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General Binary Detection: Gaussian Processes

In this Chapter we generalize the model of Chapter 2 to include other Gaussian problems that we encounter frequently in practice. After developing the generalized model in Section 3.1, we study the optimum receiver and its performance for the remainder of the chapter.

3.1 MODEL AND PROBLEM CLASSIFICATION

An obvious generalization is suggested by the digital communication system on page 26. In this case we transmit a different signal on each hypothesis. Typically we transmit

$$\sqrt{2P}\sin(\omega_1 t), \qquad T_i \le t \le T_f : H_1 \tag{1}$$

and

$$\sqrt{2P}\sin(\omega_0 t), \qquad T_i \le t \le T_f : H_0. \tag{2}$$

If the channel is the simple multiplicative channel shown in Fig. 2.9, the received waveforms on the two hypotheses are

$$r(t) = \sqrt{2P} b(t) \sin(\omega_1 t) + w(t), \qquad T_i \le t \le T_f : H_1, \qquad (3)$$

$$r(t) = \sqrt{2P \, b(t) \sin(\omega_0 t)} + w(t), \qquad T_i \le t \le T_f : H_0, \tag{4}$$

where b(t) is a sample function of Gaussian random process. This is just a special case of the general problem in which the received waveforms on the two hypotheses are

$$r(t) = s_1(t) + w(t), \qquad T_i \le t \le T_f: H_1, r(t) = s_0(t) + w(t), \qquad T_i \le t \le T_f: H_0,$$
(5)

where $s_1(t)$ and $s_0(t)$ are Gaussian processes with mean-value functions $m_1(t)$ and $m_0(t)$ and covariance functions $K_1(t, u)$ and $K_0(t, u)$, respectively. In many cases, we also have a colored noise term, $n_c(t)$, present on both hypotheses. Then

$$r(t) = s_1(t) + n_c(t) + w(t), \qquad T_i \le t \le T_f : H_1, r(t) = s_0(t) + n_c(t) + w(t), \qquad T_i \le t \le T_f : H_0.$$
(6)

We can include both these problems and many others in the general formulation,

$$r(t) = r_1(t), \quad T_f \le t \le T_i: H_1, \\
 r(t) = r_0(t), \quad T_i \le t \le T_i: H_0.$$
(7)

On H_1 , r(t) is a sample function from a Gaussian random process with mean-value function $m_1(t)$ and covariance function $K_{H_1}(t, u)$. On H_0 , r(t)is a sample function from a Gaussian random process with mean-value function $m_0(t)$ and covariance function $K_{H_0}(t, u)$. For algebraic simplicity, we assume that r(t) is zero-mean on both hypotheses in our initial discussion. The results regarding mean-value functions in Chapter 2 generalize in an obvious manner and are developed in Section 3.4.

Some of our discussion will be for the general problem in (7). On the other hand, many results are true only for subclasses of this problem. For bookkeeping purposes we define these classes by the table in Fig. 3.1. In all cases, the various processes are statistically independent. The subscript w implies that the same white noise component is present on both hypotheses. There may also be other processes present on both H_1 and H_0 . The absence of the subscript means that a white noise component is not necessarily present. The class inclusions are indicated by solid lines. Thus,

$$B_w \subset A_w \subset A \subset GB, \tag{8}$$

$$B_w \subset B. \tag{9}$$

Two additional subscripts may be applied to any of the above classes. The additional subscript s means that all of the processes involved have a finite-dimensional state representation. The additional subscript m means that some of the processes involved have a nonzero mean. The absence of the subscript m implies that all processes are zero-mean. We see that the simple binary problem in Chapter 2 is the special case of class B_w , in which $n_c(t)$ is not present. This class structure may seem cumbersome, but it enables us to organize our results in a clear manner.

As in the simple binary problem, we want to find the optimum receiver and evaluate its performance. The reason the calculation of the likelihood ratio was easy in the simple binary case was that only white noise was



present on H_0 . Thus, we could choose our coordinate system based on the covariance function of the signal process on H_1 . As a result of this choice, we had statistically independent coefficients on both hypotheses. Now the received waveform may have a nonwhite component on both hypotheses. Therefore, except for the trivial case in which the nonwhite components have the same eigenfunctions on both hypotheses, the technique in Section 2.1 will give correlated coefficients. There are several ways around this difficulty. An intuitively appealing method is the whitening approach, which we encountered originally in Chapter I-4 (page I-290). We shall use this approach in the text.

In Section 3.2 we derive the likelihood ratio test and develop various receiver structures for the class A_w problem. In Section 3.3 we study the performance for the class A_w problem. In Section 3.4 we discuss four important special situations: the binary symmetric problem, the non-zeromean problem, the bandpass problem, and the binary symmetric bandpass problem. In Section 3.5 we look at class *GB* problems and discuss the singularity problem briefly. We have deliberately postponed our discussion of the general case because almost all physical situations can be modeled by a class A_w system. Finally, in Section 3.6, we summarize our results and discuss some related issues.

3.2 RECEIVER STRUCTURES

In this section, we derive the likelihood ratio test for problems in class A_w and develop several receiver configurations. Looking at Fig. 3.1, we see that class A_w implies that the same white noise process is present on both hypotheses. Thus,

$$r(t) = s_1(t) + w(t), \qquad T_i \le t \le T_f : H_1,$$

$$r(t) = s_0(t) + w(t), \qquad T_i \le t \le T_f : H_0.$$
(10)

In addition, we assume that both $s_1(t)$ and $s_0(t)$ are zero-mean Gaussian processes with finite mean-square values. They are statistically independent of w(t) and have continuous covariance functions $K_1(t, u)$ and $K_0(t, u)$, respectively. The spectral height of the Gaussian white noise is $N_0/2$. Therefore, the covariance functions of r(t) on the two hypotheses are

$$E[r(t)r(u) \mid H_1] \stackrel{\Delta}{=} K_{H_1}(t, u) = K_1(t, u) + \frac{N_0}{2}\delta(t - u), \quad (11)$$

$$E[r(t)r(u) \mid H_0] \stackrel{\Delta}{=} K_{H_0}(t, u) = K_0(t, u) + \frac{N_0}{2} \,\delta(t - u). \tag{12}$$

We now derive the likelihood ratio test by a whitening approach.

3.2.1 Whitening Approach

The basic idea of the derivation is straightforward. We whiten r(t) on one hypothesis and then operate on the whitened waveform using the techniques of Section 2.1. As long as the whitening filter is reversible, we know that the over-all system is optimum (see page I-289). (Notice that realizability is *not* an issue.)

The whitening filter is shown in Fig. 3.2. We choose $h_{w_0}(t, u)$ so that $r_*(t)$ is white on H_0 and has a unity spectral height. Thus,

$$E[r_{*}(t)r_{*}(u) \mid H_{0}] = \delta(t-u), \qquad T_{i} \le t, u \le T_{f}.$$
(13)

On pages I-290-I-297 we discussed construction of the whitening filter.

$$r(t) \longrightarrow h_{w_0}(t, u) \xrightarrow{r_*(t)}$$

Fig. 3.2 Whitening filter.

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From that discussion we know that we can always find a filter such that (13) is satisfied. Because

$$r_{*}(t) = \int_{T_{i}}^{T_{f}} h_{w_{0}}(t, u) r(u) \, du, \qquad (14)$$

(13) implies that

$$\iint_{T_i}^{T_f} h_{w_0}(t,\alpha) h_{w_0}(u,\beta) K_{H_0}(\alpha,\beta) \, d\alpha \, d\beta = \delta(t-u). \tag{15}$$

The covariance function of $r_*(t)$ on H_1 is

$$E[r_{*}(t)r_{*}(u) \mid H_{1}] = \int_{T_{i}}^{T_{f}} h_{w_{0}}(t, \alpha)h_{w_{0}}(u, \beta)K_{H_{1}}(\alpha, \beta) \, d\alpha \, d\beta \, \Delta \, K_{1}^{*}(t, u).$$
(16)

We now expand $r_*(t)$ using the eigenfunctions of $K_1^*(t, u)$, which are specified by the equation

$$\lambda_i^* \varphi_i(t) = \int_{T_i}^{T_f} K_1^*(t, u) \varphi_i(u) \, du, \qquad T_i \le t \le T_f. \tag{17}$$

Proceeding as in Section 2.1, we find that

$$l_R = -\frac{1}{2} \sum_{i=1}^{\infty} \left(\frac{1}{\lambda_i^*} - 1 \right) r_i^2.$$
 (18)

(Remember that the whitened noise on H_0 has unity spectral height.) As before we define an inverse kernel, $Q_1^*(t, u)$,

$$\int_{T_i}^{T_f} Q_1^*(t, u) K_1^*(u, z) \, du = \delta(t - z), \qquad T_i < t, z < T_f. \tag{19}$$

Then we can write

$$l_{R} = -\frac{1}{2} \iint_{T_{i}}^{T_{f}} dt \ du \ r_{*}(t) [Q_{1}^{*}(t, u) - \delta(t - u)] r_{*}(u).$$
(20a)

It is straightforward to verify that the kernel in (20a) is always squareintegrable (see Problem 3.2.11). Using (14), we can write this in terms of r(t).

$$l_{R} = -\frac{1}{2} \iint_{T_{i}}^{T_{f}} r(\alpha) \Biggl\{ \iint_{T_{i}}^{T_{f}} h_{w_{0}}(t, \alpha) [Q_{1}^{*}(t, u) - \delta(t-u)] h_{w_{0}}(u, \beta) dt du \Biggr\} r(\beta) d\alpha d\beta.$$
(20b)

We want to examine the term in the braces. The term contributed by the impulse is just $Q_{H_0}(\alpha, \beta)$, the inverse kernel of $K_{H_0}(\alpha, \beta)$ [see (I-4.152)]. We now show that the remaining term is $Q_{H_1}(\alpha, \beta)$. We must show that

$$Q_{H_1}(\alpha,\beta) = \iint_{T_i}^{T_f} h_{w_0}(t,\alpha) Q_1^*(t,u) h_{w_0}(u,\beta) \, dt \, du.$$
(21)

This result is intuitively obvious from the relationship between $K_{H_1}(\alpha, \beta)$ and $K_1^*(t, u)$ expressed in (16). It can be verified by a few straightforward manipulations. [Multiply both sides of (16) by

$$Q_1^*(z_1, t)h_{w_0}^{-1}(z_2, u)Q_{H_1}(z_2, z_3)h_{w_0}(z_1, z_4).$$

Integrate the left side with respect to u, β , z_2 , and α , in that order. Integrate the right side with respect to t, z_1 , u, and z_2 , in that order. At each step simplify by using known relations.] The likelihood function in (19) can now be written as

$$l_R = -\frac{1}{2} \iint_{T_i}^{T_f} d\alpha \ d\beta \ r(\alpha) r(\beta) [Q_{H_1}(\alpha, \beta) - Q_{H_0}(\alpha, \beta)].$$
(22)

In a moment we shall see that the impulses in the inverse kernels cancel, so that kernel is a square-integrable function. This can also be written formally as a difference of two quadratic forms,

$$l_{R} = \frac{1}{2} \iint_{T_{i}}^{T_{f}} d\alpha \ d\beta \ r(\alpha) Q_{H_{0}}(\alpha, \beta) r(\beta) - \frac{1}{2} \iint_{T_{i}}^{T_{f}} d\alpha \ d\beta \ r(\alpha) Q_{H_{1}}(\alpha, \beta) r(\beta).$$
(23)

The reader should note the similarity between (23) and the LRT for the finite-dimensional general Gaussian problem in (I-2.327). This similarity enables one to guess both the form of the test for nonzero means and the form of the bias terms. Several equivalent forms of (22) are also useful.

3.2.2 Various Implementations of the Likelihood Ratio Test

To obtain the first equivalent form, we write $Q_{H_1}(\alpha, \beta)$ and $Q_{H_0}(\alpha, \beta)$ in terms of an impulse and a well-behaved function,

$$Q_{H_i}(\alpha, \beta) = \frac{2}{N_0} [\delta(\alpha - \beta) - h_i(\alpha, \beta)], \quad i = 0, 1, \quad (24)$$

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where $h_i(\alpha, \beta)$ satisfies the equation

$$\frac{N_0}{2}h_i(\alpha,\beta) + \int_{T_i}^{T_f} h_i(\alpha,x) K_i(x,\beta) \, dx = K_i(\alpha,\beta),$$
$$T_i \le \alpha, \beta \le T_f, \quad i = 0, 1. \quad (25)$$

Using (24) in (22) gives

$$l_R = l_{R_1} - l_{R_0}, (26)$$

where

$$l_{R_i} = \frac{1}{N_0} \iint_{T_i}^{T_f} r(\alpha) r(\beta) h_i(\alpha, \beta) \, d\alpha \, d\beta, \qquad i = 0, 1.$$
(27)

It is easy to verify (see Problem 3.2.1) that the bias term can be written as

$$l_B = l_{B_1} - l_{B_0}, (28)$$

where [by analogy with (2.73)]

$$l_{B_i} = -\frac{1}{N_0} \int_{T_i}^{T_f} \xi_{P_{s_i}}(t) \, dt, \qquad i = 0, \, 1.$$
 (29)

The complete LRT is

$$l_{R_1} + l_{B_1} - l_{R_0} - l_{B_0} \mathop{\gtrless}_{H_0}^{H_1} \ln \eta.$$
(30)

We see that the receiver can be realized as two *simple* binary receivers in parallel, with their outputs subtracted. Thus, any of the four canonic realizations developed in Section 2.1 (Figs. 2.2–2.7) can be used in each path. A typical structure using Realization No. 1 is shown in Fig. 3.3. This parallel processing structure is frequently used in practice.

A second equivalent form of (22) is also useful. We define a function

$$h_{\Delta}(t, u) \triangleq Q_{H_0}(t, u) - Q_{H_1}(t, u). \tag{31}$$

Then,

$$l_R = \frac{1}{2} \iint_{T_i}^{T_f} r(t) h_{\Delta}(t, u) r(u) dt du.$$
(32)

To eliminate the inverse kernels in (31), we multiply by the two covariance





Fig. 3.3 Parallel processing realization of general binary receiver (class A_w).

functions and integrate. The result is an integral equation specifying $h_{\Delta}(t, u)$.

$$\iint_{T_{i}}^{T_{f}} dt \, du K_{H_{0}}(x, t) h_{\Delta}(t, u) K_{H_{1}}(u, z) = K_{H_{1}}(x, z) - K_{H_{0}}(x, z),$$

$$T_{i} \leq x, z \leq T_{f}.$$
(33)

This form of the receiver is of interest because the white noise level does not appear *explicitly*. Later we shall see that (32) and (33) specify the receiver for class *GB* problems. The receiver is shown in Fig. 3.4.

Two other forms of the receiver are useful for class B_w problems. In this case, the received waveform contains the same noise process on both hypotheses and an additional signal process on H_1 . Thus,

$$r(t) = s(t) + n_c(t) + w(t), \qquad T_i \le t \le T_f: H_1, r(t) = n_c(t) + w(t), \qquad T_i \le t \le T_f: H_0,$$
(34)

where s(t), $n_c(t)$, and w(t) are zero-mean, statistically independent Gaussian random processes with covariance functions $K_s(t, u)$, $K_c(t, u)$, and $(N_0/2)\delta(t - u)$, respectively. On H_0 ,

$$K_{H_0}(t, u) = K_c(t, u) + \frac{N_0}{2} \,\delta(t - u) \stackrel{\Delta}{\to} K_n(t, u), \qquad T_i \le t, u \le T_f.$$
(35)



Fig. 3.4 Receiver for class A_w problem.

For this particular case the first form is an alternative realization that corresponds to the estimator-correlator of Fig. 2.3. We define a new function $h_o(t, z)$ by the relation

$$h_o(x, u) = \int_{T_i}^{T_f} h_{\Delta}(t, u) K_{H_0}(x, t) dt.$$
 (36)

Using (36) and the definition of $h_{\Delta}(t, u)$ in (33), we have

$$\int_{T_i}^{T_f} h_o(t, x) [K_s(x, u) + K_u(x, u)] \, dx = K_s(t, u), \qquad T_i \le x, z \le T_f.$$
(37)

This equation is familiar from Chapter 1-6 as the equation specifying the optimum linear filter for estimating s(t) from an observation r(t) assuming that H_1 is true. Thus,

$$\hat{s}(t) = \int_{T_i}^{T_f} h_o(t, u) r(u) \, du.$$
(38)

We now implicitly define a function $r_q(t)$,

$$r(t) = \int_{T_i}^{T_f} K_{H_0}(t, x) r_g(x) \, dx, \qquad T_i \le t \le T_f.$$
(39)

Equivalently,

$$r_{g}(t) = \int_{T_{i}}^{T_{f}} Q_{H_{0}}(t, x) r(x) \, dx, \qquad T_{i} \le t \le T_{f}. \tag{40}$$

This type of function is familiar from Chapter I-5 (I-5.32). Then, from (36) and (40), we have

$$l_{R} = \frac{1}{2} \int_{T_{i}}^{T_{f}} \hat{s}(t) r_{g}(t) dt.$$
(41)

The resulting receiver structure is shown in Fig. 3.5. We see that this has the same structure as the optimum receiver for known signals in colored noise (Fig. I-4.38c) except that a MMSE estimate $\hat{s}(t)$ has replaced the



Fig. 3.5 Estimator-correlator realization for class B_w problems.

known signal in the correlation operation. This configuration is analogous to the estimator-correlator in Fig. 2.3.

The second receiver form of interest for class B_w is the filter-squarer realization. For this class a functional square root exists,

$$h_{\Delta}(t, u) = \int_{T_i}^{T_f} h_{\Delta}^{[\frac{1}{2}]}(z, t) h_{\Delta}^{[\frac{1}{2}]}(z, u) \, dz, \qquad T_i \le t, u \le T_f.$$
(42)

The existence can be shown by verifying that one solution to (42) is

$$h_{\Delta}^{[\frac{1}{2}]}(t, u) = \int_{T_i}^{T_f} h_1^{*[\frac{1}{2}]}(t, z) h_{w_0}(z, u) \, dz, \qquad T_i \le t, u \le T_f, \qquad (43)$$

since both functions in the integrand exist (see Problem 3.2.10). This filter-squarer realization is shown in Fig. 3.6. For class A_w problems a functional square root of $h_{\Delta}(t, u)$ may not exist, and so a filter-squarer realization is not always possible (see Problem 3.2.10).

3.2.3 Summary: Receiver Structures

In this section we have derived the likelihood ratio test for the class A_w problem. The LRT was given in (23). We then looked at various receiver configurations. The parallel processing configuration is the one most commonly used. All of the canonical receiver configuration developed for the simple binary problem can be used in each path. For class B_w problems, the filter-squarer realization shown in Fig. 3.6 is frequently used.

The next problem of interest is the performance of the optimum receiver



Fig. 3.6 Filter-squarer realization for class B_w problems.

3.3 PERFORMANCE

All of our performance discussion in Section 2.2 is valid for class A_{w} problems with the exception of the closed-form expressions for $\mu(s)$. In this section we derive an expression for $\mu(s)$. Just as in the derivation of the optimum receiver, there is a problem due to the nonwhite process that is present on both hypotheses. As before, one way to avoid this is to prewhiten the received signal on H_0 . It is possible to carry out this derivation, but it is too tedious to have much pedagogical appeal. Of the various alternatives available at this point the sampling approach seems to be the simplest. In Section 3.5, we study the performance question again. The derivation of $\mu(s)$ at that point is much neater.

In the problems on pages I-231-233 of Chapter I-3, we discussed how many of the continuous waveform results could be derived easily using a sampling approach. The received waveforms on the two hypotheses are given by (5). We sample r(t) every T/N seconds. This gives us an *N*dimensional vector **r** whose mean and covariance matrix are sampled versions of the mean-value function and covariance function of the process. We can then use the $\mu(s)$ expression derived in Section I-2.7. Finally, we let $N \rightarrow \infty$ to get the desired result. For algebraic simplicity, we go through the details for the zero-mean case.

Denote the sample at t_i as r_i . The covariances between the samples are

$$E[r_i r_j \mid H_{\alpha}] = K_{H_{\alpha}}(t_i, t_j) = K_{\alpha, ij}, \quad i, j = 1, \dots, N, \, \alpha = 0, \, 1.$$
(44)

The set of samples is denoted by the vector \mathbf{r} . The covariance matrix of \mathbf{r} is

$$E[\mathbf{r}\mathbf{r}^T \mid H_{\alpha}] = \mathbf{K}_{\alpha}, \quad \alpha = 0, 1.$$
(45)

The matrices in (45) are $N \times N$ covariance matrices. The elements of the matrices on the two hypotheses are

$$K_{1,ij} = K_{s_1,ij} + \frac{N_0}{2} \,\delta_{ij},\tag{46}$$

$$K_{0,ij} = K_{s_0,ij} + \frac{N_0}{2} \,\delta_{ij}.\tag{47}$$

Notice that

$$K_{s_1,ij} = K_{s_1}(t_i, t_j)$$
(48)

and

$$K_{s_0,ij} = K_{s_0}(t_i, t_j).$$
⁽⁴⁹⁾

We can write (46) and (47) in matrix notation as

$$\mathbf{K}_{1} \stackrel{\Delta}{=} \mathbf{K}_{s_{1}} + \frac{N_{0}}{2} \mathbf{I}, \tag{50}$$

$$\mathbf{K}_{\mathbf{0}} \stackrel{\Delta}{=} \mathbf{K}_{\mathbf{s}_{\mathbf{0}}} + \frac{N_{\mathbf{0}}}{2} \mathbf{I}.$$
 (51)

We can now use the $\mu(s)$ expression derived in Chapter I-2. From the solution to Problem I-2.7.3,

$$\mu_N(s) = -\frac{1}{2} \ln \left(|\mathbf{K}_1|^{s-1} |\mathbf{K}_0|^{-s} |\mathbf{K}_0 s + \mathbf{K}_1(1-s)| \right), \quad 0 \le s \le 1.$$
(52)

Notice that $|\cdot|$ denotes the determinant of a matrix. Substituting (50) and (51) into (52), we have

$$\mu_{N}(s) = -\frac{1}{2} \ln \left\{ \left| \frac{N_{0}}{2} \mathbf{I} + \mathbf{K}_{s_{1}} \right|^{s-1} \left| \frac{N_{0}}{2} \mathbf{I} + \mathbf{K}_{s_{0}} \right|^{-s} \times \left| \left(\frac{N_{0}}{2} \mathbf{I} + \mathbf{K}_{s_{0}} \right) s + \left(\frac{N_{0}}{2} \mathbf{I} + \mathbf{K}_{s_{1}} \right) (1-s) \right| \right\}.$$
(53)

The matrices in (53) cannot be singular, and so all of the indicated operations are valid. Collecting $N_0/2$ from the various terms and rewriting (53) as a sum of logarithms, we have

$$\mu_{N}(s) = \frac{1}{2} \left\{ (1-s) \ln \left| \mathbf{I} + \frac{2}{N_{0}} \mathbf{K}_{s_{1}} \right| + s \ln \left| \mathbf{I} + \frac{2}{N_{0}} \mathbf{K}_{s_{0}} \right| - \ln \left| \mathbf{I} + \frac{2}{N_{0}} (s \mathbf{K}_{s_{0}} + (1-s) \mathbf{K}_{s_{1}}) \right| \right\}.$$
 (54)

Now each term is the logarithm of the determinant of a matrix and can be rewritten as the sum of the logarithms of the eigenvalues of the matrix by using the Cayley-Hamilton theorem. For example,

$$\ln \left| \mathbf{I} + \frac{2}{N_0} \mathbf{K}_{s_1} \right| = \sum_{i=1}^N \ln \left(1 + \frac{2}{N_0} \lambda_{s_1,i} \right), \tag{55}$$

where $\lambda_{s_1,i}$ is the *i*th eigenvalue of \mathbf{K}_{s_1} . As $N \to \infty$, this function of the eigenvalues of the matrix, \mathbf{K}_{s_1} , will approach the same function of the eigenvalues of the kernel, $K_{s_1}(t, u)$.[†] We denote the eigenvalues of $K_{s_1}(t, u)$ by $\lambda_i^{s_1}$. Thus,

$$\lim_{N \to \infty} \sum_{i=1}^{N} \ln\left(1 + \frac{2}{N_0} \lambda_{s_1, i}\right) = \sum_{i=1}^{\infty} \ln\left(1 + \frac{2}{N_0} \lambda_i^{s_1}\right).$$
(56)

[†] We have not proved that this statement is true. It is shown in various integral equation texts (e.g., Lovitt [1, Chapter III]).

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The sum on the right side is familiar from (2.73) as

$$\sum_{i=1}^{\infty} \ln\left(1 + \frac{2\lambda_i^s}{N_0}\right) = \frac{2}{N_0} \int_{T_i}^{T_f} \xi_P\left(t \mid s_1(\cdot), \frac{N_0}{2}\right) dt.$$
(57)

Thus, the first term in $\mu(s)$ can be expressed in terms of the realizable mean-square error for the problem of filtering $s_1(t)$ in the presence of additive white noise. A similar interpretation follows for the second term. To interpret the third term we define a new composite signal process,

$$s_{\rm com}(t,s) \stackrel{\Delta}{=} \sqrt{s} \, s_0(t) + \sqrt{(1-s)} \, s_1(t).$$
 (58)

This is a fictitious process constructed by generating two sample functions $s_0(t)$ and $s_1(t)$ from statistically independent random processes with covariances $K_0(t, u)$ and $K_1(t, u)$ and then forming a weighted sum. The resulting composite process has a covariance function

$$K_{\rm com}(t, u:s) = sK_0(t, u) + (1 - s)K_1(t, u), \qquad T_i \le t, u \le T_f.$$
(59)

We denote the realizable mean-square filtering error in the presence of white noise as $\xi_P(t \mid s_{\text{com}}(\cdot), N_0/2)$. The resulting expression for $\mu(s)$ is

$$\mu(s) = \frac{1}{N_0} \int_{T_i}^{T_f} dt \left[(1 - s) \xi_P \left(t \mid s_1(\cdot), \frac{N_0}{2} \right) + s \xi_P \left(t \mid s_0(\cdot), \frac{N_0}{2} \right) - \xi_P \left(t \mid s_{\rm com}(\cdot), \frac{N_0}{2} \right) \right].$$
(60)

We see that for the general binary problem, we can express $\mu(s)$ in terms of three different realizable filtering errors.

To evaluate the performance, we use the expression for $\mu(s)$ in (60) in the Chernoff bounds in (2.127), or the approximate error expressions in (2.164), (2.166), (2.173), and (2.174). We shall look at some specific examples in Chapter 4. We now look at four special situations.

3.4 FOUR SPECIAL SITUATIONS

In this section, we discuss four special situations that arise in practice:

- 1. The binary symmetric problem.
- 2. The non-zero-mean problem.
- 3. The stationary independent bandpass problem.
- 4. The binary symmetric bandpass problem.

We define each of these problems in detail in the appropriate subsection.

3.4.1 Binary Symmetric Case

In this case the received waveforms on the two hypotheses are

$$r(t) = s_1(t) + w(t), \qquad T_i \le t \le T_f : H_1,$$

$$r(t) = s_0(t) + w(t), \qquad T_i \le t \le T_f : H_0.$$
(61)

We assume that the signal processes $s_1(t)$ and $s_0(t)$ have identical eigenvalues and that their eigenfunctions are essentially disjoint. For stationary processes, this has the simple interpretation illustrated by the spectra in Fig. 3.7. The two processes have spectra that are essentially disjoint in frequency and are identical except for a frequency shift. The additive noise w(t) is white with spectral height $N_0/2$. This class of problems is encountered frequently in binary communications over a fading channel and is just the waveform version of Case 2 on page I-114. We shall discuss the physical channel in more detail in Chapter 11 and see how this mathematical model arises. The receiver structure is just a special case of Fig. 3.3. We can obtain $\mu_{BS}(s)$ from (60) by the following observations (the subscript denotes binary symmetric):

1. The minimum mean-square filtering error only depends on the *eigenvalues* of the process. Therefore,

$$\xi_P\left(t \mid s_1(\cdot), \frac{N_0}{2}\right) = \xi_P\left(t \mid s_0(\cdot), \frac{N_0}{2}\right). \tag{62}$$

2. If two processes have *no* eigenfunctions in common, then the minimum mean-square error in filtering their sum is the sum of the minimum mean-square errors for filtering the processes individually. Therefore,

$$\xi_{P}\left(t \mid s_{\text{com}}(\cdot), \frac{N_{0}}{2}\right) = \xi_{P}\left(t \mid \sqrt{s} \ s_{0}(\cdot), \frac{N_{0}}{2}\right) + \xi_{P}\left(t \mid \sqrt{1-s} \ s_{0}(\cdot), \frac{N_{0}}{2}\right).$$
(63)

Fig. 3.7 Disjoint processes (spectrum is symmetric around $\omega = 0$; only positive frequencies are shown).

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Using (62) and (63) in (64), we have

$$\mu_{BS}(s) = \frac{1}{N_0} \int_{T_i}^{T_f} dt \bigg[(1-s)\xi_P \bigg(t \mid s_0(\cdot), \frac{N_0}{2} \bigg) - \xi_P \bigg(t \mid \sqrt{1-s} s_0(\cdot), \frac{N_0}{2} \bigg) \\ + s\xi_P \bigg(t \mid s_0(\cdot), \frac{N_0}{2} \bigg) - \xi_P \bigg(t \mid \sqrt{s} s_0(\cdot), \frac{N_0}{2} \bigg) \bigg].$$
(64)

This can be rewritten in two different forms. Looking at the expression for $\mu(s)$ in (2.139) for the simple binary problem, we see that (64) can be written as

$$\mu_{BS}(s) = \mu_{SIB}(s) + \mu_{SIB}(1-s), \tag{65}$$

where the subscript SIB denotes simple binary, and, from (2.139),

$$\mu_{SIB}(s) = \frac{1}{N_0} \int_{T_i}^{T_f} dt \left[(1-s)\xi_P \left(t \mid s_0(\cdot), \frac{N_0}{2} \right) - \xi_P \left(t \mid \sqrt{1-s} \, s_0(\cdot), \frac{N_0}{2} \right) \right].$$
(66)

From (65), it is clear that $\mu_{BS}(s)$ is symmetric about s = 1/2. A second form of $\mu_{BS}(s)$ that is frequently convenient is

$$\mu_{BS}(s) = \frac{1}{N_0} \int_{T_i}^{T_f} dt \bigg[\xi_P \bigg(t \mid s_0(\cdot), \frac{N_0}{2} \bigg) \\ - (1 - s) \xi_P \bigg(t \mid s_0(\cdot), \frac{N_0}{2(1 - s)} \bigg) - s \xi_P \bigg(t \mid s_0(\cdot), \frac{N_0}{2s} \bigg) \bigg].$$
(67)

The binary symmetric model is frequently encountered in communication systems. In most cases the a-priori probabilities of the two hypotheses are equal,

$$\Pr[H_0] = \Pr[H_1] = \frac{1}{2}, \tag{68}$$

and the criterion is minimum $Pr(\epsilon)$,

$$\Pr\left(\epsilon\right) = \frac{1}{2}P_{F'} + \frac{1}{2}P_{M'}.$$
(69)

Under these conditions the threshold, $\ln \eta$, equals zero. All of our bounds and performance expressions require finding the value of s where

$$\dot{u}(s) = \ln \eta. \tag{70}$$

In this case, we want the value of s where

$$\dot{\mu}_{BS}(s) = 0. \tag{71}$$

† This particular form was derived in [2].

From the symmetry it is clear that

$$\dot{\mu}_{BS}(s)\Big|_{s=1/2} = 0.$$
 (72)

Thus, the important quantity is $\mu_{BS}(1/2)$. From (65),

$$\mu_{BS}(\frac{1}{2}) = 2\mu_{SIB}(\frac{1}{2}). \tag{73}$$

From (66),

$$\mu_{BS}(\frac{1}{2}) = \frac{1}{N_0} \int_{T_i}^{T_f} dt \left[\xi_P \left(t \mid s_0(\cdot), \frac{N_0}{2} \right) - \xi_P(t \mid s_0(\cdot), N_0) \right].$$
(74)

Using (I-2.473), we have a bound on $Pr(\epsilon)$,

$$\Pr\left(\epsilon\right) \leq \frac{1}{2} \exp\left[\mu_{BS}\left(\frac{1}{2}\right)\right]. \tag{75}$$

In order to get an approximate error expression, we proceed in exactly the same manner as in (2.164) and (2.173). The one-term approximation is

$$\Pr(\epsilon) \simeq \left[\operatorname{erfc}_{\ast}\left(\frac{\sqrt{\mu_{BS}(\frac{1}{2})}}{2}\right)\right] \exp\left(\mu_{BS}(\frac{1}{2}) + \frac{\mu_{BS}(\frac{1}{2})}{8}\right).$$
(76)

When the argument of $\operatorname{erfc}_*(\cdot)$ is greater than two, this can be approximated as

$$\Pr\left(\epsilon\right) \simeq \left[\frac{2}{\pi \ddot{\mu}_{BS}(\frac{1}{2})}\right]^{\frac{1}{2}} \exp\left(\mu_{BS}(\frac{1}{2})\right). \tag{77}$$

As before, the coefficient is frequently needed in order to get a good estimate of the Pr (ϵ). On page 79 we shall revisit this problem and investigate the accuracy of (77) in more detail.

Two other observations are appropriate:

1. From our results in (2.72) and (2.74), we know that $\mu_{BS}(s)$ can be written in terms of Fredholm determinants. Using these equations, we have

$$u_{BS}(s) = \frac{1}{2} \ln \left[\frac{D_{\mathscr{F}}(2/N_0)}{D_{\mathscr{F}}([2(1-s)]/N_0)D_{\mathscr{F}}(2s/N_0)} \right]$$
(78)

and

$$\mu_{BS}(\frac{1}{2}) = \frac{1}{2} \ln \left[\frac{D_{\mathscr{F}}([2(1-s)]/N_0)}{D_{\mathscr{F}}^2(1/N_0)} \right].$$
(79)

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2. The negative of $\mu(\frac{1}{2})$ has been used as criterion for judging the quality of a test by a number of people. It was apparently first introduced by Hellinger [3] in 1909. It is frequently referred to as the Bhattacharyya distance [4]. (Another name used less frequently is the Kakutani distance [5].) It is essential to observe that the importance of $\mu(\frac{1}{2})$ arises from both the symmetry of the problem *and* the choice of the threshold. If either of these elements is changed, $\mu(s)$ for some $s \neq \frac{1}{2}$ will provide a better measure of performance. It is easy to demonstrate cases in which ordering tests by their $\mu(\frac{1}{2})$ value or designing signals to minimize $\mu(\frac{1}{2})$ gives incorrect results because the model is asymmetric.

The formulas derived in this section are essential in the analysis of binary symmetric communication systems. In Chapter 5 we shall derive corresponding results for M-ary systems. The next topic of interest is the effect of nonzero means.

3.4.2 Non-zero Means

All of our discussion of the general binary problem up to this point has assumed that the processes were zero-mean on both hypotheses. In this section we consider a class A_{wm} problem and show how nonzero means affect the optimum receiver structure and the system performance. The received waveforms on the two hypotheses are

$$r(t) = s_1(t) + w(t), \qquad T_i \le t \le T_f : H_1,$$

$$r(t) = s_0(t) + w(t), \qquad T_i \le t \le T_f : H_0,$$
(80)

where

$$E[s_1(t)] = m_1(t)$$
(81)

and

$$E[s_0(t)] = m_0(t).$$
 (82)

The covariance functions of $s_1(t)$ and $s_0(t)$ are $K_{s_1}(t, u)$ and $K_{s_0}(t, u)$, respectively. The additive zero-mean white Gaussian noise is independent of the signal processes and has spectral height $N_0/2$. As in the simple binary problem, we want to obtain an expression for l_D and $\mu_D(s)$. [Recall the definition of these quantities in (2.32) and (2.147).] Because of the similarity of both the derivation and the results to the simple binary case, we simply state the answers and leave the derivations as an exercise (see Problem 3.4.1).

Modifying (23), we obtain

$$l_D = \int_{T_i}^{T_f} r(u) \, du \bigg[\int_{T_i}^{T_f} [m(t)Q_{H_1}(t, u) - m_0(t)Q_{H_0}(t, u)] \, dt \bigg].$$
(83)

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This can be written as

$$l_D = \int_{T_i}^{T_f} r(u) [g_1(u) - g_0(u)] \, du, \tag{84}$$

where

$$g_1(u) \triangleq \int_{T_i}^{T_f} m_1(t) Q_{H_1}(t, u) \, dt, \qquad T_i < u < T_f$$
(85)

and

$$g_0(u) \stackrel{\Delta}{=} \int_{T_i}^{T_f} m_0(t) Q_{H_0}(t, u) \, dt, \qquad T_i < u < T_f.$$
(86)

The functions $g_1(u)$ and $g_0(u)$ can also be defined implicitly by the relations

$$m_1(t) = \int_{T_i}^{T_f} K_{H_1}(t, u) g_1(u) \, du, \qquad T_i \le t \le T_f \tag{87}$$

and

$$m_0(t) = \int_{T_i}^{T_f} K_{H_0}(t, u) g_0(u) \, du, \qquad T_i \le t \le T_f.$$
(88)

The resulting test is

$$l_R + l_D \underset{H_0}{\stackrel{H_1}{\gtrsim}} \gamma', \tag{89}$$

where l_R is given by (23) or (32) and γ' is the threshold that includes the bias terms. An alternative expression for the test derived in Problem 3.4.1 is

$$l \triangleq \int_{T_{i}}^{T_{f}} r(t)g(t) dt + \frac{1}{2} \iint_{T_{i}}^{T_{f}} [r(t) - m_{1}(t)]h_{\Delta}(t, u)[r(u) - m_{1}(u)] dt du \underset{H_{0}}{\overset{H_{1}}{\gtrless}} \gamma'',$$
(90)

where g(t) satisfies the equation

$$\int_{T_i}^{T_f} K_{H_0}(t, u) g(u) \, du = m_1(t) - m_0(t), \qquad T_i \le t \le T_f \qquad (91)$$

and $h_{\Delta}(t, u)$ satisfies (33). The advantage of the form in (90) is that it requires solving two integral equations rather than three.

The derivation of $\mu_D(s)$ is a little more involved (see Problem 3.4.2). We define a function

$$m_{\Delta}(t) = m_0(t) - m_1(t)$$
 (92)

and a composite signal process

$$s_{\rm com}(t,s) = \sqrt{s} s_0(t) + \sqrt{1-s} s_1(t),$$
 (93)

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whose covariance is denoted by $K_{com}(t, u)$. In (93), $s_0(t)$ and $s_1(t)$ are assumed to be statistically independent processes. Thus,

$$K_{\rm com}(t, u) = sK_0(t, u) + (1 - s)K_1(t, u).$$
(94)

This process was encountered previously in (58). Finally we define $g_{\Delta com}(t \mid N_0/2)$ implicitly by the integral equation

$$m_{\Delta}(t) = \int_{T_i}^{T_f} \left[K_{\text{com}}(t, u) + \frac{N_0}{2} \,\delta(t - u) \right] g_{\Delta_{\text{com}}}\left(u \left| \frac{N_0}{2} \right) \, du. \tag{95}$$

Then we can show that

$$\mu_D(s) = -\frac{s(1-s)}{2} \int_{T_i}^{T_f} m_{\Delta}(t) g_{\Delta \text{com}}\left(t \left| \frac{N_0}{2} \right) dt.$$
(96)

To get the $\mu(s)$ for the entire problem, we add the $\mu(s)$ from l_R [denoted now by $\mu_R(s)$ and defined in (60)] and $\mu_D(s)$.

$$\mu(s) = \mu_R(s) + \mu_D(s).$$
(97)

The results in (84), (90), and (96) specify the non-zero-mean problem. Some typical examples are developed in the problems.

3.4.3 Stationary "Carrier-symmetric" Bandpass Problems

Many of the processes that we encounter in practice are bandpass processes centered around a carrier frequency. In Chapter 11, we shall explore this class of problem in detail. By introducing suitable notation we shall be able to study the general bandpass process efficiently. In this section we consider a special class of bandpass problems that can be related easily to the corresponding low-pass problem. We introduce this special class at this point because it occurs frequently in practice. Thus, it is a good vehicle for discussing some of the solution techniques in Chapter 4.

The received waveforms on the two hypotheses are

$$r(t) = s_1(t) + w(t), \qquad T_i \le t \le T_f : H_1,$$

$$r(t) = s_0(t) + w(t), \qquad T_i \le t \le T_f : H_0.$$
(98)

The signal $s_1(t)$ is a segment of a sample function of a zero-mean stationary Gaussian process whose spectrum is narrow-band and symmetric about a carrier ω_1 . The signal $s_0(t)$ is a segment of a sample function of a zero-mean stationary Gaussian process whose spectrum is narrow-band and symmetric about a carrier ω_0 . The two spectra are essentially disjoint, as illustrated



Fig. 3.8 Disjoint bandpass spectra.

in Fig. 3.8. This problem differs from that in Section 3.4.1 in that we do *not* require that the two processes have the same eigenvalues in the present problem.

To develop the receiver structure, we multiply r(t) by the four carriers shown in Fig. 3.9 and pass the resulting outputs through ideal low-pass filters. These low-pass filters pass the frequency-shifted versions of $s_1(t)$ and $s_0(t)$ without distortion. We now have four waveforms, $r_{c_1}(t)$, $r_{s_1}(t)$, $r_{c_0}(t)$, and $r_{s_0}(t)$, to use as inputs for our likelihood ratio test. The four waveforms on the two hypotheses are

$$r_{c_{1}}(t) = s_{c_{1}}(t) + w_{c_{1}}(t), \qquad T_{i} \leq t \leq T_{f} \\ r_{s_{1}}(t) = s_{s_{1}}(t) + w_{s_{1}}(t), \qquad T_{i} \leq t \leq T_{f} \\ r_{c_{0}}(t) = w_{c_{0}}(t), \qquad T_{i} \leq t \leq T_{f} \\ r_{s_{0}}(t) = w_{s_{0}}(t), \qquad T_{i} \leq t \leq T_{f} \\ r_{c_{1}}(t) = w_{c_{1}}(t), \qquad T_{i} \leq t \leq T_{f} \\ r_{s_{1}}(t) = w_{s_{1}}(t), \qquad T_{i} \leq t \leq T_{f} \\ r_{c_{0}}(t) = s_{c_{0}}(t) + w_{c_{0}}(t), \qquad T_{i} \leq t \leq T_{f} \\ r_{s_{0}}(t) = s_{s_{0}}(t) + w_{s_{0}}(t), \qquad T_{i} \leq t \leq T_{f} \\ r_{s_{0}}(t) = s_{s_{0}}(t) + w_{s_{0}}(t), \qquad T_{i} \leq t \leq T_{f} \\ r_{s_{0}}(t) = s_{s_{0}}(t) + w_{s_{0}}(t), \qquad T_{i} \leq t \leq T_{f} \\ r_{s_{0}}(t) = s_{s_{0}}(t) + w_{s_{0}}(t), \qquad T_{i} \leq t \leq T_{f} \\ r_{s_{0}}(t) = s_{s_{0}}(t) + w_{s_{0}}(t), \qquad T_{i} \leq t \leq T_{f} \\ r_{s_{0}}(t) = s_{s_{0}}(t) + w_{s_{0}}(t), \qquad T_{i} \leq t \leq T_{f} \\ r_{s_{0}}(t) = s_{s_{0}}(t) + w_{s_{0}}(t), \qquad T_{i} \leq t \leq T_{f} \\ r_{s_{0}}(t) = s_{s_{0}}(t) + w_{s_{0}}(t), \qquad T_{i} \leq t \leq T_{f} \\ r_{s_{0}}(t) = s_{s_{0}}(t) + w_{s_{0}}(t), \qquad T_{i} \leq t \leq T_{f} \\ r_{s_{0}}(t) = s_{s_{0}}(t) + w_{s_{0}}(t), \qquad T_{i} \leq t \leq T_{f} \\ r_{s_{0}}(t) = s_{s_{0}}(t) + w_{s_{0}}(t), \qquad T_{i} \leq t \leq T_{f} \\ r_{s_{0}}(t) = s_{s_{0}}(t) + w_{s_{0}}(t), \qquad T_{i} \leq t \leq T_{f} \\ r_{s_{0}}(t) = s_{s_{0}}(t) + w_{s_{0}}(t), \qquad T_{i} \leq t \leq T_{f} \\ r_{s_{0}}(t) = s_{s_{0}}(t) + w_{s_{0}}(t), \qquad T_{i} \leq t \leq T_{f} \\ r_{s_{0}}(t) = s_{s_{0}}(t) + w_{s_{0}}(t), \qquad T_{i} \leq t \leq T_{f} \\ r_{s_{0}}(t) = s_{s_{0}}(t) + w_{s_{0}}(t), \qquad T_{i} \leq t \leq T_{f} \\ r_{s_{0}}(t) = s_{s_{0}}(t) + w_{s_{0}}(t), \qquad T_{i} \leq t \leq T_{f} \\ r_{s_{0}}(t) = s_{s_{0}}(t) + w_{s_{0}}(t), \qquad T_{i} \leq t \leq T_{f} \\ r_{s_{0}}(t) = s_{s_{0}}(t) + w_{s_{0}}(t), \qquad T_{i} \leq t \leq T_{f} \\ r_{s_{0}}(t) = s_{s_{0}}(t) + w_{s_{0}}(t) \\ r_{s_{0}}(t) = s_{s_{0}}(t) \\ r_{s_{0}}(t) = s_{s_{0}}(t)$$

Because of the assumed symmetry of the spectra, *all* of the processes are statistically independent (e.g., Appendix A.3.1). The processes $s_{c_1}(t)$ and $s_{s_1}(t)$ have identical spectra, which we denote by $S_{L_1}(\omega)$. It is just the low-pass component of the bandpass spectrum after it has been shifted to the origin. Similarly, $s_{c_0}(t)$ and $s_{s_0}(t)$ have identical spectra, which we denote by $S_{L_0}(\omega)$. In view of the statistical independence, we can write the LRT by inspection. By analogy with (30), the LRT is

$$l_{R_{c_1}} + l_{R_{s_1}} + l_{B_{c_1}} + l_{B_{s_1}} - l_{R_{c_0}} - l_{R_{s_0}} - l_{B_{c_0}} - l_{B_{s_0}} \overset{H_1}{\underset{H_0}{\gtrsim}} \ln \eta, \quad (100)$$



Fig. 3.9 Generation of low-pass waveforms.



Fig. 3.10 Optimum processing of low-pass waveforms.

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where the definitions of the various terms are parallel to (27) and (29). A filter-squarer version of the optimum receiver is shown in Fig. 3.10. (Notice that $l_{B_{c1}} = l_{B_{c1}}$.) In most cases, the filters before the squarer are low-pass, so that the ideal low-pass filters in Fig. 3.9 can be omitted. In Chapter 11, we develop a more efficient realization using bandpass filters and square-law envelope detectors.

To evaluate the performance, we observe that the sine components provide exactly the same amount of information as the cosine components. Thus, we would expect that

$$\mu_{BP}(s) = 2\mu_{LP}(s), \tag{101}$$

where the subscript *BP* denotes the actual bandpass problem and the subscript *LP* denotes a low-pass problem with inputs $r_{c_1}(t)$ and $r_{c_0}(t)$. Notice that the power (or energy) in the low-pass problem is one-half the power (or energy) in the bandpass problem.

$$P_{LP} = \frac{P_{BP}}{2}, \qquad (102)$$

$$E_{r_{LP}} = \frac{E_{r_{BP}}}{2}.$$
(103)

It is straightforward to verify that (101)-(103) are correct (see Problem 3.4.8). Notice that since the bandpass process generates two statistically independent low-pass processes, we can show that the eigenvalues of the bandpass process occur in pairs.

The important conclusion is that, for this *special class* of bandpass problems, there is an equivalent low-pass problem that can be obtained by a simple scale change. Notice that three assumptions were made:

- 1. The signal processes are stationary.
- 2. The signal spectra on the two hypotheses are essentially disjoint.
- 3. The signal spectra are symmetric about their respective carriers.

Later we shall consider asymmetric spectra and nonstationary spectra. In those cases the transition will be more involved and it will be necessary to develop a more efficient notation.

3.4.4 Error Probability for the Binary Symmetric Bandpass Problem

In this section we consider the binary symmetric bandpass problem. The model for this problem satisfies the assumptions of both Sections 3.4.1 and 3.4.3. We shall derive tight upper and lower bounds for the Pr (ϵ).[†]

Because we have assumed equally likely hypotheses and a minimum total probability of error criterion, we have

$$\Pr[\epsilon] = \Pr[\epsilon \mid H_0]$$

$$= \int_0^\infty p_{\iota \mid H_0}(L) \, dL$$

$$= \int_0^\infty \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} M_{\iota \mid H_0}(w) e^{-wL} \, dw \, dL \quad \text{for} \quad 0 \le \sigma \le 1.$$
(104)

Notice that w is a complex variable, \ddagger

$$w = \sigma + jv. \tag{105}$$

Interchanging the order of integration and evaluating the results in the integral, we obtain

$$\Pr\left[\epsilon\right] = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} M_{l|H_0}(w) \int_0^\infty e^{-wL} dL dw$$
$$= \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} \frac{1}{w} M_{l|H_0}(w) dw$$
$$= \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} \frac{e^{\mu(w)}}{w} dw, \qquad 0 < \sigma \le 1.$$
(106)

For our specific problem, $\mu(w)$ follows immediately from (57) and (67),

$$\mu(w) = \sum_{i=1}^{\infty} \left[\ln \left(1 + \frac{2\lambda_i}{N_0} \right) - \ln \left(1 + \frac{2w\lambda_i}{N_0} \right) - \ln \left(1 + \frac{2(1-w)\lambda_i}{N_0} \right) \right].$$
(107)

Notice that we have used (101) to eliminate the one-half factor in $\mu(s)$. As pointed out earlier, this is because the eigenvalues appear in pairs in the bandpass problem. From (107),

$$e^{\mu(w)} = \prod_{i=1}^{\infty} \frac{(1 + (2\lambda_i/N_0))}{(1 + (2w\lambda_i/N_0))(1 + (2(1 - w)\lambda_i/N_0))}.$$
 (108)

Thus,

$$\Pr\left[\epsilon\right] = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} \frac{1}{w} \prod_{i=1}^{\infty} \frac{(1+(2\lambda_i/N_0)) dw}{(1+(2w\lambda_i/N_0))(1+(2(1-w)\lambda_i/N_0))}$$
for $0 < \sigma \le 1$. (109)

† Our discussion in this section follows [2]. The original results are due to Pierce [6]. ‡ All of our previous discussions assumed that the argument of $M_{l|H_0}(\cdot)$ was real. The necessary properties are also valid for complex arguments with the restriction $0 \le \text{Re}[w] \le 1$. The result in (109) is due to Turin ([7], Eq. 27). Pierce [6] started with (109) and derived tight upper and lower bounds on Pr (ϵ). Since his derivation is readily available, we omit it and merely state the result. (A simple derivation using our notation is given in [2].) We can show that

$$\operatorname{erfc}_{*}\left(-\frac{\sqrt{\mu(\frac{1}{2})}}{2}\right) \exp\left[\mu(\frac{1}{2}) + \frac{\mu(\frac{1}{2})}{8}\right] \le \Pr\left[\epsilon\right] \le \frac{\exp\left[\mu(\frac{1}{2})\right]}{2(1+\sqrt{\mu(\frac{1}{2})/8})}.$$
 (110)

The lower bound can be further relaxed to yield

$$\frac{\exp\left[\mu(\frac{1}{2})\right]}{2(1+\sqrt{(\frac{\pi}{8})\ddot{\mu}(\frac{1}{2})})} \le \Pr\left[\epsilon\right] \le \frac{\exp\left[\mu(\frac{1}{2})\right]}{2(1+\sqrt{(\frac{1}{8})\ddot{\mu}(\frac{1}{2})})},$$
(111)

which is Pierce's result. Notice that the upper and lower bounds differ at most by $\sqrt{\pi}$. From (76), we see that our first-order approximation to Pr (ϵ) is identical with the lower bound in (110). Thus, for the *binary-symmetric* bandpass case, our approximate error expression is always within a factor of $\sqrt{\pi}$ of the exact Pr (ϵ). Notice that our result assumes that the spectra are symmetric about their carriers. The results in (110) and (111) are also valid for asymmetric spectra. We shall prove this in Chapter 11.

We have not been able to extent Pierce's derivation to the asymmetric case in order to obtain tight bounds on P_D and P_F . However several specific examples indicate that our approximate error expressions give accurate results.

In this section we have examined four special models of the Gaussian problem. In the next section we return to the general problem and look at the effect of removing the white noise assumption.

3.5 GENERAL BINARY CASE: WHITE NOISE NOT NECESSARILY PRESENT: SINGULAR TESTS

In this section, we discuss the general binary problem briefly. The received waveforms on the two hypotheses are

$$r(t) = r_1(t), T_i \le t \le T_f: H_1, r(t) = r_0(t), T_i \le t \le T_f: H_0. (112)$$

The processes are zero-mean Gaussian processes with covariance functions $K_{H_1}(t, u)$ and $K_{H_0}(t, u)$, respectively. We assume that both processes are (strictly) positive-definite.

3.5.1 Receiver Derivation

To solve the problem we first pass r(t) through a whitening filter to generate an output $r_*(t)$, which is white on H_0 . Previously we have denoted the whitening filter by $h_w(t, z)$. In our present discussion we shall denote it by $K_{H_0}^{[-\frac{1}{2}]}(t, z)$. The reason for this notation will become obvious shortly.

$$r_{*}(t) = \int_{T_{i}}^{T_{i}} K_{H_{0}}^{[-\frac{1}{2}]}(t, z) r(z) \, dz.$$
(113)

The whitening requirement implies

$$E[r_{*}(t)r_{*}(u) \mid H_{0}] = \delta(t - u)$$

=
$$\iint_{T_{i}}^{T_{f}} K_{H_{0}}^{[-1/2]}(t, z) K_{H_{0}}(z, y) K_{H_{0}}^{[-1/2]}(u, y) dz dy. \quad (114)$$

On H_1 , the covariance of $r_*(t)$ is

$$E[r_{*}(t)r_{*}(u) \mid H_{1}] = \iint_{T_{i}}^{T_{f}} K_{H_{0}}^{[-\frac{1}{2}]}(t,z) K_{H_{1}}(z,y) K_{H_{0}}^{[-\frac{1}{2}]}(u,y) \, dz \, dy \, \Delta K_{1}^{*}(t,u).$$
(115)

We can now expand $r_*(t)$ using the eigenfunctions of $K_1^*(t, u)$,

$$\lambda_i^* \phi_i(t) = \int_{T_i}^{T_f} K_1^*(t, u) \phi_i(u) \, du, \qquad T_i \le t \le T_f. \tag{116}$$

The coefficients are

$$r_i \Delta \int_{T_i}^{T_f} r_*(t) \phi_i(t) dt, \qquad (117)$$

and the waveform is

$$r_{*}(t) = \lim_{K \to \infty} \sum_{i=1}^{K} r_{i} \phi_{i}(t), \qquad T_{i} \le t \le T_{f}.$$
(118)

The coefficients are zero-mean. Their covariances are

$$E[r_i r_j \mid H_1] = \lambda_i^* \delta_{ij} \tag{119}$$

and

$$E[r_i r_j \mid H_0] = \delta_{ij}. \tag{120}$$

Notice that we could also write (116) as

$$\lambda_{i}^{*}\phi_{i}(t) = \int_{T_{i}}^{T_{f}} \left[\iint_{T_{i}}^{T_{f}} K_{H_{0}}^{[-\nu_{2}]}(t,z) K_{H_{1}}(z,y) K_{H_{0}}^{[-\nu_{2}]}(u,y) \, dz \, dy \right] \phi_{i}(u) \, du,$$
$$T_{i} \leq t \leq T_{f}. \tag{121}$$

Now we define the function $K_{H_0}^{[\frac{1}{2}]}(z, t)$ implicitly by the relation

$$K_{H_0}(t, u) = \int_{T_i}^{T_f} K_{H_0}^{[\frac{1}{2}]}(z, t) K_{H_0}^{[\frac{1}{2}]}(z, u) \, dz.$$
(122)

We see that $K_{H_0}^{[1/2]}(z,t)$ is just the functional square root of the covariance function $K_{H_0}(t, u)$. Observe that

$$\delta(t-u) = \int_{T_i}^{T_f} K_{H_0}^{[\frac{1}{2}]}(t,z) K_{H_0}^{[-\frac{1}{2}]}(z,u) \, dz.$$
(123)

The result in (123) can be verified by writing each term in an orthogonal series expansion. Multiplying both sides of (121) by $K_{H_0}^{[-1/2]}(\cdot, \cdot)$, integrating, and using (123), we obtain

$$\lambda_{i}^{*} \int_{T_{i}}^{T_{f}} K_{H_{0}}(t, u) \, du \left[\int_{T_{i}}^{T_{f}} K_{H_{0}}^{[-1/2]}(u, z) \phi_{i}(z) \, dz \right]$$
$$= \int_{T_{i}}^{T_{f}} K_{H_{1}}(t, u) \, du \left[\int_{T_{i}}^{T_{f}} K_{H_{0}}^{[-1/2]}(u, z) \phi_{i}(z) \, dz \right].$$
(124)

If we define

$$\psi_i(u) \triangleq \int_{T_i}^{T_f} K_{H_0}^{[-1/_2]}(u, z) \phi_i(z) \, dz, \qquad (125)$$

(124) becomes

$$\lambda_{i}^{*} \int_{T_{f}}^{T_{f}} K_{H_{0}}(t, u) \psi_{i}(u) \, du = \int_{T_{i}}^{T_{f}} K_{H_{1}}(t, u) \psi_{i}(u) \, du.$$
(126)

Notice that we could also write the original waveform r(t) as

$$r(t) = \lim_{K \to \infty} \sum_{i=1}^{K} r_i \left\{ \int_{T_i}^{T_f} K_{H_0}(t, u) \psi_i(u) \, du \right\}.$$
 (127)

Thus we have available a decomposition that gives statistically independent coefficients on both hypotheses. The likelihood ratio test is

$$\Lambda(\mathbf{R}) = \frac{\prod_{i=1}^{K} \frac{1}{(2\pi)^{K/2} (\lambda_i^*)^{\frac{1}{2}}} \exp\left(-\frac{1}{2} \frac{R_i^2}{\lambda_i^*}\right)}{\prod_{i=1}^{K} \frac{1}{(2\pi)^{K/2}} \exp\left(-\frac{1}{2} R_i^2\right)} \stackrel{H_1}{\gtrsim} \eta.$$
(128)

If we let $K \rightarrow \infty$, this reduces to

$$l_R \stackrel{\Delta}{=} \frac{1}{2} \sum_{i=1}^{\infty} R_i^2 \left(\frac{\lambda_i^* - 1}{\lambda_i^*} \right) \stackrel{H_1}{\underset{H_0}{\gtrsim}} \ln \eta + \frac{1}{2} \sum_{i=1}^{\infty} \ln \lambda_i^* \stackrel{\Delta}{=} \gamma.$$
(129)

We now define a kernel,

$$h_{*}(t, u) \stackrel{\Delta}{=} \sum_{i=1}^{\infty} \left(\frac{\lambda_{i}^{*} - 1}{\lambda_{i}^{*}} \right) \phi_{i}(t) \phi_{i}(u), \qquad T_{i} \leq t, u \leq T_{f}, \qquad (130)$$

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that satisfies the integral equation

$$\int_{T_i}^{T_f} h_*(t, z) K_1^*(z, u) \, dz = K_1^*(t, u) - \delta(t - u), \qquad T_i \le t, u \le T_f. \tag{131}$$
Then

Then

$$l_R = \frac{1}{2} \iint_{T_i}^{T_f} r_*(t) h_*(t, u) r_*(u) \, dt \, du.$$
(132)

Using (113), we have

$$l_{R} = \frac{1}{2} \iint_{T_{i}}^{T_{f}} dx \, dy \, r(x) \Biggl\{ \iint_{T_{i}}^{T_{f}} K_{H_{0}}^{[-\frac{1}{2}]}(t, x) h_{*}(t, u) K_{H_{0}}^{[-\frac{1}{2}]}(u, y) \, dt \, du \Biggr\} r(y).$$
(133)

Defining

$$h_{\Delta}(x, y) \triangleq \iint_{T_{i}}^{T_{f}} K_{H_{0}}^{[-\frac{1}{2}]}(t, x) h_{*}(t, u) K_{H_{0}}^{[-\frac{1}{2}]}(u, y) dt du, \qquad (134)$$

we have

$$l_{R} = \frac{1}{2} \iint_{T_{i}}^{T_{f}} dx \, dy \, r(x) h_{\Delta}(x, y) r(y).$$
(135)

Starting with (134), it is straightforward to show that $h_{\Delta}(x, y)$ satisfies the equation

$$\iint_{T_{i}}^{T_{f}} K_{H_{0}}(t, x) h_{\Delta}(x, y) K_{H_{1}}(y, u) \, dx \, dy = K_{H_{1}}(t, u) - K_{H_{0}}(t, u),$$

$$T_{i} \leq t, u \leq T_{f}.$$
(136)

As we would expect, the result in (136) is identical with that in (33). The next step is to evaluate the performance of the optimum receiver.

3.5.2 Performance: General Binary Case

To evaluate the performance in the general binary case, we use (2.126) to evaluate $\mu(s)$.

$$\mu(s) = \lim_{K \to \infty} \sum_{i=1}^{K} \ln \left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi} (\lambda_{i}^{*})^{s/2}} \exp\left(-\frac{sR_{i}^{2}}{2\lambda_{i}^{*}}\right) \exp\left(-\frac{(1-s)R_{i}^{2}}{2}\right) dR_{i} \right].$$
(137)

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Evaluating the integral, we have

$$\mu(s) = \sum_{i=1}^{\infty} \ln \left\{ \frac{(\lambda_i^*)^{(1-s)/2}}{(s+(1-s)\lambda_i^*)^{\frac{1}{2}}} \right\},$$
(138)

where the λ_i^* are the eigenvalues of the kernel,

$$K_1^*(t, u) = \int_{T_i}^{T_f} K_{H_0}^{[-\frac{1}{2}]}(t, z) K_{H_1}(z, x) K_{H_0}^{[-\frac{1}{2}]}(u, x) \, dz \, dx.$$
(139)

In our discussion of performance for the case of a known signal in Gaussian noise, we saw that when there was no white noise present it was possible to make perfect decisions under some circumstances (see pages I-303–I-306). We now consider the analogous issue for the general Gaussian problem.

3.5.3 Singularity

The purpose of our singularity discussion is to obtain a necessary and sufficient condition for the test to be nonsingular. The derivation is a sequence of lemmas. As before, we say that a test is singular if $Pr(\epsilon) = 0$. Notice that we do not assume equally likely hypotheses, and

$$\Pr(\epsilon) = P_1 \Pr(\epsilon \mid H_1) + P_0 \Pr(\epsilon \mid H_0).$$
(140)

The steps in the development are the following:

1. We show that the Pr (ϵ) is greater than zero iff $\mu(\frac{1}{2})$ is finite.

2. We then derive a necessary and sufficient condition for $\mu(\frac{1}{2})$ to be finite.

Finally we consider two simple examples of singular tests.

Lemma 1. The Pr (ϵ) can be bounded by

$$\frac{1}{2} \{ \min \left[P_{H_1}, P_{H_0} \right] \} e^{2\mu(\frac{1}{2})} \le \Pr(\epsilon) \le \frac{1}{2} e^{\mu(\frac{1}{2})}$$
(141)

Therefore the $Pr(\epsilon)$ will equal zero if P_0 or P_1 or $e^{\mu(\frac{1}{2})}$ equals zero. If we assume that P_1 and P_0 are positive then $Pr(\epsilon)$ will be greater than zero *iff* $\mu(\frac{1}{2})$ is finite. In other words, a singular test will occur *iff* $\mu(\frac{1}{2})$ diverges.

The upper bound is familiar. The proof of the lower bound is straightforward.

Proof.† Let

$$\alpha \triangleq e^{\mu(\frac{1}{2})} = \int_{-\infty}^{\infty} [p_{r|H_1}(\mathbf{R} \mid H_1)p_{r|H_0}(\mathbf{R} \mid H_0)]^{\frac{1}{2}} d\mathbf{R}.$$
 (142a)

† This result is similar to that in [8].

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Now observe, from the Schwarz inequality, that for any set S,

$$\int_{S} [p_{\mathbf{r}|H_{1}}(\mathbf{R} \mid H_{1})p_{\mathbf{r}|H_{0}}(\mathbf{R} \mid H_{0})]^{\frac{1}{2}} d\mathbf{R} \leq \left[\int_{S} p_{\mathbf{r}|H_{1}}(\mathbf{R} \mid H_{1}) d\mathbf{R} \int_{S} p_{\mathbf{r}|H_{0}}(\mathbf{R} \mid H_{0}) d\mathbf{R}\right]^{\frac{1}{2}} \leq \left[\int_{S} p_{\mathbf{r}|H_{m}}(\mathbf{R} \mid H_{m}) d\mathbf{R}\right]^{\frac{1}{2}}, \quad m = 0, 1. \quad (142b)$$

We recall from page I-30 that the probability of error using the optimum test is

$$Pr(\epsilon) = P_1 \int_{Z_0} p_{r|H_1}(\mathbf{R} \mid H_1) \, d\mathbf{R} + P_0 \int_{Z_1} p_{r|H_0}(\mathbf{R} \mid H_0) \, d\mathbf{R}$$

$$\geq \min \left[P_1, P_0 \right] \left\{ \int_{Z_0} p_{r|H_1}(\mathbf{R} \mid H_1) \, d\mathbf{R} + \int_{Z_1} p_{r|H_0}(\mathbf{R} \mid H_0) \, d\mathbf{R} \right\} \quad (143a)$$

Using the result in (142b) on each integral in (143a) gives

$$\begin{aligned} \Pr(\epsilon) &\geq \min \left[P_{1}, P_{0} \right] \left\{ \left(\int_{Z_{0}} [p_{r|H_{1}}(\mathbf{R} \mid H_{1})p_{r|H_{0}}(\mathbf{R} \mid H_{0})]^{\frac{1}{2}} d\mathbf{R} \right)^{2} \\ &+ \left(\int_{Z_{1}} [p_{r|H_{1}}(\mathbf{R} \mid H_{1})p_{r|H_{0}}(\mathbf{R} \mid H_{0})]^{\frac{1}{2}} d\mathbf{R} \right)^{2} \right\} \\ &= \min \left[P_{1}, P_{0} \right] \left\{ \left(\int_{Z_{0}} [p_{r|H_{1}}(\mathbf{R} \mid H_{1})p_{r|H_{0}}(\mathbf{R} \mid H_{0})]^{\frac{1}{2}} d\mathbf{R} \right)^{2} \\ &+ \left(\alpha - \int_{Z_{0}} [P_{r|H_{1}}(\mathbf{R} \mid H_{1})p_{r|H_{0}}(\mathbf{R} \mid H_{0})]^{\frac{1}{2}} d\mathbf{R} \right)^{2} \right\} \\ &= \min \left[P_{1}, P_{0} \right] \{x^{2} + (\alpha - x)^{2}\}, \end{aligned}$$
(143b)

where

$$x \triangleq \int_{Z_0} [p_{\mathsf{r}|H_1}(\mathsf{R} \mid H_1)p_{\mathsf{r}|H_0}(\mathsf{R} \mid H_0)]^{\frac{1}{2}} d\mathsf{R}, \qquad (143c)$$

and x will lie somewhere in the range $[0, \alpha]$.

The term in the brackets in (143b) could be minimized by setting

$$x = \frac{\alpha}{2} \tag{143d}$$

and the minimum value is

$$\frac{\alpha^2}{2} = \frac{1}{2} e^{2\mu(1/2)}$$
(143e)

Thus,

$$\Pr(\epsilon) \ge \min[P_1, P_0] \cdot \frac{1}{2} e^{2\mu(1/2)}$$
(144)

which is the desired result. We should emphasize that the lower bound in (141) is used for the purpose of our singularity discussion and so it does not need to be a tight bound.

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Lemma 2. From (138),

$$\mu(\frac{1}{2}) = \frac{1}{4} \sum_{i=1}^{\infty} \ln\left(\frac{4\lambda_i^*}{(1+\lambda_i^*)^2}\right).$$
 (145)

In order for $\mu(\frac{1}{2})$ to be finite, all λ_i^* must be greater than zero. If this is true, then, in order for $\mu(\frac{1}{2})$ to be finite, it is necessary and sufficient that

$$\sum_{i=1}^{\infty} (1 - \lambda_i^*)^2 < \infty.$$
(146)

Proof (from [9]). The convergence properties of the following sums can be demonstrated.

$$\sum_{i=1}^{\infty} \ln\left(\frac{4\lambda_i^*}{(1+\lambda_i^*)^2}\right) < \infty$$
(147)

iff

$$\sum_{i=1}^{\infty} \left(1 - \frac{4\lambda_i^*}{(1+\lambda_i^*)^2} \right) = \sum_{i=1}^{\infty} \frac{(1-\lambda_i^*)^2}{(1+\lambda_i^*)^2} < \infty$$
(148)

iff

$$\sum_{i=1}^{\infty} (1-\lambda_i^*)^2 < \infty.$$
(149)

These equivalences can be verified easily.

Lemma 3. Define

$$\lambda_i^{**} \stackrel{\Delta}{-} \lambda_i^* - 1 \tag{150}$$

and a kernel,

$$Y(t, u) \triangleq \iint_{T_{i}}^{T_{f}} K_{H_{0}}^{[-\frac{1}{2}]}(t, x) K_{H_{1}}(x, z) K_{H_{0}}^{[-\frac{1}{2}]}(u, z) \, dx \, dz - \delta(t - u),$$
$$T_{i} \leq t, u \leq T_{f}.$$
 (151)

The λ_i^{**} are the eigenvalues of Y(t, u). Notice that Y(t, u) is not necesarily positive-definite (i.e., some of the λ_i^{**} may be negative).

Lemma 4. The value of $\mu(\frac{1}{2})$ will be finite iff:

(i) All $\lambda_i^{**} > -1$, (ii) The sum $\sum_{i=1}^{\infty} (\lambda_i^{**})^2$ is finite.

Assuming the first condition is satisfied, then, in order for

$$\sum_{i=1}^{\infty} (\lambda_i^{**})^2 < \infty \tag{152}$$

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it is necessary and sufficient that

$$\iint_{T_i}^{T_f} Y^2(t, u) \, dt \, du < \infty. \tag{153}$$

The equation (151) can also be written as

$$K_{\Delta}(t, u) \stackrel{\Delta}{=} K_{H_{1}}(t, u) - K_{H_{0}}(t, u) = \int_{T_{i}}^{T_{f}} K_{H_{0}}(t, x) Y(x, u) \, dx,$$

$$T_{i} \leq t, u \leq T_{f}.$$
 (154)

This equation must have a square-integrable solution.

Summarizing, a necessary and sufficient condition for a nonsingular test is that the function Y(t, u) defined by (151) or (154) be square-integrable and not have -1 as an eigenvalue.

Several observations are useful.

1. The result in (150)–(154) has a simple physical interpretation. The covariance function of the whitened waveform $r_*(t)$ on H_1 must consist of an impulse with unit area and a positive-definite square-integrable component.

2. The problem is symmetric, so that the entire discussion is valid with the subscripts 0 and 1 interchanged. Thus we can check the conditions given in (151) and (153) for whichever case is the simplest. Notice that it is not necessary to check both.

3. The function $\mu(s)$ can be written in terms of the eigenvalues of Y(t, u). Using (138) and (150),

$$\mu(s) = \sum_{i=1}^{\infty} \ln\left[\frac{(1+\lambda_i^{**})^{(1-s)/2}}{(1+(1-s)\lambda_i^{**})^{1/2}}\right],$$
(155)

where the λ_i^{**} are the eigenvalues of Y(t, u), which may be either positive or negative. Notice that in order for $\mu(s)$ to be finite, it is sufficient, but not necessary, for the logarithm of the numerator and denominator of (155) to converge individually (see Problem 3.5.11).

We now consider two simple examples of singular tests.

Example 1. Let

$$K_{H_0}(t, u) = K(t, u)$$
 (156)

$$K_{H_1}(t, u) = \alpha K(t, u). \tag{157}$$

and

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Then

$$\int_{-\infty}^{\infty} K_{H_0}^{[-\frac{1}{2}]}(t,x) K_{H_1}(x,y) K_{H_0}^{[-\frac{1}{2}]}(u,y) \, dx \, dy = \alpha \delta(t-u) \tag{158}$$

and

$$Y(t, u) = (\alpha - 1)\delta(t - u), \qquad (159)$$

which is not square-integrable unless $\alpha = 1$.

m.

Thus, when the covariance functions on the two hypotheses are identical except for an amplitude factor, the test is singular.

Example 2. Let

$$K_{H_0}(t, u) = P_0 \exp(-\alpha |t - u|)$$
(160)

anđ

$$K_{H_1}(t, u) = P_1 \exp\left(-\beta |t - u|\right).$$
(161)

For this particular example, the simplest procedure is to construct the whitening filter From page I-312,

$$r_{*}(t) = \frac{1}{\sqrt{2\alpha P_{0}}} [\dot{r}(t) + \alpha r(t)], \qquad (162)$$

or

$$K_{H_0}^{[-1/2]}(t, u) = \frac{1}{\sqrt{2\alpha P_0}} \left[\delta^{[1]}(t-u) + \alpha \delta(t-u) \right]$$
(163)†

The covariance function of $r_*(t)$ on H_1 is

$$K_1^{*}(t,u) = \frac{1}{2\alpha P_0} \left\{ \frac{\partial^2 K_{H_1}(t,u)}{\partial t \ \partial u} + \alpha \frac{\partial K_{H_1}(t,u)}{\partial t} + \alpha \frac{\partial K_{H_1}(t,u)}{\partial u} + \alpha^2 K_{H_1}(t,u) \right\}.$$
 (164)

Only the first term contains an impulse,

$$\frac{\partial^2 K_{H_1}(t, u)}{\partial t \, \partial u} = 2\beta P_1 \,\delta(t-u) - \beta^2 P_1 \exp\left(-\beta \left|t-u\right|\right). \tag{165}$$

In order for the test to be nonsingular, we require

$$\frac{\beta P_1}{\alpha P_0} = 1. \tag{166}$$

Otherwise (153) cannot be satisfied.

Example 2 suggests a simple test for singularity that can be used when the random processes on the two hypotheses are stationary with rational spectra. In this case, a necessary and sufficient condition for a nonsingular test is

$$\lim_{\omega \to \infty} \left(\frac{S_{H_1}(\omega)}{S_{H_0}(\omega)} \right) = 1$$
(167)

(see Problem 3.5.12).

[†] The symbol $\delta^{[1]}(\tau)$ denotes a doublet at $\tau = 0$.

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Several other examples of singular tests are discussed in the problems. For a rigorous and more detailed discussion, the interested reader should consult Root [9], Feldman [10]–[11], Hájek [12] or Shepp [15]. As we commented in our discussion of the class A_w problem, we can guarantee that the test is nonsingular by including the same white noise component on both hypotheses. Since the inclusion of the white noise component usually can be justified on physical grounds, we can avoid the singularity problem in this manner. We now summarize our results for the general binary detection problem.

3.6 SUMMARY: GENERAL BINARY PROBLEM

In this chapter we have discussed the general binary problem. In our initial discussion we considered class A_w problems. In this class, the same white noise process was present on both hypotheses. The likelihood ratio test can be implemented as a parallel processing receiver that computes l_{R_s} and l_{B_s} ,

$$l_{R_i} = \frac{1}{N_0} \iint_{T_i}^{T_f} r(\alpha) h_i(\alpha, \beta) r(\beta) \, d\alpha \, d\beta, \qquad i = 0, 1, \tag{168}$$

where $h(\alpha, \beta)$ is defined by (25), and

$$l_{B_i} = -\frac{1}{N_0} \int_{T_i}^{T_f} \xi_{P_i} \left(t \left| \frac{N_0}{2} \right) dt, \quad i = 0, 1,$$
 (169)

where $\xi_{P_i}(t \mid N_0/2)$ is a MMSE defined on page 22. The receiver then performs the test

$$l_{R_1} + l_{B_1} - l_{R_0} - l_{B_0} \underset{H_0}{\stackrel{H_1}{\geq}} \ln \eta.$$
(170)

The processing indicated in (168) can be implemented using the one of the canonical realizations developed in Section 2.1. The performance was calculated by computing $\mu(s)$ defined in (60).

$$\mu(s) = \frac{1}{N_0} \int_{T_i}^{T_f} dt \left[(1-s)\xi_P \left(t \mid s_1(\cdot), \frac{N_0}{2} \right) + s\xi_P \left(t \mid s_0(\cdot), \frac{N_0}{2} \right) - \xi_P \left(t \mid s_{\rm com}(\cdot), \frac{N_0}{2} \right) \right], (171)$$

where the individual terms are defined on page 68.

For the general binary case, the test is

$$l_R + l_B \underset{H_0}{\gtrsim} \ln \eta, \tag{172}$$

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where

$$l_{R} = \frac{1}{2} \iint_{T_{i}}^{T_{f}} dx \, dy \, r(x) h_{\Delta}(x, y) r(y)$$
(173)

and

$$l_{B} = -\frac{1}{2} \sum_{i=1}^{\infty} \ln (\lambda_{i}^{*}).$$
 (174)

The kernel $h_{\Delta}(x, y)$ satisfies the equation

$$\iint_{T_{i}}^{T_{f}} K_{H_{0}}(t, x) h_{\Delta}(x, y) K_{H_{1}}(y, u) \, dx \, dy = K_{H_{1}}(t, u) - K_{H_{0}}(t, u),$$

$$T_{i} \leq t, u \leq T_{f}. \quad (175)$$

The λ_i^* are the eigenvalues of the kernel

$$K_{1}^{*}(t, u) = \iint_{T_{t}}^{T_{f}} K_{H_{0}}^{[-\frac{1}{2}]}(t, z) K_{H_{1}}(z, x) K_{H_{0}}^{[-\frac{1}{2}]}(u, x) \, dz \, dx.$$
(176)

When we remove the white noise assumption, we have to be careful that our model does not lead to a singular test. We demonstrated that a necessary and sufficient condition for a nonsingular test is that

$$Y(t, u) \stackrel{\Delta}{=} K_1^*(t, u) - \delta(t - u), \qquad T_i \le t, u \le T_f$$
(177)

be a square-integrable function which does not have -1 as an eigenvalue. The performance could be evaluated by computing $\mu(s)$:

$$\mu(s) = \sum_{i=1}^{\infty} \ln \left\{ \frac{(1+\lambda_i^{**})^{(1-s)/2}}{(1+(1-s)\lambda_i^{**})^{\frac{1}{2}}} \right\},$$
(178)

where the λ_i^{**} are the eigenvalues of Y(t, u).

In addition to our general discussion, we considered several special situations. The first was the binary symmetric problem. The most important result was the relationship of $\mu_{BS}(s)$ to $\mu_{SIB}(s)$ for the simple binary problem of Section 3.4,

$$\mu_{BS}(s) = \mu_{SIB}(s) + \mu_{SIB}(1-s). \tag{179}$$

We also observed that when $\ln \eta = 0$, $\mu_{BS}(\frac{1}{2})$ was the appropriate quantity for the Pr (ϵ) bounds.

The second situation was the non-zero-mean case. This resulted in two new terms,

$$l_D = \int_{T_i}^{T_f} r(u) [g_1(u) - g_0(u)] \, du, \tag{180}$$

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in the LRT. The functions $g_i(u)$ were specified by

$$m_i(t) = \int_{T_i}^{T_f} K_{H_i}(t, u) g_i(u) \, du, \qquad T_i \le t \le T_f, \quad i = 0, 1.$$
(181)

In the performance calculation we added a term $\mu_D(s)$, which was specified by (96).

We then observed that for bandpass processes whose spectra are symmetric about the carrier there is a simple relationship between the actual bandpass problem and an equivalent low-pass problem. Finally, for the binary symmetric bandpass problem, a tight bound on the Pr (ϵ) was derived.

$$\frac{\exp\left[\mu(\frac{1}{2})\right]}{2(1+\sqrt{(\pi/8)\ddot{\mu}(\frac{1}{2})})} \le \Pr\left(\epsilon\right) \le \frac{\exp\left[\mu(\frac{1}{2})\right]}{2(1+\sqrt{(1/8)\ddot{\mu}(\frac{1}{2})})} .$$
 (182)

This bound was useful for this particular problem. In addition, it provided a good estimate of the accuracy of our approximate expression. There are large numbers of problems in which we can evaluate the approximate expression but have not been able to find tight bounds.

Throughout our discussion in Chapters 2 and 3, we have encountered linear filters, estimates of random processes, and mean-square error expressions that we had to find in order to specify the optimum receiver and its performance completely. In many cases we used processes with finite state representations as examples, because the procedure for finding the necessary quantities was easy to demonstrate. In the next chapter we consider three other categories of problems for which we can obtain a complete solution.

3.7 PROBLEMS

P.3.2 Receiver Structures

Problem 3.2.1.

- 1. Verify the result in (21) by following the suggested approach.
- 2. Verify that the bias term can be written as in (28).

Comment: In many of the problems we refer to a particular class of problems. These classes were defined in Fig. 3.1.

Problem 3.2.2. Consider the class A_w problem in which both $s_0(t)$ and $s_1(t)$ are Wiener processes,

$$E[s_0^2(t)] = \sigma_0^2 t, \qquad 0 \le t$$

and

 $E[s_1^{\ 2}(t)] = \sigma_1^{\ 2}t, \qquad 0 \le t.$

Find the optimum receiver. Use a parallel processing configuration initially and then simplify the result. Describe the filters using state equations.

Problem 3.2.3. Consider the class A_w problem in which

$$K_{s_1}(t, u) = \alpha K_{s_0}(t, u).$$
 (P.1)

1. Use the condition in (P.1) to simplify the optimum receiver.

2. Derive the optimum receiver directly for the case in (P.1). (Do not use the results of Chapter 3. You may use the results of Chapter 2.)

Problem 3.2.4. Consider the class A_w problem in which $s_0(t)$ is a Wiener process and $s_1(t)$ is a sample function from a stationary Gaussian random process whose spectrum is

$$S_{s_1}(\omega) = \frac{2kP}{\omega^2 + k^2}.$$

Find the optimum receiver.

Problem 3.2.5. Consider the class B_w problem in which both s(t) and $n_c(t)$ have finitedimensional state representations. Derive a state-variable realization for the optimum receiver. The receiver should contain $\hat{s}_r(t)$, the MMSE realizable estimate, as one of the internal waveforms. (Notice that the parallel processing receiver in Fig. 3.3 will satisfy this requirement if we use Canonical Realization No. 4S in each path. The desired receiver is analogous to that in Fig. 3.5.)

Problem 3.2.6 (continuation). Consider the special case of Problem 3.2.5 in which $n_c(t)$ is a Wiener process and s(t) is a stationary process whose spectrum is

$$S_s(\omega) = \frac{2kP}{\omega^2 + k^2}.$$

Specify the optimum receiver in Problem 3.2.5 completely.

Problem 3.2.7. In the vector version of the class A_w problem, the received waveforms are

$$\mathbf{r}(t) = \mathbf{s}_1(t) + \mathbf{w}(t), \qquad T_i \le t \le T_f: H_1,$$

$$\mathbf{r}(t) = \mathbf{s}_0(t) + \mathbf{w}(t), \qquad T_i \le t \le T_t: H_0.$$

The signal processes are sample functions from N-dimensional, zero-mean, vector Gaussian random processes with covariance function matrices $K_{s_1}(t, u)$ and $K_{s_0}(t, u)$. The additive noise process w(t) is a sample function from a statistically independent, zero-mean, vector Gaussian random process whose covariance function matrix is $(N_0/2)\delta(t - u)I$.

- 1. Find the optimum receiver.
- 2. Derive the vector versions of (32) and (33).
- 3. Consider the special case in which

$$\mathbf{K}_{s_1}(t, u) = K_{s_1}(t, u)\mathbf{I}$$

and

$$\mathbf{K}_{\boldsymbol{s}_{o}}(t, u) = K_{\boldsymbol{s}_{o}}(t, u)\mathbf{I}.$$

Simplify the optimum receiver.

Problem 3.2.8. Consider the model in Problem 3.2.7. Assume

$$E[\mathbf{w}(t)\mathbf{w}^T(u)] = \mathbf{N}\delta(t-u),$$

where N is a nonsingular matrix. Repeat Problem 3.2.7.

Problem 3.2.9 (continuation). Consider the vector version of the class B_w problem. Derive the vector analog to (41).

Problem 3.2.10.

1. Prove the result in (43).

2. Consider the functional square root defined in (42). Give an example of a class A_w problem in which $h_{L^{(2)}}^{L^{(2)}}(t, u)$ does not exist.

Problem 3.2.11. Consider the development in (16)–(23). The output of the whitening filter is a waveform $r_{\star}(t)$, whose covariance function on H_1 is $K_1^{\star}(t, u)$. Suppose that we write

$$K_1^*(t, u) = \delta(t - u) + Y(t, u).$$

1. Show that Y(t, u) is not necessarily non-negative-definite.

2. Prove that Y(t, u) is a square-integrable function. [*Hint*: Write $\iint_{T_i}^{T_i}(K_1^*(t, u) dt du$ as a 6-fold integral using (16). Simplify the result by using the fact that the same white noise is present on both hypotheses.]

P.3.3 Performance

Problem 3.3.1. Derive the result in (60) by using a whitening approach.

Problem 3.3.2. Consider the composite process defined in (58). Assume that both $s_1(t)$ and $s_0(t)$ have finite-dimensional state representations. Write the state equations for $s_{com}(t)$. What is the dimension of the resulting system?

Problem 3.3.3 (continuation). Specialize your results in Problem 3.3.2 to the case in which

$$K_1(t, u) = \alpha K_0(t, u).$$

Problem 3.3.4. Consider the class A_w problem in which both $s_0(t)$ and $s_1(t)$ are Wiener processes, where $E[s_0^2(t)] = \sigma_0^2 t, \quad t \ge 0$

$$E[s_1^2(t)] = \sigma_1^2 t, \quad t \ge 0.$$

Evaluate $\mu(s)$.

Problem 3.3.5. Define

$$\mu(s,t) = \frac{1}{N_0} \int_{T_i}^t du \left[(1-s)\xi_P\left(u \mid s_1(\cdot), \frac{N_0}{2}\right) + s\xi_P\left(u \mid s_0(\cdot), \frac{N_0}{2}\right) - \xi_P\left(u \mid s_{\rm com}(\cdot), \frac{N_0}{2}\right) \right].$$

Assume that $s_0(t)$ and $s_1(t)$ have finite-dimensional state representations.

- 1. Write a differential equation for $\mu(s, t)$.
- 2. Define the Bhattacharyya distance as

$$B(T_f) = -\mu(\frac{1}{2}, T_f).$$

Write a differential equation for B(t).

P.3.4 Special Situations

NON-ZERO MEANS

Problem 3.4.1. In the class A_{wm} problem, the received waveforms on the two hypotheses are are $r(t) = s_1(t) + w(t), \qquad T_i \le t \le T_f: H_1$

and

 $r(t) = s_0(t) + w(t), \qquad T_i \le t \le T_f: H_0,$

where

and

$$E[s_0(t)] = m_0(t)$$

 $E[s_1(t)] = m_1(t)$

1. Derive (83)-(86).

2. Assume that a Bayes test with threshold η is desired. Evaluate the threshold γ' in (89).

3. Derive (90).

4. Find the threshold γ'' in (90) in terms of η .

5. Check your results in parts 1-4 for the case in which

$$K_{H_1}(t, u) = K_{H_0}(t, u).$$

Problem 3.4.2. Consider the model in Problem 3.4.1. Derive the expression for $\mu_D(s)$ in (96).

Problem 3.4.3. Consider the class A_{wm} problem in which $s_1(t)$ and $s_0(t)$ have finite dimensional state representations.

- 1. Derive a state-variable realization for l_D .
- 2. Derive a state equation for $\mu_D(s)$.

Problem 3.4.4. Consider the class A_{wm} problem in which

$$m_{1}(t) = +m, \qquad T_{i} \leq t \leq T_{f},$$

$$m_{0}(t) = -m, \qquad T_{i} \leq t \leq T_{f},$$

$$\mathscr{F}[K_{s_{1}}(\tau)] = \frac{2\beta P_{1}}{\omega^{2} + \beta^{2}}$$

$$\mathscr{F}[K_{s_{0}}(\tau)] = \frac{2\beta P_{0}}{\omega^{2} + \beta^{2}}$$

and

$$E[w(t)w(u)] = \frac{N_0}{2}\delta(t-u)$$

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- 1. Find the optimum receiver for this problem.
- 2. Find $\mu_D(s)$ and $\mu_R(s)$.

Problem 3.4.5. Consider the modification of Problem 3.4.4 in which

$$\mathscr{F}[K_{s_0}(\tau)] = \frac{2\alpha P_0}{\omega^2 + \alpha^2}$$

where

$$\frac{\beta P_1}{\alpha P_0} = 1.$$

- 1. Evaluate $\mu_D(s)$ and $\mu_R(s)$ for the case in which $N_0 = 0$.
- 2. Find the optimum receiver for this case.

Problem 3.4.6. Consider the class A problem in which $s_0(t)$ and $s_1(t)$ are sample functions from stationary random processes whose spectra are

$$S_{1}(\omega) = S_{x}(\omega) + \frac{\beta^{2}}{2}u_{0}(\omega - \omega_{1}) + \frac{\beta^{2}}{2}u_{0}(\omega + \omega_{1})$$
(P.1)

and

$$S_0(\omega) = S_y(\omega), \tag{P.2}$$

where $S_x(\omega)$ and $S_y(\omega)$ are rational functions of ω .

- 1. Find the optimum receiver.
- 2. Find $\mu(s)$.

3. How does the model in (P.1) differ from the case in which

$$E[s_1(t)] = m_1(t) = \sqrt{2} \beta \cos(\omega_1 t)?$$

BANDPASS PROBLEMS

Problem 3.4.7. Consider the model described in (98).

1. Verify that a necessary and sufficient condition for $r_{c_1}(t)$ and $r_{s_1}(t)$ to be statistically independent is that $S_1(\omega)$ be symmetric around the carrier.

2. Verify the result in (102) and (103).

Problem 3.4.8. Consider the model in (99). This is a four-dimensional vector problem that is a special case of the model in Problem 3.2.8.

1. Use the results of Problem 3.2.8 to verify that (100) is correct. Write out the terms on the left side of (100).

2. Verify that (101) is correct.

Problem 3.4.9. Whenever the spectra are not symmetric around the carrier, the low-pass processes are not statistically independent. The most efficient way to study this problem is to introduce a complex signal. We use this technique extensively, starting in Chapter 9.

In this problem we carry out the analysis using vector techniques. Perhaps the prime benefit of doing the problem will be an appreciation for the value of the complex representation when we reach Chapter 9.

1. Consider the model in (98). Initially, we assume

$$s_0(t) = 0,$$

so that we have the simple binary problem. Evaluate the cross-correlation function between $s_{c_1}(t)$ and $s_{s_1}(t)$. Evaluate the corresponding cross-spectrum. Notice that we do not assume that $S_{s_1}(\omega)$ is symmetric about ω_1 . Check your answer with (A.67) and (A.70).

2. Use the results of Problem 3.2.8 to find the optimum receiver.

3. Derive an expression for $\mu(s)$.

4. Generalize your results to include the original model in (98). Allow $s_0(t)$ to have an asymmetric spectrum about ω_0 .

Problem 3.4.10.

1. Read [6] and verify that (110) is correct.

2. Discuss the difficulties that arise when the criterion is not minimum $Pr(\epsilon)$ (i.e., the threshold changes).

P.3.5 Singularity

Problem 3.5.1. Draw a block diagram of the receiver operations needed to generate the r_i in (117).

Problem 3.5.2. Consider the integral equation in (126). Assume

$$K_{H_0}(t, u) = \sigma^2 \min [t,$$

and

$$K_{H_1}(t, u) = P e^{-\alpha |t-u|}.$$

Find the eigenfunctions and eigenvalues of (126).

Problem 3.5.3. Consider the integral equation in (126). Assume

$$K_{H_0}(t, u) = e^{-\beta |t-u|}$$

and

$$K_{H_1}(t, u) = e^{-\alpha |t-u|}.$$

Find the eigenfunctions and eigenvalues of (126).

Problem 3.5.4. Assume that

$$K_{H_0}(t, u) = K_0(t, u) + \frac{N_0}{2}\delta(t - u)$$
(P.1)

u]

and

$$K_{H_1}(t, u) = K_1(t, u) + \frac{N_0}{2}\delta(t - u).$$
 (P.2)

How does this assumption affect the eigenvalues and eigenfunctions of (126)?

Problem 3.5.5. Assume that $r_0(t)$ and $r_1(t)$ in (112) have finite-dimensional state representations. Extend the technique in the Appendix to Part II to find the solution to (126).

Problem 3.5.6. Assume that $K_{H_0}(t, u)$ and $K_{H_1}(t, u)$ are both separable:

$$K_{H_1}(t, u) = \sum_{i=1}^{N_1} \sigma_{1i}^2 f_i(t) f_i(u)$$

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and

$$K_{H_0}(t, u) = \sum_{j=1}^{N_2} \sigma_{0j}^2 g_j(t) g_j(u),$$

where

$$\int_{T_i}^{T_f} f_i(t) f_j(t) dt = \int_{T_i}^{T_f} g_i(t) g_j(t) dt = \delta_{ij}.$$

1. Assume

$$\int_{T_i}^{T_f} f_i(t) g_j(t) \, dt = 0, \quad \text{all } i, j.$$
(P.1)

Solve (126).

2. Assume that $f_i(t)$ and $g_j(t)$ do not necessarily satisfy (P.1) for all *i* and *j*. Solve (126). How many eigenvalues does (126) have?

Problem 3.5.7. Assume

$$K_{H_0}(t, u) = \begin{cases} 1 - |t - u|, & |t - u| < 1, \\ 0, & \text{elsewhere,} \end{cases}$$

and

$$K_{H_1}(t, u) = \sigma^2 \min [t, u]$$

Solve (126).

Problem 3.5.8. Consider the definition of $h_A(x, y)$ in (134). Verify that (136) is valid.

Problem 3.5.9. Verify the equivalences in (147)-(149).

Problem 3.5.10.

- 1. Can a class A problem be singular? Prove your answer.
- 2. Can a class B problem be singular? Prove your answer.

Problem 3.5.11. Give an example of a case in which the logarithm of neither the numerator nor the denominator of (155) converges but the sum in (155) does.

Problem 3.5.12. Verify the result in (167). Is the result also true for nonrational spectra?

Problem 3.5.13. Assume that

$$S_{H_0}(\omega) = \frac{1}{c_n \omega^{2n} + c_{n-1} \omega^{2(n-1)} + \dots + c_0}$$

Assume that $r_1(t)$ has a finite-dimensional state representation and that the detection problem is nonsingular.

- 1. Find a state-variable realization of the optimum receiver.
- 2. Find a differential equation specifying $\mu(s)$.

Problem 3.5.14 (continuation). Assume that

$$S_{H_0}(\omega) = \frac{1}{c_1 \omega^2 + c_0}$$

and that $r_1(t)$ is a segment of a stationary process with a finite-dimensional state representation. Assume that the detection problem is nonsingular.

- 1. Draw a block diagram of the optimum receiver. Specify all components completely.
- 2. Evaluate $\mu(s)$.

Problem 3.5.15.

1. Generalize the result in Problem 3.5.14 to the case in which

$$S_{H_0}(\omega) = \frac{2nP}{k} \frac{\sin(\pi/2n)}{1 + (\omega/k)^{2n}} \qquad n = 1, 2, \dots$$

2. How must $S_{H_1}(\omega)$ behave as ω approaches infinity in order for the test to be nonsingular?

Problem 3.5.16. Assume that both $r_1(t)$ and $r_0(t)$ are sample functions from stationary processes with flat bandlimited spectra. Under what conditions will the test be non-singular?

Problem 3.5.17. In Section I-4.3.7 we discussed the sensitivity problem for the known signal case. Read [13, page 420] and discuss the sensitivity problem for the general binary case.

Problem 3.5.18. Extend the discussion in Section 3.5 to the general vector case. Specifically, find the vector versions of (126), (135), (136), (138), (139), (151), (154), and (167).

Problem 3.5.19 [14]. Consider the integral equation in (126). Assume

$$K_{H_0}(t, u) = 1 - \frac{|t - u|}{2T}, \ -T \le t, u \le T$$

and

$$K_{H_1}(t, u) = e^{-|t-u|/T}.$$

Let $T_i = -T$ and $T_f = +T$. Find the eigenvalues and eigenfunctions of (126).

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