty} (1 + z\lambda_i^s) \right\} \right]_{z=2/N_0} \triangleq \ln D_{\mathcal{F}} \left(\frac{2}{N_0} \right). \quad (74)$$

Now, unless we can find $D_{\mathcal{F}}(2/N_0)$ effectively, we have not made any progress. One procedure is to evaluate $\xi_{Ps}(t)$ and use the integral expression on the right side of (73). A second procedure for evaluating $D_{\mathcal{F}}(\cdot)$ is a by-product of our solution procedure for Fredholm equations for certain signal processes (see the Appendix in Part II). A third procedure is to use the relation

$$\sum_{i=1}^{\infty} \ln \left(1 + \frac{2\lambda_i^s}{N_0} \right) = \int_0^{2/N_0} dz \int_{T_i}^{T_f} h_1(t, t | z) dt, \quad (75)$$

where $h_1(t, t | z)$ is the solution to (23) when $N_0/2$ equals z . Notice that this is the optimum unrealizable filter. This result is derived in Problem 2.1.2. The choice of which procedure to use depends on the specific problem.

Up to this point in our discussion we have not made any detailed assumptions about the signal process. We now look at Realization No. 4 for signal processes that can be generated by exciting a finite-dimensional linear system with white noise. We refer to the corresponding receiver as Realization No. 4S (“S” denotes “state”).

2.1.5 Canonical Realization No. 4S: State-variable Realization

The class of signal processes of interest was described in detail in Section I-6.3 (see pages I-516–I-538). The process is described by a state equation,

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{G}(t)\mathbf{u}(t), \quad (76)$$

where $\mathbf{F}(t)$ and $\mathbf{G}(t)$ are possibly time-varying matrices, and by an observation equation,

$$s(t) = \mathbf{C}(t)\mathbf{x}(t), \quad (77)$$

where $\mathbf{C}(t)$ is the modulation matrix. The input, $\mathbf{u}(t)$, is a sample function from a zero-mean vector white noise process,

$$E[\mathbf{u}(t)\mathbf{u}^T(\tau)] = \mathbf{Q} \delta(t - \tau), \quad (78)$$

and the initial conditions are

$$E[\mathbf{x}(0)] = \mathbf{0}, \quad (79)$$

$$E[\mathbf{x}(0)\mathbf{x}^T(0)] \triangleq \mathbf{P}_0. \quad (80)$$

From Section I-6.3.2 we know that the MMSE realizable estimate of $s(t)$ is given by the equations

$$\dot{\hat{s}}_r(t) = \mathbf{C}(t)\hat{\mathbf{x}}(t), \quad (81)$$

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{F}(t)\hat{\mathbf{x}}(t) + \boldsymbol{\xi}_P(t)\mathbf{C}^T(t) \frac{2}{N_0} [r(t) - \mathbf{C}(t)\hat{\mathbf{x}}(t)]. \quad (82)$$

The matrix $\boldsymbol{\xi}_P(t)$ is the error covariance matrix of $\mathbf{x}(t) - \hat{\mathbf{x}}(t)$.

$$\boldsymbol{\xi}_P(t) \triangleq E[(\mathbf{x}(t) - \hat{\mathbf{x}}(t))(\mathbf{x}^T(t) - \hat{\mathbf{x}}^T(t))]. \quad (83)$$

It satisfies the nonlinear matrix differential equations,

$$\dot{\boldsymbol{\xi}}_P(t) = \mathbf{F}(t)\boldsymbol{\xi}_P(t) + \boldsymbol{\xi}_P(t)\mathbf{F}^T(t) - \boldsymbol{\xi}_P(t)\mathbf{C}^T(t) \frac{2}{N_0} \mathbf{C}(t)\boldsymbol{\xi}_P(t) + \mathbf{G}(t)\mathbf{Q}\mathbf{G}^T(t). \quad (84)$$

The mean-square error in estimating $s(t)$ is

$$\xi_{P_s}(t) = \mathbf{C}(t)\boldsymbol{\xi}_P(t)\mathbf{C}^T(t). \quad (85)$$

Notice that $\boldsymbol{\xi}_P(t)$ is the error covariance matrix for the state vector and $\xi_{P_s}(t)$ is the scalar mean-square error in estimating $s(t)$. Both (84) and (85) can be computed either before $r(t)$ is received or simultaneously with the computation of $\hat{\mathbf{x}}(t)$.

The system needed to generate l_R and $l_B^{[1]}$ follows easily and is shown in Fig. 2.8. The state equation describing l_R is obtained from (65),

$$\dot{l}_R(t) = \frac{1}{N_0} [2r(t)\hat{s}_r(t) - \hat{s}_r^2(t)], \quad (86)$$

where $\hat{s}_r(t)$ is defined by (81)–(84) and

$$l_R \triangleq l_R(T). \quad (87)$$

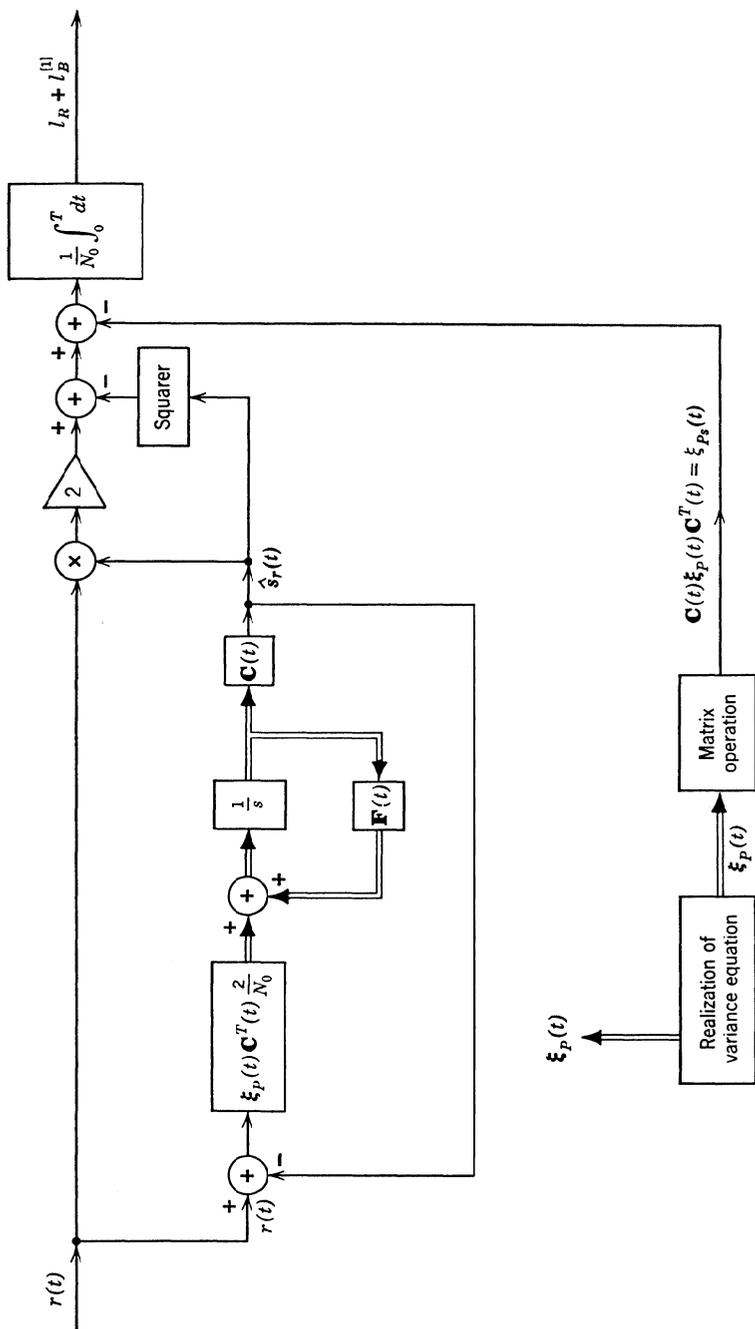


Fig. 2.8 State-variable receiver (Canonical Realization No. 4S).

The important feature of this realization is that there are *no* integral equations to solve. The likelihood ratio is generated as the output of a dynamic system. We now consider a simple example to illustrate the application of these ideas.

Example. In Fig. 2.9 we show a hypothetical communication system that illustrates many of the important features encountered in actual systems operating over fading channels. In Chapter 10, we shall develop models for fading channels and find that the models are generalizations of the system in this example. When H_1 is true, we transmit a *deterministic* signal $f(t)$. When H_0 is true, we transmit nothing. The channel affects the received signal in two ways. The transmitted signal is *multiplied* by a sample function of a Gaussian random process $b(t)$. In many cases, this channel process will be stationary over the time intervals of interest. The output of the multiplicative part of the channel is corrupted by additive white Gaussian noise $w(t)$, which is statistically independent of $b(t)$. Thus the received waveforms on the two hypotheses are

$$\begin{aligned} r(t) &= f(t)b(t) + w(t), & 0 \leq t \leq T: H_1, \\ r(t) &= w(t), & 0 \leq t \leq T: H_0. \end{aligned} \tag{88}$$

We assume that the channel process has a state representation

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{G}(t)\mathbf{u}(t), \tag{89}$$

where $\mathbf{u}(t)$ satisfies (78) and

$$b(t) = \mathbf{C}_b(t)\mathbf{x}(t). \tag{90}$$

The signal process on H_1 is $s(t)$, where

$$s(t) \triangleq f(t)b(t). \tag{91}$$

Notice that, unless $f(t)$ is constant over the interval $[0, T]$, the process, $s(t)$, will be nonstationary even though $b(t)$ is stationary. Clearly, $s(t)$ has the same state equation as $b(t)$, (89). Combining (90) and (91) gives the observation equation,

$$s(t) = f(t)\mathbf{C}_b(t)\mathbf{x}(t) \triangleq \mathbf{C}(t)\mathbf{x}(t). \tag{92}$$

We see that the transmitted signal $f(t)$ appears only in the modulation matrix, $\mathbf{C}(t)$.

It is instructive to draw the receiver for the simple case in which $b(t)$ has a one-dimensional state equation with constant coefficients. We let

$$\mathbf{F}(t) = -k_b, \tag{93}$$

$$\mathbf{G}(t) = 1, \tag{94}$$

$$Q = 2k_b\sigma_b^2, \tag{95}$$

$$\mathbf{C}_b(t) = 1, \tag{96}$$

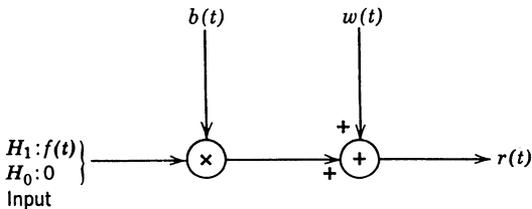
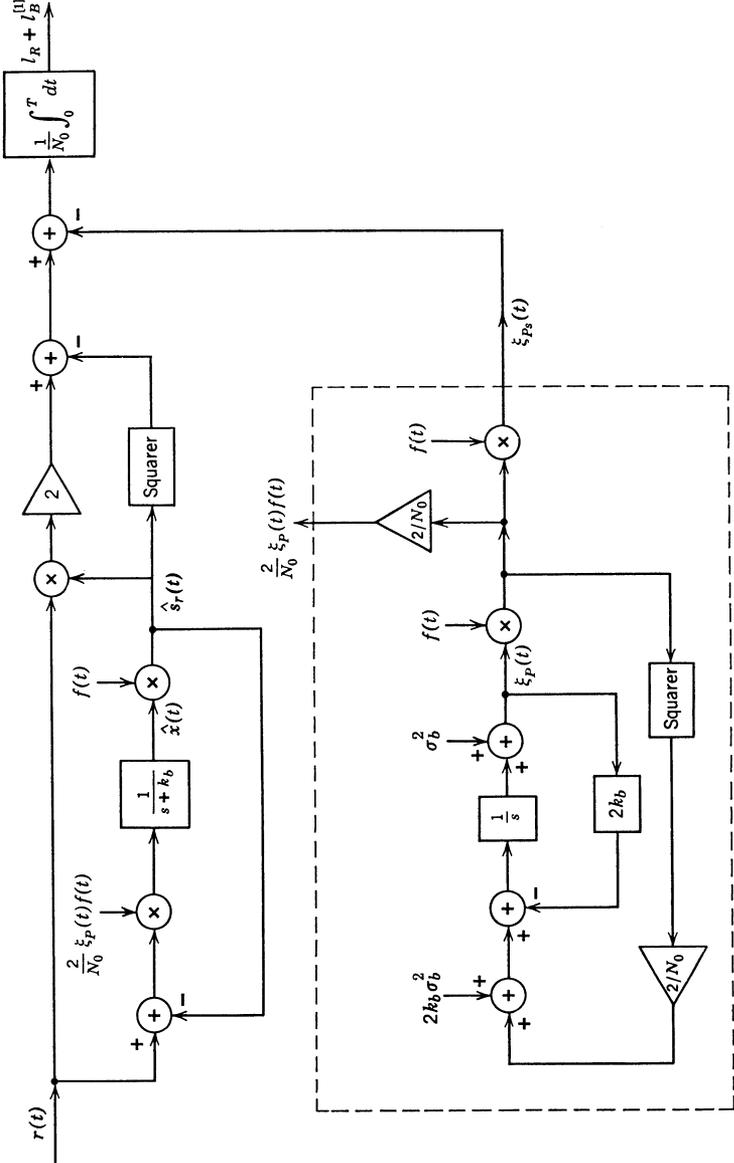


Fig. 2.9 A simple multiplicative channel.



Realization of variance equation (either in real time or precomputed)
Fig. 2.10 Feedback realization for generating $I_R + I_B^{[1]}$.

and

$$\xi_P(0) = \sigma_b^2. \quad (97)$$

Then (82) and (84) reduce to

$$\dot{\hat{x}}(t) = -k_b \hat{x}(t) + \frac{2}{N_0} \xi_P(t) f(t) [r(t) - f(t) \hat{x}(t)], \quad (98)$$

$$\dot{\xi}_P(t) = -2k_b \xi_P(t) - \frac{2}{N_0} f^2(t) \xi_P^2(t) + 2k_b \sigma_b^2, \quad (99)$$

and

$$\xi_{Ps}(t) = f^2(t) \xi_P(t). \quad (100)$$

The resulting receiver structure is shown in Fig. 2.10.

We shall encounter other examples of Canonical Realization No. 4S as we proceed. Before leaving this realization, it is worthwhile commenting on the generation of l_D , the component in the likelihood ratio that arises because of the mean value in the signal process. If the process has a finite state representation, it is usually easier to generate l_D using the optimum realizable filter. The derivation is identical with that in (54)–(66). From (22) and (26)–(28) we have

$$l_D(T) = \frac{2}{N_0} \int_0^T \left[m(\tau) - \int_0^T h_1(\tau, u: T) m(u) du \right] r(\tau) d\tau. \quad (101)$$

As before,

$$l_D(T) = \int_0^T \frac{dl_D(t)}{dt} dt, \quad (102)$$

and

$$\begin{aligned} \frac{dl_D(t)}{dt} = \frac{2}{N_0} \left[r(t) \left(m(t) - \int_0^t h_1(t, u:t) m(u) du \right) - m(t) \int_0^t h_1(\tau, t:t) r(\tau) d\tau \right. \\ \left. + \int_0^t \int_0^t h_1(\tau, t:t) h_1(t, u:t) m(u) r(\tau) d\tau du \right]. \quad (103) \end{aligned}$$

The resulting block diagram is shown in Fig. 2.11. The output of the bottom path is just a deterministic function, which we denote by $K(t)$,

$$K(t) \triangleq m(t) - \int_0^t h_1(t, u:t) m(u) du, \quad 0 \leq t \leq T. \quad (104)$$

Because $K(t)$ does not depend on $r(t)$, we can generate it before any data are received. This suggests the two equivalent realizations in Fig. 2.12.

Notice that (101) (and therefore Figs. 2.11 and 2.12) does not require that the processes be state-representable. If the processes have a finite state, the optimum realizable linear filter can be derived easily using state-variable techniques. Using the state representation in (76)–(80) gives

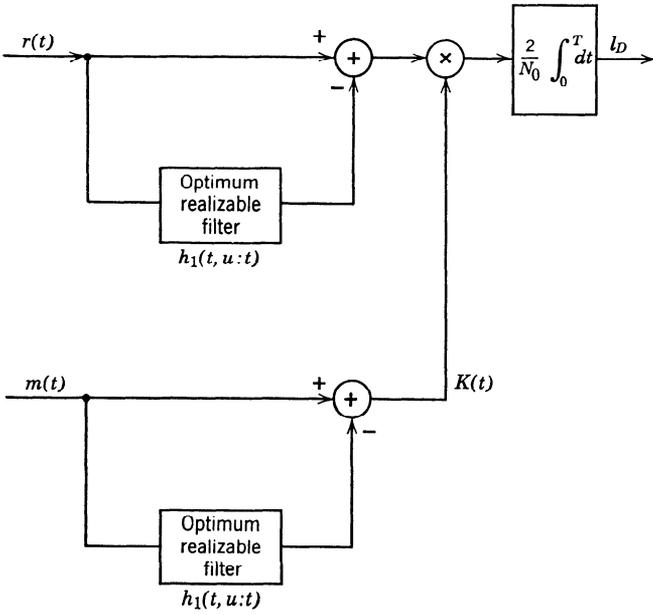


Fig. 2.11 Generation of l_D using optimum realizable filters.

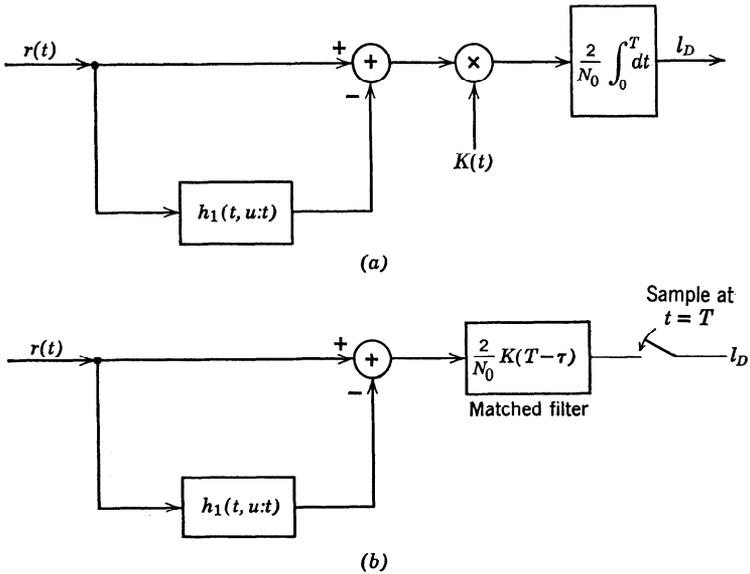


Fig. 2.12 Generation of l_D .

the realization in Fig. 2.13a.† Notice that the state vector in Fig. 2.13a is *not* $\hat{\mathbf{x}}(t)$, because $r(t)$ has a nonzero mean. We denote it by $\check{\mathbf{x}}(t)$.

The block diagram in Fig. 2.13a can be simplified as shown in Fig. 2.13b. We can also write $l_D(t)$ in a canonical state-variable form:

$$\begin{bmatrix} \dot{l}_D(t) \\ \check{\mathbf{x}}(t) \end{bmatrix} = \begin{bmatrix} 0 & -\frac{2}{N_0} K(t)C(t) \\ \mathbf{0} & \mathbf{F}(t) - \frac{2}{N_0} \boldsymbol{\xi}_P(t)C^T(t)C(t) \end{bmatrix} \begin{bmatrix} l_D(t) \\ \check{\mathbf{x}}(t) \end{bmatrix} + \begin{bmatrix} \frac{2}{N_0} K(t) \\ \frac{2}{N_0} \boldsymbol{\xi}_P(t)C^T(t) \end{bmatrix} r(t), \tag{105}$$

where $K(t)$ is defined in Fig. 2.11 and $\boldsymbol{\xi}_P(t)$ satisfies (84).

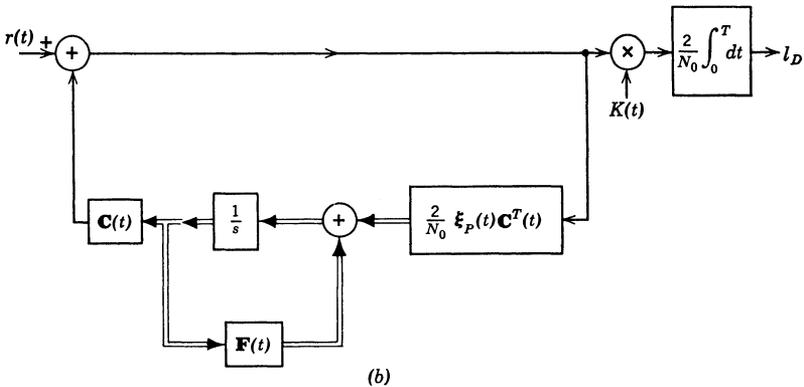
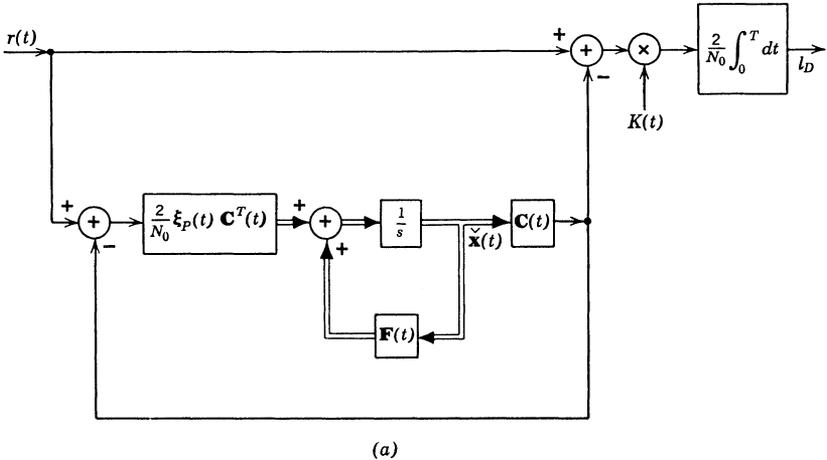


Fig. 2.13 State-variable realizations to generate l_D .

† As we would expect, the system in Fig. 3.12 is identical with that obtained using a whitening approach (e.g., Collins [9] or Problem 2.1.3).

Looking at (32) and (34), we see that their structure is identical. Thus, we can generate $l_B^{[2]}$ by driving the dynamic system in (105) with $(-m(t)/2)$ instead of $r(t)$.

It is important to emphasize that the presence of $m(t)$ does not affect the generation of $l_R(t)$ in (86). The only difference is that $\check{s}_r(t)$ and $\check{\mathbf{x}}(t)$ are no longer MMSE estimates, and so we denote them by $\check{s}_r(t)$ and $\check{\mathbf{x}}(t)$, respectively. The complete set of equations for the non-zero-mean case may be summarized as follows:

$$\dot{l}_R(t) = \frac{1}{N_0} [-\check{s}_r^2(t) + 2r(t)\check{s}_r(t)], \quad (106)$$

$$\dot{l}_D(t) = \left(-\frac{2}{N_0} K(t)C(t) \right) l_D(t) + \frac{2}{N_0} K(t)r(t), \quad (107)$$

$$\check{s}_r(t) = C(t)\check{\mathbf{x}}(t), \quad (108)$$

$$\dot{\check{\mathbf{x}}}(t) = \mathbf{F}(t)\check{\mathbf{x}}(t) + \frac{2}{N_0} \boldsymbol{\xi}_P(t)C^T(t)[r(t) - C(t)\check{\mathbf{x}}(t)], \quad (109)$$

with initial conditions

$$\check{\mathbf{x}}(0) = \mathbf{0} \quad (110)$$

and

$$l_D(0) = 0. \quad (111)$$

The matrix $\boldsymbol{\xi}_P(t)$ is specified by (84). The biases are described in (73) and a modified version of (105).

This completes our discussion of state-variable realizations of the optimum receiver for the Gaussian signal problem. We have emphasized structures based on realizable estimators. An alternative approach based on unrealizable estimator structures can also be developed (see Problem I-6.6.4 and Problem 2.1.4). Before discussing the performance of the optimum receiver, we briefly summarize our results concerning receiver structures.

2.1.6 Summary: Receiver Structures

In this section we derived the likelihood ratio test for the simple binary detection problem in which the received waveforms on the two hypotheses were

$$\begin{aligned} r(t) &= w(t), & T_i \leq t \leq T_f: H_0, \\ r(t) &= s(t) + w(t), & T_i \leq t \leq T_f: H_1. \end{aligned} \quad (112)$$

The result was the test

$$l_R + l_D \underset{H_0}{\overset{H_1}{\gtrless}} \ln \eta - l_B^{[1]} - l_B^{[2]}, \quad (113)$$

where the various terms were defined by (31)–(34).

We then looked at four receivers that could be used to implement the likelihood ratio test. The first three configurations were based on the optimum unrealizable filter and required the solution of a Fredholm integral equation (23). In Chapter 4 we shall consider problems where this equation can be easily solved. The fourth configuration was based on an optimum *realizable* filter. For this realization we had to solve (44). For a large class of processes, specifically those with a finite state representation, we have already developed an efficient technique for solving this problem (the Kalman-Bucy technique). It is important to re-emphasize that all of the receivers implement the likelihood ratio test and therefore must have identical error probabilities. By having alternative configurations available, we may choose the one easiest to implement.† In the next section we investigate the performance of the likelihood ratio test.

2.2 PERFORMANCE

In this section we analyze the performance of the optimum receivers that we developed in Section 2.1. All of these receivers perform the test indicated in (35) as

$$l = l_R + l_D + l_B \underset{H_0}{\overset{H_1}{\gtrless}} \ln \eta = \gamma, \quad (114)$$

where

$$l_R = \frac{1}{N_0} \iint_{T_i}^{T_f} r(t) h_1(t, u) r(u) dt du, \quad (115)$$

$$l_D = \int_{T_i}^{T_f} g_1(u) r(u) du, \quad (116)$$

and

$$l_B \triangleq l_B^{[1]} + l_B^{[2]}. \quad (117)$$

† The reader may view the availability of alternative configurations as a mixed blessing, because it requires some mental bookkeeping to maintain the divisions between realizable and unrealizable filters, the zero-mean and non-zero-mean cases, and similar separations. The problems at the end of Chapter 4 will help in remembering the various divisions.

From (33) and (34), we recall that

$$l_B^{[1]} \triangleq -\frac{1}{2} \sum_{i=1}^{\infty} \ln \left(1 + \frac{2\lambda_i^s}{N_0} \right), \quad (118)$$

$$l_B^{[2]} \triangleq -\frac{1}{2} \int_{T_i}^{T_f} g_1(u) m(u) du. \quad (119)$$

To compute P_D and P_F , we must find the probability that l will exceed γ on H_1 and H_0 , respectively. These probabilities are

$$P_D = \int_{\gamma}^{\infty} p_{l|H_1}(L | H_1) dL \quad (120)$$

and

$$P_F = \int_{\gamma}^{\infty} p_{l|H_0}(L | H_0) dL. \quad (121)$$

The l_D component is a linear function of a Gaussian process, so that it is a Gaussian random variable whose mean and variance can be computed easily. However, l_R is obtained by a nonlinear operation on $r(t)$, and so its probability density is difficult to obtain. To illustrate the difficulty, we look at the first term in (25). Because this term corresponds to l_R before we let $K \rightarrow \infty$, we denote it by l_R^K ,

$$l_R^K \triangleq \frac{1}{N_0} \sum_{i=1}^K \frac{\lambda_i^s}{\lambda_i^s + N_0/2} R_i^2. \quad (122)$$

We see that l_R^K is a weighted sum of *squared* Gaussian random variables. The expression in (122) is familiar from our work on the general Gaussian problem, Section I-2.6. In fact, if the R_i were zero-mean, (122) would be identical with (I-2.420). At that point we observed that if the λ_i^s were all equal, l_R^K had a chi-square density with K degrees of freedom (e.g., I-2.406). On the other hand, for unequal λ_i^s , we could write an expression for the probability density but it was intractable for large K . Because of the independence of the R_i , the characteristic function and moment-generating function of l_R^K followed easily (e.g., Problem I-4.4.2). Given the characteristic function, we could, in principle at least, find the probability density by computing the Fourier transform numerically. In practice, we are usually interested in small error probabilities, and so we must know the tails of $p_{l|H_i}(L | H_i)$ accurately. This requirement causes the amount of computation required for accurate numerical inversion to be prohibitive. This motivated our discussion of performance bounds and approximations in Section I-2.7. In this section we carry out an analogous discussion for the case in which $K \rightarrow \infty$.

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We recall† that the function $\mu_K(s)$ played the central role in our discussion. From (I-2.444),

$$\mu_K(s) \triangleq \ln [\phi_{l(\mathbf{R})|H_0}(s)], \quad (123)$$

[The subscript K is added to emphasize that we are dealing with K -term approximation to $r(t)$.] Where $l(\mathbf{R})$ is the logarithm of the likelihood ratio

$$l(\mathbf{R}) = \ln \Lambda(\mathbf{R}) = \ln \left(\frac{p_{r|H_1}(\mathbf{R} | H_1)}{p_{r|H_0}(\mathbf{R} | H_0)} \right), \quad (124)$$

and $\phi_{l(\mathbf{R})|H_0}(s)$ is its moment-generating function,

$$\phi_{l(\mathbf{R})|H_0}(s) = E[e^{s l(\mathbf{R})} | H_0], \quad (125)$$

for real s . Using the definition of $l(\mathbf{R})$ in (124),

$$\mu_K(s) = \ln \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} [p_{r|H_1}(\mathbf{R} | H_1)]^s [p_{r|H_0}(\mathbf{R} | H_0)]^{1-s} d\mathbf{R}. \quad (126)$$

We then developed upper bounds on P_F and P_M .‡

$$\begin{aligned} P_F &\leq \exp [\mu_K(s) - s\dot{\mu}_K(s)], \\ P_M &\leq \exp [\mu_K(s) + (1-s)\dot{\mu}_K(s)], \end{aligned} \quad 0 \leq s \leq 1, \quad (127)$$

where $\dot{\mu}_K(s) = \gamma_K$, the threshold in the LRT. By varying the parameter s , we could study threshold settings anywhere between $E[l | H_1]$ and $E[l | H_0]$. The definition of $l(\mathbf{R})$ in (124) guaranteed that $\mu_K(s)$ existed for $0 \leq s \leq 1$.

We now define a function $\mu(s)$,

$$\mu(s) \triangleq \lim_{K \rightarrow \infty} \mu_K(s). \quad (128)$$

If we can demonstrate that the limit exists, our bounds in (127) will still be valid. However, in order to be useful, the expression for $\mu(s)$ must be in a form that is practical to evaluate. Thus, our first goal in this section is to find a convenient closed-form expression for $\mu(s)$.

The second useful set of results in Section I-2.7 was the approximate error expressions in (I-2.480) and (I-2.483),

$$P_F \simeq \frac{1}{\sqrt{2\pi s^2 \ddot{\mu}(s)}} e^{\mu(s) - s\dot{\mu}(s)}, \quad s \geq 0, \quad (129)$$

and

$$P_M \simeq \frac{1}{\sqrt{2\pi(1-s)^2 \ddot{\mu}(s)}} e^{\mu(s) + (1-s)\dot{\mu}(s)}, \quad s \leq 1. \quad (130)$$

† Our discussion assumes a thorough understanding of Section I-2.7, so that a review of that section may be appropriate at this point.

‡ $\Pr(\varepsilon)$ bounds were also developed. Because they are more appropriate to the general binary problem in the next section, we shall review them then.

As we pointed out on page I-124, the exponents in these expressions were identical with the Chernoff bounds in (127), but the multiplicative factor was significant in many applications of interest to us. In order to derive (129) and (130), we used a central limit theorem argument. For the problems considered in Section I-2.7 (e.g., Examples 2, 3, and 3A on pages I-127–I-132), it was easy to verify that the central limit theorem is applicable. However, for the case of interest in most of this chapter, the sum defining I_R in (122) violates a necessary condition for the validity of the central limit theorem. Thus, we must use a new approach in order to find an approximate error expression. This is the second goal of this section.

In addition to these two topics, we develop an alternative expression for computing $\mu(s)$ and analyze a typical example in detail. Thus, there are four subsections:

- 2.2.1. Closed-form expressions for $\mu(s)$.
- 2.2.2. Approximate error expressions.
- 2.2.3. An alternative expression for $\mu(s)$.
- 2.2.4. Performance for a typical example.

2.2.1 Closed-form Expression for $\mu(s)$

We first evaluate $\mu_K(s)$ for finite K . Substituting (18) into (126) gives

$$\begin{aligned} \mu_K(s) = & \ln \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left\{ \left[\prod_{i=1}^K \frac{1}{\sqrt{2\pi(N_0/2 + \lambda_i^s)}} \exp \left(-\frac{1}{2} \sum_{i=1}^K \frac{(R_i - m_i)^2}{(\lambda_i^s + N_0/2)} \right) \right]^s \right. \\ & \times \left. \left[\prod_{i=1}^K \frac{1}{\sqrt{2\pi(N_0/2)}} \right] \exp \left(-\frac{1}{2} \sum_{i=1}^K \frac{R_i^2}{N_0/2} \right) \right\}^{1-s} dR_1 \cdots dR_K. \end{aligned} \quad (131)$$

Performing the integration, we have

$$\begin{aligned} \mu_K(s) = & \frac{1}{2} \sum_{i=1}^K \left[(1-s) \ln \left(1 + \frac{2\lambda_i^s}{N_0} \right) - \ln \left(1 + \frac{2(1-s)\lambda_i^s}{N_0} \right) \right] \\ & - \frac{s}{2} \sum_{i=1}^K \left(\frac{m_i^2}{N_0/2(1-s) + \lambda_i^s} \right), \quad 0 \leq s \leq 1. \end{aligned} \quad (132)$$

From our discussion on page 13, we know the first sum on the right side of (132) is well behaved as $K \rightarrow \infty$. The convergence of the second sum follows easily.

$$\sum_{i=1}^K \frac{m_i^2}{(N_0/2(1-s) + \lambda_i^s)} \leq \sum_{i=1}^K \frac{m_i^2}{N_0/2(1-s)} \leq \frac{2(1-s)}{N_0} \int_{T_i}^{T_i'} m^2(t) dt. \quad (133)$$

We now take the limit of (132) as $K \rightarrow \infty$. The first sum is due to the randomness in $s(t)$, and so we denote it by $\mu_R(s)$. The second sum is due to the deterministic component in $r(t)$, and so we denote it by $\mu_D(s)$.

$$\mu_R(s) \triangleq \frac{1}{2} \sum_{i=1}^{\infty} \left[(1-s) \ln \left(1 + \frac{2\lambda_i^s}{N_0} \right) - \ln \left(1 + \frac{2(1-s)\lambda_i^s}{N_0} \right) \right]. \quad (134)$$

$$\mu_D(s) \triangleq -\frac{s}{2} \sum_{i=1}^{\infty} \frac{m_i^2}{N_0/2(1-s) + \lambda_i^s}. \quad (135)$$

We now find a closed-form expression for the sums in (134) and (135). First we consider $\mu_R(s)$. Both of the sums in (134) are related to realizable linear filtering errors. To illustrate this, we consider the linear filtering problem in which

$$r(u) = s(u) + w(u), \quad T_i \leq u \leq t, \quad (136)$$

where $s(u)$ is a zero-mean message process with covariance function $K_s(t, u)$ and the white noise has spectral height $N_0/2$. Using our results in Chapter I-6, we can find the linear filter whose output is the MMSE point estimate of $s(\cdot)$ and evaluate the resulting mean-square error. We denote this error as $\xi_P(t | s(\cdot), N_0/2)$. (The reason for the seemingly awkward notation will be apparent in a moment.) Using (72), we can write the mean-square error in terms of a sum of eigenvalues.

$$\sum_{i=1}^{\infty} \ln \left(1 + \frac{2\lambda_i^s}{N_0} \right) = \frac{2}{N_0} \int_{T_i}^{T_f} \xi_P \left(t | s(\cdot), \frac{N_0}{2} \right) dt. \quad (137)$$

Comparing (134) and (137) leads to the desired result.

$$\mu_R(s) = \frac{1-s}{N_0} \int_{T_i}^{T_f} dt \left[\xi_P \left(t | s(\cdot), \frac{N_0}{2} \right) - \xi_P \left(t | s(\cdot), \frac{N_0}{2(1-s)} \right) \right]. \quad (138)$$

Thus, to find $\mu_R(s)$, we must find the mean-square error for two realizable linear filtering problems. In the first, the signal is $s(\cdot)$ and the noise is white with spectral height $N_0/2$. In the second, the signal is $s(\cdot)$ and the noise is white with spectral height $N_0/2(1-s)$. An alternative expression for $\mu_R(s)$ also follows easily.

$$\mu_R(s) = \frac{1}{N_0} \int_{T_i}^{T_f} \left[(1-s) \xi_P \left(t | s(\cdot), \frac{N_0}{2} \right) - \xi_P \left(t | \sqrt{1-s} s(\cdot), \frac{N_0}{2} \right) \right] dt. \quad (139)$$

Here the noise level is the same in both calculations, but the amplitude of the signal process is changed. These equations are the first key results in our performance analysis. Whenever we have a signal process such that we can calculate the realizable mean-square filtering error for the problem of estimating $s(t)$ in the presence of white noise, then we can find $\mu_R(s)$.

The next step is to find a convenient expression for $\mu_D(s)$. To evaluate the sum in (135), we recall the problem of detecting a *known* signal in colored noise, which we discussed in detail in Section I-4.3. The received waveforms on the two hypotheses are

$$\begin{aligned} r(t) &= m(t) + n_c(t) + w(t), & T_i \leq t \leq T_f: H_1, \\ r(t) &= n_c(t) + w(t), & T_i \leq t \leq T_f: H_0. \end{aligned} \quad (140)$$

By choosing the covariance function of $n_c(t)$ and $w(t)$ appropriately, we can obtain the desired interpretation. Specifically, we let

$$E[n_c(t)n_c(u)] = K_s(t, u), \quad T_i \leq t, u \leq T_f \quad (141)$$

and

$$E[w(t)w(u)] = \frac{N_0}{2(1-s)} \delta(t-u), \quad T_i \leq t, u \leq T_f. \quad (142)$$

Then, from Chapter I-4 (page I-296), we know that the optimum receiver correlates $r(t)$ with a function $g(t | N_0/2(1-s))$, which satisfies the equation†

$$m(t) = \int_0^T \left[K_s(t, u) + \frac{N_0}{2(1-s)} \delta(t-u) \right] g\left(u \mid \frac{N_0}{2(1-s)}\right) du. \quad (143)$$

We also recall that we can write $g(t | \cdot)$ explicitly in terms of the eigenfunctions and eigenvalues of $K_s(t, u)$. Writing

$$g\left(u \mid \frac{N_0}{2(1-s)}\right) = \sum_{i=1}^{\infty} g_i \phi_i(u), \quad T_i \leq u \leq T_f, \quad (144)$$

substituting into (143), and solving for the g_i gives

$$g_i = \frac{m_i}{\lambda_i^s + N_0/2(1-s)}, \quad (145)$$

where

$$m_i \triangleq \int_{T_i}^{T_f} m(t) \phi_i(t) dt. \quad (146)$$

† This notation is used to emphasize that $g(t | \cdot)$ depends on both N_0 and s , in addition to $K_s(t, u)$.

Substituting (145) and (146) into (135) and using Parseval's theorem, we have

$$\mu_D(s) = -\frac{s}{2} \int_{T_i}^{T_f} m(t)g\left(t \left| \frac{N_0}{2(1-s)} \right.\right) dt. \quad (147)$$

We observe that the integral in (147) is just d^2 for the known signal in colored noise problem described in (140) [see (I-4.198)]. We shall encounter several equivalent expressions for $\mu_D(s)$ later.

We denote the limit of the right side of (132) as $K \rightarrow \infty$ as $\mu(s)$. Thus,

$$\mu(s) = \mu_R(s) + \mu_D(s). \quad (148)$$

Using (138) and (147) in (148) gives a closed-form expression for $\mu(s)$. This enables us to evaluate the Chernoff bounds in (127) when $K \rightarrow \infty$. In the next section we develop approximate error expressions similar to those in (129) and (130).

2.2.2 Approximate Error Expressions

In order to derive an approximate error expression, we return to our derivation in Section I-2.7 (page I-123). After tilting the density and standardizing the tilted variable, we have the expression for P_F given in (I-2.477). The result is

$$P_F = e^{\mu(s) - s\dot{\mu}(s)} \int_0^\infty e^{-s\sqrt{\dot{\mu}(s)}Y} p_Y(Y) dY, \quad (149)$$

where Y is a zero-mean, unit-variance, random variable and we assume that $\dot{\mu}(s)$ equals γ . Recall that

$$y = \frac{x_s - \dot{\mu}(s)}{\sqrt{\dot{\mu}(s)}}, \quad (150)$$

where

$$p_{x_s}(X) = e^{sX - \mu(s)} p_{l|H_0}(X | H_0), \quad (151)$$

and l is the log likelihood ratio which can be written as

$$l = l_R + l_D + l_B^{[1]} + l_B^{[2]}. \quad (152)$$

[Notice that the threshold is γ as defined in (35).] The quantity l is also the limit of the sum in (19) as $K \rightarrow \infty$. If the weighted variables in the first sum in (19) were identically distributed, then, as $K \rightarrow \infty$, $p_Y(Y)$ would approach a Gaussian density. An example of a case of this type was given in Example 2 on page I-127. In that problem,

$$\lambda_i^s = \sigma_s^2, \quad i = 1, 2, \dots, N, \quad (153)$$

so that the weighting in the first term of (19) was uniform and the variables were identically distributed. In the model of this chapter, we assume that

$s(t)$ has finite average power [see the sentence below (4)]. Thus $\sum_{i=1}^{\infty} \lambda_{i_s}$ is finite. Whenever the sum of the variances of the component random variables is finite, the central limit theorem cannot hold (see [10]). This means that we must use some other argument to get an approximate expression for P_F and P_M .

A logical approach is to expand $p_y(Y)$ in an Edgeworth series. The first term in the expansion is a Gaussian density. The remaining terms take into account the non-Gaussian nature of the density. On the next few pages we carry out the details of the analysis. The major results are approximations to P_F and P_M ,

$$\begin{aligned}
 P_F &\simeq \frac{1}{\sqrt{2\pi s^2 \dot{\mu}(s)}} e^{\mu(s) - s\dot{\mu}(s)}, & 0 \leq s \leq 1 & \quad (154) \\
 &\text{and} \\
 P_M &\simeq \frac{1}{\sqrt{2\pi(1-s)^2 \dot{\mu}(s)}} e^{\mu(s) + (1-s)\dot{\mu}(s)}, & 0 \leq s \leq 1. & \quad (155)
 \end{aligned}$$

We see that (154) and (155) are identical with (129) and (130). Thus, our derivation leads us to the same result as before. The important difference is that we get to (154) and (155) without using the central limit theorem.

Derivation of Error Approximations† The first term in the Edgeworth series is the Gaussian density,

$$\phi(Y) \triangleq \frac{1}{\sqrt{2\pi}} e^{-Y^2/2}. \quad (156)$$

The construction of the remaining terms in the series and the ordering of terms are discussed in detail on pages 221–231 of Cramèr [12]). The basic functions are

$$\phi^{(k)}(Y) \triangleq \frac{d^k}{dY^k} \left[\frac{1}{\sqrt{2\pi}} e^{-Y^2/2} \right]. \quad (157)$$

We write

$$\begin{aligned}
 p_y(Y) = \phi(Y) &- \left[\frac{\gamma_3}{6} \phi^{(3)}(Y) \right] \\
 &+ \left[\frac{\gamma_4}{4!} \phi^{(4)}(Y) + \frac{10\gamma_3^2}{6!} \phi^{(6)}(Y) \right] \\
 &- \left[\frac{\gamma_5}{5!} \phi^{(5)}(Y) + \frac{35\gamma_3\gamma_4}{7!} \phi^{(7)}(Y) + \frac{280\gamma_3^3}{9!} \phi^{(9)}(Y) \right] \\
 &+ \left[\frac{\gamma_6}{720} \phi^{(6)}(Y) + \left(\frac{\gamma_4^2}{1152} + \frac{\gamma_3\gamma_5}{720} \right) \phi^{(8)}(Y) \right. \\
 &\left. + \frac{\gamma_3^2\gamma_4}{1728} \phi^{(10)}(Y) + \frac{\gamma_3^4}{31104} \phi^{(12)}(Y) \right] + \dots, \quad (158)
 \end{aligned}$$

† This derivation was done originally in [11].

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where

$$\gamma_n \triangleq \frac{d^n/ds^n [\mu(s)]}{[\dot{\mu}(s)]^{n/2}}, \quad n = 3, 4, \dots \quad (159)$$

We see that all of the coefficients can be expressed in terms of $\mu(s)$ and its derivatives. We now substitute (158) into the integral in (149). The result is a sum of integrals of the form

$$I_k(\alpha) = \int_0^\infty \phi^{(k)}(Y) e^{-\alpha Y} dY, \quad (160)$$

where

$$\alpha \triangleq s(\dot{\mu}(s))^{1/2}. \quad (161)$$

Repeated integration by parts gives an expression for $I_k(\alpha)$ in terms of $\text{erfc}_*(\alpha)$. The integrals are

$$I_0(\alpha) = \text{erfc}_*(\alpha) e^{\alpha^2/2} \quad (162)$$

and

$$I_k(\alpha) = \alpha I_{k-1}(\alpha) - \phi^{(k-1)}(0), \quad k \geq 1. \quad (163)$$

If we use just the first term in the series,

$$P_F \simeq P_F^{[1]} \triangleq \exp\left(\mu(s) - s\dot{\mu}(s) + \frac{s^2\ddot{\mu}(s)}{2}\right) \text{erfc}_*\left(s\sqrt{\dot{\mu}(s)}\right) \quad (164)$$

For large $s(\dot{\mu}(s))^{1/2} (\geq 2)$, we may use the approximation to $\text{erfc}_*(X)$ given in Fig. 2.10 of Part I.

$$\text{erfc}_*(X) \simeq \frac{1}{\sqrt{2\pi} X} e^{-X^2/2}, \quad X \geq 2. \quad (165)$$

Then (164) reduces to

$$P_F \simeq P_F^{[1]} \triangleq \frac{1}{\sqrt{2\pi s^2 \dot{\mu}(s)}} e^{\mu(s) - s\dot{\mu}(s)}. \quad (166)$$

This, of course, is the same answer we obtained when the central limit theorem was valid. The second term in the approximation is obtained by using $I_3(\alpha)$ from (163).

$$P_F^{[2]} = -\frac{\gamma_3}{6} e^{\mu(s) - s\dot{\mu}(s)} \left[(s\sqrt{\dot{\mu}(s)})^3 I_0(s\sqrt{\dot{\mu}(s)}) + \frac{1}{\sqrt{2\pi}} (1 - s^2\ddot{\mu}(s)) \right]. \quad (167)$$

Now,

$$I_0(s\sqrt{\dot{\mu}(s)}) = \text{erfc}_*(s\sqrt{\dot{\mu}(s)}) \exp\left(\frac{s^2\ddot{\mu}(s)}{2}\right). \quad (168)$$

In Problem I-2.2.15 on page I-137, we showed that

$$\frac{1}{\sqrt{2\pi} X} \left(1 - \frac{1}{X^2}\right) e^{-X^2/2} < \text{erfc}_*(X) < \frac{1}{\sqrt{2\pi} X} \left(1 - \frac{1}{X^2} + \frac{3}{X^4}\right) e^{-X^2/2}. \quad (169)$$

We can now place an upper bound on the magnitude of $P_F^{[2]}$.

$$\begin{aligned}
 |P_F^{[2]}| &\leq \left| \frac{\gamma_3}{6} e^{\mu(s)-s\dot{\mu}(s)} \left[(s\sqrt{\ddot{\mu}(s)})^3 \frac{1}{\sqrt{2\pi} s\sqrt{\ddot{\mu}(s)}} \left(1 - \frac{1}{s^2\dot{\mu}(s)} + \frac{3}{s^4(\dot{\mu}(s))^2} \right) \right. \right. \\
 &\qquad \qquad \qquad \left. \left. + \frac{1}{\sqrt{2\pi}} (1 - s^2\dot{\mu}(s)) \right] \right| \\
 &= \left| \frac{\gamma_3}{2} \frac{1}{\sqrt{2\pi} s^2 \ddot{\mu}(s)} e^{\mu(s)-s\dot{\mu}(s)} \right| \\
 &= \left| \frac{\gamma_3}{2} \frac{1}{s\sqrt{\ddot{\mu}(s)}} \right| P_{F^*}^{[1]}. \tag{170}
 \end{aligned}$$

Using (159),

$$|P_F^{[2]}| \leq \left| \frac{\mu^{(3)}(s)}{2s[\ddot{\mu}(s)]^2} \right| P_{F^*}^{[1]}. \tag{171}$$

Thus, for any particular $\mu(s)$, we can calculate a bound on the size of the second term in relation to a bound on the first term. By using more terms in the series in (169), we can obtain bounds on the other terms in (158). Notice that this is not a bound on the percentage error in P_F ; it is just a bound on the magnitude of the successive terms. In most of our calculations we shall use just the first-order term $P_F^{[1]}$. We calculated $P_F^{[2]}$ for a number of examples, and it was usually small compared to $P_F^{[1]}$. The bound on $P_F^{[2]}$ is computed for several typical systems in the problems.

To derive an approximate expression for P_M , we go through a similar argument. The starting point is (172), which is obtained from (I-2.465) by a change of variables.

$$P_M = e^{\mu(s)+(1-s)\dot{\mu}(s)} \int_{-\infty}^0 e^{(1-s)\sqrt{\ddot{\mu}(s)}Y} P_y(Y) dY. \tag{172}$$

The first-term approximation is

$$\boxed{P_M \simeq P_M^{[1]} = \left[\exp \left[\mu(s) + (1-s)\dot{\mu}(s) + \frac{(1-s)^2}{2} \ddot{\mu}(s) \right] \right] \text{erfc}_* [(1-s)\sqrt{\ddot{\mu}(s)}],} \tag{173}$$

0 ≤ s ≤ 1.

Using the approximation in (165) gives

$$\boxed{P_M \simeq P_M^{[1]} \triangleq \frac{1}{\sqrt{2\pi(1-s)^2\ddot{\mu}(s)}} e^{\mu(s)+(1-s)\dot{\mu}(s)}, \quad 0 \leq s \leq 1.} \tag{174}$$

The higher-order terms are derived exactly as in the P_F case.

The results in (164), (166), (173), and (174), coupled with the closed-form expression for $\mu(s)$ in (138) and (147), give us the ability to calculate the approximate performance of the optimum test in an efficient manner. A disadvantage of our approach is that for the general case we cannot bound the error in our approximation. Later, we shall obtain bounds for

some special cases and shall see that our first-order approximation is accurate in those cases.

We now return to the problem of calculating $\mu(s)$ and develop an alternative procedure.

2.2.3 An Alternative Expression for $\mu_R(s)$ †

The expressions in (138) and (139) depend on the realizable mean-square estimation error. If we are going to build the optimum receiver using a state-variable realization, we will have $\xi_P(t | s(\cdot), N_0/2)$ available. On the other hand, there are many cases in which we want to compute the performance for a number of systems in order to select one to build. In this case we want an expression for $\mu_R(s)$ that requires the *least* amount of computation. Specifically, we would like to find an expression for $\mu(s)$ that does not require the computation of $\xi_P(t | s(\cdot), N_0/2)$ at *each* point in $[T_i, T_f]$. Whenever the random process has a finite-dimensional state representation, we can find a much simpler expression for $\mu(s)$. The new expression is based on an alternative computation of the integral‡

$$\frac{2}{N_0} \int_{T_i}^{T_f} \xi_P \left(t | s(\cdot), \frac{N_0}{2} \right) dt. \quad (175)$$

Derivation. We use the state model in (76)–(80),

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{G}(t)u(t), \quad (176)$$

$$s(t) = \mathbf{C}(t)\mathbf{x}(t), \quad (177)$$

and the initial conditions

$$E[\mathbf{x}(T_i)] = \mathbf{0}, \quad (178)$$

$$E[\mathbf{x}(T_i)\mathbf{x}^T(T_i)] = \boldsymbol{\xi}_P(T_i) \triangleq \mathbf{P}_0. \quad (179)$$

Recall that the error covariance matrix is

$$\boldsymbol{\xi}_P(t) = E[(\mathbf{x}(t) - \hat{\mathbf{x}}(t))(\mathbf{x}^T(t) - \hat{\mathbf{x}}^T(t))]. \quad (180)$$

Using (177),

$$\xi_P \left(t | s(\cdot), \frac{N_0}{2} \right) = \mathbf{C}(t)\boldsymbol{\xi}_P(t)\mathbf{C}^T(t). \quad (181)$$

We first recall several results from Chapter I-6 and introduce some simplifying notation. From Property 16 on page I-545, we know that the variance equation (84) can be related to two simultaneous linear equations (I-6.335 or I-6.336),

$$\frac{d}{dt} \begin{bmatrix} \mathbf{v}_1(t) \\ \mathbf{v}_2(t) \end{bmatrix} = \begin{bmatrix} \mathbf{F}(t) & \mathbf{G}(t)\mathbf{Q}\mathbf{G}^T(t) \\ \mathbf{C}^T(t)\frac{2}{N_0}\mathbf{C}(t) & -\mathbf{F}^T(t) \end{bmatrix} \begin{bmatrix} \mathbf{v}_1(t) \\ \mathbf{v}_2(t) \end{bmatrix}. \quad (182)$$

† This section may be omitted on the first reading.

‡ This derivation is due to Collins [13].

The transition matrix of (182), $\mathbf{T}(t, T_i)$, satisfies the differential equation

$$\frac{d}{dt} [\mathbf{T}(t, T_i)] = \begin{bmatrix} \mathbf{F}(t) & \mathbf{G}(t)\mathbf{Q}\mathbf{G}^T(t) \\ \mathbf{C}^T(t)\frac{2}{N_0}\mathbf{C}(t) & -\mathbf{F}^T(t) \end{bmatrix} \mathbf{T}(t, T_i), \quad (183)$$

with initial conditions $\mathbf{T}(T_i, T_i) = \mathbf{I}$. In addition, from (I-6.338), the error covariance matrix is given by

$$\xi_P(t) = [\mathbf{T}_{11}(t, T_i)\xi_P(T_i) + \mathbf{T}_{12}(t, T_i)][\mathbf{T}_{21}(t, T_i)\xi_P(T_i) + \mathbf{T}_{22}(t, T_i)]^{-1}. \quad (184)$$

The inverse of the second matrix always exists because it is the transition matrix of a linear dynamical system. For simplicity, we define two new matrices,

$$\mathbf{\Gamma}_1(t) = \mathbf{T}_{11}(t, T_i)\xi_P(T_i) + \mathbf{T}_{12}(t, T_i), \quad (185)$$

$$\mathbf{\Gamma}_2(t) = \mathbf{T}_{21}(t, T_i)\xi_P(T_i) + \mathbf{T}_{22}(t, T_i).$$

Thus,

$$\xi_P(t) = \mathbf{\Gamma}_1(t)\mathbf{\Gamma}_2^{-1}(t), \quad (186)$$

and $\mathbf{\Gamma}_1(t)$ and $\mathbf{\Gamma}_2(t)$ satisfy

$$\frac{d}{dt} \begin{bmatrix} \mathbf{\Gamma}_1(t) \\ \mathbf{\Gamma}_2(t) \end{bmatrix} = \begin{bmatrix} \mathbf{F}(t) & \mathbf{G}(t)\mathbf{Q}\mathbf{G}^T(t) \\ \mathbf{C}^T(t)\frac{2}{N_0}\mathbf{C}(t) & -\mathbf{F}^T(t) \end{bmatrix} \begin{bmatrix} \mathbf{\Gamma}_1(t) \\ \mathbf{\Gamma}_2(t) \end{bmatrix}, \quad (187)$$

with initial conditions

$$\mathbf{\Gamma}_1(T_i) = \xi_P(T_i) \triangleq \mathbf{P}_0 \quad (188)$$

and

$$\mathbf{\Gamma}_2(T_i) = \mathbf{I}. \quad (189)$$

We now proceed with the derivation. Multiplying both sides of (181) by $2/N_0$ and integrating gives

$$\begin{aligned} \frac{2}{N_0} \int_{T_i}^{T_f} \xi_P \left(t \mid s(\cdot), \frac{N_0}{2} \right) dt &= \frac{2}{N_0} \int_{T_i}^{T_f} \mathbf{C}(t)\xi_P(t)\mathbf{C}^T(t) dt \\ &= \frac{2}{N_0} \int_{T_i}^{T_f} \mathbf{C}(t)[\mathbf{\Gamma}_1(t)\mathbf{\Gamma}_2^{-1}(t)]\mathbf{C}^T(t) dt. \end{aligned} \quad (190)$$

Now recall that

$$\mathbf{x}^T\mathbf{B}\mathbf{x} = \text{Tr} [\mathbf{x}\mathbf{x}^T\mathbf{B}] \quad (191)$$

for any vector \mathbf{x} . Thus,

$$\frac{2}{N_0} \int_{T_i}^{T_f} \xi_P \left(t \mid s(\cdot), \frac{N_0}{2} \right) dt = \int_{T_i}^{T_f} \text{Tr} \left[\left(\mathbf{C}^T(t)\frac{2}{N_0}\mathbf{C}(t)\mathbf{\Gamma}_1(t)\mathbf{\Gamma}_2^{-1}(t) \right) \right] dt \quad (192)$$

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Using (187) to eliminate $\mathbf{\Gamma}_1(t)$, we have

$$\begin{aligned} \frac{2}{N_0} \int_{T_i}^{T_f} \xi_P \left(t \mid s(\cdot), \frac{N_0}{2} \right) dt &= \int_{T_i}^{T_f} \text{Tr} \left[\left(\frac{d\mathbf{\Gamma}_2(t)}{dt} + \mathbf{F}^T(t)\mathbf{\Gamma}_2(t) \right) \mathbf{\Gamma}_2^{-1}(t) \right] dt \\ &= \int_{T_i}^{T_f} \text{Tr} \left[\frac{d\mathbf{\Gamma}_2(t)}{dt} \mathbf{\Gamma}_2^{-1}(t) \right] dt + \int_{T_i}^{T_f} \text{Tr} [\mathbf{F}^T(t)] dt \\ &= \int_{T_i}^{T_f} \text{Tr} [\mathbf{\Gamma}_2^{-1}(t) d\mathbf{\Gamma}_2(t)] + \int_{T_i}^{T_f} \text{Tr} [\mathbf{F}(t)] dt. \end{aligned} \quad (193)$$

From (9.31) of [14],

$$\begin{aligned} \int_{T_i}^{T_f} \text{Tr} [\mathbf{\Gamma}_2^{-1}(t) d\mathbf{\Gamma}_2(t)] &= \int_{T_i}^{T_f} d[\ln \det \mathbf{\Gamma}_2(t)] \\ &= \ln \det \mathbf{\Gamma}_2(T_f) - \ln \det \mathbf{\Gamma}_2(T_i) \\ &= \ln \det \mathbf{\Gamma}_2(T_f). \end{aligned} \quad (194)$$

Thus,

$$\boxed{\frac{2}{N_0} \int_{T_i}^{T_f} \xi_P \left(t \mid s(\cdot), \frac{N_0}{2} \right) dt = \ln \det \mathbf{\Gamma}_2(T_f) + \int_{T_i}^{T_f} \text{Tr} [\mathbf{F}(t)] dt,} \quad (195)$$

which is the desired result.†

We see that we have to compute $\mathbf{\Gamma}_2(T_f)$ at only one point rather than over an entire interval. This is particularly important when an analytic expression for $\mathbf{\Gamma}_2(T_f)$ is available. If we have to find $\mathbf{\Gamma}_2(T_f)$ by numerically integrating (187), there is no significant saving in computation.

The expression in (195) is the desired result. In the next section we consider a simple example to illustrate the application of the result we have derived.

2.2.4 Performance for a Typical System

In this section we analyze the performance of the system described in the example of Section 2.1.5. It provides an immediate application of the performance results we have just developed. In Chapter 4, we shall consider the performance for a variety of problems.

Example. We consider the system described in the example on page 26. We assume that the channel process $b(t)$ is a stationary zero-mean Gaussian process with a spectrum

$$S_b(\omega) = \frac{2k\sigma_b^2}{\omega^2 + k^2}. \quad (196)$$

† This result was first obtained by Baggeroer as a by-product of his integral equation work [15]. See Siegert [16] for a related result.

We assume that the transmitted signal is a rectangular pulse,

$$f(t) = \begin{cases} \sqrt{\frac{\bar{E}_t}{T}}, & 0 \leq t \leq T, \\ 0, & \text{elsewhere} \end{cases} \quad (197)$$

As we pointed out in our earlier discussion, this channel model has many of the characteristics of models of actual channels that we shall study in detail in Chapter 10. The optimum receiver is shown in Fig. 2.10. To illustrate the techniques involved, we calculate $\mu(s)$ using both (138) and (195). [Notice that $\mu_D(s)$ is zero.] To use (138), we need the realizable mean-square filtering error. The result for this particular spectrum was derived in Example 1 on pages I-546–I-548. From (I-6.353),

$$\xi_P \left(t \mid s(\cdot), \frac{N_0}{2} \right) = \frac{2\bar{E}_r}{T} \frac{1}{(1+\alpha)} \left\{ \frac{1 - [(1-\alpha)/(1+\alpha)]e^{-2kat}}{1 - [(1-\alpha)^2/(1+\alpha)^2]e^{-2kat}} \right\}, \quad 0 \leq t \leq T, \quad (198)$$

where

$$\bar{E}_r \triangleq \sigma_b^2 E_t \quad (199)$$

is the average received energy and

$$\alpha \triangleq \sqrt{1 + \frac{4\bar{E}_r}{kTN_0}}. \quad (200)$$

Integrating, we obtain

$$\int_{T_i}^{T_f} \xi_P \left(t \mid s(\cdot), \frac{N_0}{2} \right) dt = \frac{N_0}{2} \left\{ \ln \left[\frac{(1+\alpha)^2 e^{2kaT} - (1-\alpha)^2}{4\alpha} \right] - (\alpha+1)kT \right\}. \quad (201)$$

We now derive (201) using the expression in (195). The necessary quantities are

$$\begin{aligned} \mathbf{F}(t) &= -k, \\ \mathbf{G}(t)\mathbf{Q}\mathbf{G}^T(t) &= 2k\sigma_b^2, \\ \mathbf{C}(t) &= 1, \\ \mathbf{P}_0 &= \sigma_b^2. \end{aligned} \quad (202)$$

The transition matrix is given in (I-6.351) as

$$\mathbf{T}(T + T_i, T_i) = \begin{bmatrix} \cosh(\gamma T) - \frac{k}{\gamma} \sinh(\gamma T) & \frac{2k\sigma_b^2}{\gamma} \sinh(\gamma T) \\ \frac{2}{N_0\gamma} \sinh(\gamma T) & \cosh(\gamma T) + \frac{k}{\gamma} \sinh(\gamma T) \end{bmatrix}, \quad (203)$$

where

$$\gamma = k \sqrt{1 + \frac{4\sigma_b^2 E_t}{kN_0}} = k\alpha \quad (204)$$

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From the definition in (185),

$$\begin{aligned} \Gamma_2(T_f) &= \frac{2\sigma_b^2}{N_0\gamma} \sinh(\gamma T) + \cosh(\gamma T) + \frac{k}{\gamma} \sinh(\gamma T) \\ &= \cosh(\gamma T) + \frac{k}{\gamma} \left(1 + \frac{2\sigma_b^2}{kN_0}\right) \sinh(\gamma T) \\ &= e^{-k\alpha T} \left[\frac{1 - [(\alpha + 1)^2/(\alpha - 1)^2] e^{2k\alpha T}}{1 - (\alpha + 1)^2/(\alpha - 1)^2} \right]. \end{aligned} \quad (205)$$

Using (202) and (205) in (195), we have

$$\frac{2}{N_0} \int_{T_i}^{T_f} \xi_{I'} \left(t \mid s(\cdot), \frac{N_0}{2} dt \right) = \ln \left[\frac{(\alpha - 1)^2 - (\alpha + 1)^2 e^{2k\alpha T}}{(\alpha - 1)^2 - (\alpha + 1)^2} \right] - k(\alpha + 1)T, \quad (206)$$

which is identical with (201). To get the second term in (138), we define

$$\alpha_s \triangleq \sqrt{1 + \frac{4\bar{E}_r(1-s)}{kTN_0}} \quad (207)$$

and replace α by α_s in (201). Then

$$\mu(s) = \frac{1-s}{2} \left\{ \ln \left[\frac{[(1+\alpha)^2 e^{2kT\alpha} - (1-\alpha)^2] \alpha_s}{[(1+\alpha_s)^2 e^{2kT\alpha_s} - (1-\alpha_s)^2] \alpha} \right] - \frac{4\bar{E}_r}{N_0} \left[\frac{1}{\alpha - 1} - \frac{1}{\alpha_s - 1} \right] \right\}. \quad (208)$$

We see that $\mu(s)$ (and therefore the error expression) is a function of two quantities. \bar{E}_r/N_0 , the average energy divided by the noise spectral height and the kT product, The 3-db-bandwidth of the spectrum is k radians per second, so that kT is a time-bandwidth product.

To use the approximate error expressions in (154) and (155), we find $\dot{\mu}(s)$ and $\ddot{\mu}(s)$ from (208). The simplest way to display the results is to fix P_F and plot P_M versus kT for various values of $2\bar{E}_r/N_0$. We shall not carry out this calculation at this point. In Example 1 of Chapter 4, we study this problem again from a different viewpoint. At that time we plot a detailed set of performance curves (see Figs. 4.7-4.9 and Problem 4.1.21).

This example illustrates the application of our results to a typical problem of interest. Other interesting cases are developed in the problems. We now summarize the results of the Chapter.

2.3 SUMMARY: SIMPLE BINARY DETECTION

In Sections 2.1 and 2.2 we considered in detail the problem of detecting a sample function of a Gaussian random process in the presence of additive white Gaussian noise. In Section 2.1 we derived the likelihood ratio test and discussed various receiver configurations that could be used to implement the test. The test is

$$l_R + l_D + l_B^{[1]} + l_B^{[2]} \underset{H_0}{\overset{H_1}{\geq}} \ln \eta, \quad (209)$$

where

$$l_R = \frac{1}{N_0} \iint_{T_i}^{T_f} r(t)h_1(t, u)r(u) dt du, \quad (210)$$

$$l_D = \int_{T_i}^{T_f} g_1(u)r(u) du, \quad (211)$$

$$l_B^{[1]} = -\frac{1}{N_0} \int_{T_i}^{T_f} \xi_{P_s}(t) dt, \quad (212)$$

$$l_B^{[2]} = -\frac{1}{2} \int_{T_i}^{T_f} g_1(u)m(u) du. \quad (213)$$

The operation needed to generate l_R was a quadratic operation. The receiver structures illustrated different schemes for computing l_R . The three receivers of most importance in practice are the following:

1. The estimator-correlator receiver (Canonical Realization No. 1).
2. The filter-squarer receiver (Canonical Realization No. 3).
3. The optimum realizable filter receiver (Canonical Realizations Nos. 4 and 4S).

The most practical realization will depend on the particular problem of interest.

In Section 2.2 we considered the performance of the optimum receiver. In general, it was not possible to find the probability density of l_R on the two hypotheses. By extending the techniques of Chapter I-2, we were able to find good approximations to the error probabilities. The key function in this analysis was $\mu(s)$.

$$\mu(s) = \mu_R(s) + \mu_D(s), \quad (214)$$

where

$$\mu_R(s) = \frac{1-s}{N_0} \int_{T_i}^{T_f} dt \left[\xi_P \left(t \left| s(\cdot), \frac{N_0}{2} \right. \right) - \xi_P \left(t \left| s(\cdot), \frac{N_0}{2(1-s)} \right. \right) \right] \quad (215)$$

and

$$\mu_D(s) = -\frac{s}{2} \int_{T_i}^{T_f} m(t)g \left(t \left| \frac{N_0}{2(1-s)} \right. \right) dt. \quad (216)$$

The performance was related to $\mu(s)$ through the Chernoff bounds,

$$\begin{aligned} P_F &\leq e^{\mu(s) - s\hat{\mu}(s)}, \\ P_M &\leq e^{\mu(s) + (1-s)\hat{\mu}(s)}, \quad 0 \leq s \leq 1, \end{aligned} \quad (217)$$

where

$$\dot{\mu}(s) = \gamma = \ln \eta. \quad (218)$$

An approximation to the performance was obtained by an Edgeworth series expansion,

$$P_F \simeq \frac{1}{\sqrt{2\pi s^2 \ddot{\mu}(s)}} e^{\mu(s) - s\dot{\mu}(s)}, \quad (219)$$

$$P_M \simeq \frac{1}{\sqrt{2\pi(1-s)^2 \ddot{\mu}(s)}} e^{\mu(s) + (1-s)\dot{\mu}(s)}, \quad 0 \leq s \leq 1. \quad (220)$$

By varying s , we could obtain a segment of an approximate receiver operating characteristic.

We see that both the receiver structure and performance are closely related to the optimum linear filtering results of Chapter I-6. This close connection is important because it means that all of our detailed studies of optimum linear filters are useful for the Gaussian detection problem.

At this point, we have developed a set of important results but have not yet applied them to specific physical problems. We continue this development in Chapter 4, where we consider three important classes of physical problems and obtain specific results for a number of interesting examples. Many readers will find it helpful to study Section 4.1.1 before reading Chapter 3 in detail.

2.4 PROBLEMS

P.2.1 Optimum Receivers

Problem 2.1.1. Consider the model described by (1)–(6). Assume that $m(t)$ is not zero. Derive an estimator-correlator receiver analogous to that in Fig. 2.3 for this case.

Problem 2.1.2 Consider the function $h_1(t, t | z)$, which is specified by the equation

$$zh_1(t, u | z) + \int_{T_i}^{T_f} h_1(t, y | z) K_s(y, u) dy = K_s(t, u), \quad T_i \leq t, u \leq T_f.$$

Verify that (75) is true. [*Hint*: Recall (I-3.154).]

Problem 2.1.3.

1. Consider the waveform

$$r(\tau) = n_c(\tau) + w(\tau), \quad T_i \leq \tau \leq t,$$

where $n_c(\tau)$ can be generated as the output of a dynamic system,

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{G}(t)u(t),$$

$$n_c(t) = \mathbf{C}(t)\mathbf{x}(t),$$

driven by a statistically independent white noise $u(t)$. Denote the MMSE realizable estimate of $n_c(\tau)$ as $\hat{n}_c(\tau)$. Prove that the process

$$r_{\star}(t) \triangleq r(t) - \hat{n}_c(t) = r(t) - \mathbf{C}(t)\hat{\mathbf{x}}(t)$$

is white.

2. Use the result of part 1 to derive the receiver in Fig. 2.11 by inspection.

Problem 2.1.4. Read Problem I-6.6.4 and the Appendix to Part II (sect. A.4-A.6). With this background derive a procedure for generating l_R using unrealizable filters expressed in terms of vector-differential equations. For simplicity, assume zero means.

Problem 2.1.5. The received waveforms on the two hypotheses are

$$\begin{aligned} r(t) &= s(t) + w(t), & 0 \leq t \leq T: H_1, \\ r(t) &= w(t), & 0 \leq t \leq T: H_0. \end{aligned}$$

The process $w(t)$ is a sample function of a white Gaussian random process with spectral height $N_0/2$. The process $s(t)$ is a Wiener process that is statistically independent of $w(t)$.

$$\begin{aligned} s(0) &= 0, \\ E[s^2(t)] &= \sigma^2 t. \end{aligned}$$

1. Find the likelihood ratio test.
2. Draw a realization of the optimum receiver. Specify all components completely.

Problem 2.1.6. The received waveforms on the two hypotheses are

$$\begin{aligned} r(t) &= s(t) + w(t), & 0 \leq t \leq T: H_1, \\ r(t) &= w(t), & 0 \leq t \leq T: H_0. \end{aligned}$$

The process $w(t)$ is a sample function of a white Gaussian random process with spectral height $N_0/2$. The signal $s(t)$ is a sample function of a Gaussian random process and can be written as

$$s(t) = at, \quad 0 \leq t,$$

where a is a zero-mean Gaussian random variable with variance σ_a^2 . Find the optimum receiver. Specify all components completely.

Problem 2.1.7. Repeat Problem 2.1.6 for the case in which

$$s(t) = at + b, \quad 0 \leq t,$$

where a and b are statistically independent, zero-mean Gaussian random variables with variances σ_a^2 and σ_b^2 , respectively.

Problem 2.1.8.

1. Repeat Problem 2.1.7 for the case in which a and b are statistically independent Gaussian random variables with means m_a and m_b and variances σ_a^2 and σ_b^2 , respectively.
2. Consider four special cases of part 1:

- (i) $m_a = 0$,
- (ii) $m_b = 0$,
- (iii) $\sigma_a^2 = 0$,
- (iv) $\sigma_b^2 = 0$.

Verify that the receiver for each of these special cases reduces to the correct structure.

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Problem 2.1.9. Consider the model in Problem 2.1.6. Assume that $s(t)$ is a piecewise constant waveform,

$$s(t) = \begin{cases} b_1, & 0 < t \leq T_0, \\ b_2, & T_0 < t \leq 2T_0, \\ b_3, & 2T_0 < t \leq 3T_0, \\ \vdots & \\ \vdots & \\ b_n, & (n-1)T_0 < t \leq nT_0, \end{cases}$$

The b_i are statistically independent, zero-mean Gaussian random variables with variances equal to σ_b^2 . Find the optimum receiver.

Problem 2.1.10. Consider the model in Problem 2.1.6. Assume

$$s(t) = \sum_{i=1}^K a_i t^i, \quad 0 \leq t,$$

where the a_i are statistically independent random variables with variances σ_i^2 . Find the optimum receiver.

Problem 2.1.11. Re-examine Problems 2.1.6 through 2.1.10. If you implemented the optimum receiver using Canonical Realization No. 4S, go back and find an easier procedure.

Problem 2.1.12 Consider the model in Problem 2.1.5. Assume that $s(t)$ is a segment of a stationary zero-mean Gaussian process with an n th-order Butterworth spectrum

$$S_s(\omega; n) = \frac{2nP}{k} \frac{\sin(\pi/2n)}{(\omega/k)^{2n} + 1}, \quad n = 1, 2, \dots$$

1. Review the state representation for these processes in Example 2 on page I-548. Make certain that you understand the choice of initial conditions.

2. Draw a block diagram of the optimum receiver.

Problem 2.1.13. From (31), we have

$$l_R = \frac{1}{N_0} \int_{T_i}^{T_f} \int_{T_i}^{T_f} r(t)h(t, u)r(u) dt du.$$

One possible factoring of $h(t, u)$ was given in (45). An alternative factoring is

$$h(t, u) = \int_{\Omega_T} g_1(z, t)g_1(z, u) dz, \quad T_i \leq t, u \leq T_f \quad (\text{P.1})$$

where

$$[T_i, T_f] \subset \Omega_T.$$

1. Explain the physical significance of this operation. Remember that our model assumes that $r(t)$ is only observed over the interval $[T_i, T_f]$.

2. Give an example in which the factoring indicated in (P.1) is easier than that in the text.

Problem 2.1.14. Consider the expression for l_R in (31). We want to decompose $h_1(t, u)$ in terms of two new functions, $k_1(T_f, z)$ and $k_2(z, t)$, that satisfy the equation

$$h_1(t, u) = \int_{T_i}^{T_f} k_1(T_f, z)k_2(z, t)k_2(z, u) dz, \quad T_i \leq t, u \leq T_f.$$

1. Draw a block diagram of the optimum receiver in terms of these new functions.
2. Give an example in which this realization would be easier to find than Canonical Realization No. 3.
3. Discuss the decomposition

$$h_1(t, u) = \int_{\Omega_T} k_1(T_f, z)k_2(z, t)k_2(z, u) dz, \quad T_i \leq t, u \leq T_f, [T_i, T_f] \subset \Omega_T.$$

Problem 2.1.15. From (86) and (87),

$$l_R = \frac{1}{N_0} \int_0^T [2r(t)\hat{s}(t) - \hat{s}^2(t)] dt. \tag{P.1}$$

Consider the case in which

$$\dot{\hat{s}}(t) + a_0\hat{s}(t) = b_0r(t), \quad 0 \leq t$$

and

$$\hat{s}(0) = 0. \tag{P.2}$$

1. Implement the optimum receiver in the form shown in Fig. P.2.1. Specify the time-invariant filter completely.
2. Discuss the case in which

$$\ddot{\hat{s}}(t) + a_1\dot{\hat{s}}(t) + a_0\hat{s}(t) = b_0r(t).$$

Suggest some possible modifications to the structure in Fig. P.2.1.

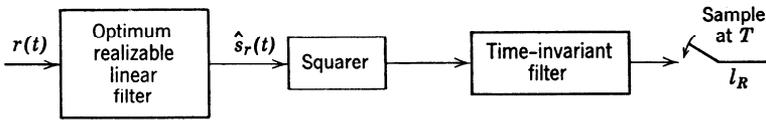


Fig. P.2.1

3. Extend your discussion to the general case in which the estimate $\hat{s}(t)$ is described by an n th-order differential equation with constant coefficients.

Problem 2.1.16. On both hypotheses there is a sample function of a zero-mean Gaussian white noise process with spectral height $N_0/2$. On H_1 , the signal is equally likely to be a sample function from any one of M zero-mean Gaussian processes. We denote the covariance function of the i th process as $K_{s_i}(t, u)$, $i = 1, \dots, M$. Thus,

$$r(t) = s_i(t) + w(t), \quad T_i \leq t \leq T_f, \quad \text{with probability } \frac{1}{M} : H_1, \quad i = 1, \dots, M.$$

$$r(t) = w(t), \quad T_i \leq t \leq T_f : H_0.$$

Find the optimum Bayes receiver to decide which hypothesis is true.

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Problem 2.1.17. Consider the vector version of the simple binary detection problem. The received waveforms on the two hypotheses are

$$\begin{aligned} \mathbf{r}(t) &= \mathbf{s}(t) + \mathbf{w}(t), & T_i \leq t \leq T_f: H_1, \\ &= \mathbf{w}(t), & T_i \leq t \leq T_f: H_0, \end{aligned} \quad (\text{P.1})$$

where $\mathbf{s}(t)$ and $\mathbf{w}(t)$ are sample functions of zero-mean, statistically independent, N -dimensional, vector Gaussian processes with covariance matrices

$$\mathbf{K}_s(t, u) \triangleq E[\mathbf{s}(t)\mathbf{s}^T(u)] \quad (\text{P.2})$$

and

$$\mathbf{K}_w(t, u) \triangleq E[\mathbf{w}(t)\mathbf{w}^T(u)] = \frac{N_0}{2} \delta(t - u)\mathbf{I}. \quad (\text{P.3})$$

1. Derive the optimum receiver for this problem. (*Hint*: Review Sections I-3.7 and I-4.5.)
2. Derive the equations specifying the four canonical realizations. Draw a block diagram of the four realizations.
3. Consider the special case in which

$$\mathbf{K}_s(t, u) = K_s(t, u)\mathbf{I}. \quad (\text{P.4})$$

Explain what the condition in (P.4) means. Give a physical situation that would lead to this condition. Simplify the optimum receiver in part 1.

4. Consider the special case in which

$$\mathbf{K}_s(t, u) = K_s(t, u) \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & & 1 \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ 1 & & & 1 \end{bmatrix}. \quad (\text{P.5})$$

Repeat part 3.

Problem 2.1.18. Consider the model in Problem 2.1.17. The covariance of $\mathbf{w}(t)$ is

$$\mathbf{K}_w(t, u) = \mathbf{N} \delta(t - u)\mathbf{I},$$

where \mathbf{N} is a nonsingular matrix.

1. Repeat parts 1 and 2 of Problem 2.1.17. (*Hint*: Review Problem I-4.5.2 on page I-408.)
2. Why do we assume that \mathbf{N} is nonsingular?
3. Consider the special case in which

$$\mathbf{K}_s(t, u) = K_s(t, u)\mathbf{I}$$

and \mathbf{N} is diagonal. Simplify the results in part 1.

Problem 2.1.19. Consider the model in Problem 2.1.17. Assume

$$E[\mathbf{s}(t)] = \mathbf{m}(t).$$

All of the other assumptions in Problem 2.1.17 are still valid. Repeat Problem 2.1.17

Problem 2.1.20. In Section 2.1.5 we considered a simple multiplicative channel. A more realistic channel model is the Rayleigh channel model that we encountered previously in Section I-4.4.2 and Chapter II-8. We shall study it in detail in Chapter 10.

On H_1 we transmit a bandpass signal,

$$s_i(t) \triangleq \sqrt{2P} f(t) \cos \omega_c t,$$

where $f(t)$ is a slowly varying function (the envelope of the signal). The received signal is

$$r(t) = \sqrt{2P} b_1(t) f(t) \cos \omega_c t + \sqrt{2P} b_2(t) f(t) \sin \omega_c t + w(t), \quad T_i \leq t \leq T_f; H_1.$$

The channel processes $b_1(t)$ and $b_2(t)$ are statistically independent, zero-mean Gaussian processes whose covariance functions are $K_b(t, u)$. The additive noise $w(t)$ is a sample function of a statistically independent, zero-mean Gaussian process with spectral height $N_0/2$. The channel processes vary slowly compared to ω_c . On H_0 , only white noise is present.

1. Derive the optimum receiver for this model of the Rayleigh channel.
2. Draw a filter-squarer realization for the optimum receiver.
3. Draw a state-variable realization of the optimum receiver. Assume that

$$S_b(\omega) = \frac{2k\sigma_b^2}{\omega^2 + k^2}.$$

Problem 2.1.21. The model for a Rician channel is the same as that in Problem 2.1.19, except that

$$E[b_1(t)] = m$$

instead of zero. Repeat Problem 2.1.19 for this case.

P.2.2. Performance

Problem 2.2.1. Consider the problem of evaluating $\mu_D(s)$, which is given by (135) or (147). Assume that $s(t)$ has a finite-dimensional state representation. Define

$$\mu_D(s, T) = -\frac{s}{2} \int_0^T m(x) g\left(x \mid \frac{N_0}{2(1-s)}\right) dx.$$

Find a finite-dimensional dynamic system whose output is $\mu_D(s, T)$.

Problem 2.2.2. Consider the model in the example in Section 2.2.4. Assume that

$$E[b(t)] = m$$

instead of zero. Evaluate $\mu_D(s)$ for this problem. [Hint: If you use (147), review pages I-320 and I-390.]

Problem 2.2.3.

1. Consider the model in Problem 2.1.5. Evaluate $\mu(s)$ for this system.
2. Define

$$\gamma \triangleq \sqrt{\frac{2\sigma^2}{N_0}}.$$

Simplify the expression in part 1 for the case in which $\gamma T \gg 1$.

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Problem 2.2.4 (continuation). Use the expression for $\mu(s)$ in part 2 of Problem 2.2.3. Evaluate $P_F^{[2]}$ and $P_M^{[2]}$ [see (167)]. Compare their magnitude with that of $P_F^{[1]}$ and $P_M^{[1]}$.

Problem 2.2.5. Consider the model in Problem 2.1.5. Assume that

$$r(t) = s(t) + m(t) + w(t), \quad 0 \leq t \leq T,$$

$$r(t) = w(t), \quad 0 \leq t \leq T,$$

where $m(t)$ is a deterministic function. The processes $s(t)$ and $w(t)$ are as described in Problem 2.1.5. Evaluate $\mu_D(s)$ for this model.

Problem 2.2.6.

1. Evaluate $\mu(s)$ for the system in Problem 2.1.6.
2. Plot the result as a function of s .
3. Find P_F and P_D .

Problem 2.2.7. Evaluate $\mu(s)$ for the system in Problem 2.1.7.

Problem 2.2.8. Evaluate $\mu(s)$ for the system in Problem 2.1.8.

Problem 2.2.9.

1. Evaluate $\mu(s)$ for the system in Problem 2.1.9.
2. Evaluate P_F and P_D .

Problem 2.2.10. Consider the system in Problem 2.1.17.

1. Assume that (P.4) in part 3 is valid. Find $\mu(s)$ for this special case.
2. Assume that (P.5) in part 4 is valid. Find $\mu(s)$ for this special case.
3. Derive an expression for $\mu(s)$ for the general case.

Problem 2.2.11. Consider the system in Problem 2.1.19. Find an expression for $\mu_D(s)$ for this system.

Problem 2.2.12. Find $\mu(s)$ for the Rayleigh channel model in Problem 2.1.20.

Problem 2.2.13. Find $\mu(s)$ for the Rician channel model in Problem 2.1.21.

REFERENCES

- [1] R. Price, "Statistical Theory Applied to Communication through Multipath Disturbances," Massachusetts Institute of Technology Research Laboratory of Electronics, Tech. Rept. 266, September 3, 1953.
- [2] R. Price, "The Detection of Signals Perturbed by Scatter and Noise," IRE Trans. PGIT-4, 163-170 (Sept. 1954).
- [3] R. Price, "Notes on Ideal Receivers for Scatter Multipath," Group Rept. 34-39, Lincoln Laboratory, Massachusetts Institute of Technology, May 12, 1955.
- [4] R. Price, "Optimum Detection of Random Signals in Noise, with Application to Scatter-Multipath Communication. I," IRE Trans. PGIT-6, 125-135 (Dec. 1956).
- [5] F. Schewpe, "Evaluation of Likelihood Functions for Gaussian Signals," IEEE Trans. IT-11, No. 1, 61-70 (Jan. 1965).
- [6] R. E. Kalman and R. S. Bucy, "New Results in Linear Filtering and Prediction Theory," ASME J. Basic Eng., 83, 95-108 (March 1961).
- [7] L. D. Collins, "An Expression for $\partial h_0(s, \tau: t) / \partial t$," Detection and Estimation Theory Group Internal Memorandum IM-LDC-6, Massachusetts Institute of Technology, April 1966.

- [8] W. Lovitt, *Linear Integral Equations*, Dover Publications, New York, 1924.
- [9] L. D. Collins, "Realizable Whitening Filters and State-Variable Realizations," *IEEE Proc.* **56**, No. 1, 100–101 (Jan. 1968).
- [10] W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. II, John Wiley, New York, 1966.
- [11] L. D. Collins, "Asymptotic Approximations to the Error Probability for Detecting Gaussian Signals," Massachusetts Institute of Technology, Department of Electrical Engineering, Sc.D. Thesis Proposal, January 1968.
- [12] H. Cramèr, *Mathematical Methods in Statistics*, Princeton University Press, Princeton, N.J., 1946.
- [13] L. D. Collins, "Closed-Form Expressions for the Fredholm Determinant for State-Variable Covariance Functions," *IEEE Proc.* **56**, No. 4 (April 1968).
- [14] M. Athans and F. C. Schweppe, "Gradient Matrices and Matrix Calculations," Technical Note 1965–53, Lincoln Laboratory, Massachusetts Institute of Technology, 1965.
- [15] A. B. Baggeroer, "A State-Variable Approach to the Solution of Fredholm Integral Equations," November 15, 1967.
- [16] A. J. F. Siegert, "A Systematic Approach to a Class of Problems in the Theory of Noise and Other Random Phenomena. II. Examples," *IRE Trans. IT-3*, No. 1, 38–43 (March 1957).
- [17] D. Middleton, "On the Detection of Stochastic Signals in Additive Normal Noise. I," *IRE Trans. Information Theory IT-3*, 86–121 (June 1957).
- [18] D. Middleton, "On the Detection of Stochastic Signals in Additive Normal Noise. II," *IRE Trans. Information Theory IT-6*, 349–360 (June 1960).
- [19] D. Middleton, "On Singular and Nonsingular Optimum (Bayes) Tests for the Detection of Normal Stochastic Signals in Normal Noise," *IRE Trans. Information Theory IT-7*, 105–113 (April 1961).
- [20] D. Middleton, *Introduction to Statistical Communication Theory*, McGraw-Hill, New York, 1960.
- [21] R. L. Stratonovich and Y. G. Sosulin, "Optimal Detection of a Markov Process in Noise," *Eng. Cybernet.* **6**, 7–19 (Oct. 1964).
- [22] R. L. Stratonovich and Y. G. Sosulin, "Optimal Detection of a Diffusion Process in White Noise," *Radio Eng. Electron. Phys.* **10**, 704–713 (May 1965).
- [23] R. L. Stratonovich and Y. G. Sosulin, "Optimum Reception of Signals in Non-Gaussian Noise," *Radio Eng. Electron. Phys.* **11**, 497–507 (April 1966).
- [24] Y. G. Sosulin, "Optimum Extraction of Non-Gaussian Signals in Noise," *Radio Eng. Electron. Phys.* **12**, 89–97 (Jan. 1967).
- [25] R. L. Stratonovich, "A New Representation for Stochastic Integrals," *J. SIAM Control* **4**, 362–371 (1966).
- [26] J. L. Doob, *Stochastic Processes*, Wiley, New York, 1953.
- [27] K. Itô, *Lectures on Stochastic Processes*, Tata Institute for Fundamental Research, Bombay, 1961.
- [28] T. Duncan, "Probability Densities for Diffusion Processes with Applications to Nonlinear Filtering Theory and Detection Theory," *Information Control* **13**, 62–74 (July 1968).
- [29] T. Kailath and P. A. Frost, "Mathematical Modeling of Stochastic Processes," *JACC Control Symposium* (1969).
- [30] T. Kailath, "A General Likelihood-Ratio Formula for Random Signals in Noise," *IEEE Trans. Information Theory IT-5*, No. 3, 350–361 (May 1969).