

# 12

## *Range-Spread Targets and Channels*

In Chapters 9 and 10, we studied slowly fluctuating point targets. In Chapter 11, we studied point targets that could fluctuate at an arbitrary rate. In this chapter, we consider slowly fluctuating targets that are spread in range.

A typical case is shown in Fig. 12.1. We transmit a short pulse as shown in Fig. 12.1*a*. The target configuration is shown in Fig. 12.1*b*. The surface is rough, so that energy is reflected in the direction of the receiver. The target has length  $L$  (measured in seconds of travel time). To characterize the reflected signal, we divide the target in  $\Delta\lambda$  increments. The return from each increment is a superposition of a number of reflections, and so we can characterize it as a complex Gaussian random variable. Thus the return from the first increment is

$$\sqrt{E_t} \tilde{b}(\lambda_0) \tilde{f}(t - \lambda_0) \Delta\lambda, \tag{1}$$

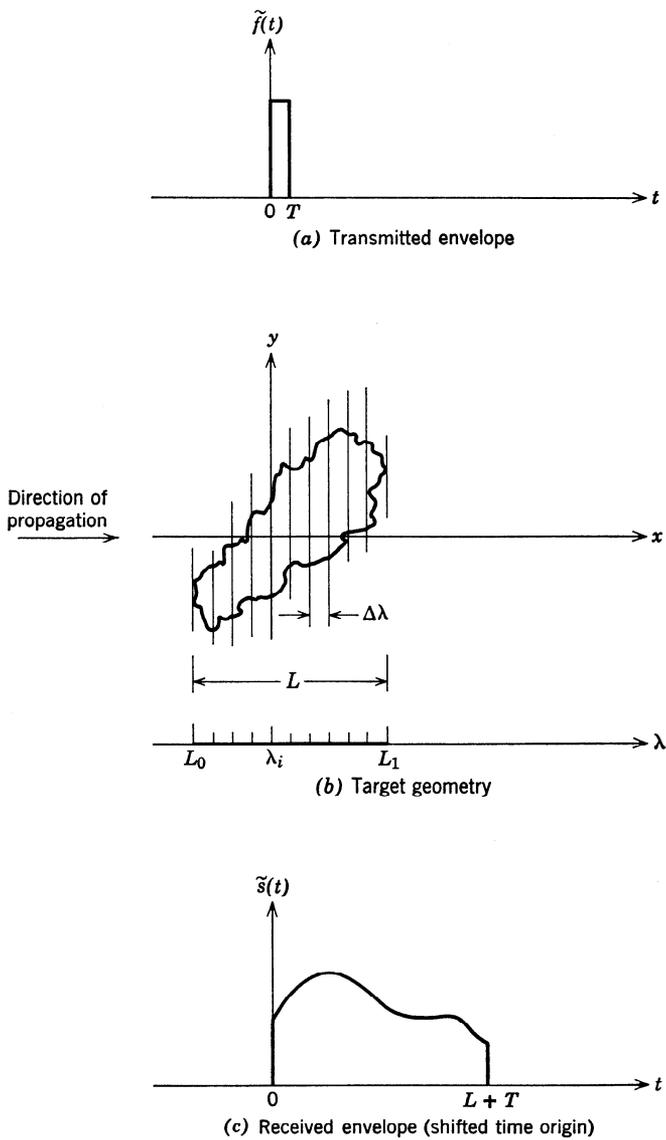
the return from the second increment is

$$\sqrt{E_t} \tilde{b}(\lambda_1) \tilde{f}(t - \lambda_1) \Delta\lambda, \tag{2}$$

and so forth. The total return is

$$\tilde{s}(t) = \sqrt{E_t} \sum_{i=0}^N \tilde{b}(\lambda_i) \tilde{f}(t - \lambda_i) \Delta\lambda. \tag{3}$$

We see that it consists of delayed versions of the signal, which are weighted with complex Gaussian variables and summed. A typical returned signal is shown in Fig. 12.1*c*. We see that the signal is spread out in time (or range), and so we refer to this type of target as a *range-spread target*. Other adjectives commonly used are *delay-spread* and *dispersive*.



**Fig. 12.1 Range-spread model.**

In this chapter we study detection and parameter estimation for range-spread targets. In Section 12.1 we develop a quantitative model for range-spread targets and channels and show how this type of target causes *frequency-selective fading*. In Section 12.2, we discuss optimum receiver configurations briefly. In Section 12.3, we develop the concept of time-frequency duality. This development enables us to translate all range-spread channels into equivalent Doppler-spread channels. We can then use all of the results in Chapter 11 directly. We also discuss a number of applications in Section 12.3. Finally, in Section 12.4, we summarize our results.

### 12.1 MODEL AND INTUITIVE DISCUSSION

We begin our model development with the relation in (3). The increments are useful for explanatory purposes, but the reflections actually occur from a continuous range of  $\lambda$ . As  $\Delta\lambda \rightarrow 0$ , the sum in (3) becomes the integral

$$\tilde{s}(t) = \sqrt{E_t} \int_{L_0}^{L_1} \tilde{f}(t - \lambda) \tilde{b}_R(\lambda) d\lambda. \tag{4}$$

Now  $\tilde{b}_R(\lambda)$  is a sample function from a zero-mean complex Gaussian process whose independent variable is the spatial variable  $\lambda$ . Notice that  $\tilde{b}_R(\lambda)$  is *not* a function of time. We see that a range-spread target behaves exactly as a linear time-invariant filter with a random complex impulse response  $\tilde{b}_R(\lambda)$ . To characterize  $\tilde{b}_R(\lambda)$  completely, we need the two complex covariance functions

$$\tilde{K}_{\tilde{b}_R}(\lambda, \lambda_1) = E[\tilde{b}_R(\lambda) \tilde{b}_R^*(\lambda_1)] \tag{5}$$

and

$$E[\tilde{b}_R(\lambda) \tilde{b}_R(\lambda_1)] = 0, \quad \text{for all } \lambda, \lambda_1, \tag{6}$$

where the result in (6) is a restriction we impose.

We shall assume that the returns from different ranges are statistically independent. To justify this assumption, we return to the incremental model in Fig. 12.1. The value of  $\tilde{b}_R(\lambda_i)$  will be determined by the relative phases and strengths of the component reflections in the  $i$ th interval. Assuming that the surface is rough compared to the carrier wavelength, the values of  $\tilde{b}_R(\lambda_i)$  in different intervals will not be related. In the continuous model this implies that

$$\tilde{K}_{\tilde{b}_R}(\lambda, \lambda_1) = \delta(\lambda - \lambda_1) E\{|\tilde{b}_R(\lambda)|^2\}. \tag{7}$$

Notice that the relation in (7) is an idealization analogous to white noise in the time domain. The reflected signal is given by the convolution in (4).

As long as the correlation distance of  $\tilde{b}_R(\lambda)$  is much shorter than the reciprocal of the bandwidth of  $\tilde{f}(t)$ , then (7) will be a good approximation.

Physically, the expectation in (7) is related to the expected value of energy returned (or scattered) from an incremental element located at  $\lambda$ . We define

$$\tilde{S}_R(\lambda) \triangleq E\{|\tilde{b}(\lambda)|^2\}, \quad -\infty < \lambda < \infty \quad (8)$$

and refer to it as the *range-scattering function*. For convenience, we shall always define  $\tilde{S}_R(\lambda)$  for an infinite range. The finite target length will be incorporated in the functional definition.

The covariance of the received signal in the absence of additive noise is

$$\begin{aligned} \tilde{K}_s(t, u) &= E[\tilde{s}_r(t)\tilde{s}_r^*(u)] \\ &= E\left\{E_t \int_{\Omega_L} d\lambda \tilde{f}(t - \lambda)\tilde{b}_R(\lambda) \int_{\Omega_L} d\lambda_1 \tilde{f}^*(u - \lambda_1)\tilde{b}_R^*(\lambda_1)\right\}. \end{aligned} \quad (9)$$

Using (7) and (8) in (9) gives

$$\tilde{K}_s(t, u) = E_t \int_{-\infty}^{\infty} \tilde{f}(t - \lambda)\tilde{S}_R(\lambda)\tilde{f}^*(u - \lambda) d\lambda. \quad (10)$$

The relation in (10) completely characterizes the signal returned from a range-spread target.

Notice that the total received energy is

$$\bar{E}_r = \int_{-\infty}^{\infty} \tilde{K}_s(t, t) dt = E_t \int_{-\infty}^{\infty} \tilde{S}_R(\lambda) d\lambda \int_{-\infty}^{\infty} |\tilde{f}(t - \lambda)|^2 dt = E_t \int_{-\infty}^{\infty} \tilde{S}_R(\lambda) d\lambda. \quad (11)$$

We see that

$$\tilde{S}_R(\lambda) d\lambda = \frac{\text{expected value of the energy returned from } (\lambda, \lambda + d\lambda)}{E_t}. \quad (12)$$

The result in (12) is a quantitative statement of the idea expressed in (8). In order to be consistent with the point target model, we assume that

$$\int_{-\infty}^{\infty} \tilde{S}_R(\lambda) d\lambda = 2\sigma_b^2. \quad (13)$$

This completes our specification of the reflection model. Before beginning our optimum receiver development, it is useful to spend some time on an intuitive discussion. In Chapter 11 we saw that a Doppler-spread target causes time-selective fading. Now we want to demonstrate that a range-spread target causes frequency-selective fading.

The Fourier transform of  $\tilde{s}(t)$  is a well-defined quantity when the target length is finite. Thus,

$$\begin{aligned} \tilde{S}\{f\} &\triangleq \int_{-\infty}^{\infty} \tilde{s}(t)e^{-j2\pi f_1 t} dt \\ &= \int_{-\infty}^{\infty} e^{-j2\pi f_1 t} dt \int_{-\infty}^{\infty} \tilde{f}(t - \lambda)\tilde{b}_R(\lambda) d\lambda. \end{aligned} \tag{14}$$

Notice that  $\tilde{S}\{f\}$  is a sample function of a complex Gaussian process.

We want to compute the cross-correlation between  $\tilde{S}\{f\}$  at two different frequencies.

$$\begin{aligned} E[\tilde{S}\{f_1\}\tilde{S}^*\{f_2\}] &= E\left\{ \int_{-\infty}^{\infty} e^{-j2\pi f_1 t_1} dt_1 \int_{-\infty}^{\infty} \tilde{f}(t_1 - \lambda_1)\tilde{b}_R(\lambda_1) d\lambda_1 \right. \\ &\quad \left. \times \int_{-\infty}^{\infty} e^{j2\pi f_2 t_2} dt_2 \int_{-\infty}^{\infty} \tilde{f}^*(t_2 - \lambda_2)\tilde{b}_R^*(\lambda_2) d\lambda_2 \right\}. \end{aligned} \tag{15}$$

Bringing the expectation inside the integrals, using (7) and (8), we obtain

$$E[\tilde{S}\{f_1\}\tilde{S}^*\{f_2\}] = \tilde{F}\{f_1\}\tilde{F}^*\{f_2\} \int_{-\infty}^{\infty} e^{-j2\pi\lambda(f_1-f_2)}\tilde{S}_R(\lambda) d\lambda, \tag{16}$$

where  $\tilde{F}\{f_1\}$  is the Fourier transform of  $\tilde{f}(t)$ . To interpret (16), we define

$$\tilde{K}_R\{v\} \triangleq \int_{-\infty}^{\infty} e^{-j2\pi\lambda v}\tilde{S}_R(\lambda) d\lambda. \tag{17}$$

Using (17) in (16), we obtain

$$E[\tilde{S}\{f_1\}\tilde{S}^*\{f_2\}] = \tilde{F}\{f_1\}\tilde{F}^*\{f_2\}\tilde{K}_R\{f_1 - f_2\}, \tag{18}$$

or

$$\tilde{K}_R\{f_1 - f_2\} = \frac{E[\tilde{S}\{f_1\}\tilde{S}^*\{f_2\}]}{\tilde{F}\{f_1\}\tilde{F}^*\{f_2\}}. \tag{19}$$

The function  $\tilde{K}_R\{v\}$  is called the *two-frequency correlation function*. It measures the correlation between the fading at different frequencies. Notice that it is the Fourier transform of the range-scattering function. Therefore,

$$\tilde{S}_R(\lambda) = \int_{-\infty}^{\infty} e^{j2\pi\lambda v}\tilde{K}_R\{v\} dv, \tag{20}$$

and we can use either  $\tilde{K}_R\{v\}$  or  $\tilde{S}_R(\lambda)$  to characterize the target.

To illustrate the implication of the result in (18), we consider the scattering function in Fig. 12.2a,

$$\tilde{S}_R(\lambda) = \begin{cases} \frac{2\sigma_b^2}{L}, & -\frac{L}{2} \leq \lambda \leq \frac{L}{2}, \\ 0, & \text{elsewhere.} \end{cases} \tag{21}$$

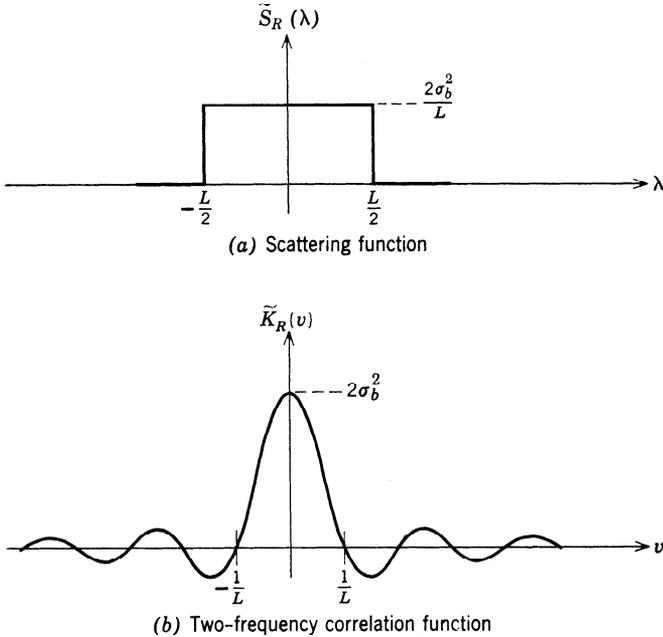


Fig. 12.2 Functions for a uniform range-spread target.

Thus,

$$\tilde{K}_R\{v\} = 2\sigma_b^2 \frac{\sin \pi Lv}{\pi Lv}, \quad -\infty < v < \infty, \quad (22)$$

as shown in Fig. 12.2b. We see that frequency components separated by more than  $1/L$  cps will be essentially uncorrelated (and statistically independent, because they are jointly Gaussian).

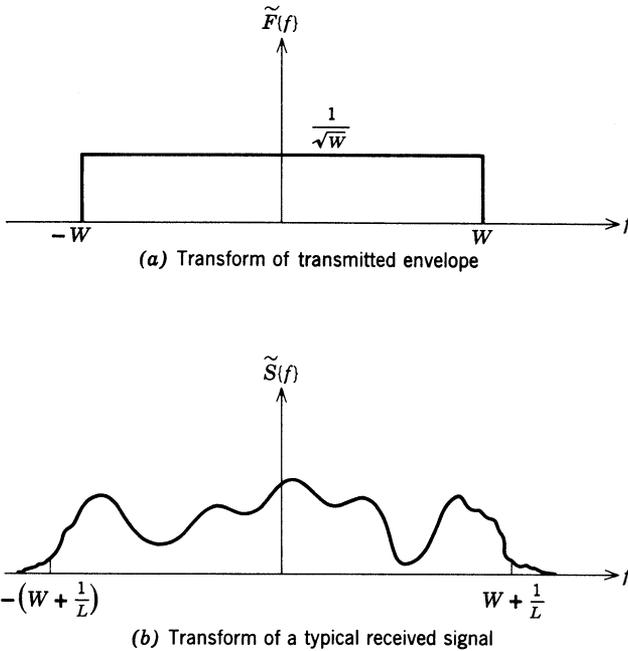
Now assume that we transmit a signal whose Fourier transform is

$$\tilde{F}\{f\} = \begin{cases} \frac{1}{\sqrt{W}}, & -\frac{W}{2} \leq f \leq \frac{W}{2}, \\ 0, & \text{elsewhere.} \end{cases} \quad (23)$$

In Fig. 12.3a, we show the case in which

$$W \gg \frac{1}{L}. \quad (24)$$

In Fig. 12.3b, we show the transform of a typical sample function of  $\tilde{s}(t)$ . The amplitudes at frequencies separated by more than  $1/L$  cps are essentially statistically independent, and so we refer to this behavior as frequency-selective fading.



**Fig. 12.3 Functions to illustrate frequency-selective fading.**

The function in Fig. 12.3b is very similar to that in Fig. 11.2b, except that the axis is frequency instead of time. We shall exploit this similarity (or duality) in detail in Section 12.3.

Notice that if the signal bandwidth is such that

$$W \ll \frac{1}{L}, \tag{25}$$

the returned signal will be undistorted. This is, of course, the slowly fluctuating point target model of Chapters 9 and 10. The relation in (25) tells us when we can model the target as a point target.

We now have a quantitative model for range-spread targets and an intuitive understanding of how they affect the transmitted signal. The next step is to consider the problem of optimum receiver design.

## 12.2 DETECTION OF RANGE-SPREAD TARGETS

In this section we consider the binary detection problem, in which the complex envelopes of the received waveforms on the two hypotheses are

$$\tilde{r}(t) = \tilde{s}(t) + \tilde{w}(t), \quad -\infty < t < \infty: H_1, \tag{26}$$

and

$$\tilde{r}(t) = \tilde{w}(t), \quad -\infty < t < \infty : H_0. \quad (27)$$

The signal is a sample function from a zero-mean complex Gaussian process,

$$\tilde{s}(t) = \sqrt{E_t} \int_{-\infty}^{\infty} \tilde{f}(t - \lambda) \tilde{b}_R(\lambda) d\lambda, \quad (28)$$

whose covariance function is

$$\tilde{K}_s(t, u) = E_t \int_{-\infty}^{\infty} \tilde{f}(t - \lambda) \tilde{S}_R(\lambda) \tilde{f}^*(u - \lambda) d\lambda. \quad (29)$$

The additive noise,  $\tilde{w}(t)$ , is a sample function from a statistically independent, zero-mean, complex white Gaussian process with spectral height  $N_0$ . We have assumed an infinite observation interval for simplicity.

The expression for the optimum test follows directly from (11.33) as

$$l = \frac{1}{N_0} \iint_{-\infty}^{\infty} \tilde{r}^*(t) \tilde{h}(t, u) \tilde{r}(u) dt du \underset{H_0}{\overset{H_1}{\geq}} \gamma, \quad (30)$$

where  $\tilde{h}(t, u)$  satisfies the equation

$$N_0 \tilde{h}(t, u) + \int_{-\infty}^{\infty} \tilde{h}(t, z) \tilde{K}_s(z, u) dz = \tilde{K}_s(t, u), \quad -\infty < t < \infty. \quad (31)$$

The difficulty arises in solving (31). There are two cases in which the solution is straightforward, the separable kernel case and the low-energy-coherence case. The separable kernel analysis is obvious, and so we relegate it to the problems. The LEC condition leads to an interesting receiver configuration, however, and so we discuss it briefly.

When the LEC condition is valid, the solution to (31) may be written as

$$\tilde{h}(t, u) = \frac{1}{N_0} \tilde{K}_s(t, u). \quad (32)$$

Using (29) in (32) and the result in (30) gives

$$l = \frac{E_t}{N_0^2} \iiint_{-\infty}^{\infty} \tilde{r}^*(t) \tilde{f}(t - \lambda) \tilde{S}_R(\lambda) \tilde{f}^*(u - \lambda) \tilde{r}(u) dt du d\lambda \underset{H_1}{\overset{H_1}{\geq}} \gamma. \quad (33)$$

This can be rewritten as

$$l_1 = \int_{-\infty}^{\infty} \tilde{S}_R(\lambda) |\tilde{x}(\lambda)|^2 d\lambda \underset{H_0}{\overset{H_1}{\geq}} \gamma_1, \quad (34a)$$

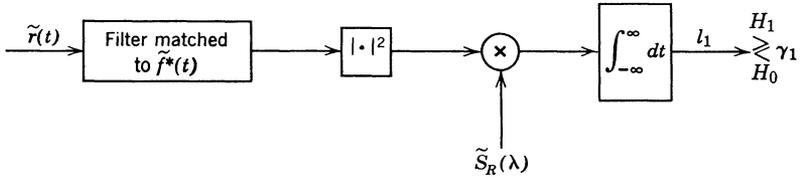


Fig. 12.4 Two-filter radiometer: optimum detector under the LEC condition.

where

$$\tilde{x}(\lambda) \triangleq \int_{-\infty}^{\infty} \tilde{r}(u) \tilde{f}^*(u - \lambda) du, \tag{34b}$$

and we have absorbed the constant in the threshold. The operation in (34a) may be realized as shown in Fig. 12.4. This receiver is called a two-filter radiometer and is due to Price [1].

When the LEC condition is valid, the performance is specified by (11.65). Using (10) in (11.65) and integrating gives

$$\tilde{\mu}(s) = - \frac{s(1-s)E_t^2}{N_0^2} \iint_{-\infty}^{\infty} \tilde{S}_R(\lambda_1) \theta\{\lambda_1 - \lambda_2, 0\} \tilde{S}_R(\lambda_2) d\lambda_1 d\lambda_2, \tag{35}$$

which is the desired result.

When the LEC condition is not valid, it is difficult to solve (31) directly. In the next section we develop a procedure for solving it by transforming it into an equivalent Doppler-spread target problem.

### 12.3 TIME-FREQUENCY DUALITY

The utility of the duality concept is well known in classical network theory. Bello [2] has developed the concept of time-frequency duality in a more general framework and applied it to communications problems. In Section 12.3.1, we develop the basic duality concepts. In Section 12.3.2, we consider the dual relations in range-spread and Doppler-spread channels. In Section 12.3.3, we apply the results to specific cases.

One comment is worthwhile before beginning our development. We shall develop a number of properties and formal relationships. These are useful in solving specific problems. The reader who only learns these properties and applies them blindly will miss what we think is a major benefit of duality. This benefit is the guidance it offers in thinking about a particular problem. Often by just thinking about the duality, one can solve a problem directly without going through the formal manipulations.

### 12.3.1 Basic Duality Concepts

Our discussion consists of a series of definitions and properties with some examples interspersed to illustrate the ideas. Notice that throughout the discussion all functions are defined over infinite intervals.

**Definition 1.** Consider the two complex time functions  $\tilde{y}_1(t)$  and  $\tilde{y}_2(t)$ . If

$$\tilde{y}_2(f) = \mathcal{F}[\tilde{y}_1(t)] \triangleq \tilde{Y}_1\{f\} \triangleq \int_{-\infty}^{\infty} \tilde{y}_1(t)e^{-j2\pi ft} dt, \quad (36)$$

then  $\tilde{y}_2(t)$  is the *dual* of  $\tilde{y}_1(t)$ . If

$$\tilde{y}_2(t) = \mathcal{F}^{-1}[\tilde{y}_1(f)] \triangleq \int_{-\infty}^{\infty} \tilde{y}_1(f)e^{j2\pi ft} df, \quad (37)$$

then  $\tilde{y}_2(t)$  is the *inverse dual* of  $\tilde{y}_1(t)$ .

**Example 1.** Let

$$\tilde{y}_1(t) = \begin{cases} 1, & -T \leq t \leq T, \\ 0, & \text{elsewhere.} \end{cases} \quad (38)$$

The dual of  $\tilde{y}_1(t)$  is

$$\tilde{y}_2(t) = \frac{\sin 2\pi Tt}{\pi t}, \quad -\infty < t < \infty. \quad (39)$$

**Definition 2. Dual Processes.** The complex Gaussian process  $\tilde{y}_2(t)$  is the *statistical dual* of the complex Gaussian process  $\tilde{y}_1(t)$  if

$$\begin{aligned} \tilde{K}_{\tilde{y}_2}\{f_1, f_2\} &\triangleq E[\tilde{y}_2(f_1)\tilde{y}_2^*(f_2)] \\ &= \mathcal{F}[\tilde{K}_{\tilde{y}_1}(t_1, t_2)] \\ &\triangleq \iint_{-\infty}^{\infty} \exp[-j2\pi f_1 t_1 + j2\pi f_2 t_2] \tilde{K}_{\tilde{y}_1}(t_1, t_2) dt_1 dt_2. \end{aligned} \quad (40)$$

Note the sign convention in the direct Fourier transform. The complex Gaussian process  $\tilde{y}_2(t)$  is the *statistical inverse dual* of the complex Gaussian process  $\tilde{y}_1(t)$  if

$$\begin{aligned} \tilde{K}_{\tilde{y}_2}(t_1, t_2) &= \mathcal{F}^{-1}[\tilde{K}_{\tilde{y}_1}\{f_1, f_2\}] \\ &\triangleq \iint_{-\infty}^{\infty} \exp[+j2\pi f_1 t_1 - j2\pi f_2 t_2] \tilde{K}_{\tilde{y}_1}\{f_1, f_2\} df_1 df_2. \end{aligned} \quad (41)$$

**Property 2 [3].** Assume that  $\tilde{y}_2(t)$  is the statistical dual of  $\tilde{y}_1(t)$ , which is a *nonstationary* process whose expected energy is finite. We expand both

processes over the infinite interval. The eigenvalues of  $\tilde{y}_2(t)$  are identical with those of  $\tilde{y}_1(t)$ , and the eigenfunctions of  $\tilde{y}_2(t)$  are Fourier transforms of the eigenfunctions of  $\tilde{y}_1(t)$ . This property follows by direct substitution.

**Example 2.** Let

$$\tilde{K}_{\tilde{y}_1}(t_1, t_2) = \sum_{i=1}^{\infty} \tilde{\lambda}_i \tilde{\varphi}_i(t_1) \tilde{\varphi}_i^*(t_2), \quad -\infty < t_1, t_2 < \infty. \quad (42)$$

The expansion of the dual process is

$$\tilde{K}_{\tilde{y}_2}\{f_1, f_2\} = \sum_{i=1}^{\infty} \tilde{\lambda}_i \tilde{\Phi}_i\{f_1\} \tilde{\Phi}_i^*\{f_2\}, \quad -\infty < f_1, f_2 < \infty. \quad (43)$$

At this point the reader should see why we are interested in dual processes. The performance of detection and estimation systems depends on eigenvalues, *not* eigenfunctions. Thus, systems in which the various processes are dual will perform in an identical manner.

**Property 3.** White complex Gaussian noise is a statistically self-dual process.

**Property 4.** If  $\tilde{y}_2(t)$  is the dual of  $\tilde{y}_1(t)$ , where  $\tilde{y}_1(t)$  is any sample function from a zero-mean random process, then  $\tilde{K}_{\tilde{y}_2}\{f_1, f_2\}$  is the double Fourier transform of  $K_{\tilde{y}_1}(t_1, t_2)$ .

**Definition 3.** Consider the two deterministic functionals

$$\tilde{z}_1(t) = g_1(\tilde{y}_1(\cdot), t) \quad (44)$$

and

$$\tilde{z}_2(t) = g_2(\tilde{y}_2(\cdot), t). \quad (45)$$

Assume that  $\tilde{y}_2(t)$  is the dual of  $\tilde{y}_1(t)$ . If this always implies that  $\tilde{z}_2(t)$  is the dual of  $\tilde{z}_1(t)$ , then  $g_2(\cdot, \cdot)$  is the *dual operation* of  $g_1(\cdot, \cdot)$ .

To illustrate this idea, we consider a simple example of a dual operation.

**Example 3.** Let  $g_1(\cdot, \cdot)$  correspond to a delay line with a delay of  $a$  seconds. Thus,

$$\tilde{z}_1(t) = \tilde{y}_1(t - a). \quad (46)$$

The dual operation is the frequency translation

$$\tilde{z}_2(t) = \tilde{y}_2(t) e^{-j2\pi at}. \quad (47)$$

To verify this, we observe that

$$\tilde{Z}_1\{f\} = \tilde{Y}_1\{f\} e^{-j2\pi fa}. \quad (48)$$

**Property 5.** The operations listed in Table 12.1 are duals (see Problems 12.3.4–12.3.10).

**Table 12.1**

Operation	Dual operation
Delay line	Frequency translation
Time-varying gain	Filter
Gate	Low-pass or bandpass filter
Adder	Adder
Convolution	Multiplier
Aperiodic correlator	Square-law envelope detector

Thus, if

$$\tilde{y}_2(t) = \tilde{Y}_1\{t\}, \tag{49}$$

then

$$\tilde{z}_2(t) = \tilde{Z}_1\{t\}, \tag{50}$$

which is the required result.

**Property 6.** Assume that the input to  $g_1(\tilde{y}_1(\cdot), t)$  is a sample function of a complex Gaussian random process and that the input to  $g_2(\tilde{y}_2(\cdot), t)$  is a sample function of a dual process. If  $g_1(\cdot, \cdot)$  is the dual operation of  $g_1(\cdot, \cdot)$ , then  $\tilde{z}_2(t)$  is the dual process of  $\tilde{z}_1(t)$ .

This completes our introductory discussion. We now turn to the specific problem of interest.

### 12.3.2 Dual Targets and Channels

In this section we introduce the idea of a dual target or channel and demonstrate that a nonfluctuating dispersive target is the dual of a fluctuating point target.

To motivate the definition, we recall the relations for the Doppler-spread and range-spread targets. The reflected signal from a Doppler-spread target at zero range is a signal

$$\tilde{s}_D(t) = \sqrt{E_t} \tilde{b}_D(t) \tilde{f}_D(t), \quad -\infty < t < \infty, \tag{51}$$

whose covariance function is

$$\tilde{K}_{\tilde{s}_D}(t, u) = E_t \tilde{f}_D(t) \tilde{K}_D(t - u) \tilde{f}_D^*(u), \quad -\infty < t, u < \infty. \tag{52}$$

The reflected signal from a range-spread target is a signal

$$\tilde{s}_R(t) = \sqrt{E_t} \int_{-\infty}^{\infty} \tilde{f}_R(t - \lambda) \tilde{b}_R(\lambda) d\lambda, \quad -\infty < t < \infty, \tag{53}$$

whose covariance function is

$$\tilde{K}_{\tilde{s}_R}(t, u) = E_t \int_{-\infty}^{\infty} \tilde{f}_R(t - \lambda) \tilde{S}_R(\lambda) \tilde{f}_R^*(u - \lambda) d\lambda, \quad -\infty < t, u < \infty. \tag{54}$$

We may now define dual targets and channels.

**Definition 4.** Let  $\tilde{f}_1(t)$  denote the transmitted signal in system 1, and  $\tilde{z}_1(t)$  the returned signal. Let  $\tilde{f}_2(t)$  denote the transmitted signal in system 2, and  $\tilde{z}_2(t)$  the returned signal.

If the condition that  $\tilde{f}_2(t)$  is the dual of  $\tilde{f}_1(t)$  implies that  $\tilde{z}_2(t)$  is the statistical dual of  $\tilde{z}_1(t)$ , system 2 is the *dual system* of system 1. (Notice that “systems” have randomness in them whereas the “operations” in Definition 3 were deterministic.)

We now apply this definition to the targets of interest.

**Property 7.** If

$$\tilde{K}_D(\tau) = \tilde{K}_R\{\tau\} \tag{55}$$

or, equivalently,

$$\tilde{S}_D\{\lambda\} = \tilde{S}_R(-\lambda), \tag{56}$$

then the Doppler-spread target (or channel) is a *dual system* with respect to the range-spread target (or channel).

*Proof.* We must prove that

$$\tilde{K}_{\tilde{s}_D}\{f_1, f_2\} = \mathcal{F}[\tilde{K}_{\tilde{s}_R}(t_1, t_2)]. \tag{57}$$

Now

$$\begin{aligned} \mathcal{F}[\tilde{K}_{\tilde{s}_R}(t_1, t_2)] &\triangleq \iint_{-\infty}^{\infty} e^{-j2\pi[f_1 t_1 - f_2 t_2]} \tilde{K}_{\tilde{s}_R}(t_1, t_2) dt_1 dt_2 \\ &= \iiint_{-\infty}^{\infty} e^{-j2\pi[f_1 t_1 - f_2 t_2]} \tilde{f}_R(t_1 - \lambda) \tilde{S}_R(\lambda) \tilde{f}_R^*(t_2 - \lambda) dt_1 dt_2 d\lambda \\ &= \int_{-\infty}^{\infty} \tilde{S}_R(\lambda) e^{-j2\pi\lambda[f_1 - f_2]} d\lambda \int_{-\infty}^{\infty} \tilde{f}_R(t_1 - \lambda) e^{-j2\pi f_1(t_1 - \lambda)} dt_1 \\ &\quad \times \int_{-\infty}^{\infty} \tilde{f}_R(t_2 - \lambda) e^{j2\pi f_2(t_2 - \lambda)} dt_2. \end{aligned} \tag{58}$$

Using (17), this reduces to

$$\mathcal{F}[\tilde{K}_{\tilde{s}_R}(t_1, t_2)] = \tilde{K}_R\{f_1 - f_2\} \tilde{F}_R\{f_1\} \tilde{F}_R^*\{f_2\}. \tag{59}$$

If

$$\tilde{f}_D(\cdot) = \tilde{F}_R\{\cdot\} \tag{60}$$

and

$$\tilde{K}_D(\cdot) = \tilde{K}_R\{\cdot\}, \quad (61)$$

then

$$\tilde{K}_{s_D}\{f_1, f_2\} = \tilde{F}_R\{f_1\}\tilde{K}_R\{f_1 - f_2\}\tilde{F}_R^*\{f_2\}, \quad (62)$$

which is the desired result.

This result is of fundamental importance, because it implies that we can work with the most tractable target model. We formalize this idea with the following definition.

**Definition 5. Dual Detection Problems.** The received waveforms on the two hypotheses in system  $A$  are

$$\tilde{r}_{A1}(t), \quad -\infty < t < \infty : H_1 \quad (63)$$

and

$$\tilde{r}_{A0}(t), \quad -\infty < t < \infty : H_0. \quad (64)$$

The received waveforms on the two hypotheses in system  $B$  are

$$\tilde{r}_{B1}(t), \quad -\infty < t < \infty : H_1 \quad (65)$$

and

$$\tilde{r}_{B0}(t), \quad -\infty < t < \infty : H_0. \quad (66)$$

All waveforms are sample functions of complex Gaussian processes.

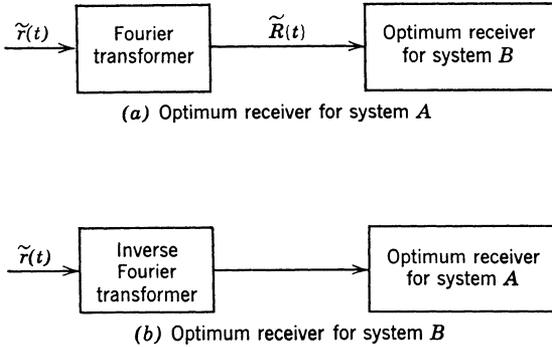
If  $\tilde{r}_{B1}(t)$  is the dual process to  $\tilde{r}_{A1}(t)$  and  $\tilde{r}_{B0}(t)$  is the dual process to  $\tilde{r}_{A0}(t)$ , problem  $B$  is the dual detection problem of problem  $A$ .

The following properties are straightforward to verify.

**Property 8.** If the a-priori probabilities and costs are the same in both systems, the Bayes risks in equal detection problems are identical.

**Property 9.** We can always realize the optimum receiver for system  $A$  as shown in Fig. 12.5a. We can always realize the optimum receiver for system  $B$  as shown in Fig. 12.5b.

Property 9 means that being able to construct the optimum receiver for either one of the two dual systems is adequate. Techniques for implementing the Fourier transformer are discussed in numerous references (e.g., [4–7]). There is some approximation involved in this operation, but we shall ignore it in our discussion. In Section 12.3.3 we shall discuss direct implementations by using dual operations.



**Fig. 12.5** Optimum receivers for dual detection problems.

**Property 10.** Consider the problem of detecting a range-spread target. The received signals on the two hypotheses are

$$\tilde{r}(t) = \tilde{s}_R(t) + \tilde{w}(t), \quad -\infty < t < \infty: H_1 \quad (67)$$

and

$$\tilde{r}(t) = \tilde{w}(t), \quad -\infty < t < \infty: H_0. \quad (68)$$

Consider the problem of detecting a Doppler-spread target. The received signals on the two hypotheses are

$$\tilde{r}(t) = \tilde{s}_D(t) + \tilde{w}(t), \quad -\infty < t < \infty: H_1 \quad (69)$$

and

$$\tilde{r}(t) = \tilde{w}(t), \quad -\infty < t < \infty: H_0. \quad (70)$$

In both cases  $\tilde{w}(t)$  is a sample function from a complex Gaussian white noise process with spectral height  $N_0$ .

If the Doppler-spread target is a dual system to the range-spread target, the second detection problem is the dual of the first detection problem.

This property follows by using Properties 3 and 7 in Definition 5. It is important because it enables us to apply all of the results in Chapter 11 to the range-spread problem.

The result in Definition 5 concerned binary detection. The extension to  $M$ -ary problems and estimation problems is straightforward.

At this point in our discussion we have a number of general results available. In the next section we consider some specific cases.

### 12.3.3 Applications

In this section we apply the results of our duality theory discussion to some specific cases.

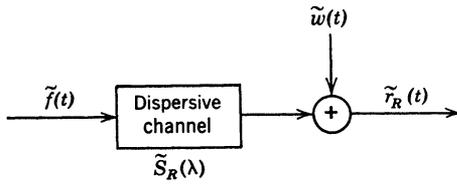
**Case 1. Dual of a Finite-State Doppler-spread Target.** The spectrum of the reflection process for a finite-state Doppler-spread target is the rational function

$$\tilde{S}_D\{f\} = \frac{a_n f^{2n-2} + \dots + a_0}{b_n f^{2n} + \dots + b_0} \quad (71)$$

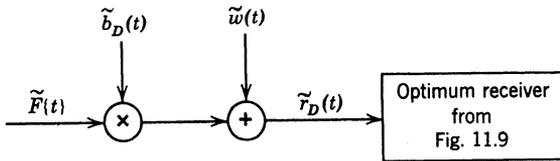
Notice that it is a real, non-negative, not necessarily even function of frequency. To obtain dual systems, the range-scattering function must be the rational function of  $\lambda$ ,

$$\tilde{S}_R(\lambda) = \tilde{S}_D\{-\lambda\} = \frac{a_n(-\lambda)^{2n-2} + \dots + a_0}{b_n(-\lambda)^{2n} + \dots + b_0} \quad (72)$$

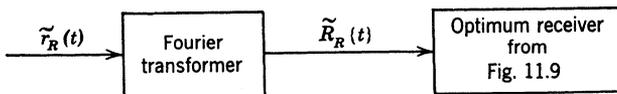
If the transmitted signal for the dispersive target is  $\tilde{f}\{t\}$ , the Doppler-spread target system which is its dual will transmit  $\tilde{F}\{t\}$ . The optimum receiver is shown in Fig. 12.6.



(a) Actual channel



(b) Dual system and optimum receiver



(c) Optimum receiver for dispersive channel

**Fig. 12.6 Finite-state range-spread target.**

To illustrate this case, we consider an example.

**Example 1.** Consider the range-spread target detection problem in which

$$\tilde{S}_R\{\lambda\} = \frac{4k\sigma_b^2}{(2\pi\lambda)^2 + k^2}, \quad -\infty < \lambda < \infty \quad (73)$$

and

$$\tilde{f}_R\{t\} = \begin{cases} \frac{1}{\sqrt{T}}, & -\frac{T}{2} \leq t \leq \frac{T}{2}, \\ 0, & \text{elsewhere.} \end{cases} \quad (74)$$

The dual of this is the Doppler-spread problem in which

$$\tilde{S}_D\{f\} = \frac{4k\sigma_b^2}{(2\pi f)^2 + k^2}, \quad -\infty < f < \infty \quad (75)$$

and

$$\tilde{f}_D\{t\} = \sqrt{T} \frac{\sin \pi T t}{\pi T t}, \quad -\infty < t < \infty. \quad (76)$$

Combining the results in Fig. 12.6 and (11.38)–(11.49) (see also Prob. 11.2.4) gives the optimum receiver in Fig. 12.7. The performance is obtained from the result in Section 11.2.3.

We should observe that the dual of a finite-state Doppler-spread target is a range-spread target that is infinite in extent. This is never true in practice, but frequently we obtain an adequate approximation to  $\tilde{S}_R(\lambda)$  with a rational function.

**Case 2. SPLOT Condition.** In the Doppler-spread case we obtained simple results when

$$\tilde{f}_D(t) = \begin{cases} \frac{1}{\sqrt{T}}, & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0, & \text{elsewhere} \end{cases} \quad (77)$$

and  $T$  was large compared to the correlation time of  $\tilde{b}_D(t)$  as measured by the covariance function  $\tilde{K}_D(\tau)$ . The dual of this case arises when

$$\tilde{F}_R\{f\} = \begin{cases} \frac{1}{\sqrt{W}}, & -\frac{W}{2} \leq f \leq \frac{W}{2} \\ 0, & \text{elsewhere} \end{cases} \quad (78)$$

and  $W$  is large compared to the two-frequency correlation distance as measured by  $K_R\{v\}$ .

A filter-squarer-integrator receiver for the Doppler-spread case is shown in Fig. 12.8. The gating operation is added to take into account the finite

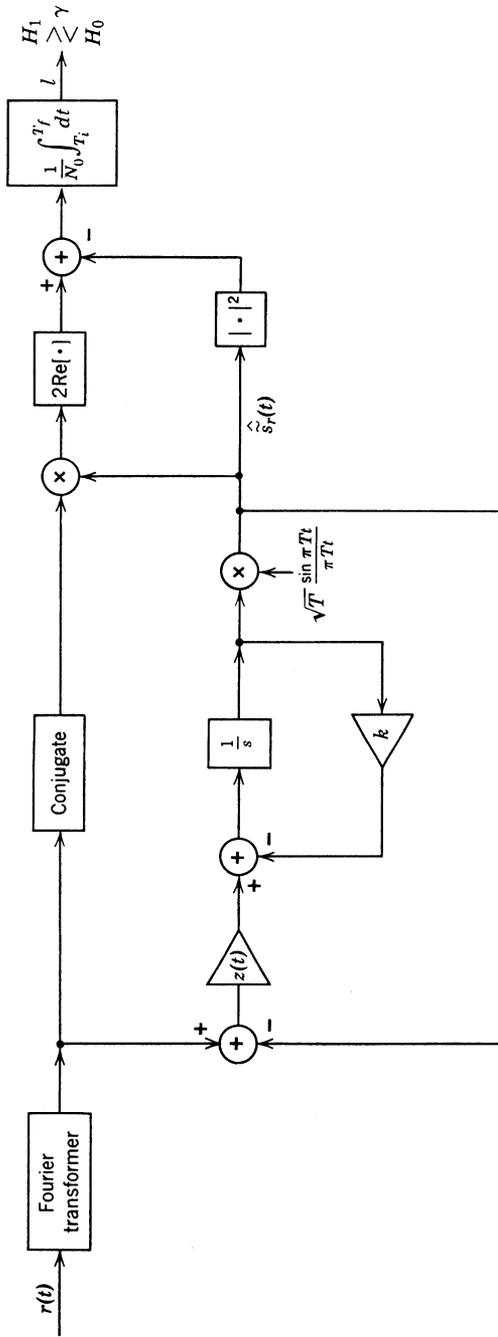


Fig. 12.7 Optimum receiver: finite-state dispersive channel.

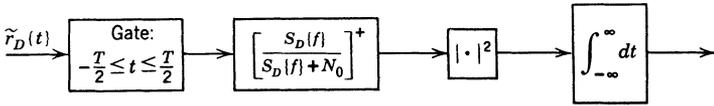


Fig. 12.8 Optimum receiver for Doppler-spread target: SPLIT condition.

observation time. We could implement the optimum receiver by using a Fourier transformer cascaded with the system in Fig. 12.8. In this particular problem it is easier just to implement the inverse dual of Fig. 12.8. Using the properties in Table 12-1, we obtain the system in Fig. 12.9. (We reversed the two zero-memory operations to avoid factoring the spectrum.) Notice that the transmitted signal is

$$\hat{f}(t) = \sqrt{W} \frac{\sin \pi W t}{\pi W t}, \quad -\infty < t < \infty. \tag{79}$$

This pulse will never be used exactly. However, if the transmitted pulse has a transform that is relatively flat over a frequency band, the receiver in Fig. 12.9 should be close to optimum.

**Case 3. LEC Condition.** When the LEC condition is valid, we can solve the problem directly for either the range-spread or Doppler-spread target. In Fig. 12.10 we show the two receivers. It is easy to verify that they are duals.

**Case 4. Resolvable Multipath.** The resolvable multipath problem corresponds to a scattering function,

$$\tilde{S}_R\{\lambda\} = \sum_{i=1}^K \tilde{b}_i \delta\{\lambda - \lambda_i\}, \tag{80}$$

where the  $\lambda_i$  are sufficiently separated so that the output due to each path may be identified. This is the dual of the Doppler channel with the scattering function

$$\tilde{S}_D\{f\} = \sum_{i=1}^K \tilde{b}_i \delta\{f - f_i\}. \tag{81}$$

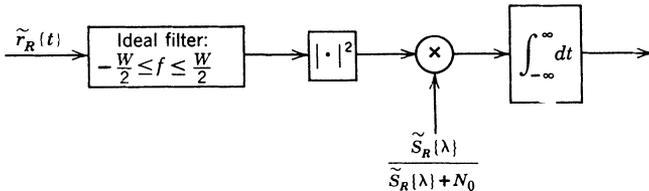


Fig. 12.9 Optimum receiver for range-spread target: SPLIT condition.

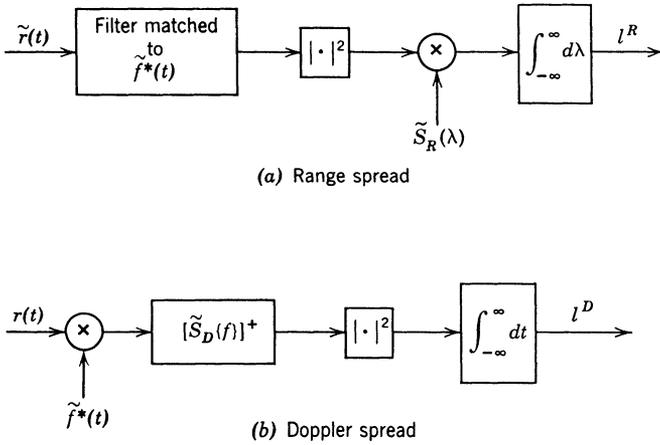


Fig. 12.10 Optimum receiver: LEC condition.

Notice that (81) does not describe a frequency-diversity system. It corresponds to a set of slowly fluctuating point targets moving at different velocities.

**Case 5. Optimum Binary Communication.** The model for a binary communication system operating over a range-spread channel is analogous to that in Section 11.3. The transmitted signals are given by (11.68). The receiver consists of two simple binary receivers referenced to  $\omega_1$  and  $\omega_0$ . The actual implementation will depend on the physical situation, but it will correspond to one of the structures developed in this chapter.

The point of current interest is the performance. The derivation in Section 11.3.1 did not rely on the channel characteristics. Thus, the bound in (11.91),

$$\Pr(\epsilon) \leq \frac{1}{2} \exp \left[ -0.1488 \frac{\bar{E}_r}{N_0} \right], \tag{82}$$

is also valid for range-spread channels. We now consider two examples of signal design to show how we can approach the bound.

**Example 2.** Let

$$\tilde{S}_R(\lambda) = \begin{cases} \frac{\sigma_b^2}{L}, & |\lambda| \leq L, \\ 0, & |\lambda| > L. \end{cases} \tag{83}$$

This is the dual of the channel in (11.94). From the results of that example, we know that if  $\bar{E}_r/N_0$  is large we can achieve the bound by transmitting

$$\tilde{f}(t) = \sqrt{W} \frac{\sin \pi W t}{\pi W t}, \quad -\infty < t < \infty, \quad (84)$$

with  $W$  chosen properly. From (11.96) the optimum value of  $W$  is

$$W_o = \frac{\bar{E}_r/N_0}{3.07(2L)} \quad (85)$$

Notice that results assume

$$W_o L \gg 1. \quad (86)$$

The signal in (84) is not practical. However, any signal whose transform is reasonably constant over  $[-W_o, W_o]$  should approach the performance in (82).

**Example 3.** Let

$$\tilde{S}_R\{\lambda\} = \frac{4k\sigma_b^2}{(2\pi\lambda)^2 + k^2}, \quad -\infty < \lambda < \infty. \quad (87)$$

This is the dual of the channel in Examples 2 and 3 on pages 382 and 384. The dual of the signal in Fig. 11.16 is

$$\tilde{F}\{f\} = \sum_{i=1}^{D_o} a \tilde{Y}(f - iW_p), \quad (88)$$

where

$$\tilde{Y}\{f\} \triangleq \begin{cases} \frac{1}{\sqrt{W_s}}, & 0 \leq f \leq W_s, \\ 0, & \text{elsewhere,} \end{cases} \quad (89)$$

and  $D_o$  satisfies (11.111).

If

$$W_s \ll \frac{2\pi}{k} \quad (90)$$

and

$$W_p \gg \frac{2\pi}{k}, \quad (91)$$

then we approach the bound in (82).

The signal in (88) corresponds to transmitting  $D_o$  frequency-shifted pulses simultaneously. An identical result can be obtained by transmitting them sequentially (see Problem 12.3.14). The shape in (89) is used to get an exact dual. Clearly, the shape is unimportant as long as (90) is satisfied.

These results deal with binary communication. The results in Section 11.3.4 on  $M$ -ary systems carry over to range-spread channels in a similar manner.

**Case 6. Suboptimum Receiver No. 1.** In Section 11.3.3 we developed a suboptimum receiver for the Doppler-spread channel by using a piecewise

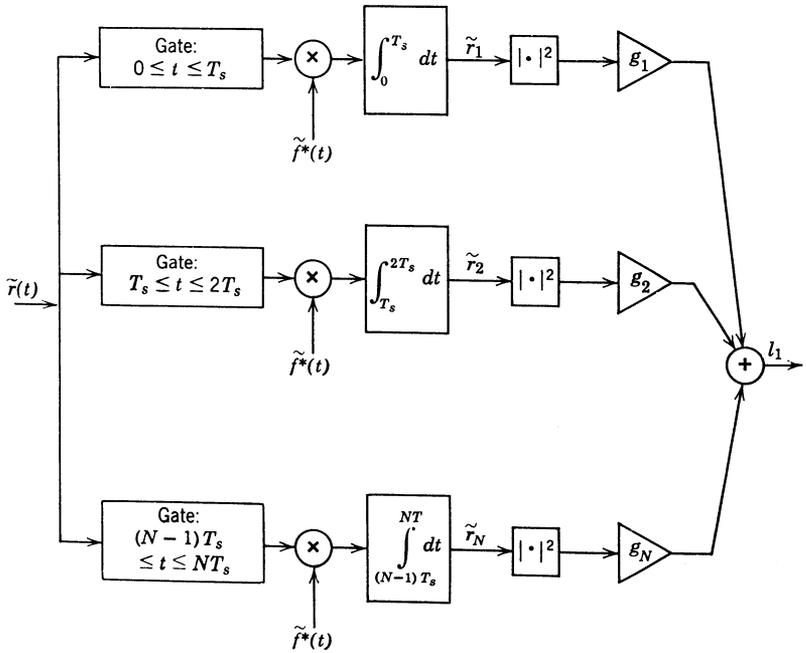


Fig. 12.11 Suboptimum receiver No. 1 for Doppler-spread channel.

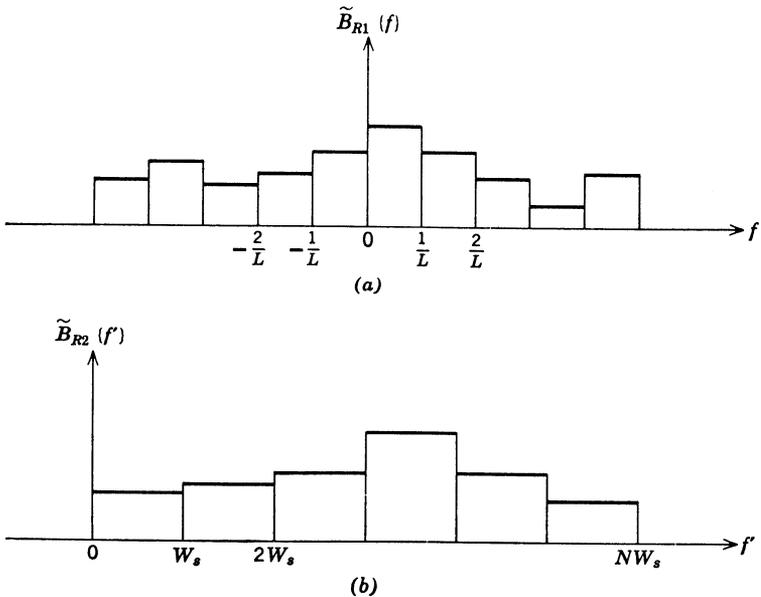


Fig. 12.12 Piecewise constant approximations to the transform of  $\tilde{b}_R(t)$ .

constant approximation to the channel fading process. We have repeated Fig. 11.18 (redrawn slightly) in Fig. 12.11. In Fig. 12.12a and b, we show two types of piecewise approximations to the transform of the channel fading process. In the first approximation, we use segments of  $L^{-1}$  and have a total of

$$D_R \triangleq WL \tag{92}$$

segments. In the second approximation, we let the segment length equal  $W_s$  and regard it as a design parameter. We also shift the origin for notational simplicity. The resulting receiver is shown in Fig. 12.13. This is the dual of the receiver in Fig. 12.11. The performance can be analyzed in exactly the same manner as in Section 11.3.3.

**Case 7. Suboptimum Receiver No. 2.** The dual of the suboptimum FSI receiver in Fig. 11.20 is the two-filter radiometer in Fig. 12.14. The multiplier  $\tilde{G}(\lambda)$  is a function that we choose to optimize the performance. In the LEC case

$$\tilde{G}(\lambda) = \tilde{S}_R(\lambda) \tag{93}$$

(see Case 3), while in the SPLOT case

$$\tilde{G}(\lambda) = \frac{\tilde{S}_R(\lambda)}{\tilde{S}_R(\lambda) + N_0} \tag{94}$$

This type of receiver is analyzed in [8].

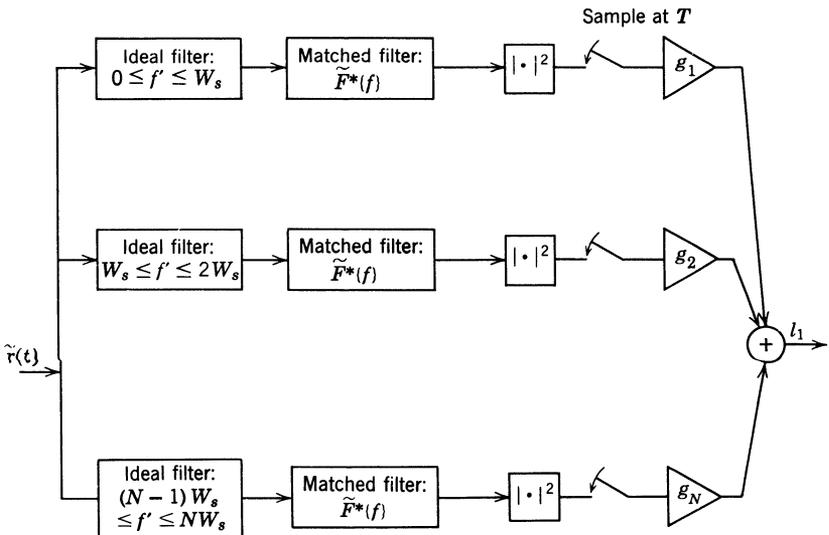


Fig. 12.13 Suboptimum receiver No. 1 for range-spread channel.

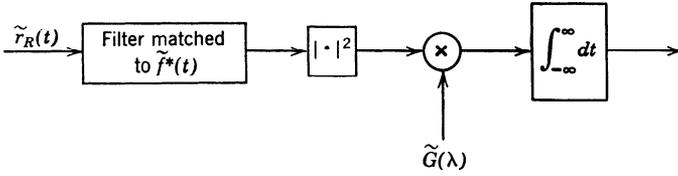


Fig. 12.14 Suboptimum two-filter radiometer.

**Case 8. Dual Estimation Problems.** In Section 11.4, we introduced the problem of estimating the range and mean Doppler of a fluctuating point target. The dual problem is that of estimating the Doppler and mean range of a nonfluctuating range-spread target.

We assume that the target is a nonfluctuating range-spread target whose mean range is

$$m_R \triangleq \frac{1}{2\sigma_b^2} \int_{-\infty}^{\infty} \lambda \bar{S}_R(\lambda) d\lambda. \tag{95}$$

It is moving at a constant velocity corresponding to a Doppler shift  $f$ . The complex envelope of the returned waveform is

$$\tilde{r}(t) = \sqrt{E_t} e^{j2\pi ft} \int_{-\infty}^{\infty} \tilde{f}(t - \lambda) \tilde{b}_R(\lambda) d\lambda + \tilde{w}(t), \quad -\infty < t < \infty. \tag{96}$$

The covariance function of the first term is

$$E[\tilde{s}(t)\tilde{s}^*(u)] = E_t e^{j2\pi f[t-u]} \int_{-\infty}^{\infty} \tilde{f}(t - \lambda_1 - m_R) \bar{S}_{R_0}(\lambda_1) \tilde{f}^*(u - \lambda_1 - m_R) d\lambda_1, \quad -\infty < t, u < \infty, \tag{97}$$

where

$$\bar{S}_{R_0}(\lambda) \triangleq \bar{S}_R(\lambda - m_R). \tag{98}$$

This problem is the dual of that in Section 11.4. The various results of interest are developed in the problems.

This completes our discussion of the applications of time-frequency duality. Our interesting examples are developed in the problems. Before leaving the subject, several observations are important:

1. The discussion assumes infinite limits, so that there is an approximation involved.
2. If the system is implemented with a Fourier transformer, there is an approximation.
3. The concept as a guide to thinking about problems is as useful as the formal manipulations. The result of the manipulations should always be checked to see whether they operate as intended.

If one remembers these points, duality theory provides a powerful tool for solving and understanding problems. We now summarize the results of the chapter.

12.4 SUMMARY: RANGE-SPREAD TARGETS

In this chapter we have considered range-spread targets. The returned signal is modeled as a sample function of a zero-mean complex Gaussian process that is described by the relation

$$\tilde{s}(t) = \sqrt{E_t} \int_{-\infty}^{\infty} \tilde{b}_R(\lambda) \tilde{f}(t - \lambda) d\lambda. \tag{99}$$

The covariance function of the returned signal is

$$\tilde{K}_{\tilde{s}}(t, u) = E_t \int_{-\infty}^{\infty} \tilde{f}(t - \lambda) \tilde{S}_R(\lambda) \tilde{f}^*(u - \lambda) d\lambda. \tag{100}$$

We observed that a range-spread target caused frequency-selective fading.

The detection problem was once again that of detecting a sample function of a Gaussian random process in the presence of additive noise. The

Table 12.2 Singly spread target results  
 Doppler-spread target                      Range-spread target

Reflected signal	$\tilde{s}(t) = \sqrt{E_t} \tilde{b}_D(t) \tilde{f}(t - \lambda)$	$\tilde{s}(t) = \sqrt{E_t} \int_{-\infty}^{\infty} \tilde{f}(t - \lambda) \tilde{b}_R(\lambda) d\lambda$
Covariance function	$E_t \tilde{f}(t - \lambda) \tilde{K}_D(t - u) \tilde{f}^*(u - \lambda)$	$E_t \int_{-\infty}^{\infty} \tilde{f}(t - \lambda) \tilde{S}_R(\lambda) \tilde{f}^*(u - \lambda) d\lambda$
Scattering functions	$\tilde{S}_D\{f\}$	$\tilde{S}_R(\lambda)$
Correlation functions	$\tilde{K}_D(\tau)$	$\tilde{K}_R\{v\}$ Two-frequency correlation function
Type of fading	Time-selective	Frequency-selective
Approximate diversity, $WT \approx 1$	$(B + W)T \simeq BT + 1$	$(L + T)W \simeq WL + 1$
Condition for flat fading	$T \ll \frac{1}{B}$	$W \ll \frac{1}{L}$

structure of the optimum receiver followed easily, but the integral equation of interest was difficult to solve. For the LEC case and the separable kernel case we could obtain a complete solution.

In order to obtain solutions for the general case, we introduced the concept of time-frequency duality. This duality theory enabled us to apply all of our results for Doppler-spread targets to range-spread targets. In retrospect, we can think of Doppler-spread and range-spread targets as examples of *single-spread targets* (either time or frequency, but not both). In Chapter 13, we shall encounter other examples of singly-spread targets. In Table 12.2, we have collected some of the pertinent results for singly-spread targets.

Our discussion has concentrated on the detection problem in the presence of white noise. Other interesting topics, such as parameter estimation, detection in the presence of colored noise, and the resolution problem, are developed in the problems of Section 12.5. We now turn our attention to targets and channels that are spread in both range and Doppler.

## 12.5 PROBLEMS

### P.12.2 Detection of Range-Spread Targets

**Problem 12.2.1.** Consider the covariance function in (10). Prove that  $\tilde{K}_{\tilde{s}}(t, u)$  can be written as

$$K_{\tilde{s}}(t, u) = \iint_{-\infty}^{\infty} df_1 df_2 e^{j2\pi f_1 t} \tilde{F}\{f_1\} K_R\{f_1 - f_2\} \tilde{F}^*\{f_2\} e^{-j2\pi f_2 u}. \quad (\text{P.1})$$

**Problem 12.2.2.** Assume that

$$\tilde{S}_R(\lambda) = \sum_{k=1}^N 2\sigma_k^2 \delta(\lambda - \lambda_k).$$

1. Find  $\tilde{K}_{\tilde{s}}(t, u)$ .
2. Find the optimum receiver. Specify all components completely.

**Problem 12.2.3.** Assume that  $\tilde{f}(t)$  is bandlimited to  $\pm W/2$  cps. We approximate  $\tilde{S}_R(\lambda)$  as

$$\tilde{S}_R(\lambda) = \sum_{k=1}^N \frac{1}{W} \tilde{S}_R\left(\frac{k}{W}\right) \delta\left(\lambda - \frac{k}{W}\right), \quad (\text{P.1})$$

where

$$N = LW,$$

which is assumed to be an integer.

1. Draw a block diagram of the optimum receiver.
2. Justify the approximation in (P.1) in the following way:
  - a. Use finite limits on the expression in (P.1) in Problem 12.2.1.
  - b. Expand  $K_R\{\cdot\}$  using Mercer's theorem.
  - c. Use the asymptotic properties of the eigenfunctions that were derived in Section I-3.4.6 (page I-206).

**Problem 12.2.4.** Assume that

$$\tilde{S}_R(\lambda) = \frac{2\sigma_b^2}{\sqrt{2\pi}\sigma_R} e^{-\lambda^2/2\sigma_R^2}, \quad -\infty < \lambda < \infty,$$

and that

$$\tilde{f}(t) = \left(\frac{1}{\pi T^2}\right)^{1/4} e^{-t^2/2T^2} \quad -\infty < t < \infty.$$

The LEC condition is valid.

1. Evaluate  $\tilde{\mu}(s)$ .
2. What choice of  $T$  minimizes  $\tilde{\mu}(s)$ ? Explain your result intuitively.

**Problem 12.2.5.**

1. Prove that the expression in (35) can also be written as

$$\tilde{\mu}(s) = -\frac{s(1-s)E_t^2}{N_0} \int_{-\infty}^{\infty} \theta\{x, 0\} \beta_R(x) dx,$$

where

$$\beta_R(x) = \int_{-\infty}^{\infty} \tilde{S}_R(\lambda_2 + x) \tilde{S}_R(\lambda_2) d\lambda_2.$$

2. Express  $\beta_R(x)$  in terms of  $\tilde{K}_R\{v\}$ .
3. Combine parts 1 and 2 to obtain another expression for  $\tilde{\mu}(s)$ .

**Problem 12.2.6.** Assume that

$$\tilde{S}_R(\lambda) = \begin{cases} \frac{2\sigma_b^2}{L}, & |\lambda| \leq \frac{L}{2}, \\ 0, & \text{elsewhere} \end{cases}$$

and

$$\tilde{f}(t) = \begin{cases} \frac{1}{\sqrt{T}}, & |t| \leq \frac{T}{2} \\ 0, & \text{elsewhere.} \end{cases}$$

The LEC condition is valid. Evaluate  $\tilde{\mu}(s)$ .

### P. 12.3 Time-Frequency Duality

**Problem 12.3.1.** The signal  $\tilde{f}(t)$  is

$$\tilde{f}(t) = a \sum_{i=1}^K \tilde{u}(t - iT_p),$$

where  $\tilde{u}(t)$  is defined in (10.29). Find the dual signal.

**Problem 12.3.2.** The signal  $\tilde{f}(t)$  is

$$\tilde{f}(t) = \left( \frac{1}{\pi T^2} \right)^{1/4} e^{-t^2/2T^2}, \quad -\infty < t < \infty.$$

Find the dual signal.

**Problem 12.3.3.** Find the duals of the Barker codes in Table 10.1.

**Problem 12.3.4. Time-Varying Gain.** Let

$$\tilde{z}_1(t) = \tilde{a}(t)\tilde{y}_1(t),$$

where  $\tilde{a}(t)$  is a known function. Find the dual operation.

**Problem 12.3.5. Filter.** Let

$$\tilde{z}_1(t) = \int_{-\infty}^{\infty} \tilde{h}(t - \tau)\tilde{y}_1(\tau) d\tau,$$

where  $\tilde{h}(\cdot)$  is a known function. Find the dual operation.

**Problem 12.3.6. Gate.** Let

$$\tilde{z}_1(t) = \begin{cases} \tilde{y}_1(t), & T_1 \leq t \leq T_2, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the dual operation.

**Problem 12.3.7. Ideal Filters.** Let

$$\tilde{z}_1(t) = \int_{-\infty}^{\infty} \tilde{h}(t - \tau)\tilde{y}_1(\tau) d\tau,$$

where

$$\tilde{H}\{f\} = \begin{cases} 1, & F_0 \leq f \leq F_1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the dual operation.

**Problem 12.3.8 Aperiodic Cross-Correlator.** Let

$$\tilde{z}_1(t) = \int_{-\infty}^{\infty} \tilde{g}_1^*(t + \tau)\tilde{g}_2(\tau) d\tau.$$

Find the dual operation.

**Problem 12.3.9.** Let

$$\tilde{z}_1(t) = \int_{-\infty}^{\infty} \tilde{g}_1^*(t + \tau)\tilde{g}_1(\tau) d\tau.$$

Find the dual operation.

**Problem 12.3.10.**

1. Let

$$\tilde{z}_1(t) = \int_{t-T}^t \tilde{y}_1(u) du.$$

Find the dual operation.

2. Let

$$\tilde{z}_1(t) = \int_0^t \tilde{y}_1(u) du.$$

Find the dual operation.

**Problem 12.3.11.** Consider the detection problem specified in (67) and (68).

$$\tilde{S}_R(\lambda) = \frac{2\sqrt{2} P/k}{(2\pi f/k)^4 + 1}, \quad -\infty < \lambda < \infty$$

and

$$\tilde{f}(t) = c \sin^2\left(\frac{2\pi t}{T}\right), \quad 0 \leq t \leq T.$$

Draw a block diagram of the optimum receiver.

**Problem 12.3.12.** Consider Case 2 and the signal in (78). Derive the receiver in Fig. 12.9 directly from (29)–(31) without using duality theory.

**Problem 12.3.13.** Consider the two systems in Fig. 12.10. Verify that the receiver in Fig. 12.10b is the dual of the receiver in Fig. 12.10a.

**Problem 12.3.14.** Consider the example on page 433. Assume that we transmit

$$\tilde{f}(t) = \sum_{k=1}^{D_0} a\tilde{u}(t - kT_0)e^{j2\pi k W_p},$$

where  $\tilde{u}(t)$  satisfies (11.113b).

1. Describe  $\tilde{f}(t)$ .

2. Verify that this signal achieves the same performance as that in (88) when the parameters are chosen properly.

**Problem 12.3.15.** Assume that  $L = 200 \mu\text{sec}$  in (83). The available signal power-to-noise level ratio at the channel output is

$$\frac{P_R}{N_0} = 10^5.$$

The required  $\text{Pr}(\epsilon)$  is  $10^{-4}$ . We use a binary FSK system.

1. What is the maximum rate in bits per second that we can communicate over this channel with a binary system satisfying the above constraints?

2. Design a signaling scheme to achieve the rate in part 1.

**Problem 12.3.16.** Consider Case 6. Derive all of the expressions needed to analyze the performance of suboptimum receiver No. 1.

**Problem 12.3.17.** Consider Case 7. Derive all of the expressions needed to analyze the performance of suboptimum receiver No. 2.

**Problem 12.3.18.** In Case 8 (95)–(98), we formulated the problem of estimating the Doppler shift and mean range of a nonfluctuating range-spread target.

1. Starting with the general definition in (11.181), show that the *range-spread ambiguity function* is

$$\theta_{\Omega_R}\{m_{R_a}, m_R; f_a, f\} = \frac{E_t}{N_0} \iint_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_{ou}(t, u; m_R, f) e^{-j2\pi f_a(t-u)} \tilde{f}^*(t - \lambda_1 - m_{R_a}) \tilde{S}_{R_0}^*(\lambda_1) \times \tilde{f}(u - \lambda_1 - m_{R_a}) d\lambda_1 dt du.$$

2. When the LEC condition is valid, the expression can be simplified. Show that one expression is

$$\theta_{\Omega_R, \text{LEC}}\{m'_R; f'\} = \frac{E_t^2}{N_0^2} \int_{-\infty}^{\infty} dx \theta\{x + m_R, f'\} \int_{-\infty}^{\infty} \tilde{S}_{R_0}(x + \lambda) \tilde{S}_{R_0}^*(\lambda) d\lambda,$$

where

$$m'_R \triangleq m_R - m_{Ra}$$

and

$$f' \triangleq f - f_a.$$

3. Express  $\theta_{\Omega_R, \text{LEC}}\{\cdot, \cdot\}$  in several different ways.

**Problem 12.3.19.** Prove that  $\theta_{\Omega_D, \text{LEC}}\{\lambda, m_D\}$ , is the dual of  $\theta_{\Omega_D, \text{LEC}}\{m_R, f\}$ . Specifically, if

$$\check{f}_D(t) = \check{F}_R(t)$$

and

$$\check{S}_D\{\lambda\} = \check{S}_R(-\lambda),$$

then

$$\theta_{\Omega_D, \text{LEC}}\{y, x\} = \theta_{\Omega_R, \text{LEC}}\{x, y\}.$$

**Problem 12.3.20.** Derive the elements in the information matrix  $\mathbf{J}$  in terms of  $\theta_{\Omega_R, \text{LEC}}\{\cdot, \cdot\}$  and its derivatives.

**Problem 12.3.21.** Assume that

$$\check{f}(t) = \left(\frac{1}{\pi T^2}\right)^{1/4} e^{-t^2/2T^2}, \quad -\infty < t < \infty$$

and

$$\check{S}_{R_0}\{\lambda\} = \frac{2\sigma_b^2}{\sqrt{2\pi}\sigma_R} e^{-\lambda^2/2\sigma_R^2}, \quad -\infty < \lambda < \infty.$$

1. Evaluate  $\theta_{\Omega_R, \text{LEC}}\{m_R, f\}$ .
2. Calculate the  $\mathbf{J}$  matrix.

**Problem 12.3.22.** Consider the problem of estimating the amplitude of a scattering function. Thus,

$$\check{S}_R(\lambda: A) = A\check{S}_R(\lambda)$$

and  $\check{S}_R(\lambda)$  is assumed known. Assume that the LEC condition is valid.

1. Find a receiver to generate  $\hat{a}_{ml}$ .
2. Is  $\hat{a}_{ml}$  unbiased?
3. Assume that the bias on  $\hat{a}_{ml}$  is negligible. Calculate

$$E[(\hat{a}_{ml} - A)^2].$$

4. Calculate a bound on the normalized variance of any unbiased estimate of  $A$ . Compare this bound with the result in part 3.

5. Compare the results of this problem with those in Problem 11.4.8.

**Problem 12.3.23.** Assume that

$$\check{S}_R\{\lambda: A\} = \check{S}_{R_1}\left\{\frac{\lambda}{A}\right\},$$

where  $\check{S}_{R_1}\{\cdot\}$  is known. We want to estimate  $A$ , the scale of the range axis. Assume that the LEC condition is valid.

1. Draw a block diagram of a receiver to generate  $\hat{a}_{ml}$ .
2. Evaluate the Cramér-Rao bound.

**Problem 12.3.24.** Assume that

$$\tilde{S}_R(\lambda) = 2\sigma_1^2 \delta(\lambda - \lambda_1) + 2\sigma_2^2 \delta(\lambda - \lambda_2).$$

We want to estimate  $\lambda_1$  and  $\lambda_2$ .

1. Find a receiver to generate  $\hat{\lambda}_{1,ml}$  and  $\hat{\lambda}_{2,ml}$ .
2. Evaluate the Cramér-Rao bound.
3. How does this problem relate to the discrete resolution problem of Section 10.5?

**Problem 12.3.25.** Assume that we design the optimum receiver to detect a slowly fluctuating point target located at  $\tau = 0, f = 0$ , in the presence of white noise. We want to calculate the effect of various types of interfering targets. Recall from (9.49) that  $\Delta$  characterizes the performance. Calculate the decrease in  $\Delta$  due to the following:

1. A slowly fluctuating point target located at  $(\tau, f)$ .
2. A range-spread target with scattering function  $\tilde{S}_R(\lambda)$  and Doppler shift of  $f$  cps.
3. A Doppler-spread target with scattering function  $\tilde{S}_D\{f\}$  and range  $\lambda$ .
4. Interpret the above results in terms of the ambiguity function. Discuss how you would design signals to minimize the interference.

**Problem 12.3.26.** Assume that we design the optimum LEC receiver to detect a range-spread target in the presence of white noise. We want to calculate the effect of various types of interfering targets. For simplicity, assume that the desired target has zero velocity and zero mean range. Calculate the degradation due to the following:

1. A slowly fluctuating point target at  $(\tau, f)$ .
2. A range-spread target with scattering function  $\tilde{S}_R(\lambda)$  and Doppler shift of  $f$  cps.
3. A Doppler-spread target with scattering function  $\tilde{S}_D\{f\}$  and range  $\lambda$ .
4. Can the results in parts 1, 2, and 3 be superimposed to give a general result?

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