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# 11

# Doppler-Spread Targets and Channels

In Chapters 9 and 10 we confined our attention to slowly fluctuating point targets. They were characterized by a "perfect" reflection of the envelope of the incident signal. The returned signal differed from the transmitted signal in four ways:

- 1. Random amplitude.
- 2. Random phase angle.
- 3. Doppler shift.
- 4. Delay.

The amplitude and phase were due to the reflective characteristics of the target and could be modeled as random variables. The Doppler shift and delay were due to the velocity and range of the target and were modeled as unknown nonrandom variables.

In this chapter we consider point targets that cannot be modeled as slowly fluctuating targets. We begin our development with a qualitative discussion of the target model.

A simple example is shown in Fig. 11.1. The geometry could represent the reflective structure of an airplane, a satellite, or a submarine. The direction of signal propagation is along the x-axis. The target orientation changes as a function of time. Three positions are shown in Fig. 11.1a-c. As the orientation changes, the reflective characteristics change.

Now assume that we illuminate the target with a long pulse whose complex envelope is shown in Fig. 11.2*a*. A typical returned signal envelope is shown in Fig. 11.2*b*. We see that the effect of the changing orientation of the target is a time-varying attenuation of the envelope, which is usually referred to as *time-selective fading*.

Notice that if we transmit a short pulse as shown in Fig. 11.2c, the received signal envelope is undistorted (Fig. 11.2d) and the target can be modeled as a slowly fluctuating target. Later we shall see that all of our



Fig. 11.1 Target orientations.



Fig. 11.2 Signals illustrating time-selective fading.

results in Chapters 9 and 10 can be viewed as limiting cases of the results from the more general model in this chapter.

The energy spectrum of the long transmitted pulse is shown in Fig. 11.3*a*. Since the time-varying attenuation is an amplitude modulation, the spectrum of the returned signal is spread in frequency, as shown in Fig. 11.3*b*. The amount of spreading depends on the rate at which the target's reflective characteristics are changing. We refer to this type of target as a *frequency-spread* or *Doppler-spread* target. (Notice that frequency spreading and time-selective fading are just two different ways of describing the same phenomenon.)

Our simple example dealt with a radar problem. We have exactly the same mathematical problem when we communicate over a channel whose reflective characteristics change during the signaling interval. We refer to such channels as *Doppler-spread channels*, and most of our basic results will be applicable to both the radar/sonar and communications problems.

At this point we have an intuitive understanding of how a fluctuating target causes Doppler spreading. In Section 11.1 we develop a mathematical model for a fluctuating target. In Section 11.2 we derive the optimum receiver to detect a Doppler-spread target and evaluate its performance. In Section 11.3 we study the problem of digital communication systems



Fig. 11.3 Energy spectra of transmitted and returned signals.

operating over Doppler-spread channels. In Section 11.4 we consider the problem of estimating the parameters of a Doppler-spread target. Finally, in Section 11.5, we summarize our results.

#### 11.1 MODEL FOR DOPPLER-SPREAD TARGET (OR CHANNEL)

The model for the point target with arbitrary fluctuations is a straightforward generalization of the slow-fluctuation model.<sup>†</sup> Initially we shall discuss the model in the context of an active radar or sonar system. If a sinusoidal signal

$$\sqrt{2}\cos\omega_c t = \sqrt{2} \operatorname{Re}\left[e^{j\omega_c t}\right] \tag{1}$$

is transmitted, the return from a target located at a point  $\lambda$  (measured in units of round-trip travel time) is

$$b(t) = \sqrt{2} \left[ b_c \left( t - \frac{\lambda}{2} \right) \cos \left[ \omega_c (t - \lambda) \right] + b_s \left( t - \frac{\lambda}{2} \right) \sin \left[ \omega_c (t - \lambda) \right] \right].$$
(2)

The  $\lambda/2$  arises because the signal arriving at the receiver at time t left the transmitter at  $t - \lambda$  and was reflected from the target at  $t - \lambda/2$ . We assume that  $b_c(t)$  and  $b_s(t)$  are sample functions from low-pass, zero-mean, stationary, Gaussian random processes and that b(t) is a stationary bandpass process.

Defining

$$\tilde{b}_D(t) \stackrel{\Delta}{=} b_c(t) - jb_s(t), \tag{3}$$

we have

$$b(t) = \sqrt{2} \operatorname{Re}\left[\tilde{b}_D\left(t - \frac{\lambda}{2}\right)e^{j\omega_c(t-\lambda)}\right],\tag{4}$$

where  $\tilde{b}_D(t)$  is a sample function from a complex Gaussian process. (The subscript *D* denotes Doppler.) We assume that  $\tilde{b}_D(t)$  varies slowly compared to the carrier frequency  $\omega_c$ . Because  $\tilde{b}_D(t)$  has a uniform phase at any time, this assumption allows us to write (4) as

$$b(t) = \sqrt{2} \operatorname{Re}\left[\tilde{b}_D\left(t - \frac{\lambda}{2}\right)e^{j\omega_c t}\right].$$
(5)

† This model has been used by a number of researchers (e.g., Price and Green [1] and Bello [2]).

The random process  $\tilde{b}_D(t)$  is completely characterized by its complex covariance function,

$$E[\tilde{b}_D(t)\tilde{b}_D(u)] \stackrel{\Delta}{=} \tilde{K}_D(t-u) = \tilde{K}_D(\tau).$$
(6)

From our development in the Appendix,

$$E[\dot{b}_D(t)\dot{b}_D(u)] = 0 \quad \text{for all } t \text{ and } u.$$
(7)

In all of our discussion we assume that  $\tilde{K}_D(\tau)$  is known. In Section 11.4 we discuss the problem of measuring the parameters of  $\tilde{K}_D(\tau)$ .

Notice that if we assume

$$K_D(\tau) = K_D(0) \quad \text{for all } \tau, \tag{8}$$

we would have the slowly fluctuating model of Chapter 9 (see page 242). To be consistent with that model, we assume

$$\tilde{K}_D(0) = 2\sigma_b^2. \tag{9}$$

Because the target reflection process is assumed to be stationary, we can equally well characterize it by its spectrum:<sup>†</sup>

$$\widetilde{S}_D\{f\} = \int_{-\infty}^{\infty} \widetilde{K}_D(\tau) e^{-j2\pi f\tau} d\tau.$$
(10)

We refer to  $\tilde{S}_D\{f\}$  as the Doppler scattering function. From (A.56) we know that  $\tilde{S}_D\{f\}$  is a real function and that the spectrum of the actual bandpass signal is

$$S_D\{f\} = \frac{1}{2}\tilde{S}_D\{f - f_e\} + \frac{1}{2}\tilde{S}_D\{-f - f_e\}.$$
 (11)

Some typical spectra are shown in Fig. 11.4. We assume that the transmitted signal is a long pulse with a rectangular envelope. It has a narrow energy spectrum, as shown in Fig. 11.4*a*. In Fig. 11.4*b*, we show the energy spectrum of the returned signal when the target is fluctuating and has a zero average velocity. In Fig. 11.4*c*, we show the spectrum corresponding to a target that has a nonzero average velocity but is not

 $\dagger$  In most of the discussion in the next three chapters it is convenient to use f as an argument in the spectrum and Fourier transform. The braces  $\{ \}$  around the argument imply the f notation. The f notation is used throughout Chapters 11-13, so that the reader does not need to watch the  $\{\cdot\}$ . Notice that for deterministic signals,

$$\tilde{F}{f} \stackrel{\Delta}{=} \int_{-\infty}^{\infty} \tilde{f}(t) e^{-j2\pi ft} dt.$$



Fig. 11.4 Typical energy spectra.

fluctuating. This is the model of Chapter 9. In Fig. 11.4d, we show the spectrum for a fluctuating target with a nonzero average velocity.

We introduce two quantities to describe the gross behavior of the target. The first is the *mean Doppler shift*, which is defined as

$$m_D \stackrel{\Delta}{=} \frac{1}{2\sigma_b^2} \int_{-\infty}^{\infty} f \tilde{S}_D \{f\} df.$$
(12)

We next define

$$\overline{f_D}^2 \stackrel{\Delta}{=} \frac{1}{2\sigma_b^2} \int_{-\infty}^{\infty} f^2 S_D\{f\} df.$$
(13)

Combining (12) and (13) gives a quantity that we refer to as a mean-square Doppler spread,

$$\sigma_D^2 \stackrel{\Delta}{=} \overline{f_D^2} - m_D^2 = \frac{1}{2\sigma_b^2} \int_{-\infty}^{\infty} f^2 S_D\{f\} df - m_D^2.$$
(14)

We see that  $m_D$  and  $\sigma_D^2$  are identical with the mean and variance of a random variable.

Our discussion up to this point has a sinusoidal transmitted signal. However, because we assume that the reflection process is linear and frequency-independent, (2) characterizes the target behavior. Therefore, if we assume that the transmitted waveform is a known narrow-band signal,

$$f(t) = \sqrt{2} \operatorname{Re} \left[ \sqrt{E}_t \tilde{f}(t) e^{j\omega_c t} \right], \quad -\infty < t < \infty, \quad (15)$$

the returned signal in the absence of noise is

$$s(t) = \sqrt{2} \operatorname{Re}\left[\sqrt{E_t}\tilde{f}(t-\lambda)\tilde{b}\left(t-\frac{\lambda}{2}\right)e^{i\omega_c t}\right].$$
(16)

The complex envelope is

$$\tilde{s}(t) \triangleq \sqrt{E_t} \tilde{f}(t-\lambda) \tilde{b}\left(t-\frac{\lambda}{2}\right), \qquad (17)$$

and the actual signal can be written as

$$s(t) = \sqrt{2} \operatorname{Re} \left[ \sqrt{E_t} \, \tilde{s}(t) e^{j \omega_c t} \right]. \tag{18}$$

The complex covariance function of the signal process is

$$\tilde{K}_{\tilde{s}}(t, u) = E[\tilde{s}(t)\tilde{s}^*(u)], \qquad (19)$$

or

$$\tilde{K}_{\tilde{s}}(t,u) = E_{\iota}\tilde{f}(t-\lambda)\tilde{K}_{D}(t-u)\tilde{f}^{*}(u-\lambda).$$
<sup>(20)</sup>

#### 364 11.1 Model for Doppler-spread Target (or Channel)

Now (20) completely specifies the characteristics of the received signal.

The total received waveform is s(t) plus an additive noise. Thus,

$$r(t) = \sqrt{2} \operatorname{Re} \left[\tilde{s}(t)e^{j\omega_{c}t}\right] + \sqrt{2} \operatorname{Re} \left[\tilde{w}(t)e^{j\omega_{c}t}\right], \qquad T_{i} \leq t \leq T_{f}, \quad (21)$$

or

$$\tilde{t}(t) = \sqrt{2} \operatorname{Re}\left[\tilde{r}(t) e^{j\omega_{c}t}\right], \qquad (22)$$

where

$$\tilde{r}(t) = \tilde{s}(t) + \tilde{w}(t).$$
(23)

The complete model is shown in Fig. 11.5.

We assume that the additive noise is a sample function from a zeromean, stationary Gaussian process that is statistically independent of the reflection process and has a flat spectrum of height  $N_0/2$  over a band wide compared to the signals of interest. Then

$$E[\tilde{w}(t)\tilde{w}^*(u)] = N_0 \,\delta(t-u), \qquad (24)$$

and the covariance function of  $\tilde{r}(t)$  is

$$\widetilde{K}_{\widetilde{r}}(t,u) = E_t \widetilde{f}(t-\lambda) \widetilde{K}_D(t-u) \widetilde{f}^*(u-\lambda) + N_0 \,\delta(t-u), \qquad T_i \le t, \, u \le T_f.$$
(25)

The covariance function in (25) completely characterizes the received waveform that we have available for processing.

Whenever the reflection process  $\tilde{b}_D(t)$  has a rational spectrum, we may also characterize it using complex state variables.<sup>†</sup> The state equation is

$$\dot{\mathbf{x}}(t) = \mathbf{\tilde{F}}\mathbf{\tilde{x}}(t) + \mathbf{\tilde{G}}\mathbf{\tilde{u}}(t), \quad t \ge T_i,$$
(26)

where

$$E[\tilde{u}(t)\tilde{u}(\tau)] = \tilde{Q}\,\delta(t-\tau) \tag{27}$$

and

$$E[\tilde{\mathbf{x}}(T_i)\tilde{\mathbf{x}}^{\dagger}(T_i)] = \tilde{\mathbf{P}}_0.$$
(28)



Fig. 11.5 Model for Doppler-spread target problem.

† Complex state variables are discussed in Section A.3.3.



Fig. 11.6 State-variable model for Doppler-spread target (or channel).

The process  $\tilde{b}_D(t)$  is obtained by the relation

$$\tilde{b}_D(t) = \tilde{\mathbf{C}}\tilde{\mathbf{x}}(t), \qquad t > T_i.$$
<sup>(29)</sup>

We shall find this representation useful in many problems of interest. This model is shown in Fig. 11.6 for the special case in which  $\lambda = 0$ .

This completes our formulation of the model for a Doppler-spread target. All of our work in the text deals with this model. There are two simple generalizations of the model that we should mention:

1. Let  $\tilde{b}_D(t)$  be a non-zero-mean process. This corresponds to a fluctuating Rician channel.

2. Let  $\tilde{b}_D(t)$  be a nonstationary complex Gaussian process.

Both these generalizations can be included in a straightforward manner and are discussed in the problems. There are targets and channels that do not fit the Rayleigh or Rician model (recall the discussion on page 243). The reader should consult the references cited earlier for a discussion of these models. We now turn our attention to the optimum detection problem.

# **11.2 DETECTION OF DOPPLER-SPREAD TARGETS**

In this section we consider the problem of detecting a Doppler-spread target. The complex envelope of the received waveform on the two 366 11.2 Detection of Doppler-spread Targets

hypotheses is

$$\tilde{r}(t) = \sqrt{E_t} \tilde{f}(t-\lambda) \tilde{b}_D\left(t-\frac{\lambda}{2}\right) + \tilde{w}(t), \qquad T_i \le t \le T_f : H_1, \quad (30)$$

and

$$\tilde{r}(t) = \tilde{w}(t), \qquad \qquad T_i \le t \le T_f : H_0. \tag{31}$$

The signal process is a sample function from a zero-mean complex Gaussian process whose covariance function is

$$\widetilde{K}_{\tilde{s}}(t,u) = E_t \widetilde{f}(t-\lambda) \widetilde{K}_D(t-u) \widetilde{f}^*(u-\lambda), \qquad T_i \le t, \ u \le T_f. \quad (32)$$

The additive noise  $\tilde{w}(t)$  is a sample function of a statistically independent, zero-mean complex white Gaussian process with spectral height  $N_0$ . The range parameter  $\lambda$  is known.

We see that this problem is just the complex version of the Gaussian signal in Gaussian noise problem that we discussed in detail in Chapter 2.† Because of this strong similarity, we state many of our results without proof. The four issues of interest are:

1. The likelihood ratio test.

2. The canonical receiver realizations to implement the likelihood ratio test.

3. The performance of the optimum receiver.

4. The classes of spectra for which complete solutions can be obtained. We discuss all of these issues briefly.

### 11.2.1 Likelihood Ratio Test

The likelihood ratio test can be derived by using series expansion as in (A.116), or by starting with (2.31) and exploiting the bandpass character of the processes (e.g., Problems 11.2.1 and 11.2.2, respectively). The result is

$$l = \frac{1}{N_0} \iint_{T_i}^{T_f} \tilde{r}^*(t) \tilde{h}(t, u) \tilde{r}(u) dt du \overset{H_1}{\underset{H_0}{\gtrless}} \gamma,$$
(33)

† As we pointed out in Chapter 2, the problem of detecting Gaussian signals in Gaussian noise has been studied extensively. References that deal with problems similar to that of current interest include Price [8]-[10], Kailath [11], [12], Turin [13], [14], and Bello [15]. The fundamental Gaussian signal detection problem is discussed by Middleton [16]-[18]. Book references include Helstrom [19, Chapter 11] and Middleton [20, Part 4].

where  $\tilde{h}(t, u)$  satisfies the integral equation

$$N_0\tilde{h}(t,u) + \int_{T_i}^{T_f} \tilde{h}(t,z)\tilde{K}_{\tilde{s}}(z,u) dz = \tilde{K}_{\tilde{s}}(t,u), \qquad T_i \le t, u \le T_f$$
(34)

and

$$\widetilde{K}_{\overline{s}}(t, u) = E_t \widetilde{f}(t-\lambda) \widetilde{K}_D(t-u) \widetilde{f}^*(u-\lambda), \qquad T_i \le t, u \le T_f.$$
(35)

The threshold  $\gamma$  is determined by the costs and a-priori probabilities in a Bayes test and by the desired  $P_F$  in a Neyman-Pearson test. In the next section we discuss various receiver realizations to generate l.

# 11.2.2 Canonical Receiver Realizations

The four realizations of interest were developed for real processes in Chapter 2 (see pages 15–32). The extension to the complex case is straightforward. We indicate the resulting structures for reference.

**Estimator-correlator Receiver.** Realization No. 1 is shown in complex notation in Fig. 11.7*a*. The filter  $\tilde{h}(t, u)$  is the optimum unrealizable filter for estimating  $\tilde{s}(t)$  and satisfies (34). The actual bandpass realization is shown in Fig. 11.7*b*. Notice that the integrator eliminates the high-frequency component of the multiplier output.

**Filter-squarer-integrator (FSI) Receiver.** To obtain this realization, we factor  $\tilde{h}(t, u)$  as

$$\int_{T_i}^{T_f} \tilde{g}^*(z,t)\tilde{g}(z,u) dz = \tilde{h}(t,u), \qquad T_i \le t, u \le T_f.$$
(36)

Then

$$l = \frac{1}{N_0} \int_{T_i}^{T_f} dz \left| \int_{T_i}^{T_f} \tilde{g}(z, t) \tilde{r}(t) dt \right|^2.$$
(37)

The complex operations are indicated in Fig. 11.8a, and the actual receiver is shown in Fig. 11.8b.

**Optimum Realizable Filter Receiver.** For this realization, we rewrite the LRT as

$$l = \frac{1}{N_0} \int_{T_i}^{T_f} \{ 2 \operatorname{Re} \left[ \tilde{r}^*(t) \hat{\bar{s}}_r(t) \right] - |\hat{\bar{s}}_r(t)|^2 \} dt \stackrel{H_1}{\underset{H_0}{\overset{\underset{}}{\underset{}}}} \gamma,$$
(38)



(b) Actual operations

Fig. 11.7 Estimator-correlator receiver (Canonical Realization No. 1).



(b) Actual operations

Fig. 11.8 Filter-squarer-integrator receiver (Canonical Realization No. 3).

where  $\hat{s}_r(t)$  is the realizable MMSE estimate of  $\bar{s}(t)$  when  $H_1$  is true (e.g., Problem 11.2.3). It is obtained by passing  $\tilde{r}(t)$  through a filter  $\tilde{h}_{or}(t, u)$ , whose impulse response is specified by

$$N_0 \tilde{h}_{or}(t, u) + \int_{T_i}^t \tilde{h}_{or}(t, z) \tilde{K}_{\tilde{s}}(z, u) dz = \tilde{K}_{\tilde{s}}(t, u), \qquad T_i \le u \le t \quad (39)$$

and

$$\hat{\tilde{s}}_{r}(t) \ \Delta \int_{T_{i}}^{t} \tilde{h}_{or}(t, u) \tilde{r}(u) \ du.$$
(40)

The complex receiver is shown in Fig. 11.9.

**State-Variable Realization.**<sup>†</sup> When  $\tilde{b}_D(t)$  has a finite-dimensional complex state representation, it is usually more convenient to obtain  $\hat{s}_r(t)$  through the use of state-variable techniques. Recall that we are considering a point target and  $\lambda$  is assumed known. Therefore, for algebraic simplicity, we can let  $\lambda = 0$  with no loss of generality.

If we denote the state vector of  $\tilde{b}_D(t)$  as  $\tilde{\mathbf{x}}(t)$ , then

$$\tilde{b}_D(t) = \tilde{\mathbf{C}}\tilde{\mathbf{x}}(t) \tag{41}$$

and

$$\tilde{s}(t) = \tilde{f}(t)\tilde{\mathbf{C}}\tilde{\mathbf{x}}(t) \triangleq \tilde{\mathbf{C}}_{s}(t)\tilde{\mathbf{x}}(t).$$
(42)

The state vector  $\mathbf{\tilde{x}}(t)$  satisfies the differential equation

$$\dot{\tilde{\mathbf{x}}}(t) = \tilde{\mathbf{F}}(t)\tilde{\mathbf{x}}(t) + \tilde{\mathbf{G}}(t)\tilde{\boldsymbol{u}}(t), \quad t > T_i,$$
(43)

where

$$E[\tilde{u}(t)\tilde{u}^*(\sigma)] = \tilde{Q}\,\,\delta(t-\sigma) \tag{44}$$

and

$$E[\tilde{\mathbf{x}}(T_i)] = \mathbf{0},\tag{45}$$

$$E[\tilde{\mathbf{x}}(T_i)\tilde{\mathbf{x}}^{\dagger}(T_i)] = \tilde{\mathbf{P}}_0$$
(46)

(see page 590).



Fig. 11.9 Optimum receiver: optimum realizable filter realization (Canonical Realization No. 4).

† This section assumes familiarity with Section A.3.3.

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The optimum estimate is specified as a solution of the differential equations

$$\dot{\hat{\mathbf{x}}}(t) = \tilde{\mathbf{F}}(t)\hat{\hat{\mathbf{x}}}(t) + \tilde{\mathbf{\xi}}_{P}(t)\tilde{\mathbf{C}}_{s}^{\dagger}(t) \frac{1}{N_{0}}[\tilde{r}(t) - \tilde{\mathbf{C}}_{s}(t)\hat{\hat{\mathbf{x}}}(t)], \quad t > T_{i}$$
(47a)

and

$$\hat{\hat{s}}_{r}(t) = \tilde{f}(t)\tilde{\mathbf{C}}\hat{\hat{\mathbf{x}}}(t) = \tilde{\mathbf{C}}_{s}(t)\hat{\hat{\mathbf{x}}}(t).$$
(47b)

The variance equation is

$$\dot{\tilde{\mathbf{\xi}}}_{P}(t) = \tilde{\mathbf{F}}(t)\tilde{\mathbf{\xi}}_{P}(t) + \tilde{\mathbf{\xi}}_{P}(t)\tilde{\mathbf{F}}^{\dagger}(t) - \tilde{\mathbf{\xi}}_{P}(t)\tilde{\mathbf{C}}_{s}^{\dagger}(t)\frac{1}{N_{0}}\tilde{\mathbf{C}}_{s}(t)\tilde{\mathbf{\xi}}_{P}(t) + \tilde{\mathbf{G}}(t)\tilde{\mathbf{Q}}\tilde{\mathbf{G}}(t),$$
$$t > T_{i}, \quad (48a)$$

and

$$\tilde{\xi}_{P}(t,\,\tilde{s}(t),\,N_{0}) \stackrel{\Delta}{=} E[|\tilde{s}(t) - \hat{\tilde{s}}_{r}(t)|^{2}] = \tilde{\mathbf{C}}_{s}(t)\tilde{\mathbf{\xi}}_{P}(t)\tilde{\mathbf{C}}_{s}^{\dagger}(t).$$
(48b)

Notice that the covariance matrix  $\mathbf{\xi}_{P}(t)$  is a Hermitian matrix. Substituting (42) into (48*a*) gives

$$\dot{\tilde{\mathbf{\xi}}}_{P}(t) = \tilde{\mathbf{F}}(t)\tilde{\mathbf{\xi}}_{P}(t) + \tilde{\mathbf{\xi}}_{P}(t)\tilde{\mathbf{F}}^{\dagger}(t) - \tilde{\mathbf{\xi}}_{P}(t)\tilde{\mathbf{C}}^{\dagger}\left[\frac{|\tilde{f}(t)|^{2}}{N_{0}}\right]\tilde{\mathbf{C}}\tilde{\mathbf{\xi}}_{P}(t) + \tilde{\mathbf{G}}(t)\tilde{\mathcal{Q}}\tilde{\mathbf{G}}^{\dagger}(t),$$
$$t > T_{i}. \quad (49)$$

We see that the mean-square error is only affected by the *envelope* of the transmitted signal. When we discuss performance we shall find that it can be expressed in terms of the mean-square error. Thus, performance is not affected by phase-modulating the signal. If the target has a mean Doppler shift, this is mathematically equivalent to a phase modulation of the signal. Thus, a mean Doppler shift does not affect the performance.

It is important to observe that, even though the reflection process is stationary, the returned signal process will be nonstationary unless  $\tilde{f}(t)$  is a real pulse with constant height. This nonstationarity makes the state-variable realization quite important, because we can actually find the necessary functions to implement the optimum receiver.

This completes our initial discussion of receiver configurations. We now consider the performance of the receiver.

# 11.2.3 Performance of the Optimum Receiver

To evaluate the performance, we follow the same procedure as in Chapter 2 (pages 32-42). The key function is  $\tilde{\mu}(s)$ . First assume that there are K complex observables, which we denote by the vector  $\tilde{\mathbf{r}}$ . Then

$$\tilde{\mu}_{K}(s) \triangleq \ln \int_{\infty}^{\infty} [p_{\tilde{r} \mid H_{1}}(\tilde{\mathbf{R}} \mid H_{1})]^{s} [p_{\tilde{r} \mid H_{0}}(\tilde{\mathbf{R}} \mid H_{0})]^{1-s} d\tilde{\mathbf{R}}.$$
 (50)

Using (A.116) and (A.117), we have

$$p_{\tilde{\mathbf{r}}\mid H_1}(\tilde{\mathbf{R}}\big|H_1) = \prod_{i=1}^K \frac{1}{\pi(\tilde{\lambda}_i + N_0)} \exp\left(-\frac{|\tilde{R}_i|^2}{(\tilde{\lambda}_i + N_0)}\right), \quad (51)$$

where  $\tilde{\lambda}_i$  is the eigenvalue of the signal process  $\tilde{s}(t)$ , and

$$p_{\tilde{r}|H_0}(\tilde{\mathbf{R}}|H_0) = \prod_{i=1}^K \frac{1}{\pi N_0} \exp\left(-\frac{|\tilde{R}_i|^2}{N_0}\right).$$
(52)

Substituting (51) and (52) into (50), evaluating the integral (or comparing with 2.131), and letting  $K \rightarrow \infty$ , we obtain

$$\tilde{\mu}(s) = \sum_{i=1}^{\infty} \left[ (1-s) \ln \left( 1 + \frac{\tilde{\lambda}_i}{N_0} \right) - \ln \left( 1 + (1-s) \frac{\tilde{\lambda}_i}{N_0} \right) \right], \qquad (53)$$
$$0 \le s \le 1.$$

Notice that  $\tilde{\mu}(s)$  is a real function and is identical with (2.134), except for factors of 2. This can be expressed in a closed form in several ways. As in (2.138), it is the integral of two mean-square realizable filtering errors.

$$\tilde{\mu}(s) = \frac{1-s}{N_0} \int_{T_i}^{T_f} dt \left( \xi_P(t, \tilde{s}(t), N_0) - \xi_P\left(t, \tilde{s}(t), \frac{N_0}{1-s}\right) \right].$$
(54)

It can also be expressed by a formula like that in (2.195) if the signal process has a finite-dimensional state equation. For complex state-variable processes the appropriate formulas are

$$\ln \tilde{D}_{\mathcal{F}}(z) = \sum_{i=1}^{\infty} \ln (1 + z\tilde{\lambda}_i) = \ln \det \widetilde{\mathbf{\Gamma}}_2(T_f) + \operatorname{Re} \int_{T_i}^{T_f} \operatorname{Tr} [\widetilde{\mathbf{F}}(t)] dt, \quad (55)$$

where  $\mathbf{\Gamma}_2(t)$  is specified by

$$\frac{d}{dt} \begin{bmatrix} \widehat{\mathbf{\Gamma}}_{1}(t) \\ \widehat{\mathbf{\Gamma}}_{2}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{\tilde{F}}(t) & \widehat{\mathbf{G}}(t)\widetilde{\mathbf{Q}}\widehat{\mathbf{G}}^{\dagger}(t) \\ z\widetilde{\mathbf{C}}^{\dagger}(t)\widetilde{\mathbf{C}}(t) & -\mathbf{\tilde{F}}^{\dagger}(t) \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{\Gamma}}_{1}(t) \\ \widehat{\mathbf{\Gamma}}_{2}(t) \end{bmatrix}$$
(56)

with initial conditions

$$\widetilde{\mathbf{\Gamma}}_1(T_i) = \widetilde{\mathbf{P}}_0,\tag{57}$$

$$\hat{\Gamma}_2(T_i) = \mathbf{I}.$$
(58)

Notice that  $\tilde{D}_{\mathcal{F}}(z)$  is a real function.

To evaluate the performance, we use (53) in (2.166) and (2.174) to obtain

$$P_F \simeq \left[\sqrt{2\pi s^2 \ddot{\tilde{\mu}}(s)}\right]^{-1} e^{\tilde{\mu}(s) - s\dot{\tilde{\mu}}(s)}, \qquad 0 \le s \le 1 \qquad (59)$$

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and

$$P_{M} \simeq \left[\sqrt{2\pi(1-s)^{2}\ddot{\tilde{\mu}}(s)}\right]^{-1} e^{\tilde{\mu}(s) + (1-s)\dot{\tilde{\mu}}(s)} \qquad 0 \le s \le 1.$$
(60)

As a final topic we discuss the classes of returned signal processes for which complete solutions for the optimum receiver and its performance can be obtained.

### 11.2.4 Classes of Processes

There are four cases in which complete results can be obtained.

**Case 1. Reflection Process with Finite State Representation.** In this case  $\tilde{b}_D(t)$  can be described by differential equations, as in (41)-(46). Because we have limited ourselves to stationary processes, this is equivalent to the requirement that the spectrum  $\tilde{S}_D\{f\}$  be rational. In this case, (38) and (47)-(49) apply directly, and the receiver can be realized with a feedback structure. The performance follows easily by using (54) in (59) and (60).

**Case 2.** Stationary Signal Process: Long Observation Time. This case is the bandpass analog of the problem discussed in Section 4.1. Physically it could arise in several ways. Two of particular interest are the following:

1. The complex envelope of the transmitted signal is a real rectangular pulse whose length is appreciably longer than the correlation time of the reflection process.

2. In the passive detection problem the signal is generated by the target, and if this process is stationary, the received envelope is a stationary process.

For this case, we can use asymptotic formulas and obtain much simpler expressions. We solve (34) using transforms. The result is

$$\tilde{H}\{f\} = \frac{\tilde{S}_{\tilde{s}}\{f\}}{\tilde{S}_{\tilde{s}}\{f\} + N_0}.$$
(61)

A common realization in this case is the filter-squarer realization [see (36) and (37)]. We can find a solution to (36) that is a realizable filter:

$$\tilde{G}\lbrace f\rbrace = \left[\frac{\tilde{S}_{\tilde{s}}\lbrace f\rbrace}{\tilde{S}_{\tilde{s}}\lbrace f\rbrace + N_{0}}\right]^{+}.$$
(62)

Recall that the superscript "+" denotes the term containing the lefthalf-plane poles and zeros. **Case 3.** Separable Kernels. In this case, the reflection process has a finite number of eigenvalues (say K). Looking at (20), we see that this means that the received signal process must also have K eigenvalues. In this case the problem is mathematically identical with a reflection from K slowly fluctuating targets.

**Case 4.** Low-Energy-Coherence Case. In this case, the largest eigenvalue is much smaller than the white noise level. Then, as on pages 131–137, we can obtain a series solution to the integral equation specifying  $\tilde{h}(t, u)$ . By an identical argument, the likelihood ratio test becomes

$$l = \frac{1}{N_0^2} \iint_{T_i}^{T_f} \tilde{r}^*(t) \tilde{K}_{\tilde{s}}(t, u) \tilde{r}(u) \, du \stackrel{H_1}{\underset{H_0}{\gtrless}} \gamma.$$
(63a)

Using (35),

$$l = \frac{E_t}{N_0^2} \iint_{T_i} \tilde{r}^*(t) \tilde{f}(t-\lambda) \tilde{K}_D(t-u) \tilde{f}^*(u-\lambda) \tilde{r}(u) dt du.$$
(63b)

We can write  $\tilde{K}_D(t-u)$  in a factored form as

$$\tilde{K}_D(t-u) = \int_{T_i}^{T_f} \tilde{k}^*(z,t) \tilde{k}(z,u) \, dz, \qquad T_i \le t, \, u \le T_f. \tag{64a}$$

Using (64a) in (63b) gives

$$l = \frac{E_t}{N_0^2} \int_{T_i}^{T_f} dz \left| \int_{T_i}^{T_f} k(z, u) \tilde{f}^*(u - \lambda) \tilde{r}(u) \, du \right|^2.$$
(64b)

This realization is shown in Fig. 11.10. (The reader should verify that the receiver in Fig. 11.10*b* generates the desired output. The bandpass filter at  $\omega_{\Delta}$  is assumed to be ideal.)

For an arbitrary time interval, the factoring indicated in (64a) may be difficult to carry out. However, in many cases of interest the time interval is large and we can obtain an approximate solution to (64a) by using Fourier transforms.

$$\tilde{K}_{\infty}\{f\} = [\tilde{S}_D\{f\}]^+.$$
(64c)

In this case we obtain the receiver structure shown in Fig. 11.11. Notice that we do not require  $\tilde{f}(t)$  to be a constant, and so the result is more general than the SPLOT condition in Case 2.

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(b) Actual implementation

Fig. 11.10 Optimum LEC receiver.

The performance in the LEC case is obtained from (53). Expanding the logarithm in a series and neglecting the higher-order terms, we have

$$\tilde{\mu}(s) = -\frac{s(1-s)}{2N_0^2} \sum_{i=1}^{\infty} (\tilde{\lambda}_i)^2 = -\frac{s(1-s)}{2N_0^2} \iint_{-\infty}^{\infty} |\tilde{K}_s(t,u)|^2 dt du.$$
(65)

Using (20) in (65) gives

$$\tilde{\mu}(s) = -\frac{s(1-s)E_t^2}{2N_0^2} \int_{-\infty}^{\infty} |\tilde{f}(t-\lambda)|^2 |\tilde{K}_D(t-u)|^2 |\tilde{f}(u-\lambda)|^2 dt du, \quad (66a)$$

which may also be written as

$$\tilde{\mu}(s) = -\frac{s(1-s)E_t^2}{2N_0^2} \iint_{-\infty}^{\infty} \tilde{S}_D\{f_1\}\theta\{0, f_1 - f_2\}\tilde{S}_D\{f_2\} df_1 df_2.$$
(66b)

Thus the performance can be obtained by performing a double integration. We use  $\tilde{\mu}(s)$  in (59) and (60) to find  $P_F$  and  $P_M$ .



Fig. 11.11 Optimum LEC receiver: long observation time.

# 11.2.5 Summary

In this section we have discussed the detection of Doppler-spread targets. The four important issues were the derivation of the likelihood ratio test, canonical realizations of the optimum receiver, performance of the optimum receiver, and the classes of spectra that permit complete solutions. All the results are the complex versions of the results in Chapters 2-4.

Throughout our discussion we have tried to emphasize the similarities between the current problem and our earlier work. The reader should realize that these similarities arise because we have introduced complex notation. It is difficult to solve the bandpass problem without complex notation unless the quadrature components of the signal process are statistically independent (recall Problem 3.4.9). Using (20) in (A.67), we see that

$$\operatorname{Im}\left[\tilde{K}_{s}(t, u)\right] = \operatorname{Im}\left[\tilde{f}(t)\tilde{K}_{D}(t-u)\tilde{f}^{*}(u)\right] = 0$$
(67)

must be satisfied in order for the quadrature components to be statistically independent. The restriction in (67) would severely limit the class of targets and signals that we could study [e.g., a linear FM signal would violate (67)].

A second observation is that we are almost always dealing with nonstationary signal processes in the current problem. This means that the complex state-variable approach will prove most effective in many problems.

The problem that we have studied is a simple binary problem. All of the results can be extended to the bandpass version of the general binary problem of Chapter 3. Some of these extensions are carried out in the problems at the end of this chapter.

We next consider the problem of digital communication over Dopplerspread channels. This application illustrates the use of the formulas in this section in the context of an important problem.

### 11.3 COMMUNICATION OVER DOPPLER-SPREAD CHANNELS

In this section we consider the problem of digital communication over a Doppler-spread channel. In the first three subsections, we consider binary systems. In Section 11.3.1 we derive the optimum receiver for a binary system and evaluate its performance. In Section 11.3.2 we derive a bound on the performance of any binary system, and in Section 11.3.3 we study suboptimum receivers. In Section 11.3.4 we consider M-ary systems, and in Section 11.3.5 we summarize our results.

# 11.3.1 Binary Communications Systems: Optimum Receiver and Performance

We consider a binary system in which the transmitted signals on the two hypotheses are

$$\frac{\sqrt{2E_t} \operatorname{Re}\left[\tilde{f}(t)e^{j\omega_0 t}\right]:H_0}{\sqrt{2E_t} \operatorname{Re}\left[\tilde{f}(t)e^{j\omega_1 t}\right]:H_1}.$$
(68)

We assume that  $\omega_1 - \omega_0$  is large enough so that the output signal processes on the two hypotheses are in disjoint frequency bands. The received waveforms are

$$r(t) = \begin{cases} \sqrt{2E}_{t} \operatorname{Re}\left[\tilde{b}(t)\tilde{f}(t)e^{j\omega_{0}t}\right] + w(t), & T_{i} \leq t \leq T_{f}:H_{0}, \\ \sqrt{2E}_{t} \operatorname{Re}\left[\tilde{b}(t)\tilde{f}(t)e^{j\omega_{1}t}\right] + w(t), & T_{i} \leq t \leq T_{f}:H_{1}. \end{cases}$$
(69)

The hypotheses are equally likely, and the criterion is minimum probability of error. The optimum receiver consists of two parallel branches centered at  $\omega_1$  and  $\omega_0$ . The first branch computes

$$l_{1} = \frac{1}{N_{0}} \iint_{T_{i}}^{T_{f}} \tilde{r}^{*}(t) \tilde{h}(t, u) \tilde{r}(u) dt du,$$
(70)

where the complex envelopes are referenced to  $\omega_1$ . The second branch computes

$$l_{0} = \frac{1}{N_{0}} \iint_{T_{i}}^{T_{f}} \tilde{r}^{*}(t) \tilde{h}(t, u) \tilde{r}(u) dt du,$$
(71)

where the complex envelopes are referenced to  $\omega_0$ . In both cases  $\tilde{h}(t, u)$  is specified by

$$N_0 \tilde{h}(t, u) + \int_{T_i}^{T_f} \tilde{h}(t, z) \tilde{K}_{\tilde{s}}(z, u) dz = \tilde{K}_{\tilde{s}}(t, u), \quad T_i \le t, u \le T_f \quad (72)$$

where

$$\tilde{K}_{\tilde{s}}(t,u) = E_t \tilde{f}(t-\lambda)\tilde{K}_D(t-u)\tilde{f}^*(u-\lambda), \qquad T_i \le t, u \le T_f.$$
(73)

The receiver performs the test

$$l_1 \overset{H_1}{\underset{H_0}{\gtrsim}} l_0. \tag{74}$$

The receiver configuration is shown in Fig. 11.12. Notice that each branch is just the simple binary receiver of Fig. 11.7. This simple structure arises because the signal processes on the two hypotheses are in disjoint frequency bands.



 $h(t, u) = \operatorname{Re}\left[2\widetilde{h}(t, u)e^{j\omega_0(t-u)}\right]$ 

Fig. 11.12 Optimum receiver: binary FSK system operating over a Doppler-spread channel (Canonical Realization No. 1).

An alternative configuration is obtained by factoring  $\tilde{h}(t, u)$  as indicated in (36). This configuration is shown in Fig. 11.13.

Because this is a binary symmetric bandpass problem,<sup>†</sup> we may use the bounds on the probability of error that we derived in (3.111).

$$\frac{e^{\ddot{\mu}_{\rm BS}(\frac{1}{2})}}{2[1+((\pi/8)\ddot{\ddot{\mu}}_{\rm BS}(\frac{1}{2}))^{\frac{1}{2}}]} \le \Pr(\epsilon) \le \frac{e^{\ddot{\mu}_{\rm BS}(\frac{1}{2})}}{2[1+((\frac{1}{8})\ddot{\ddot{\mu}}_{\rm BS}(\frac{1}{2}))^{\frac{1}{2}}]} \le \frac{e^{\ddot{\mu}_{\rm BS}(\frac{1}{2})}}{2}, \quad (75)$$

where

$$\tilde{\mu}_{BS}(s) = \tilde{\mu}_{SIB}(s) + \tilde{\mu}_{SIB}(1-s)$$
$$= \sum_{i=1}^{\infty} \left[ \ln \left( 1 + \frac{\tilde{\lambda}_i}{N_0} \right) - \ln \left( 1 + \frac{s\tilde{\lambda}_i}{N_0} \right) - \ln \left( 1 + \frac{(1-s)\tilde{\lambda}_i}{N_0} \right) \right] \quad (76)$$

and

$$\tilde{\mu}_{\rm BS}(\frac{1}{2}) = \sum_{i=1}^{\infty} \left[ \ln \left( 1 + \frac{\tilde{\lambda}_i}{N_0} \right) - 2 \ln \left( 1 + \frac{\tilde{\lambda}_i}{2N_0} \right) \right]. \tag{77}$$

The  $\tilde{\lambda}_i$  are the eigenvalues of the output signal process  $\tilde{s}(t)$ , whose covariance is given by (73). We can also write  $\tilde{\mu}_{BS}(s)$  in various closed-form expressions such as (54) and (55).

<sup>†</sup> Notice that "binary symmetric" refers to the hypotheses. The processes are not necessarily symmetric about their respective carriers.

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Fig. 11.13 Optimum receiver for binary communication over a Doppler-spread channel: filter squarer-integrator realization.

There are three questions of interest with respect to the binary communication problem:

1. What is the performance of the optimum system when the signal  $\sqrt{E_t}\tilde{f}(t)$ , the channel covariance function  $\tilde{K}_D(\tau)$ , and the noise level  $N_0$  are fixed?

2. If we use a suboptimum receiver, how does its performance compare with that of the optimum receiver for a particular  $\tilde{f}(t)$ ,  $E_t$ ,  $\tilde{K}_D(\tau)$ , and  $N_0$ ?

3. If the channel covariance function  $\tilde{K}_D(\tau)$ , the noise level  $N_0$ , and the transmitted energy  $E_t$  are fixed, how can we choose  $\tilde{f}(t)$  to minimize the probability of error?

We can answer the first question for a large class of problems by evaluating  $\tilde{\mu}_{BS}(s)$  and using (75). Specific solution techniques were discussed on pages 35-44. We can answer the second question by using the techniques of Section 5.1.2. We shall discuss this question in Section 11.3.3. We now consider the third question.

# 11.3.2 Performance Bounds for Optimized Binary Systems

We assume that the channel covariance function  $\tilde{K}_D(\tau)$ , the noise level  $N_0$ , and the transmitted energy  $E_t$  are fixed. We would like to choose  $\tilde{f}(t)$  to minimize the probability of error. In practice it is much simpler to

minimize  $\tilde{\mu}_{BS}(\frac{1}{2})$ . This minimizes the exponent in the bound in (75). Our procedure consists of two steps:

1. We consider the covariance function of the output signal process  $\tilde{K}_{s}(t, u)$  and its associated eigenvalues  $\tilde{\lambda}_{i}$ . We find the set of  $\tilde{\lambda}_{i}$  that will minimize  $\tilde{\mu}_{BS}(\frac{1}{2})$ . In this step we do not consider whether a transmitted signal exists that would generate the optimum set of  $\tilde{\lambda}_{i}$  through the relation in (73). The result of this step is a bound on the performance of any binary system.

2. We discuss how to choose  $\tilde{f}(t)$  to obtain performance that is close to the bound derived in the first step.

We first observe that a constraint on the input energy implies a constraint on the expected value of the output energy. From (73), the expected value of the total received signal energy is

$$E[|\tilde{s}(t)|^{2}] = \int_{T_{i}}^{T_{f}} \tilde{K}_{\tilde{s}}(t, t) dt = 2E_{t}\sigma_{b}^{2} = \bar{E}_{r}.$$
 (78*a*)

(Recall that

$$\int_{T_i}^{T_f} |\tilde{f}(t)|^2 dt = 1.$$
 (78b)

Notice that this constraint is independent of the signal shape. In terms of eigenvalues of the output process, the constraint is

$$\sum_{i=1}^{\infty} \tilde{\lambda}_i = \bar{E}_r.$$
(79)

We now choose the  $\tilde{\lambda}_i$ , subject to the constraint in (79), to minimize  $\tilde{\mu}_{BS}(\frac{1}{2})$ . Notice that it is not clear that we can find an  $\tilde{f}(t)$  that can generate a particular set of  $\tilde{\lambda}_i$ .

We first define normalized eigenvalues,

$$\tilde{\lambda}_{in} \stackrel{\Delta}{=} \frac{\tilde{\lambda}_i}{\bar{E}_r} \,. \tag{80}$$

We rewrite (77) as<sup>†</sup>

$$\tilde{\mu}_{\rm BS}(\frac{1}{2}) = \frac{-\bar{E}_{\mathbf{r}}}{N_0} \left\{ \left( \sum_{i=1}^{\infty} \tilde{\lambda}_{in} \tilde{g}(\tilde{\lambda}_{in}) \right) \right\},\tag{81}$$

where

$$\tilde{g}(x) \triangleq \frac{2}{\gamma x} \left[ -\frac{1}{2} \ln \left( 1 + \gamma x \right) + \ln \left( 1 + \frac{\gamma x}{2} \right) \right]$$
(82)

and

$$\gamma \triangleq \frac{\bar{E}_r}{N_0} \,. \tag{83}$$

† This derivation is due to Kennedy [3].

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We refer to the term in braces in (81) as the *efficiency factor*. (Recall the discussion on page 118.) The function  $\tilde{g}(x)$  is plotted in Fig. 11.14. We see that  $\tilde{g}(x)$  is a positive function whose unique maximum occurs at  $x = \hat{x}$ , which is the solution to

$$\frac{\gamma^2 \hat{x}/4}{(1+\gamma \hat{x})(1+\gamma \hat{x}/2)} = \frac{1}{\hat{x}} \left[ -\frac{1}{2} \ln \left( 1+\gamma \hat{x} \right) + \ln \left( 1+\frac{\gamma \hat{x}}{2} \right) \right].$$
 (84)

The solution is  $\gamma \hat{x} = 3.07$ . For all positive  $\gamma$ ,

$$\hat{x} = \frac{3.07}{\gamma} \,. \tag{85}$$

We can use the result in (84) and (85) to bound  $\tilde{\mu}_{BS}(\frac{1}{2})$ . From (81),

$$-\tilde{\mu}_{\mathrm{BS}}(\frac{1}{2}) = \frac{\bar{E}_r}{N_0} \sum_{i=1}^{\infty} \tilde{\lambda}_{in} \tilde{g}(\tilde{\lambda}_{in}) \le \frac{\bar{E}_r}{N_0} \sum_{i=1}^{\infty} \lambda_{in} \tilde{g}(\hat{x}).$$
(86)

Using (79) and (80), (86) reduces to

$$-\tilde{\mu}_{\rm BS}(\frac{1}{2}) \le \frac{\bar{E}_r}{N_0} \tilde{g}(\hat{x}).$$
(87)

Thus we have a bound on how negative we can make  $\tilde{\mu}_{BS}(\frac{1}{2})$ . We can achieve this bound exactly by letting

$$\tilde{\lambda}_{in} = \begin{cases} \hat{x} = \frac{3.07}{\gamma}, & i = 1, 2, \dots, D_o, \\ 0, & i > D_o, \end{cases}$$
(88)<sup>†</sup>

where

$$D_o = \frac{\gamma}{3.07} = \frac{\bar{E}_r/N_0}{3.07}.$$
 (89)

This result says that we should choose the first  $D_o$  normalized eigenvalues to be equal to  $3.07/\gamma$  and choose the others to be zero. Using (88) in (82) and the result in (87) gives

$$\tilde{\mu}_{BS}(\frac{1}{2}) \ge -0.1488 \left(\frac{\bar{E}_r}{N_0}\right).$$
 (90)

Substituting (90) into (75) gives an upper bound of the probability of error as

$$\Pr(\epsilon) \le \frac{1}{2} \exp\left(-0.1488 \frac{\bar{E}_r}{N_0}\right).$$
(91)

† This result assumes that  $\bar{E}_r/N_0$  is a integer multiplier of 3.07. If this is not true, (86) is still a bound and the actual performance is slightly worse.



Fig. 11.14 Plot of  $\tilde{g}(x)$  versus  $\gamma x$  (from[3]).

We have encountered this type of result in some of earlier examples. In Section 4.2.3, we studied a frequency diversity system in which the received energies in all channels were required to be equal. We found that we could minimize  $\mu_{BS,BP,SK}(\frac{1}{2})$  by dividing the available energy among the channels so that

$$\frac{\bar{E}_{r1}}{N_0} = 3.07\tag{92}$$

(see 4.116). In that case we achieved the optimum performance by an *explicit diversity* system.

In the present case there is only one channel. In order to achieve the optimum performance, we must transmit a signal so that the covariance function

$$\tilde{K}_{\tilde{s}}(t, u) = \tilde{f}(t)\tilde{K}_{D}(t-u)\tilde{f}^{*}(u)$$
(93)

has  $D_o$  equal eigenvalues. We can think of this as an *implicit diversity* system.

The result in (91) is quite important, because it gives us a performance bound that is independent of the shape of the Doppler scattering functions. It provides a standard against which we can compare any particular signaling scheme. Once again we should emphasize that there is no guarantee that we can achieve this bound for all channel-scattering functions.

The next step in our development is to consider some specific Doppler scattering functions and see whether we can design a signal that achieves the bound in (90) with equality. We first review two examples that we studied in Section 4.1.2.

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**Example 1.** This example is identical with Example 4 on page 116. The channel-scattering function is

$$\tilde{S}_{D}\{f\} = \begin{cases} \frac{\sigma_{b}^{2}}{B}, & |f| \le B, \\ 0, & |f| > B. \end{cases}$$
(94)

The transmitted signal is

$$\tilde{f}(t) = \begin{cases} \frac{1}{\sqrt{T}}, & 0 \le t \le T, \\ 0, & \text{elsewhere.} \end{cases}$$
(95)

We assume that BT is large enough that we may use the SPLOT formulas. We constrain  $\bar{E}_r$  and choose T to minimize  $\tilde{\mu}_{BS,\infty}(\frac{1}{2})$ . From (4.76), the optimum value is  $T_o$ , which is specified by

$$\frac{\bar{E}_r/N_0}{2BT_o} = 3.07.$$
(96)

Then

$$[\tilde{\mu}_{\rm BS,\infty}(\frac{1}{2})]_{\rm opt} = -0.1488 \left(\frac{\bar{E}_r}{N_0}\right),\tag{97}$$

and we achieve the bound with equality. The result in (96) assumes the SPLOT condition If we require

$$BT_o \ge 5$$
 (98)

to assure the validity of the SPLOT assumption, the result in (96) requires that

$$\frac{\bar{E}_r}{N_0} \ge 30.7. \tag{99}$$

When (99) is satisfied, the optimum binary system for a channel whose scattering function is given by (94) is one that transmits a rectangular pulse of duration  $T_o$ . [Notice that the condition in (99) is conservative.]

**Example 2.** This example is identical with Example 3 on page 111. The channel-scattering function is

$$\tilde{S}_D\{f\} = \frac{4k\sigma_b^2}{(2\pi f)^2 + k^2},$$
(100)

and the transmitted signal is given in (95). Previously, we used the SPLOT assumption to evaluate  $\tilde{\mu}_{BS}(s)$ . In this example we use complex state variables. The channel state equations are

$$\widetilde{F}(t) = -k, \tag{101}$$

$$\widetilde{G}(t) = \widetilde{C}(t) = 1, \tag{102}$$

$$\tilde{Q} = 4k\sigma_b^2,\tag{103}$$

$$\widetilde{\mathbf{P}}_0 = 2\sigma_b^2. \tag{104}$$

We evaluate  $\tilde{\mu}_{BS}(\frac{1}{2})$  by using (48*a*), (48*b*), (54), and (76) and then minimize over *T*. The result is shown in Fig. 11.15. We see that if  $\bar{E}_r/N_0$  is small, we can transmit a very short pulse such that

$$kT_o \simeq 0. \tag{105}$$



Fig. 11.15 Optimum error exponent and  $kT_o$  product as a function of  $\vec{E}_r/N_0$  (first-order fading, rectangular pulse (from [4]).

A short pulse causes a single eigenvalue at the channel output, because the channel does not fluctuate during the signaling interval. (This is the model in Chapter 9.) When

$$\frac{\bar{E}_r}{N_0} = 3.07,$$
 (106)

the condition in (89) is satisfied and the bound in (90) is achieved. As long as

$$\frac{\bar{E}_r}{N_0} \le 7.8,$$
 (107)

a single eigenvalue is still optimum, but the system only achieves the bound in (90) when (106) is satisfied. As the available  $\bar{E}_r/N_0$  increases, the optimum kT product increases. For

$$\frac{\bar{E}_r}{N_0} > 13,$$
 (108)

the results coincide with the SPLOT results of Example 3 on page 111:

$$kT_{o} \simeq \frac{\bar{E}_{r}/N_{0}}{3.44}$$
(109)

and

$$\tilde{\mu}_{\rm BS}(\frac{1}{2}) = -0.118 \frac{\bar{E}_r}{N_0} \,. \tag{110}$$

The result in (110) indicates that a rectangular pulse cannot generate the equal eigenvalue distribution required to satisfy the bound.

In the next example we consider a more complicated signal in an effort to reach the bound in (90) for a channel whose scattering function is given in (100).

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**Example 3.** The channel-scattering function is given in (100). To motivate our signal choice, we recall that a short pulse generates a single eigenvalue. By transmitting a sequence of pulses whose time separation is much greater than the channel correlation time, we can obtain the desired number of equal eigenvalues at the output.

The signal of interest is shown in Fig. 11.16. It consists of a train of rectangular pulses with width  $T_s$  and interpulse spacing  $T_x$ . The number of pulses is

$$n \triangleq D_o = \frac{\bar{E}_\tau / N_0}{3.07} \,.$$
 (111)

The height of each pulse is chosen so that the average received energy per pulse is

$$\bar{E}_{ri} = 3.07N_0, \qquad i = 1, 2, \dots, D_o.$$
 (112)

We can write

$$\tilde{f}(t) = \sum_{i=1}^{D_0} c \ \tilde{u}(t - iT_p),$$
(113a)

where

$$\tilde{u}(t) = \begin{cases} \frac{1}{\sqrt{T_s}}, & 0 \le t \le T, \\ 0, & \text{elsewhere,} \end{cases}$$
(113b)

and c normalizes f(t) to have unit energy. The covariance function of the output signal process is

$$\tilde{K}_{s}(t,\tau) = \sum_{i=1}^{D_{o}} \sum_{k=1}^{D_{o}} c^{2} \tilde{u}(t-iT_{p}) \tilde{K}_{D}(t-\tau) \tilde{u}^{*}(\tau-kT_{p}).$$
(114)

We can evaluate the performance for any particular  $T_s$  and  $T_p$ . The case of current interest is obtained by letting

$$T_s \to 0$$
 (115)

and

$$T_n \to \infty$$
. (116)

In this limit the covariance function becomes the separable function

$$\tilde{K}_{\tilde{s}}(t,\tau) = \sum_{i=1}^{D_o} c^2 \tilde{K}_D(0) \hat{u}(t-iT_p) \tilde{u}^*(\tau-kT_p).$$
(117)



Fig. 11.16 Transmitted signal in Example 3.

We now have  $D_o$  equal eigenvalues whose magnitudes satisfy (88). Therefore the performance satisfies the upper bound in (90).

The limiting case is not practical, but we can frequently obtain a good approximation to it. We need to make  $T_s$  appreciably shorter than the correlation time of  $\tilde{b}_D(t)$ . This will make the amplitude of each returned pulse approximately constant. We need to make  $T_p$  appreciably longer than the correlation time of  $\tilde{b}_D(t)$ , so that the amplitude of different pulses will be statistically independent. The result approximates an optimumdiversity system. Notice that the optimum receiver reduces to two branches like that in Fig. 4.16 in the limiting case. There are no integral equations to solve.

There may be constraints that make it impossible to use this solution:

1. If there is a peak power limitation, we may not be able to get enough energy in each pulse.

2. If there is a bandwidth limitation, we may not be able to make  $T_s$  short enough to get a constant amplitude on each received pulse.

3. If there is a time restriction on the signaling interval, we may not be able to make  $T_p$  long enough to get statistically independent amplitudes.

These issues are investigated in Problems 11.3.6 and 11.3.7. If any of the above constraints makes it impossible to achieve the bound with this type of signal, we can return to the signal design problem and try a different strategy.

Before leaving this example, we should point out that a digital system using the signal in Fig. 11.16 would probably work in a time-division multiplex mode (see Section II-9.11) and interleave signals from other message sources in the space between pulses.

We should also observe that the result does not depend on the detailed shape of  $\tilde{S}_{D}\{f\}$ .

In this section we have studied the problem of digital communication over a channel that exhibits time-selective fading by using binary orthogonal signals. The basic receiver derivation and performance analysis were straightforward extensions of the results in Section 11.2.

The first important result of the section was the bound in (90). For any scattering function,

$$\tilde{\mu}_{\rm BS}(\frac{1}{2}) \ge -0.1488 \left(\frac{\bar{E}_r}{N_0}\right).$$
 (118)

In order to achieve this bound, the transmitted signal must generate a certain number of equal eigenvalues in the output signal process.

The second result of interest was the demonstration that we could essentially achieve the bound for various channel-scattering functions by the use of simple signals.

There are two topics remaining to complete our digital communication discussion. In Section 11.3.3, we study the design of suboptimum receivers. In Section 11.3.4, we discuss M-ary communication briefly.

#### 11.3.3 Suboptimum Receivers

For a large number of physical situations we can find the optimum receiver and evaluate its performance. Frequently, the optimum receiver is

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complicated to implement, and we wish to study suboptimum receiver designs. In this section we develop two logical suboptimum receiver configurations and analyze their performance.

To obtain the first configuration, we consider a typical sample function of  $\tilde{b}_D(t)$  as shown in Fig. 11.17*a*. For discussion purposes we assume that  $\tilde{b}_D(t)$  is bandlimited to  $\pm B/2$  cps. We could approximate  $\tilde{b}_D(t)$  by the piecewise constant function shown in Fig. 11.17*b*. In this approximation we have used segments equal to the reciprocal of the bandwidth. A more general approximation is shown in Fig. 11.17*c*. Here we have left the



length of the subintervals as a parameter. We would expect  $T_s$  to be less than the reciprocal of the fading bandwidth in order for the approximation to be valid.

In order to design the first suboptimum receiver, we assume that the function in Fig. 11.17c is exact and that the values in each subinterval are statistically independent. Notice that the two assumptions are somewhat contradictory. As  $T_s$  decreases, the approximation is more exact, but the values are more statistically dependent. As  $T_s$  increases, the opposite behavior occurs. The fact that the assumptions are not valid is the reason why the resulting receiver is suboptimum.

We write

$$\tilde{b}_{Da}(t) = \sum_{i=1}^{N} \tilde{b}_i \tilde{u}(t - iT_s), \qquad (119)$$

where  $\tilde{u}(t)$  is the unit pulse defined in (113b). Using (119) in (20) gives

$$\tilde{K}_{\tilde{s}}(t,v) = \sum_{i=1}^{N} \frac{2\sigma_b^2 E_i}{N} \tilde{f}(t) \tilde{u}(t-iT_s) \tilde{f}^*(v) \tilde{u}^*(v-iT_s).$$
(120)

The covariance function in (120) is separable, and so the receiver structure is quite simple.

The branch of the resulting receiver that generates  $l_1$  is shown in Fig. 11.18. A similar branch generates  $l_0$ .<sup>+</sup> The different weightings arise because

$$E_{i} = \int_{(i-1)T_{s}}^{iT_{s}} |\tilde{f}(t)|^{2} dt$$
(121)

is usually a function of i, so that the eigenvalues are unequal. From Problem 11.2.1,

$$g_i = \frac{2\sigma_b^2 E_i}{2\sigma_b^2 E_i + N_0}.$$
 (122)

The receiver in Fig. 11.18 is easy to understand but is more complicated than necessary. Each path is gating out a  $T_s$  segment of  $\tilde{r}(t)$  and operating on it. Thus we need only one path if we include a gating operation. This version is shown in Fig. 11.19. A particularly simple version of the receiver arises when  $\tilde{f}(t)$  is constant over the entire interval. Then the weightings are unnecessary and we have the configuration in Fig. 11.20. We have replaced the correlation operation with a matched filter to emphasize the interchangeability.

<sup>†</sup> In Figs. 11.18 to 11.22, we use complex notation to show one branch of various receivers. The complex envelopes in the indicated branch are referenced to  $\omega_1$  so that the output is  $l_1$ . As discussed at the beginning of Section 11.3.1, we compute  $l_0$  by using the same complex operations referenced to  $\omega_0$ .



Fig. 11.18 Suboptimum receiver No. 1 (one branch).



Fig. 11.19 Alternative version of suboptimum receiver No. 1 (one branch).



Fig. 11.20 Suboptimum receiver No. 1 for constant  $\tilde{f}(t)$  (one branch).

This completes our development of our first suboptimum receiver structure. We refer to it as a GFSS (gate-filter-square-sum) receiver. Before analyzing its performance, we develop a second suboptimum receiver configuration.

In our development of the second suboptimum receiver, we restrict our attention to channel processes with finite-dimensional state representations. The second suboptimum receiver configuration is suggested by the optimum receiver that we obtain when both the LEC condition and the long observation time assumption are valid. This receiver is shown in Fig. 11.21. (This is the receiver of Fig. 11.11 redrawn in state-variable notation with  $\lambda = 0$ .) Notice that the state-variable portion corresponds exactly to the system used to generate  $\tilde{b}_D(t)$ .

We retain the basic structure in Fig. 11.21. To obtain more design flexibility, we do not require the filter matrices to be capable of generating  $\tilde{b}_D(t)$ , but we do require them to be time-invariant. The resulting receiver is shown in Fig. 11.22. (This type of receiver was suggested in [4].) The receiver equations are

$$\dot{\mathbf{x}}_{r}(t) = \mathbf{\tilde{F}}_{r} \mathbf{\tilde{x}}_{r}(t) + \mathbf{\tilde{G}}_{r} \mathbf{\tilde{f}}^{*}(t) \mathbf{\tilde{r}}(t), \qquad (123)$$

$$\tilde{l}_i(t) = |\tilde{\mathbf{C}}_r \tilde{\mathbf{x}}_r(t)|^2, \qquad (124)$$

$$E[\tilde{\mathbf{x}}_r(T_i)\tilde{\mathbf{x}}_r^{\dagger}(T_i)] = \tilde{\mathbf{P}}_r, \qquad (125)$$

and

$$\tilde{\mathbf{x}}_r(T_i) = \mathbf{0}. \tag{126}$$

This specifies the structure of the second suboptimum receiver [we refer to it as an FSI (filter-squarer-integrator) receiver]. We must specify  $\mathbf{F}_r$ ,  $\tilde{\mathbf{G}}_r$ ,  $\tilde{\mathbf{C}}_r$ , and  $\tilde{\mathbf{P}}_r$  to maximize its performance.



Fig. 11.21 Optimum LEC receiver: long observation time (one branch).



Fig. 11.22 Suboptimum receiver No. 2 (one branch).

We now proceed as follows.

1. We derive a bound on the performance of the suboptimum receivers. This is a straightforward extension of the bound in Section 5.1.2 to include complex processes. This bound is valid for both receiver configurations.

2. We develop expressions for the quantities in the bound for the two receivers.

3. We optimize each receiver and compare its performance with that of the optimum receiver.

All the steps are straightforward, but complicated. Many readers will prefer to look at the results in Figs. 11.24 and 11.25 and the accompanying conclusions.

**Performance Bounds for Suboptimum Receivers.** Because we have a binary symmetric system with orthogonal signals, we need to modify the results of Problem 5.1.16. (These bounds were originally derived in [5].) The result is

$$\Pr\left(\epsilon\right) < \frac{1}{2}e^{\tilde{\mu}_{BS}(s)},\tag{127}$$

where

$$\tilde{\mu}_{\rm BS}(s) \triangleq \tilde{\mu}_{11}(s) + \tilde{\mu}_{01}(-s)$$
 (128)

and

$$\tilde{\mu}_{11}(s) \triangleq \ln E[e^{sl_1} \mid H_1], \qquad (129)$$

$$\tilde{\mu}_{01}(s) \triangleq \ln E[e^{sl_0} \mid H_1]. \tag{130}$$

The next step is to evaluate  $\tilde{\mu}_{BS}(s)$  for the two receivers.

Evaluation of  $\tilde{\mu}_{BS}(s)$  for Suboptimum Receiver No. 1. In this case,  $l_1$  and  $l_0$  are finite quadratic forms and the evaluation is straightforward (see Problem 11.3.9; the original result was given in [4]).

$$\tilde{\mu}_{11}(s) = -\ln \det \left( \mathbf{I} - s \widetilde{\mathbf{W}} [\widetilde{\mathbf{\Lambda}}_{s} + \widetilde{\mathbf{\Lambda}}_{n}] \right)$$
(131)

and

$$\tilde{\mu}_{01}(s) = -\ln \det \left(\mathbf{I} - s \mathbf{W} \mathbf{\Lambda}_{\mathbf{n}}\right), \tag{132}$$

where

and

$$\widetilde{\Lambda}_{\mathbf{s},ij} \triangleq \int_{(i-1)T_s}^{iT_s} \int_{(j-1)T_s}^{jT_s} |\widetilde{f}(t)|^2 \widetilde{K}_D(t-u) |\widetilde{f}(u)|^2 dt \, du.$$
(135)

Notice that we include the statistical dependence between the various subintervals in the performance analysis. Using (131)–(135) in (127) gives the performance bound for any particular system.

**Evaluation of**  $\tilde{\mu}_{BS}(s)$  for Suboptimum Receiver No. 2. In this case, we can write  $\tilde{\mu}_{11}(s)$  and  $\tilde{\mu}_{01}(s)$  as Fredholm determinants,

$$\tilde{\mu}_{11}(s) = -\sum_{i=1}^{\infty} \ln (1 - s\tilde{\lambda}_{11,i}) = \tilde{D}_{\mathcal{F}_{11}}(-s), \quad s < 0.$$
(136)

Here the  $\tilde{\lambda}_{11,i}$  are the ordered eigenvalues of  $\tilde{y}_1(t)$  when  $H_1$  is true. Notice that  $\tilde{y}_i(t)$  is the input to the squarer in the *i*th branch. Similarly,

$$\begin{split} \tilde{\mu}_{01}(s) &= -\sum_{i=1}^{\infty} \ln (1 + s\lambda_{01,i}) \\ &= \tilde{D}_{\mathscr{F}_{01}}(s), \qquad -\frac{1}{\tilde{\lambda}_{01,i}} < s \le 0, \end{split}$$
(137)



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where the  $\lambda_{01,i}$  are the ordered eigenvalues of  $\tilde{y}_0(t)$  when  $H_1$  is true. We now illustrate how to evaluate  $\tilde{D}_{\mathcal{F}_{11}}(s)$  using state-variable techniques.

**Evaluation of**  $D_{\mathcal{F}_{11}}(-s)$ . To do this we must write  $\tilde{y}_1(t)$  given  $H_1$  as the output of a linear dynamic sytesm excited by a white noise process. We assume the relations

$$\dot{\tilde{\mathbf{x}}}_{c}(t) = \tilde{\mathbf{F}}_{c}(t)\tilde{\mathbf{x}}_{c}(t) + \tilde{\mathbf{G}}_{c}(t)\tilde{\mathbf{u}}_{c}(t), \qquad t \ge T_{i},$$
(138)

$$\tilde{y}_1(t) = \tilde{C}_c(t)\tilde{\mathbf{x}}_c(t), \qquad (139)$$

$$E[\tilde{\mathbf{u}}_{c}(t)\tilde{\mathbf{u}}_{c}^{\dagger}(\tau)] = \tilde{\mathbf{Q}}_{c}, \qquad (140)$$

$$E[\tilde{\mathbf{x}}_c(T_i)\tilde{\mathbf{x}}_c^{\dagger}(T_i)] = \widetilde{\mathbf{P}}_c.$$
(141)

We must specify  $\tilde{\mathbf{F}}_{c}(t)$ ,  $\tilde{\mathbf{G}}_{c}(t)$ ,  $\tilde{\mathbf{Q}}_{c}$ ,  $\tilde{\mathbf{C}}_{c}(t)$ , and  $\tilde{\mathbf{P}}_{c}$ .

On  $H_1$ ,  $\tilde{y}_i(t)$  is generated as shown in Fig. 11.23. We must express this system in the form of (127)-(130). We do this by adjoining the state vectors  $\tilde{x}(t)$  and  $\tilde{x}_r(t)$  to obtain

$$\tilde{\mathbf{x}}_c(t) = \begin{bmatrix} \tilde{\mathbf{x}}(t) \\ \tilde{\mathbf{x}}_r(t) \end{bmatrix}.$$
(142)

The resulting system matrices are

$$\tilde{\mathbf{G}}_{c}(t) = \begin{bmatrix} \tilde{\mathbf{G}} & \mathbf{0} \\ \mathbf{0} & -\tilde{\mathbf{G}}_{r} \end{bmatrix},$$
(144)

$$\widetilde{\mathbf{C}}_{c}(t) = [\mathbf{0} \quad | \quad \widetilde{\mathbf{C}}_{r}], \tag{145}$$

$$\tilde{\mathbf{Q}}_{c} = \begin{bmatrix} \mathcal{Q} & \mathbf{0} \\ \cdots & \cdots \\ \mathbf{0} & N_{0} \end{bmatrix}, \tag{146}$$

and

$$\tilde{\mathbf{P}}_{c} = \begin{bmatrix} \tilde{\mathbf{P}} & \mathbf{0} \\ \cdots & \cdots \\ \mathbf{0} & \tilde{\mathbf{P}}_{r} \end{bmatrix}.$$
(147)

Once we have represented  $\tilde{y}_1(t)$  in this manner, we know that

$$\tilde{D}_{\mathscr{F}_{11}}(-s) = \ln \det \tilde{\Gamma}_2(T_f) + \operatorname{Re} \int_{T_i}^{T_f} \operatorname{Tr} \left[\tilde{F}_c(t)\right] dt,$$
(148)

where

and

$$\tilde{\mathbf{T}}_1(T_i) = \tilde{\mathbf{P}}_c, \tag{150}$$

$$\mathbf{T}_2(T_i) = \mathbf{I} \tag{151}$$

(see pages 42-44).

The results in (143)-(151) completely specify the first Fredholm determinant. We can carry out the actual evaluation numerically. The second Fredholm determinant can be calculated in a similar manner. Thus, we have formulated the problem so that we can investigate any set of filter matrices.

**Example** [4]. We consider a first-order Butterworth fading spectrum and a transmitted signal with a constant envelope. The scattering function is

$$\tilde{S}_D\{f\} = \frac{4k\sigma_b^2}{(2\pi f)^2 + k^2}$$
(152)

and

$$\tilde{f}(t) = \sqrt{\frac{1}{T}} \qquad 0 \le t \le T.$$
(153)

The average received energy in the signal component is

$$\bar{E}_r = 2\sigma_b^2 E_t. \tag{154}$$

To evaluate the performance of Receiver No. 1, we calculate  $\tilde{\mu}_{BS}(s)$  by using (128) and (131)–(135). We then minimize over s to obtain the tightest bound in (127). Finally we minimize over  $T_s$ , the subinterval length, to obtain the best suboptimum receiver. The result is a function

$$\min_{T_s} \left[ \min_{s} \left( \tilde{\mu}_{\rm BS}(s) \right) \right], \tag{155}$$

which is a measure of performance for Receiver No. 1.

In Receiver No. 2 we use a first-order filter. Thus,

$$\tilde{y}(t) = -k_r \tilde{y}(t) + \tilde{f}^*(t)r(t).$$
(156)

We also assume that

$$\tilde{\mathbf{P}}_r = \mathbf{0} \tag{157}$$

for simplicity.

We evaluate  $\tilde{\mu}_{BS}(s)$  as a function of  $k_r T$ . For each value of  $k_r T$  we find

$$\min\left[\tilde{\mu}_{BS}(s)\right] \tag{158}$$

to use in the exponent of (127). We then choose the value of  $k_r T$  that minimizes (158). The resulting value of

$$\min_{k_T T} \min_{s} \left[ \tilde{\mu}_{BS}(s) \right]$$
(159)

is a measure of performance of Receiver No. 2. In Figs. 11.24 and 11.25, we have plotted the quantities in (155) and (159) for the cases in which  $\bar{E}_r/N_0$  equals 5 and 20, respectively. We also show  $\tilde{\mu}_{BS}(\frac{1}{2})$  for the optimum receiver. The horizontal axis is kT, and the number in parentheses on the Receiver No. 1 curve is  $T/T_s$ , the number of subintervals used. In both cases, the performance of Receiver No. 1 approaches that of



Fig. 11.24 Normalized error-bound exponents for optimum and suboptimum receivers: Doppler-spread channel with first-order fading,  $\overline{E_r}/N_0 = 5$ . (from [4].)



Fig. 11.25 Normalized error-bound exponents for optimum and suboptimum receivers: Doppler-spread channel with first-order fading,  $\vec{E_r}/N_0 = 20$ . (From [4].)

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the optimum receiver as kT approaches zero, and the performance of Receiver No. 2 approaches that of the optimum receiver as kT becomes large. This behavior is just what we would expect. We also see that one of the receivers is within .01 of the optimum over the entire range of kT. Thus, for this particular example, the simplicity afforded by the suboptimum receivers is probably worth the slight decrease in performance.

We should comment that the above example is not adequate to verify that these suboptimum receivers will be satisfactory in all cases. A more severe test of the suboptimum receiver structure would require a nonconstant signal. Our problem formulation allows us to carry this analysis out for any desired  $\tilde{f}(t)$ . Other references that deal with suboptimum receiver analysis include [21] and [22].

#### 11.3.4 M-ary Systems

We consider an M-ary system in which the transmitted signal on the *i*th hypothesis is

$$s_{ti}(t) = \sqrt{2E_t} \operatorname{Re}\left[\tilde{f}(t)e^{i\omega_i t}\right]: H_i.$$
(160)

We assume that the  $\omega_i$  are chosen so that the output signal processes on the different hypotheses are in disjoint frequency bands. The received waveform on the *i*th hypothesis is

$$r(t) = \sqrt{2E_t} \operatorname{Re}\left[\tilde{b}(t)\tilde{f}(t)e^{j\omega_i t}\right] + w(t), \quad 0 \le t \le T : H_i.$$
(161)

The hypotheses are equally likely, and the criterion is minimum probability of error.

The optimum receiver is an obvious generalization of the binary receiver in Figs. 11.12 and 11.13. To calculate the performance, we extend (5.22) to include nonstationary complex processes. The result is

$$\Pr\left(\epsilon\right) < e^{\rho TR} \frac{\left[\tilde{D}_{\mathscr{F}}(1/N_{0})\right]^{\rho}}{\left[\tilde{D}_{\mathscr{F}}(\rho/N_{0}(1+\rho))\right]^{1+\rho}}, \qquad 0 \le \rho \le 1, \qquad (162)$$

where

$$\tilde{D}_{\mathscr{F}}(z) = \prod_{i=1}^{\infty} (1 + z\tilde{\lambda}_i), \qquad (163)$$

and the  $\tilde{\lambda}_i$  are the eigenvalues of (20). We then minimize over  $\rho$  as in (5.29-5.35) to obtain E(R). The next step is to find the distribution of eigenvalues that minimizes E(R). Kennedy has carried out this minimization, and the result is given in [3]. Once again, the minimum is obtained by using a certain number of equal eigenvalues. The optimum number depends on  $\bar{E}_r/N_0$  and the rate R. The final step is to try to find signals to give the appropriate eigenvalue distribution. The techniques of the binary case carry over directly to this problem.

This completes our discussion of the M-orthogonal signal problem. The interested reader should consult [3] for a complete discussion.

# 11.3.5 Summary: Communication over Doppler-spread Channels

In this section we have studied the problem of digital communication over Doppler-spread channels. There are several significant results that should be re-emphasized:

1. The optimum receiver can be realized exactly when the channel process has a state-variable representation.

2. Tight bounds on the probability of error are given by (75). These can be evaluated for any desired  $\tilde{f}(t)$  if  $\tilde{b}_D(t)$  has a state-variable representation.

3. There exists an upper bound on the probability of error for any  $\tilde{f}(t)$  that does not depend on  $\tilde{S}_D\{f\}$ . For any binary system,

$$\Pr\left(\epsilon\right) \le \frac{1}{2} \exp\left(-0.1488 \frac{\bar{E}_r}{N_0}\right). \tag{164}$$

4. In many cases we can choose signals that give performance close to the bound in (164).

5. Two suboptimum receiver configurations were developed that are much simpler to implement than the optimum receiver. In many cases they will perform almost as well as the optimum receiver.

6. The basic results can be extended to include systems using M-orthogonal signals.

This completes our discussion of digital communication.

The reader may wonder why we have included a detailed discussion of digital communication in the middle of a radar/sonar chapter. One obvious reason is that it is an important problem and this is the first place where we possess the necessary background to discuss it. This reason neglects an important point. The binary symmetric problem is one degree easier to analyze than the radar-detection problem, because the symmetry makes  $\tilde{\mu}(\frac{1}{2})$  the important quantity. In the radar problem we must work with  $\tilde{\mu}(s)$ ,  $0 \le s \le 1$ , until a specific threshold (or  $P_F$ ) is chosen. This means that all the signal-design ideas and optimum eigenvalue distributions are harder to develop. Now that we have developed them for the symmetric communications case, we could extend them to the asymmetric problem. The quantitative results are different, but the basic concepts are the same.

### **11.4 PARAMETER ESTIMATION: DOPPLER-SPREAD TARGETS**

The model for the estimation problem is a straightforward modification of the detection model. Once again,

$$\tilde{r}(t) = \sqrt{E_t} \tilde{f}(t-\lambda) \tilde{b}_D\left(t-\frac{\lambda}{2}\right) + \tilde{w}(t), \qquad T_i \le t \le T_f.$$
(165)

There are two cases of the parameter-estimation problem that we shall consider. In the first case, the only unknown parameters are the range to the target and its mean Doppler shift,  $m_D$ . We assume that the scattering function of  $\tilde{b}_D(t)$  is completely known except for its mean. The covariance function of the signal returned from the target is

$$\widetilde{K}_{\widetilde{s}}(t, u: \lambda, m_D) = E_t \widetilde{f}(t-\lambda) e^{j2\pi m_D t} \widetilde{K}_{D_0}(t-u) e^{-j2\pi m_D u} \widetilde{f}^*(u-\lambda), \quad (166)$$

where  $\tilde{K}_{D_0}(t-u)$  is the covariance function of  $\tilde{b}_D(t)$  with its mean Doppler removed. In other words,

$$\widetilde{K}_D(t-u) \triangleq e^{j2\pi m_D t} \widetilde{K}_{D_0}(t-u) e^{-j2\pi m_D u}.$$
(167)

We observe  $\tilde{r}(t)$  and want to estimate  $\lambda$  and  $m_D$ . Notice that the parameters of interest can be separated out of the covariance function.

In the second case, the covariance function of  $\tilde{b}_D(t)$  depends on a parameter (either scalar or vector) that we want to estimate. Thus

$$\widetilde{K}_{\tilde{s}}(t, u : \lambda, \mathbf{A}) = E_t \widetilde{f}(t-\lambda) \widetilde{K}_D(t-u : \mathbf{A}) \widetilde{f}^*(u-\lambda).$$
(168)

A typical parameter of interest might be the amplitude, or the root-meansquare Doppler spread. In this case the parameters cannot necessarily be separated out of the covariance function. Notice that the first case is included in the second case.

Most of the necessary results for both cases can be obtained by suitably combining the results in Chapters 6, 7, and 10. To illustrate some of the ideas involved, we consider the problem outlined in (166) and (167).

We assume that the target is a point target at range R, which corresponds to a round-trip travel time of  $\lambda$ . It is moving at a constant velocity corresponding to a Doppler shift of  $m_D$  cps. In addition, it has a Doppler spread characterized by the scattering function  $\tilde{S}_{D_0}\{f\}$ , where

$$\widetilde{S}_{D_0}\{f_1\} \triangleq \widetilde{S}_D\{f_1 - m_D\}.$$
(169)

The complex envelope of the received signal is

$$\tilde{r}(t) = \sqrt{E_t} \tilde{f}(t-\lambda) e^{j\omega t} \tilde{b}_{D_0}\left(t-\frac{\lambda}{2}\right) + \tilde{w}(t), \qquad -\infty < t < \infty.$$
(170)

We assume an infinite observation interval for simplicity.

The covariance function of the returned signal process is given in (166). The likelihood function is given by

$$l(\lambda, m_D) = \frac{1}{N_0} \iint_{-\infty}^{\infty} \tilde{r}^*(t) \tilde{h}_{ou}(t, u : \lambda, m_D) \tilde{r}(u) dt du, \qquad (171)$$

where  $\tilde{h}_{ou}(t, u: \lambda, m_D)$  is specified by

$$N_{0}\tilde{h}_{ou}(t, u: \lambda, m_{D}) + \int_{-\infty}^{\infty} \tilde{h}_{ou}(t, z: \lambda, m_{D})\tilde{K}_{\bar{s}}(z, u: \lambda, m_{D}) dz$$
$$= \tilde{K}_{\bar{s}}(t, u: \lambda, m_{D}), \qquad -\infty < t, u < \infty.$$
(172)

(Notice that the bias term is not a function of  $\lambda$  or  $m_D$ , and so it has been omitted.) In order to construct the likelihood function, we must solve (172) for a set of  $\lambda_i$  and  $m_{D_i}$  that span the region of the range-Doppler plane in which targets- may be located. Notice that, unlike the slowly fluctuating case in Chapter 10, we must normally use a discrete approximation in both range and Doppler. (There are other realizations for generating  $l(\lambda, m_D)$  that may be easier to evaluate, but (171) is adequate for discussion purposes.) The maximum likelihood estimates are obtained by finding the point in the  $\lambda$ ,  $m_D$  plane where  $l(\lambda, m_D)$  has its maximum.

To analyze the performance, we introduce a spread ambiguity function. As before, the ambiguity function corresponds to the output of the receiver when the additive noise  $\tilde{w}(t)$  is absent. In this case the signal is a sample function of a random process, so that the output changes each time the experiment is conducted. A useful characterization is the expectation of this output. The input in the absence of noise is

$$\tilde{r}(t) = \sqrt{E_t} \tilde{f}(t - \lambda_a) e^{j2\pi m_{Da}t} \tilde{b}_{D_0} \left(t - \frac{\lambda_a}{2}\right).$$
(173)

We substitute (173) into (171) and take the expectation over  $\tilde{b}_{D_0}(t)$ . The result is

$$\begin{aligned} \theta_{\Omega_D}\{\lambda_a,\,\lambda:m_{Da},\,m\} &= \frac{1}{N_0} \iint_{-\infty}^{\infty} \tilde{h}_{ou}(t,\,u:\lambda,\,m_D) \tilde{K}^*_{\tilde{s}}(t,\,u:\lambda_a,\,m_{Da}) \,dt \,du \\ &= \frac{E_t}{N_0} \iint_{-\infty}^{\infty} \tilde{h}_{ou}(t,\,u:\lambda,\,m_D) \tilde{f}^*(t-\lambda_a) \\ &\times e^{-j2\pi m_{Da}(t-u)} \tilde{f}(u-\lambda_a) \tilde{K}^*_{D_0}(t-u) \,dt \,du, \end{aligned}$$

(174)

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which we define to be the *Doppler-spread ambiguity function*. Notice that it is a function of four variables,  $\lambda_a$ ,  $\lambda$ ,  $m_{D_a}$ , and  $m_D$ . This function provides a basis for studying the accuracy, ambiguity, and resolution problems when the target is Doppler-spread. The local accuracy problem can be studied by means of Cramér-Rao bounds. The elements in the J matrix are of the form

$$J_{\lambda\lambda}(\lambda_a, m_{Da}) = c \left. \frac{\partial^2 \theta_{\Omega_D} \{\lambda_a, \lambda : m_{Da}, m\}}{\partial \lambda \, \partial \lambda_a} \right|_{\substack{\lambda = \lambda_a \\ m = m_{Da}}}$$
(175)

(see Problem 11.4.7). The other elements have a similar form.

We do not discuss the ambiguity and resolution issues in the text. Several properties of the Doppler-spread ambiguity function are developed in the problems. Notice that  $\theta_{\Omega_D}\{\lambda_a, \lambda: m_{D_a}, m\}$  may be written in several other forms that may be easier to evaluate.

In general, the spread ambiguity function is difficult to use. When the LEC condition is valid,

$$h_{ou}(t, u: \lambda, m) \simeq \frac{1}{N_0} K_{\bar{s}}(t, u: \lambda, m)$$
  
=  $\frac{E_t}{N_0} \tilde{f}(t-\lambda) e^{j2\pi m t} \tilde{K}_{D_0}(t-u) e^{-j2\pi m u} \tilde{f}^*(u-\lambda).$  (176)

Using (176) in (174) gives

$$\theta_{\Omega_{D}, \text{LEC}}\{\lambda_{a}, \lambda : m_{a}, m\}$$

$$= \frac{E_{t}^{2}}{N_{0}^{2}} \int_{-\infty}^{\infty} \tilde{f}(t-\lambda) \tilde{f}^{*}(t-\lambda_{a}) e^{j2\pi(m-m_{a})t} |\tilde{K}_{D_{0}}(t-u)|^{2}$$

$$\times e^{-j2\pi(m-m_{a})u} \tilde{f}^{*}(u-\lambda) \tilde{f}(u-\lambda_{a}) dt du.$$
(177)

(We suppressed the D subscript on m for notational simplicity.) This can be reduced to the two-variable function

$$\theta_{\Omega_{D}, \text{LEC}}\{\lambda_{e}, m_{e}\} = \frac{E_{t}^{2}}{N_{0}^{2}} \int_{-\infty}^{\infty} \tilde{f}\left(t - \frac{\lambda_{e}}{2}\right) \tilde{f}^{*}\left(t + \frac{\lambda_{e}}{2}\right) e^{j2\pi m_{e}t} \left|\tilde{K}_{D_{0}}(t-u)\right|^{2} \\ \times e^{-j2\pi m_{e}u} \tilde{f}^{*}\left(u - \frac{\lambda_{e}}{2}\right) \tilde{f}\left(u + \frac{\lambda_{e}}{2}\right) dt du.$$

(178)

Some of the properties of  $\theta_{\Omega_{in} \text{LEC}}\{\cdot, \cdot\}$  are developed in the problems.

A final comment concerning ambiguity functions is worthwhile. In the general parameter estimation problem, the likelihood function is

$$l(\mathbf{A}) = \frac{1}{N_0} \iint_{-\infty}^{\infty} \tilde{r}^*(t) \tilde{h}_{ou}(t, u: \mathbf{A}) \tilde{r}(u) dt du + l_B(\mathbf{A}), \qquad \mathbf{A} \in \psi_{\mathbf{a}}, \quad (179)$$

where  $\tilde{h}_{ou}(t, u: \mathbf{A})$  satisfies

$$N_{0}\tilde{h}_{ou}(t, u: \mathbf{A}) + \int_{-\infty}^{\infty} \tilde{h}_{ou}(t, z: \mathbf{A})\tilde{K}_{\tilde{s}}(z, u: \mathbf{A}) dz = \tilde{K}_{\tilde{s}}(t, u: \mathbf{A}),$$
$$-\infty < t, u < \infty, \qquad \mathbf{A} \in \psi_{\mathbf{a}}, \quad (180)$$

and  $l_B(\mathbf{A})$  is the bias. For this problem we define the generalized spread ambiguity function as

$$\theta_{\Omega}(\mathbf{A}_{a},\mathbf{A}) = \frac{1}{N_{0}} \int_{-\infty}^{\infty} \tilde{h}_{ou}(t, u; \mathbf{A}) \tilde{K}_{\tilde{s}}(t, u; \mathbf{A}_{a}) dt du, \qquad \mathbf{A}_{a}, \mathbf{A} \in \psi_{\mathbf{a}}.$$
 (181)

We shall encounter this function in Chapters 12 and 13.

This completes our discussion of the estimation problem. Our discussion has been brief because most of the basic concepts can be obtained by modifying the results in Chapters 6 and 7 in a manner suggested by our work in Chapter 10.

# 11.5 SUMMARY: DOPPLER-SPREAD TARGETS AND CHANNELS

In this chapter we have studied detection and parameter estimation in situations in which the target (or channel) caused the transmitted signal to be spread in frequency. We modeled the complex envelope of the received signal as a sample function of a complex Gaussian random process whose covariance is

$$\widetilde{K}_{\tilde{s}}(t,u) = E_t \widetilde{f}(t-\lambda) \widetilde{K}_D(t-u) \widetilde{f}^*(u-\lambda).$$
(179)

The covariance function  $\tilde{K}_D(t-u)$  completely characterized the target (or channel) reflection process. We saw that whenever the transmitted pulse was longer than the reciprocal of the reflection process, the target (or channel) caused time-selective fading. We then studied three problems.

In Section 11.2, we formulated the optimum detection problem and gave the formulas that specify the optimum receiver and its performance. This problem was just the bandpass version of the Gaussian signal in noise problem that we had solved in Chapters 2–5. By exploiting our complex representation, all of the results carried over easily. We observed that

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whenever the reflection process could be modeled as a complex finite-state process, we could find a complete solution for the optimum receiver and obtain a good approximation to the performance. This technique is particularly important in this problem, because the reflected signal process is usually nonstationary. Another special case that is important is the LEC case. Here the optimum receiver and its performance can be evaluated easily. The results for the LEC condition also suggest suboptimum receivers for other situations.

In Section 11.3, we studied binary communication over Doppler-spread channels. The first important result was a bound on the probability of error that was independent of the channel-scattering function. We then demonstrated how to design signals that approached this bound. Techniques for designing and analyzing suboptimum receivers were developed. In the particular example studied, the performance of the suboptimum receivers was close to that of the optimum receiver. The extension of the results to M-ary systems was discussed briefly.

The final topic was the parameter-estimation problem. In Section 11.4, we formulated the problem and indicated some of the basic results. We defined a new function, the spread-ambiguity function, which could be used to study the issues of accuracy, ambiguity, and resolution. A number of questions regarding estimation are discussed in the problems. We study parameter estimation in more detail in Section 13.4.

We now turn to the other type of singly-spread target discussed in Chapter 8. This is the case in which the transmitted signal is spread in range.

### **11.6 PROBLEMS**

### P.11.2 Detection of Doppler-spread Targets

Problem 11.2.1. We want to derive the result in (33) Define

$$\tilde{r}_i = \int_{T_i}^{T_f} \tilde{r}(t) \tilde{\varphi}_i^*(t) dt, \qquad (P.1)$$

where  $\tilde{\varphi}_i(t)$  is the *i*th eigenfunction of  $\tilde{K}_{\tilde{s}}(t, u)$ . Observe from (A.116) that

$$p_{\tilde{r}_i|H_1}(\tilde{R}_i \mid H_1) = \frac{1}{\pi(\tilde{\lambda}_i + N_0)} \exp\left[-\frac{|\tilde{R}_i|^2}{\tilde{\lambda}_i + N_0}\right], \quad -\infty < \tilde{R}_i < \infty.$$
(P.2)

Using (P.1) and (P.2) as a starting point, derive (33).

**Problem 11.2.2.** Derive (33) directly from (2.31) by using bandpass characteristics developed in the Appendix.

Problem 11.2.3. Derive the result in (38) in two ways:

1. Use (33) and (34) as a starting point.

2. Use (2.86) as a starting point.

Problem 11.2.4. Consider the detection problem specified below.

$$\begin{split} \tilde{r}(t) &= \sqrt{E_t} \tilde{f}(t) \tilde{b}_D(t) + \tilde{w}(t), \qquad T_i \leq t \leq T_f : H_1, \\ \tilde{r}(t) &= \tilde{w}(t), \qquad \qquad T_i \leq t \leq T_f : H_0. \end{split}$$

The Doppler scattering function is

$$\tilde{S}_D\{f\} = \frac{4k\sigma_b^2}{(2\pi f)^2 + k^2}.$$

The complex white noise has spectral height  $N_0$ .

1. Draw a block diagram of the optimum receiver. Write out explicitly the differential equations specifying the system.

2. Write out the equations that specify  $\tilde{\mu}(s)$ . Indicate how you would use  $\tilde{\mu}(s)$  to plot the receiver operating characteristic.

Problem 11.2.5. Consider the same model as in Problem 11.2.4. Assume that

$$\tilde{f}(t) = \begin{cases} \sqrt{\frac{1}{T}}, & T_i \le t \le T_f, \\ 0, & \text{elsewhere,} \end{cases}$$

where

$$T = T_f - T_i,$$

and that T is large enough that the asymptotic formulas are valid.

1. Draw the filter-squarer realization of the optimum receiver. Specify the transfer function of the filter.

2. Draw the optimum realizable filter realization of the optimum receiver. Specify the transfer function of the filter.

3. Compute  $\tilde{\mu}_{\infty}(s)$ .

**Problem 11.2.6.** Consider the same model as in Problem 11.2.4. Assume that  $\tilde{f}(t)$  is a piecewise constant signal,

$$\tilde{f}(t) = \begin{cases} c \sum_{i=1}^{k} \tilde{f}_{i} \tilde{u}(t-iT_{s}), & 0 \le t \le T_{s} \\ 0, & \text{elsewhere,} \end{cases}$$

where

$$\tilde{u}(t) = \begin{cases} \frac{1}{\sqrt{T_s}}, & 0 \le t \le T_s; \\ 0, & \text{elsewhere,} \end{cases}$$

and

$$T_s = \frac{T}{K}.$$

The  $\tilde{f}_i$  are complex weighting coefficients, and c is a constant chosen so that  $\tilde{f}(t)$  has unit energy.

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1. Draw a block diagram of the optimum receiver. Write out explicitly the differential equations specifying the system.

2. Write out the equations specifying  $\tilde{\mu}(s)$ .

Problem 11.2.7. Repeat Problem 11.2.4 for the Doppler scattering function

$$\tilde{S}_D\{f\} = \frac{4k\sigma_b^2}{[2\pi(f-m_D)]^2 + k^2}, \quad -\infty < f < \infty.$$

**Problem 11.2.8.** Repeat Problem 11.2.4 for the case in which the target reflection process is characterized by the spectrum in (A.148).

Problem 11.2.9. Consider the following detection problem:

$$\widetilde{r}(t) = \sqrt{E_t} \widetilde{f}(t) \widetilde{b}_D(t) + \widetilde{n}_c(t) + \widetilde{w}(t), \qquad 0 \le t \le T : H_1,$$
  
$$\widetilde{r}(t) = \widetilde{n}_c(t) + \widetilde{w}(t), \qquad 0 \le t \le T : H_0.$$

The colored noise is a zero-mean complex Gaussian process with covariance function  $\tilde{K}_{c}(t, u)$ . It is statistically independent of both  $\tilde{b}_{D}(t)$  and  $\tilde{w}(t)$ .

- 1. Derive the equations specifying the optimum receiver.
- 2. Derive a formula for  $\tilde{\mu}(s)$ .

**Problem 11.2.10.** Consider the model in Problem 11.2.9. Assume that  $\tilde{n}_c(t)$  has a complex finite state representation.

- 1. Write out the differential equations specifying the optimum receiver.
- 2. Write out the differential equations specifying  $\tilde{\mu}(s)$ .

Problem 11.2.11. Consider the following detection problem.

$$\begin{split} r(t) &= \sqrt{E_t} \{ \tilde{f}(t-\lambda_1) e^{j\omega_1 t} \tilde{b}_{D1}(t) + \tilde{f}(t-\lambda_2) e^{j\omega_2 t} \tilde{b}_{D2}(t) + \tilde{w}(t) \}, \quad T_i \leq t \leq T_f : H_1, \\ &= \sqrt{E_t} \tilde{f}(t-\lambda_2) e^{j\omega_2 t} \tilde{b}_{D2}(t) + \tilde{w}(t), \qquad \qquad T_i \leq t \leq T_f : H_0. \end{split}$$

The quantities  $\lambda_1$ ,  $\lambda_2$ ,  $\omega_1$ , and  $\omega_2$  are known. The two reflection processes are statistically independent, zero-mean complex Gaussian processes with covariance functions  $\tilde{K}_{D1}(\tau)$  and  $\tilde{K}_{D2}(\tau)$ . Both processes have finite state representations.

- 1. Find the optimum receiver.
- 2. Find an expression for  $\tilde{\mu}(s)$ .

**Problem 11.2.12.** Consider the model in Problem 11.2.11. Assume that  $\tilde{b}_{D2}(t)$  is a random variable instead of a random process.

$$\tilde{b}_{D2}(t) = \tilde{b}_{D2}.$$

- 1. Find the optimum receiver.
- 2. Find an expression for  $\tilde{\mu}(s)$ .

**Problem 11.2.13.** Consider the model in Problem 11.2.11. Assume that  $\tilde{b}_{DI}(t)$  is a random variable instead of a random process.

$$\tilde{b}_{D1}(t) = \tilde{b}_{D1}.$$

Assume that  $\tilde{b}_{D2}(t)$  has a finite state representation.

1. Find the optimum receiver. Specify both a correlator realization and a realizable filter realization.

2. Recall that

$$P_F = (P_D)^{1+\Delta}$$

for this type of model (see page 251). Find an integral expression for  $\Delta$ . Find the set of differential equations that specify  $\Delta$ .

3. Assume that

$$\tilde{S}_{D2}\{f\} = \frac{2kP_2}{(2\pi f)^2 + k^2} \,.$$

Write out the differential equations specifying the optimum receiver and  $\Delta$ . Problem 11.2.14. Consider the model in Problem 11.2.13.

$$\begin{split} \widetilde{r}(t) &= \sqrt{E_t} \, \widetilde{b}_{D1} \widetilde{f}(t-\lambda_1) e^{j\omega_1 t} + \sqrt{E_t} \, \widetilde{b}_{D2}(t) \, \widetilde{f}(t-\lambda_2) e^{j\omega_2 t} + \widetilde{w}(t), \quad T_i \leq t \leq T_f : H_1, \\ \widetilde{r}(t) &= \sqrt{E_t} \, \widetilde{b}_{D2}(t) \, \widetilde{f}(t-\lambda_2) e^{j\omega_2 t} + \widetilde{w}(t), \quad T_i \leq t \leq T_f : H_0. \end{split}$$

We want to design the optimum signal subject to an energy and bandwidth constraint.

$$\int_{T_i}^{T_f} |\tilde{f}(t)|^2 dt = 1,$$
$$\int_{T_i}^{T_f} f^2 |\tilde{F}\{f\}|^2 dt = B^2.$$

1. Assume that we use an optimum receiver. Find the differential equations that specify the optimum *signal* (see Section 9.5).

2. Assume that we use a conventional receiver (see Section 10.5). Find the differential equations that specify the optimum *signal*.

3. What is the fundamental difference between the equations in parts 1 and 2 and the equations in Section 9.5 (9.133)-(9.139)?

Problem 11.2.15. Consider the following detection problem:

$$\begin{split} \tilde{r}(t) &= \sqrt{E_t} \, \tilde{b}_D \, \tilde{f}(t) + \sqrt{E_t} \left\{ \sum_{i=1}^K \tilde{b}_{D_i}(t) \, \tilde{f}(t-\lambda_i) e^{j\omega_i t} \right\} + \tilde{w}(t), \qquad T_0 \leq t \leq T_f : H_1, \\ \tilde{r}(t) &= \sqrt{E_t} \left\{ \sum_{i=1}^K \tilde{b}_{D_i}(t) \, \tilde{f}(t-\lambda_i) e^{j\omega_i t} \right\} + \tilde{w}(t), \qquad T_0 \leq t \leq T_f : H_0. \end{split}$$

The  $\tilde{b}_{Di}(t)$  are statistically independent, zero-mean complex Gaussian processes with covariance functions  $\tilde{K}_{Di}(\tau)$ . The  $\lambda_i$  and  $\omega_i$  are known. The target reflection  $\tilde{b}_D$  is a complex zero-mean Gaussian variable with mean-square value  $2\sigma_b^2$ .

1. Find the optimum receiver and an expression for  $\Delta$ .

2. Assume that a conventional receiver is used (see Section 10.5). Find an expression for  $\Delta_{wo}$ . Write this expression in terms of  $\theta\{\tau, f\}$  and  $S_{Di}\{f\}$ .

Problem 11.2.16. Consider the multiple hypothesis detection problem:

$$\begin{split} &\tilde{r}(t) = \tilde{w}(t), & T_i \leq t \leq T_f : H_0, \\ &\tilde{r}(t) = \sqrt{E_t} \, \tilde{b}_{D1} \, \tilde{f}(t) + \tilde{w}(t), & T_i \leq t \leq T_f : H_1, \\ &\tilde{r}(t) = \sqrt{E_t} \, \tilde{b}_{D2}(t) \, \tilde{f}(t) + \tilde{w}(t), & T_i \leq t \leq T_f : H_2. \end{split}$$

We see that the three hypotheses correspond to noise only, noise plus a point-nonfluctuating target, and noise plus a fluctuating target.

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Assume the following cost matrix:

$$\mathbf{C} = \begin{bmatrix} 0 & C_M & C_M \\ C_F & 0 & C_X \\ C_F & C_X & 0 \end{bmatrix}.$$

1. Find the optimum Bayes receiver.

2. Consider the special case when  $C_X = 0$ . Draw the optimum receiver. Find an expression for  $\tilde{\mu}(s)$ .

3. Assume that the following criteria are used:

a.  $P_F \triangleq \{ \Pr [ \text{say } H_1 \text{ or } H_2 | H_0 \text{ is true} ] \}.$ 

b.  $P_D \triangleq \{ \Pr [ say H_1 \text{ or } H_2 | H_1 \text{ or } H_2 \text{ is true} ] \}.$ 

c. Maximize  $P_D$  subject to constraint that  $P_F \leq \alpha$ .

d. If the receiver says that a target is present, we want to classify it further. Define

$$P_{F_2} \triangleq \{ \Pr [ say H_2 | H_1 \text{ is true, target decision positive} ] \}$$

and

$$P_{D_2} \triangleq \{ \Pr [ say H_2 | H_2 \text{ is true, target decision positive} ] \}$$

Maximize  $P_{D_2}$  subject to constraint  $P_{F_2} \leq \alpha_2$ .

Explain how the over-all receiver operates. Can you write this in terms of a Bayes test?

Problem 11.2.17. Consider the detection problem in (30) and (31). Assume that

 $E[\tilde{b}_D(t)] = \tilde{m},$ 

where  $\tilde{m}$  is itself a complex Gaussian random variable with mean-square value  $2\sigma^2$ . The rest of the model remains the same. Find the optimum receiver.

#### P.11.3 Digital Communication over Doppler-Spread Channels

Problem 11.3.1. Consider the binary FSK system described in Section 11.3.1. Assume that

$$\tilde{S}_D\{f\} = rac{4k\sigma_b^2}{(2\pi f)^2 + k^2}.$$

- 1. Write out the differential equations specifying the receiver in detail.
- 2. Write out the differential equations specifying  $\tilde{\mu}_{BS}(\frac{1}{2})$ .

**Problem 11.3.2.** The performance of a binary FSK system operating over a Dopplerspread channel is given by

$$\tilde{\mu}_{\rm BS}(\frac{1}{2}) = \sum_{i=1}^{\infty} \left[ \ln \left( 1 + \frac{\tilde{\lambda}_i}{N_0} \right) - 2 \ln \left( 1 + \frac{\tilde{\lambda}_i}{2N_0} \right) \right]. \tag{P.1}$$

For constant transmitted signals and large time-bandwidth products, we can use the SPLOT formulas.

- 1. Write the SPLOT formula corresponding to (P.1).
- 2. Evaluate  $\tilde{\mu}_{BS,\infty}(\frac{1}{2})$  for

$$\tilde{S}_D\{f\} = \frac{4n\sigma_b^2}{k} \frac{\sin(\pi/2n)}{(2\pi f/k)^{2n} + 1}.$$

The transmitted signal has energy  $E_t$  and duration T.

3. Find the optimum value of kT. Show that if the optimum value of kT is used,  $\tilde{\mu}_{BS,\infty}(\frac{1}{2})$  will decrease monotonically with n.

Problem 11.3.3. Consider the binary FSK system described in Section 11.3.1. Assume that

$$\tilde{f}(t) = \left(\frac{1}{\pi T^2}\right)^{\frac{1}{4}} e^{-t^2/2T^2}, \quad -\infty < t < \infty$$

and

$$\tilde{S}_D\{f\} = \frac{2{\sigma_b}^2}{\sqrt{\pi} B_c} e^{-f^2 B_c^2}, \quad -\infty < f < \infty.$$

The observation interval is infinite.

- 1. Find the output eigenvalues. (Hint: Use Mehler's expansion [e.g., [6] or [7].)
- 2. Evaluate  $\tilde{\mu}_{BS,\infty}(\frac{1}{2})$ .

Problem 11.3.4. Consider a binary communication system operating under LEC conditions.

- 1. Show that  $\tilde{\mu}_{BS}(\frac{1}{2})$  can be expressed in terms of  $\Delta$  [see (9.49)].
- 2. Use the results of part 1 in (75) to find a bound on the probability of error.
- 3. Find an expression for  $\Delta$  in terms of  $\tilde{f}(t)$  and  $S_D\{f\}$ .

**Problem 11.3.5.** Consider a *K*-channel frequency-diversity system using orthogonal FSK in each channel. The received waveform in the *i*th channel is

$$r(t) = \begin{cases} \sqrt{\frac{2E_t}{K}} \operatorname{Re} \left[\tilde{b}_i(t)\tilde{f}(t)e^{j\omega_{1i}t}\right] + w(t), & T_0 \le t \le T_f : H_1, \\ \sqrt{\frac{2E_t}{K}} \operatorname{Re} \left[\tilde{b}_i(t)\tilde{f}(t)e^{j\omega_{0i}t}\right] + w(t), & T_0 \le t \le T_f : H_0, \quad i = 1, 2, \dots, K. \end{cases}$$

The channel fading processes are statistically independent and have identical scattering functions. Assume that the SPLOT condition is valid.

- 1. Evaluate  $\tilde{\mu}_{BS}(\frac{1}{2})$ .
- 2. Assume that

$$\tilde{S}_D\{f\} = \frac{4k{\sigma_b}^2}{(2\pi f)^2 + k^2}.$$

The single-channel system with this scattering function was discussed in Example 2 on page 382. How would you use the additional freedom of a frequency-diversity system to improve the performance over that of the system in Example 2?

**Problem 11.3.6.** Consider the model in Example 3 on page 384. We want to investigate the probability of error as a function of  $T_s$ . One of the two branches of the receiver is



Fig. P.11.1

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shown in Fig. P.11.1. This branch is referenced to  $f_1$ ; the other branch, to  $f_0$ . Assume that  $T_p$  is large enough that the outputs due to each pulse are statistically independent.

1. Find an expression for Pr ( $\epsilon$ ) as a function of  $\overline{E}_r$ ,  $N_0$ , k, and  $T_s$ . Assume that  $D_o$  pulses are used. [*Hint*: Recall the results in (I-2.434) and (I-2.516).]

2. Plot

$$\frac{\ln \Pr(\epsilon)}{-0.1488\bar{E}_r/N_0}$$

as a function of  $kT_s$ .

**Problem 11.3.7.** Consider the model in Example 3 on page 384. We want to investigate the probability of error as a function of  $T_s$  and  $T_p$ . The receiver in Problem 11.3.6 is used. Derive an expression for  $\tilde{\mu}_{BS}(\frac{1}{2})$ .

**Problem 11.3.8.** Consider the piecewise constant channel model in Fig. 11.17 and assume that  $\tilde{f}(t)$  is a rectangular pulse. We generate a set of random variables  $\tilde{r}_i$  as shown in Fig. 11.18. However, instead of using a weighted sum of their squared magnitudes, we operate on them in an optimum manner.

- 1. Find the optimum test based on the observed vector  $\tilde{r}$ .
- 2. Find an expression for  $\tilde{\mu}_{BS}(s)$  for this test.

3. Prove that the receiver in part 1 approaches the optimum receiver of Section 11.3.1 as  $T_s$  approaches zero.

**Problem 11.3.9.** The definitions of  $\tilde{\mu}_{11}(s)$  and  $\tilde{\mu}_{01}(s)$  are given in (129) and (130).

- 1. Verify that the results in (131) and (132) are correct.
- 2. Verify the result in (136).

**Problem 11.3.10.** Consider the *M*-ary problem described in Section 11.3.4. Draw a block diagram of the optimum receiver.

**Problem 11.3.11.** Consider a binary communication system operating over a discrete multipath channel. The complex envelopes of the received waveforms are

$$\tilde{r}(t) = \sqrt{E}_t \left\{ \sum_{i=1}^K \tilde{b}_{Di}(t) \tilde{f}(t-\lambda_i) \right\} + \tilde{w}(t), \qquad T_i \le t \le T_f : H_1,$$

where the complex representation is with reference to  $\omega_1$ , and

$$\tilde{r}(t) = \sqrt{E_t} \left\{ \sum_{i=1}^{K} \tilde{b}_{Di}(t) \tilde{f}(t-\lambda_i) \right\} + \tilde{w}(t), \qquad T_i \le t \le T_f : H_0,$$

where the complex representation is with reference to  $\omega_0$ . The  $\lambda_i$  are known and the  $f_{Di}(t)$  are statistically independent, zero-mean complex Gaussian random processes with rational spectra. The signal components on the two hypotheses are in disjoint frequency bands.

1. Find the optimum receiver.

2. How is the receiver simplified if f(t) and  $\lambda_i$  are such that the path outputs are disjoint in time (resolvable multipath)?

**Problem 11.3.12.** Consider the detection problem described in (30)–(32). Assume that we use a gated correlator-squarer-summer-receiver of the type shown in Fig. 11.18.

1. Modify the results of Chapter 5 to obtain formulas that can be used to evaluate suboptimum bandpass receivers.

2. Use the results of part 1 to obtain performance expressions for the above receiver.

#### P.11.4 Parameter Estimation

**Problem 11.4.1.** Consider the estimation problem described in (168)-(175). Assume that the LEC condition is valid.

- 1. Verify that the result in (178) is correct.
- 2. Evaluate  $\theta_{\Omega_p, \text{LEC}}\{0, 0\}$ .
- 3. Prove

$$\theta_{\Omega_p, \text{LEC}}\{\lambda, m\} \leq \theta_{\Omega_p, \text{LEC}}\{0, 0\}.$$

4. Is there a volume invariance relation for  $\theta_{\Omega_n, \text{LEC}}\{\lambda, m\}$ ?

Problem 11.4.2. Assume

$$\tilde{f}(t) = \left(\frac{1}{\pi T^2}\right)^{\frac{1}{4}} e^{-t^2/2T^2}, \quad -\infty < t < \infty$$

and

$$\tilde{S}_{D_0}\{f\} = \frac{2\sigma_b^2}{\sqrt{2\pi}\sigma_D} e^{-f^2/2\sigma_D^2}, \quad -\infty < f < \infty.$$

Evaluate  $\theta_{\Omega_n, \text{LEC}}\{\lambda, m\}$ .

Problem 11.4.3. Consider the LEC estimation problem discussed in Problem 11.4.1.

- 1. Derive an expression for the elements of the J matrix in terms of  $\theta_{\Omega_p, \text{LEC}}(\tau, m)$ .
- 2. Evaluate the J matrix for the signal and scattering function in Problem 11.4.2.

**Problem 11.4.4.** Go through the list of properties in Section 10.3 and see which ones can be generalized to the spread-ambiguity function,  $\theta_{\Omega_{n}, \text{LEC}}\{\tau, m\}$ .

**Problem 11.4.5.** Assume that we are trying to detect a Doppler-spread target in the presence of white noise and have designed the optimum LEC receiver.

1. In addition to the desired target, there is a second Doppler-spread target with an identical scattering function. Evaluate the effect of the second target in terms of  $\theta_{\Omega_D, \text{LEC}}(\lambda, m)$ . (Notice that the receiver is not changed from the original design.)

 $\tilde{2}$ . Extend the result to K interfering targets with identical scattering functions.

3. What modifications must be made if the Doppler scattering functions are not identical? (This is the *spread cross-ambiguity function.*)

4. In part 3, we encountered a spread cross-ambiguity function. A more general definition is

$$\psi_{\Omega}\{\lambda,f\} \stackrel{\Delta}{=} \frac{1}{N_0} \int_{-\infty}^{\infty} \tilde{g}\left(t - \frac{\lambda}{2}\right) \tilde{f}^*\left(t - \frac{\lambda}{2}\right) e^{j2\pi ft} \tilde{K}_g(t - u) \tilde{K}_{D_0}^*(t - u) e^{-j2\pi ft} \\ \times \tilde{g}^*\left(u - \frac{\lambda}{2}\right) \tilde{f}\left(u + \frac{\lambda}{2}\right) dt du. \quad (P.1)$$

Where would this function be encountered? How is it related to the ordinary crossambiguity function  $\theta_{fa}\{\lambda, f\}$ ?

**Problem 11.4.6.** Consider the degenerate case of Problem 11.4.5, in which we are trying to detect a slowly fluctuating point target in the presence of white noise and have designed the optimum receiver.

What effect will the presence of a set of a set of Doppler-spread targets have on the performance of the above receiver.?

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Problem 11.4.7. Derive the term in (175) and the other elements in the J matrix.

**Problem 11.4.8.** Consider the problem of estimating the amplitude of a scattering function. Thus,

$$\tilde{K}_D(\tau; A) = A\tilde{K}_D(\tau), \tag{P.1}$$

and  $\tilde{K}_D(\tau)$  is assumed to be known. The complex envelope of the transmitted signal is  $\sqrt{E_t} \tilde{f}(t)$ . The complex envelope of the returned waveform is

$$\widetilde{r}(t) = \sqrt{E_t} \, \widetilde{b}_D(t, A) \, \widetilde{f}(t) + \widetilde{w}(t), \qquad T_i \le t \le T_f,$$

where  $\tilde{b}_D(t, A)$  is a complex Gaussian random process whose covariance is given in (P.1). Assume that the LEC condition is valid.

- 1. Find a receiver to generate  $\hat{a}_{ml}$ .
- 2. Is  $\hat{a}_{ml}$  unbiased?
- 3. Assume that the bias of  $\hat{a}_{ml}$  is negligible. (How could you check this?) Calculate

$$E[(\hat{a}_{ml} - A)^2].$$

- 4. Calculate a bound on the normalized variance of any unbiased estimate of A.
- 5. Express the bound in part 4 in terms of  $\theta\{\tau, f\}$  and  $\tilde{S}_D\{f\}$ .
- 6. Assume that

$$\tilde{f}(t) = \left(\frac{1}{\pi T^2}\right)^{\frac{1}{4}} e^{-t^2/2T^2}$$

and

$$\tilde{S}_D\{f\} = rac{1}{\sqrt{2\pi B}} e^{-f^2/2B^2}$$

Evaluate the bound in part 4 for this case. Discuss the behavior as a function of BT. Would this behavior be the same if the LEC condition were not satisfied?

7. Express the largest eigenvalue in terms of A, B, and T.

Problem 11.4.9. The complex envelope of the received waveform is

$$\tilde{r}(t) = \sqrt{E_t} \tilde{f}(t) [e^{j\omega_1 t} + e^{j\omega_0 t}] \tilde{b}_D(t) + \tilde{w}(t), \qquad -\infty < t < \infty.$$

We want to estimate the quantity  $\omega_{\Delta} = \omega_1 - \omega_0$ . The process  $\bar{b}_D(t)$  is a zero-mean complex Gaussian process whose bandwidth is much less than  $\omega_{\Delta}$ .

- 1. Find a receiver to generate the maximum likelihood estimate of  $\omega_{\Delta}$ .
- 2. Find an expression for the Cramér-Rao bound.

Problem 11.4.10. Assume that

$$\tilde{S}_D\{f:A\} = \tilde{S}_{D_1}\left\{\frac{f}{A}\right\},\,$$

where  $\tilde{S}_{D_1}\{\cdot\}$  is known. We want to estimate A, the scale of the frequency axis. Assume that

$$\widetilde{r}(t) = \sqrt{E_t} \widetilde{b}_D(t, A) \,\widetilde{f}(t) + \widetilde{w}(t), \qquad -\infty < t < \infty,$$

and that the LEC condition is valid.

- 1. Draw a block diagram of a receiver to generate  $\hat{a}_{ml}$ .
- 2. Evaluate the Cramér-Rao bound.

**Problem 11.4.11.** Assume that the target consists of two reflectors at different ranges. The complex envelope of the returned waveform is

$$\widetilde{r}(t) = \sqrt{E_t} \sum_{i=1}^{2} \widetilde{b}_i \widetilde{f}(t - \lambda_i) + \widetilde{w}(t), \quad -\infty < t < \infty,$$

where the  $\tilde{b}_i$  are statistically independent complex Gaussian random variables  $(E |\tilde{b}_i|^2 = 2\sigma_i^2)$ . We want to estimate the mean range, which we define as

$$\lambda_r = \frac{1}{2}(\lambda_1 + \lambda_2).$$

- 1. Draw the block diagram of a receiver to generate  $\hat{\lambda}_{r,ml}$ .
- 2. Does

$$\hat{\lambda}_{r,ml} = \frac{1}{2}(\hat{\lambda}_{1,ml} + \hat{\lambda}_{2,ml})?$$

3. Evaluate the Cramér-Rao bound of the variance of the estimate.

**Problem 11.4.12.** Consider the problem of estimating the range and mean Doppler when the amplitude of the scattering function is unknown. Thus,

$$\tilde{K}_{\tilde{s}}(t, u: \mathbf{A}) = A E_t \tilde{f}(t-\lambda) e^{j2\pi m t} K_{D_0}(t-u) e^{-j2\pi m u} \tilde{f}^*(u-\lambda).$$

Assume that the LEC condition is valid and that the bias on  $\hat{a}_{ml}$  can be ignored.

1. Find  $l(\hat{a}_{ml}, \lambda, m)$ .

2. Draw a block diagram of the optimum receiver to generate  $\hat{\lambda}_{ml}$ ,  $\hat{m}_{ml}$ .

3. Evaluate J. Does the fact that A is unknown increase the bounds on the variances of  $\hat{\lambda}_{ml}$  and  $\hat{m}_{ml}$ ?

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