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9

Detection of Slowly Fluctuating Point Targets

In this chapter we discuss the problem of detecting a slowly fluctuating point target in the presence of additive noise. The first step is to develop a realistic mathematical model for the physical situations of interest. In the course of that development we shall explain the phrases "slowly fluctuating" and "point" more explicitly. Once we obtain the mathematical model, the detection problem is directly analogous to that in Sections I-4.2 and I-4.3, so that we can proceed quickly. We consider three cases:

- 1. Detection in white bandpass noise.
- 2. Detection in colored bandpass noise.
- 3. Detection in bandpass noise that has a finite state representation.

In all three cases, we use the complex notation that we develop in detail in the Appendix. We begin by developing a model for the target reflection process in Section 9.1. In Section 9.2, we study detection in white bandpass noise. In Section 9.3, we study detection in colored bandpass noise. In Section 9.4, we specialize the results of Section 9.3 to the case in which the bandpass noise has a finite state representation. In Section 9.5, we study the question of optimal signal design briefly.

9.1 MODEL OF A SLOWLY FLUCTUATING POINT TARGET

In order to develop our target model, we first assume that the radar/ sonar system transmits a cosine wave continuously. Thus,

$$s_t(t) = \sqrt{2P_t} \cos \omega_c t = \sqrt{2} \operatorname{Re}\left[\sqrt{P_t} e^{j\omega_c t}\right], \quad -\infty < t < \infty.$$
(1)

Now assume that there is a zero-velocity target located at some range R from the transmitter. We assume that the target has a physical structure that includes several reflecting surfaces. Thus the returned signal may be written as

$$s_r(t) = \sqrt{2} \operatorname{Re}\left\{\sqrt{P_t} \sum_{i=1}^{K} g_i \exp\left[j\omega_c(t-\tau) + \theta_i\right]\right\}.$$
 (2)

The attenuation g_i includes the effects of the transmitting antenna gain, the two-way path loss, the radar cross-section of the *i*th reflecting surface, and the receiving antenna aperture. The phase angle θ_i is a random phase incurred in the reflection process. The constant τ is the round-trip delay time from the target. If the velocity of propagation is c,

$$\tau \stackrel{\Delta}{=} \frac{2R}{c}.$$
 (3)

We want to determine the characteristics of the sum in (2). If we assume that the θ_i are statistically independent, that the g_i have equal magnitudes, and that K is large, we can use a central limit theorem argument to obtain

$$s_r(t) = \sqrt{2} \operatorname{Re} \left\{ \sqrt{P_t} \, \tilde{b} \exp \left(j \omega_c(t - \tau) \right] \right\},\tag{4}$$

where \tilde{b} is a complex Gaussian random variable. The envelope, $|\dot{b}|$, is a Rayleigh random variable whose moments are

$$E\{|\tilde{b}|\} = \sqrt{\frac{\pi}{2}} \sigma_b \tag{5}$$

and

$$E\{|\tilde{b}|^2\} = 2\sigma_b^2. \tag{6}$$

The value of σ_b^2 includes the antenna gains, path losses, and radar crosssection of the target. The expected value of the received power is $2P_t\sigma_b^2$. The phase of \tilde{b} is uniform. In practice, K does not have to very large in order for the complex Gaussian approximation to be valid. Slack [1] and Bennett [2] have studied the approximation in detail. It turns out that if K = 6, the envelope is essentially Rayleigh and the phase is uniform. The central limit theorem approximation is best near the mean and is less accurate on the tail of the density. Fortunately, the tail of the density corresponds to high power levels, so that it is less important that our model be exact.

We assume that the reflection process is *frequency-independent*. Thus, if we transmit

$$s_t(t) = \sqrt{2} \operatorname{Re} \left[\sqrt{P_t} \exp \left(j\omega_c t + j\omega t \right) \right], \tag{7}$$

we receive

$$s_r(t) = \sqrt{2} \operatorname{Re} \left[\sqrt{P_t} \, \tilde{b} \exp \left[j(\omega_c + \omega)(t - \tau) \right] \right]. \tag{8}$$

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We also assume that the reflection process is linear. Thus, if we transmit

$$s_{t}(t) = \sqrt{2} \operatorname{Re}\left[\sqrt{E_{t}} \tilde{f}(t) e^{j\omega_{c}t}\right]$$
$$= \sqrt{2} \operatorname{Re}\left[\sqrt{E_{t}} e^{j\omega_{c}t} \int_{-\infty}^{\infty} \tilde{F}(j\omega) e^{j\omega t} \frac{d\omega}{2\pi}\right], \qquad (9)$$

we receive

$$s_{\tau}(t) = \sqrt{2} \operatorname{Re}\left[\sqrt{E_{t}} \tilde{b} \exp\left[j\omega_{c}(t-\tau)\right] \int_{-\infty}^{\infty} \tilde{F}(j\omega) \exp\left[j\omega(t-\tau)\right] \frac{d\omega}{2\pi}\right]$$
$$= \sqrt{2} \operatorname{Re}\left[\sqrt{E_{t}} \tilde{b} \exp\left[j\omega_{c}(t-\tau)\right] \tilde{f}(t-\tau)\right]. \tag{10}$$

Since \tilde{b} has a uniform phase, we can absorb the $e^{j\omega_c\tau}$ term in the phase. Then

$$s_{\tau}(t) = \sqrt{2} \operatorname{Re}\left[\sqrt{E_{t}} \tilde{b}\tilde{f}(t-\tau)e^{i\omega_{c}t}\right].$$
(11)

The function $\tilde{f}(t)$ is the complex envelope of the transmitted signal. We assume that it is normalized:

$$\int_{-\infty}^{\infty} |\tilde{f}(t)|^2 dt = 1.$$
 (12)

Thus the transmitted energy is E_t . The expected value of the received signal energy is

$$\bar{E}_r \stackrel{\Delta}{=} 2E_t \sigma_b^2. \tag{13}$$

We next consider a target with constant radial velocity v. The range is

$$R(t) = R_0 - vt. \tag{14}$$

The signal returned from this target is

$$s_{\tau}(t) = \sqrt{2} \operatorname{Re} \left[\sqrt{E_t} \, \tilde{b} \tilde{f}(t - \tau(t)) \exp \left[j \omega_c(t - \tau(t)) \right] \right], \tag{15}$$

where $\tau(t)$ is the round-trip delay time. Notice that a signal received at t was reflected from the target at $[t - (\tau(t)/2)]$. At that time the target range was

$$R\left(t - \frac{\tau(t)}{2}\right) = R_0 - v\left(t - \frac{\tau(t)}{2}\right).$$
(16)

By definition,

$$\tau(t) = \frac{2R(t - \tau(t)/2)}{c}.$$
 (17)

Substituting (16) into (17) and solving for $\tau(t)$, we obtain

$$\tau(t) = \frac{2R_0/c}{1+v/c} - \frac{(2v/c)t}{1+v/c}.$$
(18)

For target velocities of interest,

$$\frac{v}{c} \ll 1. \tag{19}$$

Thus,

$$\tau(t) \simeq \frac{2R_0}{c} - \frac{2v}{c} t \triangleq \tau - \frac{2v}{c} t.$$
⁽²⁰⁾

Substituting (20) into (15) gives

$$s_{r}(t) = \sqrt{2} \operatorname{Re}\left[\sqrt{E_{t}} \tilde{b}\tilde{f}\left(t - \tau + \frac{2v}{c}t\right) \exp\left[j\omega_{c}\left(t + \frac{2v}{c}t\right)\right]\right].$$
(21)

(Once again, we absorbed the $\omega_c \tau$ term in \tilde{b} .) We see that the target velocity has two effects:

1. A compression or stretching of the time scale of the complex envelope.

2. A shift of the carrier frequency.

In most cases we can ignore this first effect. To demonstrate this, consider the error in plotting $\tilde{f}(t)$ instead of $\tilde{f}(t - (2v/c)t)$. The maximum difference in the arguments occurs at the end of the pulse (say T) and equals 2vT/c. The resulting error in amplitude is a function of the signal bandwidth. If the signal bandwidth is W, the signal does not change appreciably in a time equal to W^{-1} . Therefore, if

$$\frac{2vT}{c} \ll \frac{1}{W} \tag{22}$$

or, equivalently,

$$WT \ll \frac{c}{2v},\tag{23}$$

we may ignore the time-scale change. For example, if the target velocity is 5000 mph, a WT product of 2000 would satisfy the inequality.[†]

The shift in the carrier frequency is called the Doppler shift

$$\omega_D \triangleq \omega_c \left(\frac{2v}{c}\right). \tag{24}$$

 \dagger There are some sonar problems in which (23) is not satisfied. We shall comment on these problems in Section 10.6.

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Using (24) in (21), and neglecting the time compression, we obtain

$$s_{\tau}(t) = \sqrt{2} \operatorname{Re} \left[\sqrt{E_t} \, \tilde{b} \tilde{f}(t-\tau) \exp \left(j \omega_o t + \omega_D t \right) \right].$$
(25)

We shall use this expression for the received signal throughout our discussion of slowly fluctuating point targets. We have developed it in reasonable detail because it is important to understand the assumptions inherent in the mathematical model.

The next step is to characterize the additive noise process. We assume that there is an additive Gaussian noise n(t) that has a bandpass spectrum so that we can represent it as

$$n(t) = \sqrt{2} \operatorname{Re} \left[\tilde{n}(t) e^{j \omega_c t} \right].$$
(26)

(This representation of the bandpass processes is developed in the Appendix.) Thus, the total received waveform is

$$\mathbf{r}(t) = \sqrt{2E_t} \operatorname{Re} \left\{ \tilde{b}\tilde{f}(t-\tau) \exp\left(j\omega_c t + j\omega_D t\right) \right\} + \sqrt{2} \operatorname{Re} \left\{ \tilde{n}(t) \exp\left(j\omega_c t\right) \right\}$$
(27)

or, more compactly,

$$r(t) = \sqrt{2} \operatorname{Re}\left[\tilde{r}(t)e^{j\omega_{c}t}\right], \qquad (28a)$$

where

$$\tilde{r}(t) \triangleq \tilde{b}\sqrt{E_t}\tilde{f}(t-\tau)e^{j\omega_D t} + \tilde{n}(t).$$
 (28b)

Up to this point we have developed a model for the return from a target at a particular point in the range-Doppler plane. We can now formulate the detection problem explicitly. We want to examine a particular value of range and Doppler and decide whether or not a target is present at that point. This is a binary hypothesis-testing problem. The received waveforms on the two hypotheses are

$$r(t) = \sqrt{2} \operatorname{Re} \left\{ [\tilde{b}\sqrt{E_t}\tilde{f}(t-\tau)e^{i\omega_D t} + \tilde{n}(t)]e^{i\omega_c t} \right\}, \qquad T_i \le t \le T_f : H_1$$
(29a)

and

$$r(t) = \sqrt{2} \operatorname{Re} \left\{ \tilde{n}(t) e^{j\omega_c t} \right\}, \qquad T_i \le t \le T_f : H_0.$$
(29b)

Since we are considering only a particular value of τ and ω , we can assume that they are zero for algebraic simplicity. The modifications for nonzero τ and ω are obvious and will be pointed out later. Setting τ and ω_D equal

to zero gives

$$r(t) = \sqrt{2} \operatorname{Re} \left\{ [\tilde{b}\sqrt{E_t} \tilde{f}(t) + \tilde{n}(t)] e^{j\omega_c t} \right\}, \qquad T_i \le t \le T_f : H_1 \quad (30)$$

and

$$r(t) = \sqrt{2} \operatorname{Re}\left[\tilde{n}(t)e^{j\omega_{c}t}\right], \qquad T_{i} \leq t \leq T_{f}: H_{0}. \quad (31)$$

In the next three sections, we use the model described by (30) and (31) and consider the three cases outlined on page 238.

Before we begin this development, some further comments on the model are worthwhile. All of our discussion in the text will use the Rayleigh model for the envelope $|\tilde{b}|$. In practice there are target models that cannot be adequately modeled by a Rayleigh variable, and so various other densities have been introduced. Marcum's work [8]-[10] deals with *nonfluctuating* targets. Swerling [7] uses both the Rayleigh model and a probability density assuming one large reflector and a set of small reflectors. Specifically, defining

$$z \triangleq |\tilde{b}|^2,$$
 (32a)

the density given by the latter model is

$$p_{z}(Z) = \frac{Z}{\sigma_{b}^{2}} e^{-Z/\sigma_{b}}, \qquad Z \ge 0,$$
(32b)

where σ_b^2 is defined in (6). Swerling [11] also uses a chi-square density for z,

$$p_{z}(Z) = \frac{1}{(K-1)!} \frac{K}{2\sigma_{b}^{2}} \left(\frac{KZ}{2\sigma_{b}^{2}}\right)^{K-1} e^{-KZ/2\sigma_{b}^{2}}, \qquad Z \ge 0.$$
(32c)

The interested reader can consult the references cited above as well as [12, Chapter VI-5] and [13] for discussions of target models.

Most of our basic results are applicable to the problem of digital communication over slowly fluctuating point channels that exhibit Rayleigh fading. Other fading models can be used to accommodate different physical channels. The Rician channel [14] was introduced on page I-360. A more general fading model, the Nakagami channel [15], [16], models $|\tilde{b}|$ as

$$p_{|\tilde{b}|}(X) = \frac{2m^m X^{2m-1}}{\Gamma(m)(2\sigma_b^{2})^m} e^{-m X^2/2\sigma_b^2}, \qquad X \ge 0,$$
(33)

which is a generalization of (32c) to include noninteger K. Various problems using this channel model are discussed in [17]-[22].

We now proceed with our discussion of the detection of a slowly fluctuating point target.

9.2 WHITE BANDPASS NOISE

In this case, the complex envelopes of the received waveform on the two hypotheses are

$$\tilde{r}(t) = \sqrt{E_t} \, \tilde{b}\tilde{f}(t) + \tilde{w}(t), \qquad 0 \le t \le T: H_1,$$

$$\tilde{r}(t) = \tilde{w}(t), \qquad 0 \le t \le T: H_0,$$
(34)

where \tilde{b} is a zero-mean complex Gaussian random variable $(E\{|\tilde{b}|^2\} = 2\sigma_b^2)$ and $\tilde{w}(t)$ is an independent zero-mean white complex Gaussian random process,

$$E[\tilde{w}(t)\tilde{w}^*(u)] = N_0\delta(t-u).$$
(35)

The complex envelope $\tilde{f}(t)$ has unit energy. Because the noise is white, we can make the observation interval coincident with the signal duration.

The first step is to find a sufficient statistic. Since the noise is white, we can expand using any complete orthonormal set of functions and obtain statistically independent coefficients [see (A.117)]. Just as in Section I-4.2, we can choose the signal as the first orthonormal function and the resulting coefficient will be a sufficient statistic. In the complex case we correlate $\tilde{r}(t)$ with $\tilde{f}^*(t)$ as shown in Fig. 9.1. The resulting coefficient is

$$\tilde{r}_1 \triangleq \int_0^T \tilde{r}(t) \tilde{f}^*(t) dt.$$
(36)

Using (34) in (36),

$$\tilde{r}_1 = \begin{cases} \sqrt{E_t} \, \tilde{b} + \tilde{w}_1 \colon H_1 \\ & \tilde{w}_1 \colon H_0, \end{cases}$$
(37)

where \tilde{w}_1 is a zero-mean complex Gaussian random variable $(E\{|\tilde{w}_1|^2\} = N_0)$. We can easily verify that \tilde{r}_1 is a sufficient statistic. The probability density of a complex Gaussian random variable is given by (A.81).



Fig. 9.1 Generation of complex sufficient statistic.



Fig. 9.2 Correlation receiver (complex operations).

The likelihood ratio test is

$$\Lambda(\tilde{R}_{1}) = \frac{p_{\tilde{r}_{1}|H_{1}}(\tilde{R}_{1} \mid H_{1})}{p_{\tilde{r}_{1}|H_{0}}(\tilde{R}_{1} \mid H_{0})}$$
$$= \frac{[\pi(2\sigma_{b}^{2}E_{t} + N_{0})]^{-1}\exp[|\tilde{R}_{1}|^{2}/(2\sigma_{b}^{2}E_{t} + N_{0})]}{(1/\pi N_{0})\exp(-|\tilde{R}_{1}|^{2}/N_{0})} \stackrel{H_{1}}{\underset{H_{0}}{\overset{\sim}{\to}} \eta.$$
(38)

Taking the logarithm and rearranging terms, we have

$$|\tilde{R}_1|^2 \overset{H_1}{\underset{H_0}{\overset{>}{\sim}}} \frac{N_0(N_0 + 2\sigma_b^2 E_t)}{2\sigma_b^2 E_t} \left\{ \ln \eta + \ln \left(1 + \frac{2\sigma_b^2 E_t}{N_0} \right) \right\} \stackrel{\Delta}{=} \gamma.$$
(39)

A complex receiver using a correlation operation is shown in Fig. 9.2. A complex receiver using a matched filter is shown in Fig. 9.3. Here

$$\tilde{r}_1 = \int_0^T \tilde{r}(u)\tilde{h}(T-u)\,du,\tag{40}$$

where

$$\tilde{h}(u) = \tilde{f}^*(T - u). \tag{41}$$

The actual bandpass receiver is shown in Fig. 9.4. We see that it is a bandpass matched filter followed by a square-law envelope detector and sampler.

The calculation of the error probabilities is straightforward. We have solved this exact problem on page I-355, but we repeat the calculation here as a review. The false-alarm probability is

$$P_F = \Pr\left[|\tilde{r}_1|^2 > \gamma |H_0|\right]$$

= $\int_{\sqrt{\gamma}}^{\infty} \int_0^{2\pi} \frac{1}{\pi N_0} e^{-Z^2/N_0} Z \, dZ \, d\beta,$ (42)



Fig. 9.3 Matched filter receiver (complex operations).



Fig. 9.4 Optimum receiver: detection of bandpass signal in white Gaussian noise. where we have defined

 $P_F = e^{-\gamma/N_0}$

Thus,

$$\tilde{R}_1 \triangleq Z e^{j\beta}. \tag{43}$$

(44)

$$P_D = \exp\left(-\frac{\gamma}{2\sigma_b^2 E_t + N_0}\right) = \exp\left(-\frac{\gamma}{\bar{E}_r + N_0}\right),\tag{45}$$

where

$$\bar{E_r} \triangleq 2\sigma_b^2 E_t \tag{46}$$

is the expected value of the received signal energy. Combining (42) and (45) gives

$$P_F = (P_D)^{\frac{N_0 + E_r}{N_0}} = (P_D)^{1 + E_r/N_0}.$$
(47)

As we would expect, the performance is only a function of \bar{E}_r/N_0 , and the signal shape $\tilde{f}(t)$ is unimportant. We also observe that the exponent of P_D is the ratio of the expectation of $|\tilde{R}_1|^2$ on the two hypotheses:

$$\frac{N_0 + \bar{E}_r}{N_0} = \frac{E[|\tilde{R}_1|^2 | H_1]}{E[|\tilde{R}_1|^2 | H_0]}.$$
(48)

From our above development, it is clear that this result will be valid for the test in (39) whenever $\tilde{\mathcal{R}}_1$ is a zero-mean complex Gaussian random variable on both hypotheses. It is convenient to write the result in (48) in a different form.

$$\Delta \triangleq \frac{E[|\tilde{R}_1|^2 \mid H_1]}{E[|\tilde{R}_1|^2 \mid H_0]} - 1 = \frac{E[|\tilde{R}_1|^2 \mid H_1] - E[|\tilde{R}_1|^2 \mid H_0]}{E[|\tilde{R}_1|^2 \mid H_0]}.$$
 (49)

Now we can write

$$P_F = (P_D)^{1+\Delta}.$$
(50)

For the white noise case,

$$\Delta = \frac{\bar{E}_r}{N_0}.$$
 (51)

In the next section we evaluate Δ for the nonwhite-noise case.



Fig. 9.5 Optimum receiver for known Doppler shift.

The modification to include non-zero τ and ω_D is straightforward. The desired output is

$$|\tilde{R}_1(\tau,\,\omega)|^2 = \left| \int_{-\infty}^{\infty} \tilde{r}(t) \tilde{f}^*(t-\tau) e^{-j\omega_D t} \, dt \right|^2. \tag{52}$$

This could be obtained by passing the received waveform through a filter whose complex impulse response is

$$\tilde{h}(u) = \tilde{f}^*(T + \tau - u)e^{j\omega_D u} \, du, \tag{53}$$

then through a square-law envelope detector, and sampling the output at

$$t = T. \tag{54}$$

Equivalently, we can use a complex impulse response

$$\tilde{h}(u) = \tilde{f}^*(-u)e^{j\omega_D u} \, du \tag{55}$$

and sample the detector output at

$$t = \tau. \tag{56}$$

The obvious advantage of this realization is that we can test all ranges with the same filter. This operation is shown in Fig. 9.5. The complex envelope of the bandpass matched filter is specified by (55). In practice, we normally sample the output waveform at the reciprocal of the signal bandwidth. To test different Doppler values, we need different filters. We discuss this issue in more detail in Chapter 10.

We now consider the case in which $\tilde{n}(t)$ is nonwhite.

9.3 COLORED BANDPASS NOISE

In this case, the complex envelopes on the two hypotheses are

$$\tilde{r}(t) = \sqrt{E_t} \tilde{b}\tilde{f}(t) + \tilde{n}(t), \qquad T_i \le t \le T_f : H_1,$$

$$\tilde{r}(t) = \tilde{n}(t), \qquad T_i \le t \le T_f : H_0.$$
(57)

The additive noise $\tilde{n}(t)$ is a sample function from a zero-mean nonwhite complex Gaussian process. It contains two statistically independent components,

$$\tilde{n}(t) \triangleq \tilde{n}_c(t) + \tilde{w}(t).$$
(58)

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The covariance of $\tilde{n}(t)$ is

$$E[\tilde{n}(t)\tilde{n}^{*}(u)] \triangleq \tilde{K}_{\hat{u}}(t,u) = \tilde{K}_{c}(t,u) + N_{0}\,\delta(t-u), \qquad T_{i} \leq t, u \leq T_{f}.$$
(59)

Notice that the observation interval $[T_i, T_f]$ may be different from the interval over which the signal is nonzero. Any of the three approaches that we used in Section I.4.3 (pages I-287–I-301) will also work here. We use the whitening approach.[†] Let $\tilde{h}_{wu}(t, z)$ denote the impulse response of a complex whitening filter. When the filter input is $\tilde{n}(t)$, we denote the output as $\tilde{n}_*(t)$,

$$\tilde{n}_{\ast}(t) = \int_{T_i}^{T_f} \tilde{h}_{wu}(t, z) \tilde{n}(z) \, dz, \qquad T_i \le t \le T_f. \tag{60}$$

The complex impulse response $\tilde{h}_{wu}(t, z)$ is chosen so that

$$E[\tilde{n}_{*}(t)\tilde{n}_{*}^{*}(u)] = E\left[\iint_{T_{i}}^{T_{f}} \tilde{h}_{wu}(t,z)\tilde{h}_{wu}^{*}(u,y)\tilde{n}(z)\tilde{n}^{*}(y)\,dz\,dy\right]$$

= $\delta(t-u), \quad T_{i} \leq t, u \leq T_{f}.$ (61)

We define

$$\tilde{r}_*(t) = \int_{T_i}^{T_f} \tilde{h}_{wu}(t, z) \tilde{r}(z) \, dz, \qquad T_i \le t \le T_f \tag{62}$$

and

$$\tilde{f}_{*}(t) = \int_{T_{i}}^{T_{f}} \tilde{h}_{wu}(t, y) \tilde{f}(y) \, dy, \qquad T_{i} \le t \le T_{f}.$$
(63)

We may now use the results of Section 9.2 directly to form the sufficient statistic. From (36),

$$\tilde{r}_{1} = \int_{T_{i}}^{T_{f}} \tilde{r}_{*}(t) \tilde{f}_{*}^{*}(t) dt$$

$$= \int_{T_{i}}^{T_{f}} dt \int_{T}^{T_{f}} \tilde{h}_{wu}(t, z) \tilde{r}(z) dz \int_{T_{i}}^{T_{f}} \tilde{h}_{wu}^{*}(t, y) \tilde{f}^{*}(y) dy.$$
(64)

As before, we define an inverse kernel,

$$\tilde{Q}_{\tilde{n}}^{*}(z,y) \triangleq \int_{T_{i}}^{T_{f}} \tilde{h}_{wu}(t,z) \tilde{h}_{wu}^{*}(t,y) dt, \qquad T_{i} < z, y < T_{f}.$$
(65)

† The argument is parallel to that on pages I-290-I-297, and so we shall move quickly. We strongly suggest that the reader review the above pages before reading this section. ‡ In Section I-4.3, we associated the \sqrt{E} with the whitened signal. Here it is simpler to leave it out of (63) and associate it with the multiplier \tilde{b} .



Fig. 9.6 Optimum receiver: bandpass signal in nonwhite Gaussian noise (complex operations).

Using (65) in (64) gives

$$\tilde{r}_1 = \iint_{T_i}^{T_f} \tilde{r}(z) \tilde{\mathcal{Q}}_{\tilde{n}}^*(z, y) \tilde{f}^*(y) \, dz \, dy.$$
(66)

Defining

$$\tilde{g}(z) = \int_{T_i}^{T_f} \tilde{\mathcal{Q}}_{\tilde{n}}(z, y) \tilde{f}(y) \, dy, \qquad T_i \le t \le T_f, \tag{67}$$

we have

$$\tilde{r}_1 = \int_{T_i}^{T_f} \tilde{r}(z) \tilde{g}^*(z) \, dz.$$
(68)

The optimum test is

$$\left| \int_{T_i}^{T_f} \tilde{r}(z) \tilde{g}^*(z) dz \right|^2 \stackrel{H_1}{\underset{H_0}{\gtrless}} \gamma.$$
(69)

The complex receiver is shown in Fig. 9.6, and the actual bandpass receiver is shown in Fig. 9.7.

Proceeding as in Section I-4.3.1, we obtain the following relations:

$$\int_{T_i}^{T_f} \tilde{\mathcal{Q}}_{\tilde{n}}(t, x) \tilde{K}_{\tilde{n}}(x, u) \, dx = \delta(t - u), \qquad T_i < t, \, u < T_f \tag{70}$$

and

$$\tilde{Q}_{\tilde{n}}(t, u) = \frac{1}{N_0} [\delta(t - u) - \tilde{h}_{ou}(t, u)], \quad T_i < t, u < T_f, \quad (71)$$



Fig. 9.7 Optimum receiver: bandpass signal in nonwhite Gaussian noise.

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where $\tilde{h}_{ou}(t, u)$ satisfies the integral equation

$$N_0 \tilde{h}_{ou}(t, u) + \int_{T_i}^{T_f} \tilde{h}_{ou}(x, u) \tilde{K}_c(t, x) \, dx = \tilde{K}_c(t, u), \qquad T_i \le t, \, u \le T_f.$$
(72)

The function $\tilde{h}_{ou}(t, u)$ is the impulse response of the optimum unrealizable filter for estimating $\tilde{n}_o(t)$ in the presence of white noise $\tilde{w}(t)$ of spectral height N_0 . Using (70) in (67), we have

$$\tilde{f}(t) = \int_{T_i}^{T_f} \tilde{K}_{\tilde{n}}(t, u) \tilde{g}(u) \, du, \qquad T_i < t < T_f$$
(73)

or

$$\tilde{f}(t) = \int_{T_i}^{T_f} \tilde{K}_c(t, u) \tilde{g}(u) \, du + N_0 \tilde{g}(t), \qquad T_i \le t \le T_f.$$
(74)

This equation is just the complex version of (I-4.169b).† In Section I-4.3.6, we discussed solution techniques for integral equations of this form. All of these techniques carry over to the complex case. A particularly simple solution is obtained when $\tilde{n}_c(t)$ is stationary and the observation interval is infinite. We can then use Fourier transforms to solve (73),

$$\tilde{G}_{\infty}(j\omega) = \frac{\tilde{F}(j\omega)}{\tilde{S}_{\tilde{\eta}}(\omega)}.$$
(75)

For finite observation intervals we can use the techniques of Section I-4.3.6. However, when the colored noise has a finite-dimensional complex state representation (see Section A.3.3), the techniques developed in the next section are computationally more efficient.

To evaluate the performance, we compute Δ using (49). The result is

$$\Delta = \bar{E}_r \iint_{T_i}^{T_f} \tilde{f}(t) \tilde{\mathcal{Q}}_{\tilde{n}}^*(t, u) \tilde{f}^*(u) \, dt \, du \tag{76}$$

or

$$\Delta = \bar{E}_r \int_{T_i}^{T_f} \tilde{f}(t) \tilde{g}^*(t) \, dt.$$
(77)

Notice that Δ is a real quantity. Its functional form is identical with that of d^2 in the known signal case [see (I-4.198)]. The performance is obtained

[†] It is important for the reader to identify the similarities between the complex case and the known signal case. One of the advantages of the complex notation is that it emphasizes these similarities and helps us to exploit all of our earlier work.

from (50),

$$P_F = (P_D)^{1+\Delta}.$$
(78)

From (78) it is clear that increasing Δ always improves the performance. As we would expect, the performance of the system depends on the signal shape. We shall discuss some of the issues of signal design in Section 9.5.

9.4 COLORED NOISE WITH A FINITE STATE REPRESENTATION†

When the colored noise component has a finite state representation, we can derive an alternative configuration for the optimum receiver that is easy to implement. The approach is just the complex version of the derivation in the appendix in Part II. We use the same noise model as in (58).

$$\tilde{n}(t) = \tilde{n}_c(t) + \tilde{w}(t).$$
(79)

We assume that the colored noise can be generated by passing a complex white Gaussian noise process, $\tilde{\mathbf{u}}(t)$, through a finite-dimensional linear system. The state and observation equations are

$$\tilde{\mathbf{x}}(t) = \tilde{\mathbf{F}}(t)\tilde{\mathbf{x}}(t) + \tilde{\mathbf{G}}(t)\tilde{\mathbf{u}}(t),$$
(80)

$$\tilde{n}_c(t) = \tilde{\mathbf{C}}(t)\tilde{\mathbf{x}}(t) \tag{81}$$

The initial conditions are

$$E[\tilde{\mathbf{x}}(T_i)] = \mathbf{0} \tag{82}$$

and

$$E[\tilde{\mathbf{x}}(T_i)\tilde{\mathbf{x}}^{\dagger}(T_i)] = \tilde{\mathbf{P}}_0.$$
(83)

The covariance matrix of the driving function is

$$E[\tilde{\mathbf{u}}(t)\tilde{\mathbf{u}}^{\dagger}(\sigma)] = \mathbf{Q}\delta(t-\sigma).$$
(84)

In the preceding section we showed that the optimum receiver computed the statistic

$$l_o \triangleq \left| \int_{T_i}^{T_f} \tilde{r}(z) \tilde{g}^*(z) \, dz \right|^2 \tag{85}$$

 \dagger In this section, we use the results of Section A.3.3, Problem I-4.3.4, and Problem I-6.6.5. The detailed derivations of the results are included as problems. This section can be omitted on the first reading.

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and compared it with a threshold [see (69)]. The function $\tilde{g}(t)$ was specified by

$$\tilde{f}(t) = \int_{T_i}^{T_f} \tilde{K}_o(t, u) \tilde{g}(u) \, du + N_0 \tilde{g}(t), \qquad T_i \le t \le T_f.$$
(86)

From (81) we have

$$\tilde{K}_{c}(t, u) = E[\tilde{n}_{c}(t)\tilde{n}_{c}^{*}(u)] = E[\tilde{\mathbf{C}}(t)\tilde{\mathbf{x}}(t)\tilde{\mathbf{x}}^{\dagger}(u)\tilde{\mathbf{C}}^{\dagger}(u)]$$
$$= \tilde{\mathbf{C}}(t)\tilde{\mathbf{K}}_{\tilde{\mathbf{x}}}(t, u)\tilde{\mathbf{C}}^{\dagger}(u). \quad (87)$$

Using (87) in (86) gives

$$\tilde{f}(t) = \tilde{\mathbf{C}}(t) \int_{T_i}^{T_f} \tilde{\mathbf{K}}_{\tilde{\mathbf{x}}}(t, u) \tilde{\mathbf{C}}^{\dagger}(u) \tilde{\mathbf{g}}(u) \, du + N_0 \tilde{\mathbf{g}}(t), \qquad T_i \le t \le T_f. \tag{88}$$

The performance was characterized by

$$\Delta = \bar{E}_r \int_{T_i}^{T_f} \tilde{f}(t) \tilde{g}^*(t) \, dt = \bar{E}_r \int_{T_i}^{T_f} \tilde{f}^*(t) \tilde{g}(t) \, dt.$$
(89)

In this section we want to derive an expression for l_o and Δ in terms of differential equations. These expressions will enable us to specify the receiver and its performance completely without solving an integral equation. We derive two alternative expressions. The first expression is obtained by finding a set of differential equations and associated boundary conditions that specify $\tilde{g}(t)$. The second expression is based on the realizable MMSE estimate of $\tilde{n}_c(t)$.

9.4.1 Differential-equation Representation of the Optimum Receiver and Its Performance: I

We define

$$\tilde{\boldsymbol{\xi}}(t) \triangleq \int_{T_i}^{T_f} \tilde{\mathbf{K}}_{\tilde{\mathbf{x}}}(t,\tau) \tilde{\mathbf{C}}^{\dagger}(\tau) \tilde{g}(\tau) d\tau, \qquad T_i \le t \le T_f.$$
(90)‡

Using (90) in (88) gives

$$\tilde{f}(t) = \tilde{\mathbf{C}}(t)\boldsymbol{\xi}(t) + N_0 \tilde{g}(t), \qquad T_i \le t \le T_f$$
(91)

or

$$\tilde{g}(t) = \frac{1}{N_0} [\tilde{f}(t) - \tilde{\mathbf{C}}(t)\tilde{\boldsymbol{\xi}}(t)], \qquad T_i \le t \le T_f.$$
(92)

‡ Notice that $\tilde{\xi}(t)$ is defined by (90). It should not be confused with $\tilde{\xi}_{P}(t)$, the error covariance matrix.

Thus, if we can find $\xi(t)$, we have an explicit relation for $\tilde{g}(t)$. By modifying the derivation in the appendix of Part II, we can show that $\xi(t)$ is specified by the equations

$$\frac{d\mathbf{\tilde{\xi}}(t)}{dt} = \mathbf{\tilde{F}}(t)\mathbf{\tilde{\xi}}(t) + \mathbf{\tilde{G}}(t)\mathbf{\tilde{Q}}\mathbf{\tilde{G}}^{\dagger}(t)\mathbf{\tilde{\eta}}(t),$$
(93)

$$\frac{d\tilde{\boldsymbol{\eta}}(t)}{dt} = \frac{1}{N_0} \tilde{\mathbf{C}}^{\dagger}(t)\tilde{\mathbf{C}}(t)\tilde{\boldsymbol{\xi}}(t) - \tilde{\mathbf{F}}^{\dagger}(t)\tilde{\boldsymbol{\eta}}(t) - \frac{1}{N_0}\tilde{\mathbf{C}}^{\dagger}(t)\tilde{f}(t), \qquad (94)$$

$$\tilde{\boldsymbol{\xi}}(T_i) = \tilde{\mathbf{P}}_0 \tilde{\boldsymbol{\eta}}(T_i), \tag{95}$$

$$\tilde{\boldsymbol{\eta}}(T_t) = \boldsymbol{0},\tag{96}$$

and

$$\tilde{\mathbf{P}}_0 = \bar{E}_r \mathbf{I}. \tag{97}$$

This is a set of linear matrix equations that can be solved numerically. To evaluate Δ , we substitute (90) into (89) to obtain

$$\Delta = \frac{\bar{E}_{\tau}}{N_0} \bigg[1 - \int_{T_i}^{T_f} \tilde{f}^*(t) \tilde{\mathbf{C}}(t) \tilde{\mathbf{\xi}}(t) dt \bigg].$$
(98)

(Recall that we assume

$$\int_{T_i}^{T_f} |\tilde{f}(t)|^2 dt = 1.$$
(99)

The first term is the performance in the presence of white noise only. The second term is the degradation due to the colored noise, which we denote as

$$\Delta_{dg} \triangleq \int_{T_i}^{T_f} \tilde{f}^*(t) \tilde{\mathbf{C}}(t) \tilde{\mathbf{\xi}}(t) dt.$$
(100)

Later we shall discuss how to design $\tilde{f}(t)$ to minimize Δ_{dg} . Notice that Δ_{dg} is normalized and does not include the \bar{E}_r/N_0 multiplier.

We now develop an alternative realization based on the realizable estimate.

9.4.2 Differential-equation Representation of the Optimum Receiver and Its Performance: II

There are several ways to develop the desired structure. We carry out the details for two methods.

The first method is based on a whitening filter approach. In Section 9.3, we used an unrealizable whitening filter to derive $\tilde{g}(t)$. Now we use a realizable whitening filter. Let $\tilde{h}_{wr}(t, z)$ denote the impulse response of

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the complex realizable whitening filter. When the filter input is $\tilde{n}(t)$, the output is a sample function from a white noise process.

By extending the results of Problem I-4.3.4 to the complex case, we can show that

$$\tilde{h}_{wr}(t,z) = \left(\frac{1}{N_0}\right)^{\frac{1}{2}} [\delta(t-z) - \tilde{h}_o(t,\tau;t)],$$
(101)

where $\tilde{h}_o(t, \tau:t)$ is the linear filter whose output is the MMSE estimate of $\tilde{n}_o(t)$ when the input is $\tilde{n}_o(t) + \tilde{w}(t)$. The test statistic can be written as

$$l_{o} = \left| \int_{T_{i}}^{T_{f}} dt \left[\int_{T_{i}}^{t} \tilde{h}_{wr}(t, z) \tilde{r}(z) \, dz \int_{T_{i}}^{t} \tilde{h}_{wr}^{*}(t, y) \tilde{f}^{*}(y) \, dy \right] \right|^{2} \\ = \left| \int_{T_{i}}^{T_{f}} dt \tilde{r}_{wr}(t) \tilde{f}_{wr}^{*}(t) \right|^{2}.$$
(102)

The receiver is shown in Fig. 9.8. Notice that the operation inside the dashed lines does not depend on $\tilde{r}(t)$. The function $\tilde{f}_{wr}(t)$ is calculated when the receiver is designed. The operation inside the dashed lines indicates this calculation.

A state-variable implementation is obtained by specifying $\tilde{h}_o(t, \tau:t)$ in terms of differential equations. Because it can be interpreted as an optimum



Fig. 9.8 Optimum receiver realization using realizable whitening filters.

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estimator of $\tilde{n}_c(t)$, we can use (A.159)–(A.162). The estimator equation is

$$\frac{d\hat{\mathbf{x}}(t)}{dt} = \mathbf{\tilde{F}}(t)\hat{\mathbf{x}}(t) + \mathbf{\tilde{\xi}}_{P}(t)\mathbf{\tilde{C}}^{\dagger}(t)\frac{1}{N_{0}}[\tilde{r}(t) - \mathbf{\tilde{C}}(t)\hat{\mathbf{x}}(t)], \qquad T_{i} \le t, \quad (103)$$

and the variance equation is

$$\frac{d\tilde{\boldsymbol{\xi}}_{P}(t)}{dt} = \mathbf{\tilde{F}}(t)\boldsymbol{\tilde{\xi}}_{P}(t) + \boldsymbol{\tilde{\xi}}_{P}(t)\mathbf{\tilde{F}}^{\dagger}(t) - \boldsymbol{\tilde{\xi}}_{P}(t)\mathbf{\tilde{C}}^{\dagger}(t)\frac{1}{N_{0}}\mathbf{\tilde{C}}(t)\boldsymbol{\tilde{\xi}}_{P}(t) + \mathbf{\tilde{G}}(t)\mathbf{\tilde{Q}}\mathbf{\tilde{G}}^{\dagger}(t),$$
$$T_{i} \leq t, \quad (104)$$

with initial conditions

$$\hat{\tilde{\mathbf{x}}}(T_i) = E[\hat{\tilde{\mathbf{x}}}(T_i)] = \mathbf{0}$$
(105)

and

$$\boldsymbol{\tilde{\xi}}_{P}(\boldsymbol{T}_{i}) = E[\hat{\tilde{\mathbf{x}}}(\boldsymbol{T}_{i})\hat{\tilde{\mathbf{x}}}^{\dagger}(\boldsymbol{T}_{i})].$$
(106)

The estimate of $\tilde{n}_c(t)$ is

$$\hat{\tilde{n}}_{cr}(t) = \tilde{\mathbf{C}}(t)\tilde{\mathbf{x}}(t).$$
(107)

Notice that this is the MMSE realizable estimate, assuming that H_0 is true. Using (103), (107), and Fig. 9.8, we obtain the receiver shown in Fig. 9.9.

The performance expression follows easily. The output of the whitening filter in the bottom path is $\tilde{f}_{wr}(t)$. From (76),

$$\Delta = \frac{\bar{E}_r}{N_0} \int_{T_i}^{T_f} |\tilde{f}_{wr}(t)|^2 dt.$$
 (108)

From Fig. 9.8 or 9.9 we can write

$$\tilde{f}_{wr}(t) = \{\tilde{f}(t) - \tilde{f}_{r}(t)\},$$
(109)

where $\tilde{f}_r(t)$ is the output of the optimum realizable filter when its input is $\tilde{f}(t)$. Using (109) in (108), we have

$$\Delta = \frac{\bar{E}_r}{N_0} \left\{ 1 - \int_{T_i}^{T_r} \{ 2\tilde{f}^*(t)\tilde{f}_r(t) - |\tilde{f}_r(t)|^2 \} dt \right\}.$$
 (110)

From (100) we see that

$$\Delta_{dg} = \int_{T_i}^{T_f} \{2\tilde{f}^*(t)\tilde{f}_r(t) - |\tilde{f}_r(t)|^2\} dt.$$
(111)

We can also derive the optimum receiver directly from (85) and (92). Because this technique can also be used for other problems, we carry out the details.





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An Alternative Derivation.[†] This method is based on the relationship

$$\tilde{\xi}(t) = \tilde{\Sigma}(t)\tilde{\eta}(t) + \tilde{\xi}_r(t), \qquad T_i \le t \le T_f,$$
(112)

where $\tilde{\Sigma}(t)$ and $\xi_r(t)$ are matrices that we now specify. Differentiating (112) and using (93)-(96), we find that $\tilde{\Sigma}(t)$ must satisfy

$$\frac{d\tilde{\mathbf{\Sigma}}(t)}{dt} = \tilde{\mathbf{F}}(t)\tilde{\mathbf{\Sigma}}(t) + \tilde{\mathbf{\Sigma}}(t)\tilde{\mathbf{F}}^{\dagger}(t) - \frac{1}{N_0}\tilde{\mathbf{\Sigma}}(t)\tilde{\mathbf{C}}^{\dagger}(t)\tilde{\mathbf{\Sigma}}(t) + \tilde{\mathbf{G}}(t)\mathbf{Q}\tilde{\mathbf{G}}^{\dagger}(t), \quad (113)$$

with

$$\tilde{\mathbf{\Sigma}}(T_i) = \tilde{\mathbf{P}}_0,\tag{114}$$

which is familiar as the variance equation (104). [Thus, $\tilde{\Sigma}(t) = \tilde{\xi}_P(t)$.] The function $\tilde{\xi}_r(t)$ must satisfy

$$\frac{d\xi_r(t)}{dt} = \tilde{\mathbf{F}}(t)\tilde{\mathbf{\xi}}_r(t) + \frac{1}{N_0}\tilde{\mathbf{\Sigma}}(t)\tilde{\mathbf{C}}^{\dagger}(t)[\tilde{f}(t) - \tilde{\mathbf{C}}(t)\tilde{\mathbf{\xi}}_r(t)],$$
(115)

with

$$\bar{\boldsymbol{\xi}}_r(\boldsymbol{T}_i) = \boldsymbol{0}. \tag{116}$$

This has the same structure as the estimator equation, except that $\tilde{r}(t)$ is replaced by $\tilde{f}(t)$.

~

In order to carry out the next step, we introduce a notation for $\tilde{\xi}(t)$ and $\tilde{\eta}(t)$ to indicate the endpoint of the interval. We write $\tilde{\xi}(t, T_f)$ and $\tilde{\eta}(t, T_f)$. These functions satisfy (93)-(96) over the interval $T_i \leq t \leq T_f$.

The test statistic is

$$l_{o} = \left| \int_{T_{i}}^{T_{f}} \tilde{r}(t) \tilde{g}^{*}(t) dt \right|^{2}$$
$$= \left| \frac{1}{N_{0}} \int_{T_{i}}^{T_{f}} \tilde{r}(\tau) [\tilde{f}(\tau) - \tilde{C}(\tau) \tilde{\Sigma}(\tau) \tilde{\eta}(\tau, T_{f}) - \tilde{C}(\tau) \tilde{\xi}_{r}(\tau)]^{*} d\tau \right|^{2}. \quad (117)$$

To obtain the desired result, we use the familiar technique of differentiation and integration.

$$l_o = \left| \frac{1}{N_0} \int_{T_i}^{T_f} \left[\frac{d}{dt} \left\{ \int_{T_i}^t \tilde{r}(\tau) [\tilde{f}(\tau) - \tilde{C}(\tau) \tilde{\Sigma}(\tau) \tilde{\eta}(\tau, t) - \tilde{C}(\tau) \tilde{\xi}_{\tau}(\tau)]^* d\tau \right\} \right] dt \right|^2.$$
(118)

Differentiating the terms in braces gives

$$\frac{d}{dt}\left\{\cdot\right\} = \tilde{r}(t)[\tilde{f}(t) - \tilde{\mathbf{C}}(t)\tilde{\boldsymbol{\xi}}_{r}(t)]^{*} + \int_{0}^{t} \tilde{r}(\tau) \left[-\tilde{\mathbf{C}}(\tau)\tilde{\boldsymbol{\Sigma}}(\tau)\frac{\partial\tilde{\eta}(\tau, t)}{\partial t}\right]^{*} d\tau.$$
(119)

We can show (see Problem 9.4.5) that the second term reduces to

$$\int_{0}^{t} \tilde{r}(\tau) \left[-\tilde{C}(\tau)\tilde{\boldsymbol{\Sigma}}(\tau) \; \frac{\partial \tilde{\eta}(\tau,t)}{dt} \right]^{*} d\tau = [\tilde{f}(t) - \tilde{C}(t)\boldsymbol{\xi}_{r}(t)]^{*} [-\tilde{C}(t)\hat{\mathbf{x}}(t)], \quad (120)$$

† This alternative derivation can be omitted on the first reading.

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where $\hat{\mathbf{x}}(t)$ is the state vector of the optimum realizable linear filter when its input is $\tilde{r}(t)$. Using (120) in (119) and the result in (118), we obtain

$$l_o = \left| \frac{1}{N_0} \int_{T_i}^{T_f} [\tilde{r}(t) - \tilde{C}(t)\hat{\tilde{\mathbf{x}}}(t)] [\tilde{f}(t) - \tilde{C}(t)\tilde{\boldsymbol{\xi}}_r(t)]^* dt \right|^2.$$
(121)

The receiver specified by (121) is identical with the receiver shown in Fig. 9.9.

In this section we have developed two state-variable realizations for the optimum receiver to detect a bandpass signal in colored noise. The performance degradation was also expressed in terms of a differential equation. These results are important because they enable us to specify completely the optimum receiver and its performance for a large class of colored noise processes. They also express the problem in a format in which we can study the question of optimal signal design. We discuss this problem briefly in the next section.

9.5 OPTIMAL SIGNAL DESIGN

The performance in the presence of colored noise is given by (77). This can be rewritten as

$$\Delta = \bar{E}_{r} \int_{T_{i}}^{T_{f}} \tilde{f}(t) \tilde{g}^{*}(t) dt$$

$$= \bar{E}_{r} \int_{T_{i}}^{T_{f}} \tilde{f}(t) \left[\int_{T_{i}}^{T_{f}} \tilde{Q}^{*}_{\tilde{n}}(t, u) \tilde{f}^{*}(u) du \right] dt$$

$$= \bar{E}_{r} \int_{T_{i}}^{T_{f}} \tilde{f}(t) \left[\frac{1}{N_{0}} \tilde{f}^{*}(t) - \frac{1}{N_{0}} \int_{T_{i}}^{T_{f}} \tilde{h}^{*}_{ou}(t, u) \tilde{f}^{*}(u) du \right] dt$$

$$= \frac{\bar{E}_{r}}{N_{0}} \left[1 - \int_{T_{i}}^{T_{f}} \int_{T_{i}}^{T_{f}} \tilde{f}(t) \tilde{h}^{*}_{ou}(t, u) \tilde{f}^{*}(u) dt du \right].$$
(122)

In the last equality, we used (99). The integral in the second term is just Δ_{dg} , which was defined originally in (100). [An alternative expression for Δ_{dg} is given in (111).]

We want to choose $\tilde{f}(t)$ to minimize Δ_{dg} . In order to obtain a meaningful problem, we must constrain both the energy and bandwidth of $\tilde{f}(t)$ (see discussion on page I-302). We impose the following constraints. The energy constraint is

$$\int_{T_i}^{T_f} |\tilde{f}(t)|^2 dt = 1.$$
(123)

The mean-square bandwidth constraint is

$$\int_{T_i}^{T_f} \left| \frac{d\tilde{f}(t)}{dt} \right|^2 dt = B^2.$$
(124)

In addition, we require

$$\tilde{f}(T_i) = \tilde{f}(T_f) = 0 \tag{125}$$

to avoid discontinuities at the endpoints.

The function that we want to minimize is

$$J = \iint_{T_i}^{T_f} \tilde{f}(t) \tilde{h}_{ou}^*(t, u) \tilde{f}^*(u) dt du + \lambda_E \left[\int_{T_i}^{T_f} |\tilde{f}(t)|^2 dt - 1 \right] + \lambda_B \left[\int_{T_i}^{T_f} |\dot{\tilde{f}}(t)|^2 dt - B^2 \right], \quad (126)$$

where λ_E and λ_B are Lagrange multipliers. To carry out the minimization, we let

$$\tilde{f}(t) = \tilde{f}_o(t) + \varepsilon \tilde{f}_s(t)$$
(127)

and require that

$$\left. \frac{dJ}{d\varepsilon} \right|_{\varepsilon=0} = 0 \tag{128}$$

for all $\tilde{f}_{\varepsilon}(t)$ satisfying (123)–(125). Substituting (127) into (126) and carrying out the indicated steps, we obtain

$$\operatorname{Re}\left\{ \iint_{T_{i}}^{T_{f}} \tilde{f}_{\varepsilon}(t)\tilde{h}_{ou}^{*}(t,u)\tilde{f}_{o}^{*}(u) \, du \, dt + \lambda_{E} \int_{T_{i}}^{T_{f}} \tilde{f}(t)\tilde{f}_{o}^{*}(t) \, dt + \lambda_{B} \int_{T_{i}}^{T_{f}} \tilde{f}_{\varepsilon}(t)\tilde{f}(t) \, dt \right\} = 0. \quad (129)$$

Integrating the last term by parts, using (125), and collecting terms, we have

$$\operatorname{Re}\left\{\int_{T_{i}}^{T_{f}}\tilde{f}_{e}(t) dt\left[\int_{T_{i}}^{T_{f}}\tilde{h}_{ou}^{*}(t,u)\tilde{f}_{o}^{*}(u) du + \lambda_{E}\tilde{f}_{o}^{*}(t) - \lambda_{B}\tilde{f}_{o}^{*}(t)\right]\right\} = 0. \quad (130)$$

Since $\tilde{f}_{\epsilon}(t)$ is arbitrary, the term in the brackets must be identically zero. From (92) and (122), we observe that

$$\int_{T_i}^{T_f} \tilde{h}_{ou}^*(t, u) \tilde{f}_o^*(u) \, du = [\tilde{\mathbf{C}}(t)\tilde{\boldsymbol{\xi}}(t)]^* \quad \text{when } \varepsilon = 0.$$
(131)

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We define

$$\tilde{p}_f(t) = -\lambda_B \tilde{f}_o(t). \tag{132}$$

We now have the following set of differential equations that specify $\tilde{f}_o(t)$:

$$\tilde{\tilde{p}}_{f}(t) = -\lambda_{E}\tilde{f}_{o}(t) - \tilde{\mathbf{C}}(t)\tilde{\boldsymbol{\xi}}(t), \qquad (133)$$

$$\dot{f}_o(t) = -\frac{1}{\lambda_B} \tilde{p}_t(t), \qquad (134)$$

$$\dot{\tilde{\mathbf{\xi}}}(t) = \mathbf{\tilde{F}}(t)\tilde{\mathbf{\xi}}(t) + \tilde{\mathbf{G}}(t)\mathbf{Q}\tilde{\mathbf{G}}^{\dagger}(t)\tilde{\boldsymbol{\eta}}(t), \qquad (135)$$

$$\dot{\tilde{\boldsymbol{\eta}}}(t) = \frac{1}{N_0} \,\tilde{\mathbf{C}}^{\dagger}(t) \tilde{\mathbf{C}}(t) \tilde{\boldsymbol{\xi}}(t) - \tilde{\mathbf{F}}^{\dagger}(t) \tilde{\boldsymbol{\eta}}(t) - \frac{1}{N_0} \,\tilde{\mathbf{C}}^{\dagger}(t) \tilde{f}_o(t), \qquad (136)$$

with boundary conditions

$$\tilde{f}_0(T_i) = \tilde{f}_0(T_f) = 0,$$
 (137)

$$\tilde{\boldsymbol{\xi}}(T_i) = \tilde{\mathbf{P}}_0 \tilde{\boldsymbol{\eta}}(T_i), \qquad (138)$$

$$\tilde{\boldsymbol{\eta}}(T_f) = \boldsymbol{0}. \tag{139}$$

If the process state vector is *n*-dimensional, we have 2n + 2 linear equations. We must solve these as a function of λ_E and λ_B and then evaluate λ_E and λ_B by using the constraint equations (123) and (124). Since (128) is only a necessary condition, we get several solutions that satisfy (133)-(139) and (123)-(125). Therefore, we must choose the solution that gives the absolute minimum. Baggeroer [3], [4] originally derived (133)-(139) using Pontryagin's principle, and carried out the solution for some typical real-valued processes. The interested reader should consult these two references for further details.

Frequently we want to impose hard constraints on the signal instead of the quadratic constraints in (123) and (124). For example, we can require

$$|\tilde{f}(t)| < A, \qquad T_i \le t \le T_f. \tag{140}$$

In this case we can use Pontryagin's principle (cf. [5] or [6]) to find the equations specifying the optimal signal.

The purpose of this brief discussion is to demonstrate how the statevariable formulation can be used to study optimal signal design. Other signal design problems will be encountered as we proceed through the text.

9.6 SUMMARY AND RELATED ISSUES

In this chapter we have discussed the problem of detecting the return from a slowly fluctuating point target in additive noise. The derivations were all straightforward extensions of our earlier work. Several important results should be emphasized:

1. When the additive noise is white, the optimum receiver is as shown in Fig. 9.4. The received waveform is passed through a bandpass matched filter and a square-law envelope detector. The output of the envelope detector is sampled and compared with a threshold. The performance is a monotonic function of \bar{E}_r/N_0 ,

$$P_F = (P_D)^{1 + E_T/N_0}.$$
 (141)

2. When the additive noise is nonwhite, the optimum receiver is as shown in Fig. 9.7. The only difference is in the impulse response of the matched filter. The performance is a function of Δ ,

$$\Delta = \bar{E}_r \iint_{T_i}^{T_f} \tilde{f}(t) \tilde{Q}_{\hat{n}}^*(t, u) \tilde{f}^*(u) dt du.$$
(142)

Specific nonwhite noises will be studied later.

3. When the colored noise has a finite-dimensional state representation, the optimum receiver implementation is as shown in Fig. 9.9. The advantage of this implementation is that it avoids solving an integral equation.

There are several related issues that should be mentioned. In many radar/sonar systems it is necessary to illuminate the target with a number of pulses in order to achieve satisfactory performance. A typical transmitted sequence is shown in Fig. 9.10. Once again we assume that the Rayleigh reflection model developed in Section 9.1 is valid. We must now specify how the returns from successive pulses are related. There are three cases of interest.



Fig. 9.10 Typical transmitted sequence.

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In the first case, the target does not fluctuate during the time the *entire* sequence illuminates it. In this case we can write the received signal from a zero-velocity target as

$$s_r(t) = \sqrt{2} \operatorname{Re}\left\{\tilde{b}\sum_{i=1}^N \tilde{f}_s(t - iT_p - \tau)e^{j\omega_c t}\right\}.$$
(143)

Notice that there is a single complex multiplier, \tilde{b} . This model might be appropriate for a radar with a high pulse repetition rate and a target where small movements do not affect the return appreciably. Comparing (25) and (143), we see that this reduces to the problem that we just solved if we define

$$\tilde{f}(t) \triangleq \sum_{i=1}^{N} \tilde{f}_{s}(t-iT_{p}).$$
(144)

In a white noise environment, the optimum receiver has a bandpass filter matched to the subpulse. The sampled outputs are added *before* envelope detection. The performance is determined by

$$\Delta = \frac{\bar{E}_r}{N_0} = \frac{N(2\sigma^2 E_i)}{N_0},$$
(145)

where E_i is the transmitted energy in each subpulse.

In the second case we assume that $|\tilde{b}|$ has the same value on all pulses, but we model the phase of each pulse as a statistically independent, uniformly distributed random variable. This model might be appropriate in the same target environment as case 1 when the radar does not have pulse-to-pulse coherence. The optimum receiver for the problem is derived in Problem 9.6.1.

In the third case, the target fluctuates enough so that the returns from successive subpulses are *statistically independent*. Then

$$s_r(t) = \sqrt{2} \operatorname{Re}\left\{\sum_{i=1}^N \tilde{b}_i \tilde{f}_s(t - iT_p - \tau)e^{j\omega_c t}\right\}.$$
(146)

The \tilde{b}_i are zero-mean, statistically independent, complex Gaussian random variables with identical statistics. This model is appropriate when small changes in the target orientation give rise to significant changes in the reflected signal.

This model corresponds to the separable-kernel Gaussian signal-innoise problem that we discussed in Section 4.2. The optimum receiver passes the received waveform through a bandpass filter matched to the subpulse and a square-law envelope detector. The detector output is sampled every T_p seconds, and the samples are summed. The sum is compared with a threshold in order to make a decision. The performance is evaluated just as in Section 4.2 (see Problem 9.6.2). The performance for this particular model has been investigated extensively by Swerling (Case II in [7]).

A second related issue is that of digital communication over a slowly fluctuating Rayleigh channel using a binary or M-ary signaling scheme. Here the complex envelope of the received signal is

$$\tilde{r}(t) = \sqrt{E_k} \, \tilde{b} \tilde{f}_k(t) + \tilde{n}(t), \qquad T_i \le t \le T_f : H_k, \qquad k = 1, \dots, M.$$
(147)

The optimum receiver follows easily (see Problem 9.6.7). We shall return to the performance in a later chapter.

This completes our initial discussion of the detection problem. In the next chapter we consider the parameter estimation problem. Later we consider some further topics in detection.

9.7 PROBLEMS

P.9.2 Detection in White Noise

SUBOPTIMUM RECEIVERS

Problem 9.2.1. The optimum receiver in the presence of white noise is specified by (36) and (39). Consider the suboptimum receiver that computes

$$\tilde{l}_v = \int_{T_i}^{T_f} \tilde{r}(t) \tilde{v}^*(t) dt$$
(P.1)

and compares $|\tilde{l}_v|^2$ with a threshold γ . The function $\tilde{v}(t)$ is arbitrary.

1. Verify that the performance of this receiver is completely characterized by letting $\Delta=\Delta_v$ in (50), where

$$\Delta_{v} \triangleq \frac{E[|\tilde{l}_{v}|^{2} | H_{1}] - E[|\tilde{l}_{v}|^{2} | H_{0}]}{E[|\tilde{l}_{v}|^{2} | H_{0}]}.$$
(P.2)

2. Calculate Δ_v for the input

$$\widetilde{r}(t) = \widetilde{f}(t) + \widetilde{w}(t), \qquad T_i \le t \le T_f.$$

3. The results in parts 1 and 2 give an expression for Δ_v as a functional of $\tilde{v}(t)$. Find the function $\tilde{v}(t)$ that minimizes Δ_v . [This is the structured approach to the optimum receiver of (36) and (39).]

Problem 9.2.2. Assume that

$$\tilde{f}(t) = \begin{cases} \sqrt{\frac{1}{T}}, & 0 \le t \le T, \\ 0, & \text{elsewhere.} \end{cases}$$

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The complex envelope of the received waveform is

$$\tilde{r}(t) = \sqrt{E}\tilde{f}(t) + \tilde{w}(t), \quad -\infty < t < \infty.$$

1. Plot the output of the matched filter as a function of time.

2. Assume that instead of using a matched filter, we use a bandpass filter centered at ω_e , whose complex envelope's transfer function is

$$\tilde{H}(f) = \begin{cases} 1, & |f| \leq \frac{W}{2} \\ \\ 0, & |f| > \frac{W}{2} \end{cases}$$

Denote the output of this filter due to the signal as $f_0(t)$, and the output due to noise as $\tilde{w}_0(t)$. Define

$$\Delta_c = \frac{\max |f_0(t)|^2}{E[|\tilde{w}_0(t)|^2]}$$

Verify that this quantity corresponds to Δ_v as defined in (P.2) of Problem 9.2.1. Plot

$$\Lambda_{cn} = \frac{\Delta_c}{\bar{E}_r/N_0}$$

as a function of WT. What is the optimum value of WT? What is Δ_{en} in decibels at this optimum value? Is Δ_{en} sensitive to the value of WT?

Problem 9.2.3. The complex envelope of the transmitted signal is

$$\tilde{f}(t) = a \sum_{i=1}^{N} \tilde{u}(t - iT_p),$$
$$\tilde{u}(t) = \begin{cases} \frac{1}{\sqrt{T_s}}, & 0 \le t \le T_s, \\ 0, & \text{elsewhere,} \end{cases}$$

and

where

 $T_p \gg T_s$.

1. Plot the Fourier transform of $\tilde{f}(t)$.

2. The matched filter for $\tilde{f}(t)$ is sometimes referred to as a "comb filter." Consider the filter response

$$\tilde{H}\{f\} = \sum_{i=-M}^{M} \tilde{Y}\{f - iW_{p}\},$$

where

$$\widetilde{Y}{f} = \begin{cases} 1, & |f| < \frac{W_s}{2}, \\ 0, & \text{elsewhere.} \end{cases}$$

 $W_n \gg W_s$.

and

Assume that

$$W_p = \frac{1}{T_p}$$
$$W_s = \frac{2}{NT_p}$$

and that M is the smallest integer greater than T_n/T_s .

- (i) Sketch $\tilde{H}{f}$.
- (ii) Find the degradation in Δ due to this suboptimum filter.
- (iii) Why might one use $\tilde{H}\{f\}$ instead of the optimum filter?

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Problem 9.2.4. An on-off signaling scheme operates over two frequency-diversity channels. The received waveforms on the two hypotheses are

$$\begin{split} r(t) &= \sqrt{E_t} \ \mathrm{Re} \ \{ \tilde{b}_1 \tilde{f}(t) e^{j \omega_1 t} + \tilde{b}_2 \tilde{f}(t) e^{j \omega_2 t} \} + n(t), \quad 0 \le t \le T : H_1, \\ r(t) &= n(t), \quad 0 \le t \le T : H_0, \end{split}$$

The multipliers \tilde{b}_1 and \tilde{b}_2 are statistically independent, zero-mean complex Gaussian random variables

$$E[\tilde{b}_1 \tilde{b}_1^*] = E[\tilde{b}_2 \tilde{b}_2^*] = 2\sigma_b^2.$$

The frequencies ω_1 and ω_2 are such that the signals are essentially disjoint. The *total* energy transmitted is E_t . (There is $E_t/2$ in each channel.) The additive noise n(t) is a sample function from a zero-mean, white Gaussian process with spectral height $N_0/2$.

- 1. Find the optimum receiver.
- 2. Find P_D and P_F as a function of the threshold.

3. Assume a minimum probability-of-error criterion and equal a priori probabilities. Find the threshold setting and the resulting $Pr(\epsilon)$.

Problem 9.2.5. Consider the model in Problem 9.2.4. Assume that the two channels have unequal strengths and that we use unequal energies in the two channels. Thus,

$$r(t) = \sqrt{2} \operatorname{Re} \left\{ \sqrt{E_1} \tilde{b}_1 \tilde{f}(t) e^{j\omega_1 t} + \sqrt{E_2} \tilde{b}_2 \tilde{f}(t) e^{j\omega_2 t} \right\} + n(t), \quad 0 \le t \le T : H_1,$$

here

where

$$E_1 + E_2 = E_t.$$
 (P.1)

The received waveform on H_0 is the same as in Problem 9.2.4. The mean-square values of the channel variables are $E[\tilde{b}_1 \tilde{b}_1^*] = 2\sigma_1^2$

and

$$E[\tilde{b}_2 \tilde{b}_2^*] = 2\sigma_2^2.$$

1. Find the optimum receiver.

2. Find P_D and P_F as functions of the threshold.

3. Assume a minimum probability-of-error criterion. Find the threshold setting and the resulting Pr (ϵ).

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Problem 9.2.6. Consider the model in Problem 9.2.5. Now assume that the channel gains are correlated.

$$\tilde{b} \triangleq \begin{bmatrix} b_1 \\ \\ \tilde{b_2} \end{bmatrix}$$

and

$$E[\tilde{b}\tilde{b}^{\dagger}] = \tilde{\Lambda}_{\tilde{b}}.$$

Repeat parts 1-3 of Problem 9.2.5.

Problem 9.2.7. In an on-off signaling system a signal is transmitted over N Rayleigh channels when H_1 is true. The received waveforms on the two hypotheses are

$$\begin{aligned} r(t) &= \sqrt{\frac{2E_t}{N}} \operatorname{Re}\left\{\sum_{i=1}^N \tilde{b}_i \tilde{f}(t) e^{j\omega_i t}\right\} + n(t), \quad 0 \le t \le T : H_1, \\ r(t) &= n(t), \quad 0 \le t \le T : H_0, \end{aligned}$$

The channel multipliers are statistically independent, zero-mean, complex Gaussian random variables

$$E[\tilde{b}_i \tilde{b}_j^*] = 2\sigma_b^2 \delta_{ij}.$$

The frequencies are such that the signal components are disjoint. The total energy transmitted is E_t . The additive noise n(t) is a zero-mean Gaussian process with spectral height $N_0/2$.

- 1. Find the optimum receiver.
- 2. Find $\mu(s)$.

3. Assume that the criterion has a minimum probability of error. Find an approximate Pr (ϵ). (*Hint*: Review Section I-2.7.)

Problem 9.2.8. Consider the model in Problem 9.2.7. Assume that the transmitted energy in the *i*th channel is E_i , where

$$\sum_{i=1}^{N} E_i = E_t.$$

Assume that the channel multipliers are correlated:

$$E[\widetilde{b}\widetilde{b}^{\dagger}] = \widetilde{\Lambda}\widetilde{b}.$$

1. Find the optimum receiver.

2. Find $\mu(s)$.

ALTERNATIVE TARGET MODELS

Problem 9.2.9. Consider the target model given in (32*a*) and (32*b*). Assume that the phase is a uniform random variable.

- 1. Derive the optimum receiver.
- 2. Calculate P_D and P_F .

3. Assume that we require the same P_F in this system and the system corresponding to the Rayleigh model. Find an expression for the ratio of the values of P_D in the two systems.

Problem 9.2.10. Consider the target model in (32c). Repeat parts 1 and 2 of Problem 9.2.9.

P.9.3 Detection in Colored Noise

Problem 9.3.1. Consider the receiver specified in (P.1) of Problem 9.2.1. The inputs on the two hypotheses are specified by (57)–(59).

- 1. Verify that the results in part 1 of Problem 9.2.1 are still valid.
- 2. Calculate Δ_v for the model in (57)-(59).
- 3. Find the function $\tilde{v}(t)$ that minimizes Δ_v .

Problem 9.3.2. Consider the model in (57)-(59). Assume that

$$\tilde{f}(t) = \begin{cases} \frac{1}{\sqrt{T_s}}, & 0 \le t \le T_s, \\ 0, & \text{elsewhere,} \end{cases}$$

and that

$$\tilde{S}_c(\omega) = \frac{2kP_c}{\omega^2 + k^2}, \quad -\infty < \omega < \infty.$$

The observation interval is infinite.

- 1. Find $g_{\infty}(\tau)$.
- 2. Evaluate Δ_0 as a function of E_t , k, T_s , P_c , and N_0 .
- 3. What value of T_s maximizes Δ_0 ? Explain this result intuitively.

Problem 9.3.3. Assume that

$$\tilde{n}_c(t) = n_1(t) - jn_2(t), \qquad -\infty < t < \infty.$$

The function $n_1(t)$ is generated by passing $u_1(t)$ through the filter

$$H_1(j\omega) = \frac{\sqrt{2k}}{j\omega + k} \,,$$

and the function $n_2(t)$ is generated by passing $u_2(t)$ through an identical filter. The inputs $u_1(t)$ and $u_2(t)$ are sample functions of real, white Gaussian processes with unity spectral height and

$$E[u_1(t_1)u_2(t_2)] = \alpha \delta(t_1 - t_2 - \Delta).$$

1. Find $\tilde{S}_{\tilde{n}_s}(\omega)$.

2. Consider the model in (57)-(59) and assume that the observation interval is infinite. Find an expression for a realizable whitening filter whose inverse is also realizable.

Problem 9.3.4. Assume that

$$\tilde{n}_c(t) = \sum_{i=1}^N \tilde{a}_i \tilde{k}_i(t),$$

where the $\tilde{k}_i(t)$ are known functions with unit energy and the \tilde{a}_i are statistically independent, complex Gaussian random variables with

$$E[|\tilde{a}_i|^2] = 2\sigma_i^2.$$

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The observation interval is infinite. The model in (57)-(59) is assumed.

- 1. Find $\tilde{g}(t)$. Introduce suitable matrix notation to keep the problem simple.
- 2. Consider the special case in which

$$\tilde{k}_i(t) = \tilde{f}(t - \tau_i)e^{j\omega_i t},$$

where τ_i and ω_i are known constants. Draw a block diagram of the optimum receiver.

Problem 9.3.5. Assume that

$$\tilde{f}(t) = \sum_{i=1}^{N} \tilde{f}_i \tilde{u} (t - iT_s),$$
(P.1)

where

$$\widetilde{u}(t) = \begin{cases} \frac{1}{\sqrt{T_s}}, & 0 \le t \le T_s, \\ 0, & \text{elsewhere,} \end{cases}$$
(P.2)

and

$$\sum_{i=1}^{N} |\tilde{f}_i|^2 = 1.$$
 (P.3)

Assume that we use the receiver in Problem 9.2.1 and that

$$\tilde{v}(t) = \sum_{i=1}^{N} \tilde{v}_i \tilde{u}(t - iT_s), \qquad (P.4)$$

where

$$\sum_{i=1}^{N} |\tilde{v}_i|^2 = 1.$$
 (P.5)

The model in (57)-(59) is valid and the observation is infinite. Define a filter-weighting vector as

$$\mathbf{v} = \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \vdots \\ \vdots \\ \tilde{v}_N \end{bmatrix}.$$
(P.6)

1. Find an expression for Δ_v in terms of \tilde{f} , \tilde{v} , E_t , N_0 , and $\tilde{K}_c(t, u)$. Introduce suitable matrices.

2. Choose $\tilde{\mathbf{v}}$ subject to the constraint in (P.5) in order to maximize Δ_v .

Problem 9.3.6. The complex envelopes on the two hypotheses are

$$r(t) = \sqrt{E_t} \tilde{b}\tilde{f}(t) + \tilde{n}_c(t) + \tilde{w}(t), \qquad -\infty < t < \infty : H_1,$$

$$r(t) = \tilde{n}_c(t) + \tilde{w}(t), \qquad -\infty < t < \infty : H_0.$$
(P.1)

The signal has unit energy

$$\int_{-\infty}^{\infty} |\tilde{f}(t)|^2 \, dt = 1.$$
 (P.2)

The colored noise is a sample function of a zero-mean complex Gaussian process with spectrum $S_c(\omega)$, where

$$\int_{-\infty}^{\infty} \tilde{S}_c(\omega) \frac{d\omega}{2\pi} = 2\sigma_c^2.$$
 (P.3)

The white noise is a zero-mean complex Gaussian process with spectral height N_0 . The multiplier \tilde{b} is a zero-mean complex Gaussian random variable,

$$E[\tilde{b}\tilde{b}^*] = 2\sigma_b^2. \tag{P.4}$$

The various random processes and random variables are all statistically independent.

1. Find the colored noise spectrum $\tilde{S}_c(\omega)$ that satisfies the constraint in (P.3) and minimizes Δ as defined in (76). Observe that Δ can also be written as

$$\Delta = \bar{E}_r \int_{-\infty}^{\infty} \frac{|\tilde{F}(j\omega)|^2}{\tilde{S}_{\tilde{n}}(\omega)} \frac{d\omega}{2\pi}.$$

(*Hint*: Recall the technique in Chapter II-5. Denote the minimum Δ as Δ_m .)

2. Evaluate Δ_m for the signal

$$\tilde{f}(t) = \begin{cases} \alpha e^{-\alpha t}, & t \ge 0, \\ 0, & t < 0. \end{cases}$$

3. Evaluate Δ_m for the signal

$$\tilde{F}\{f\} = \begin{cases} \frac{1}{\sqrt{W}}, & |f| \le W\\ 0, & |f| > W. \end{cases}$$

Problem 9.3.7. Consider the same model as Problem 9.3.6. We want to design the optimum signal subject to an energy and bandwidth constraint. Assume that $\tilde{S}_c(\omega)$ is symmetric around zero and that we require

$$\int_{-\infty}^{\infty} \omega \tilde{F}(j\omega) = 0, \qquad (P.1)$$

$$\int_{-\infty}^{\infty} \omega^2 \, |\tilde{F}(j\omega)|^2 \le \Omega_B. \tag{P.2}$$

1. Verify that Δ depends only on the signal shape through

$$\tilde{S}_{f}(\omega) \triangleq |\tilde{F}(j\omega)|^{2}.$$

2. Find the $\tilde{S}_{f}(\omega)$ subject to the constraints in (P.1) and (P.2) of this problem and in (P.2) of Problem 9.3.6, such that Δ is maximized.

3. Is your answer to part 2 intuitively correct?

4. What is the effect of removing the symmetry requirement on $\tilde{S}_c(\omega)$ and the requirement on $\tilde{F}(j\omega)$ in (P.1)? Discuss the implications in the context of some particular spectra.

Problem 9.3.8. Consider the model in Problem 9.3.5. Assume that the complex envelope of the desired signal is

$$\tilde{f}_d(t) = \tilde{f}(t)e^{j\omega_d t}$$

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and that $\tilde{\mathbf{v}}$ is chosen to maximize Δ_v for this desired signal. Assume that $\tilde{K}_c(t, u)$ is a stationary process whose spectrum is

$$\tilde{S}_c(\omega) \triangleq \frac{E_t P_c}{N_0} |\tilde{F}(j\omega)|^2$$

1. Assume that N = 2. Find \tilde{v}_1 and \tilde{v}_2 .

2. Assume that

 $\tilde{f}_i = 1$

and N = 3. Find \hat{v}_1 , \hat{v}_2 , and \tilde{v}_3 .

P.9.4 Finite-state Noise Processes

Problem 9.4.1. Consider the model in (79)-(92). Derive the results in (93)-(96). *Hint*: Read Sections A.4-A.6 of the Appendix to Part II.

Problem 9.4.2. Assume that $\bar{n}_c(t)$ has the state representation in (A.137)–(A.140). Write out (93)–(96) in detail.

Problem 9.4.3. Assume that $\tilde{n}_c(t)$ has the state representation in (A.148)–(A.153). Write out (93)–(96) in detail.

Problem 9.4.4. Consider the model in Section 9.4.2. Assume that $\tilde{n}_c(t)$ is a complex Gaussian process whose real and imaginary parts are statistically independent Wiener processes. Find the necessary functions for the receiver in Fig. 9.9.

Problem 9.4.5. Verify the result in (120).

Problem 9.4.6. Consider the model in Section 9.4.2. Assume that

$$\tilde{n}_c(t) = \sum_{i=1}^N \tilde{b}_i(t)\tilde{f}(t-\tau_i)e^{j\omega_i t},$$

where the $\tilde{b}_i(t)$ are statistically independent, complex Gaussian processes with the state representation in (A.137)–(A.140). Draw a block diagram of the optimum receiver in Fig. 9.9. Write out the necessary equations in detail.

P.9.5 Optimum Signal Design

Problem 9.5.1. Consider the optimum signal design problem in Section 9.5. Assume that

$$\tilde{S}_c(\omega) = \frac{2\alpha}{\omega^2 + \alpha^2}, \quad -\infty < \omega < \infty.$$

Write the equations specifying the optimum signal in detail.

Problem 9.5.2. The optimal signal-design problem is appreciably simpler if we constrain the form of the signal and receiver. Assume that $\tilde{f}(t)$ is characterized by (P.1–P.3) in Problem 9.3.5, and that we require

$$\tilde{\phi}(t) = \tilde{f}(t)$$

[see (P.4)-(P.5) in Problem 9.3.5].

- 1. Express Δ_v in terms of the \tilde{f}_i , E_t , N_0 , and $\tilde{K}_c(t, u)$.
- 2. Maximize Δ_v by choosing the f_i optimally.

3. Assume that

$$\tilde{K}_c(t, u) = e^{-k|t-u|}$$

and N = 2. Solve the equations in part 2 to find the optimum value of f_1 and f_2 .

4. Consider the covariance in part 3 and assume that N = 3. Find the optimum values of f_1 , f_2 , and f_3 .

Problem 9.5.3. Consider the generalization of Problem 9.5.2, in which we let

$$\tilde{f}_i = \tilde{a}_i \ e^{j\omega_i t},$$

where the \tilde{a}_i are complex numbers such that

$$\sum_{i=1}^{N} |\tilde{a}_i|^2 = 1$$

and the ω_i may take on values

$$|\omega_i| \le 2\pi W.$$

The remainder of the model in Problem 9.5.2 is still valid.

- 1. Express Δ_v in terms of $\tilde{\mathbf{a}}$, ω_i , E_t , N_0 , and $K_c(t, u)$.
- 2. Explain how the ω_i should be chosen in order to maximize Δ_v .

3. Carry out the procedure in part 2 for the covariance function in part 3 of Problem 9.5.2 for N = 2. Is your result intuitively obvious?

Problem 9.5.4. Consider the models in Problems 9.3.5 and 9.5.2. Assume that $\tilde{\mathbf{v}}$ is chosen to maximize $\Delta_{\mathbf{v}}$. Call the maximum value $\Delta_{\mathbf{v}_a}$.

1. Express Δ_{v_0} as a function of $\tilde{\mathbf{f}}$, E_t , N_0 , and $\tilde{K}_c(t, u)$.

2. Find that value of $\tilde{\mathbf{f}}$ that maximizes $\Delta_{v_{s}}$.

3. Consider the special case in part 3 of Problem 9.5.2. Find the optimum \tilde{f} and compare it with the optimum \tilde{f} in part 3 of Problem 9.5.2.

4. Repeat part 3 for N = 3.

P.9.6 Related Issues

MULTIPLE OBSERVATIONS

Problem 9.6.1. The complex envelopes on the received waveforms on the two hypotheses are

$$\begin{split} \tilde{r}(t) &= \sqrt{\frac{E_t}{N}} \sum_{i=1}^N |\tilde{b}| \, \tilde{u}(t - iT_p) e^{j\theta_i} + \tilde{w}(t), \qquad -\infty < t < \infty, \\ \tilde{r}(t) &= \tilde{w}(t), \qquad \qquad -\infty < t < \infty, \end{split}$$

where

$$\tilde{u}(t) = \begin{cases} \frac{1}{\sqrt{T_s}}, & 0 \le t \le T_s, \\ 0, & \text{elsewhere.} \end{cases}$$

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The multiplier $|\tilde{b}|$ is a Rayleigh random variable with mean-square value $2\sigma_b^2$. The θ_i are statistically independent, uniform random variables.

1. Find the optimum receiver.

2. Evaluate P_F .

3. Set up the expressions to evaluate P_D . Extensive performance results for this model are given by Swerling [7], [10].

Problem 9.6.2. Consider the target model in (146). Review the discussion in Section 4.2.2.

1. Draw a block diagram of the optimum receiver.

2. Review the performance results in Section 4.2.2. Observe that fixing s in the $\mu_{BP,SK}(s)$ expression fixes the threshold and, therefore, P_F . Fix s and assume that

$$K\bar{E}_r \triangleq \bar{E}_r$$

is fixed. Find the value of K that minimizes $\mu_{BS,SK}(s)$ as a function of s. Discuss the implications of this result in the context of an actual radar system.

Problem 9.6.3. Consider the model in Problem 9.6.1. Define

$$z |\tilde{b}|^2$$

and assume that z has the probability density given in (32b).

- 1. Derive the optimum receiver.
- 2. Evaluate P_F .

3. Set up the expressions to evaluate P_D . Results for this model are given in [7] and [10] (Case III in those references).

Problem 9.6.4. Consider the model in (146). Write

$$\tilde{b}_i = |\tilde{b}_i| e^{j\theta_i}$$

Assume that the θ_i are statistically independent random variables with a uniform probability density, Assume that each

$$z_i \triangleq |\tilde{b}_i|^2$$

has the probability density in (32b) and the z_i are statistically independent.

- 1. Derive the optimum receiver.
- 2. Evaluate P_F .

3. Set up the expressions to evaluate P_D . See [7] and [10] for performance results (Case IV in those references). Chapter 11 of [23] has extensive performance results based on Swerling's work.

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Problem 9.6.5. The complex received waveforms on the two hypotheses in a binary communication system are

$$\begin{split} \tilde{r}(t) &= \sqrt{E_i} \, \tilde{b} \tilde{f}(t) e^{j w_\Delta t} + \tilde{w}(t), \qquad 0 \le t \le T : H_1, \\ \tilde{r}(t) &= \sqrt{E_i} \, \tilde{b} \tilde{f}(t) + \tilde{w}(t), \qquad 0 \le t \le T : H_0, \end{split}$$

where ω_{Δ} is large enough for the two signal components to be orthogonal. The hypotheses are equally likely, and the criterion is minimum probability of error.

- 1. Draw a block diagram of the optimum receiver.
- 2. Calculate the probability of error.

Problem 9.6.6. The complex received waveforms on the two hypotheses in a binary communication system are

$$\widetilde{r}(t) = \sqrt{E_t} \widetilde{b} \widetilde{f}_1(t) + \widetilde{w}(t), \qquad 0 \le t \le T : H_1,$$

$$\widetilde{r}(t) = \sqrt{E_t} \widetilde{b} \widetilde{f}_0(t) + \widetilde{w}(t), \qquad 0 \le t \le T : H_0,$$

where

$$\int_{0}^{T} \tilde{f}_{0}(t) \tilde{f}_{1}^{*}(t) dt = \tilde{\rho}_{01}.$$

- 1. Draw a block diagram of the optimum receiver.
- 2. Calculate the probability of error.

Problem 9.6.7. Consider the model in (147) and assume that

$$\int_{T_i}^{T_f} \tilde{f}_k(t) \tilde{f}_m^*(t) dt = \delta_{km}.$$

The hypotheses are equally likely, and the criterion is minimum probability of error.

- 1. Draw a block diagram of the optimum receiver.
- 2. Use the union bound on pages I-263–I-264 to approximate Pr (ϵ).

(Comment: The reader who is interested in other communications problems should look at Sections P.4.4 and P.4.5 in Part I (Pages I-394–I-416). Most of those problems could also be included at this point.)

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