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On Multidimensional Sampling

Ton Kalker
Philips Research Laboratories, Eindhoven

This chapter gives an overview of the most relevant facts of sampling theory, paying particular attention to the multidimensional aspect of the problem. It is shown that sampling theory formulated in a multidimensional setting provides insight to the supposedly simpler situation of one-dimensional sampling.

4.1 Introduction

The signals we encounter in the physical reality around us almost invariably have a continuous domain of definition. We like to model a speech signal as continuous function of amplitudes, where the domain of definition is a (finite) length interval of real numbers. A video signal is most naturally viewed as continuous function of luminance (chrominance) values, where the domain of definition is some volume in space-time.

In modern electronic systems we deal with many (in essence) continuous signals in a digital fashion. This means that we do not deal with these signals directly, but only with *sampled* versions of it: we only retain the values of these signals at a discrete set of points. Moreover, due to the inherently finite precision arithmetic capabilities of digital systems, we only record an approximated (quantized) value at every point of the sampling set. If we define sampling as the process of restricting a signal to a discrete set, explicitly without quantization of the sampled values, we can describe the contribution of this chapter as a study of the relation between continuous signals and their sampled versions.

Many textbooks start this topic by only considering sampling in the one-dimensional case. Digressions into the multidimensional case are usually made in later and more advanced sections. In this chapter we will start from the outset with the multidimensional case. It will be argued that this is the most natural setting, and that this approach will even lead to greater understanding of the one-dimensional case.

I will assume that not every reader is familiar with the concept of a *lattice*. As lattices are the most basic kind of sets onto which to sample signals, this chapter will start with a crash course on lattices in Section 4.2. After this the real work starts in Section 4.3 with an overview of the sampling theory for continuous functions. The central theme of this section is the intimate relationship between sampling and the discrete space-time Fourier transform (DSFT). In Section 4.4 we consider simultaneous sampling in both spatial and frequency domain. The central theme in this section is the relationship with the discrete fourier transform (DFT). We continue with a digression on cascaded sampling (Section 4.5), and with some useful results on changing variables (Section 4.6). We end with an application of sampling theory to HDTV-to-SDTV conversion. The proofs (or hints to it) of the stated result can be found in the Appendix.

We end this introduction with some conventions. We will refer to a signal as a function, defined on some appropriate domain. As all of our functions are in principle multidimensional, we will lighten the burden of notation by suppressing the multidimensional character of variables involved wherever possible. In particular we will use $f(x)$ to denote a function $f(x_1, \cdots, x_n)$ on some continuous domain (say $\mathbb{R}^n$). Similarly we will use $f(k)$ to denote a function $f(k_1, \cdots, k_n)$ on some discrete domain (say $\mathbb{Z}^n$). By abuse of terminology we will refer to a function defined on a continuous domain as a continuous function and to a function on discrete domain as discrete function.

4.2 Lattices

Although *sampling* of a function can in principle be done with respect to any set of points (*nonuniform sampling*), the most common form of sampling is done with respect to sets of points which have a certain algebraic structure and are known as *lattices*. They are the object of study in this section.

4.2.1 Definition

Formally, the definition of a lattice is given as

DEFINITION 4.1

A (sub)lattice $\mathcal{L}$ of $\mathbb{C}^n$ ($\mathbb{R}^n$, $\mathbb{Z}^n$) is a set of points satisfying that

1. There is a shortest nonzero element,
2. If $\lambda_1, \lambda_2 \in \mathcal{L}$, then $a\lambda_1 + b\lambda_2 \in \mathcal{L}$ for all integers a and b, and
3. $\mathcal{L}$ contains n linearly independent elements.

This definition may seem to make lattices rather abstract objects, but they can be made more tangible by representing them by *generating matrices*. Namely, one can show that every lattice $\mathcal{L}$ contains a set of linearly independent points $\{\lambda_1, \cdots, \lambda_n\}$ such that every other point $\lambda \in \mathcal{L}$ is an integer linear combination $\sum_{i=1}^{n} a_i \lambda_i$. Arranging such a set in a matrix $L = [\lambda_1, \cdots, \lambda_n]$ yields a generating matrix L of $\mathcal{L}$. It has the property that every $\lambda \in \mathcal{L}$ can be written as $\lambda = Lk$, where

$k \in \mathbb{Z}^n$ is an integer vector. At this point it is important to note that there is no such thing as *the* generating matrix L of a lattice $\mathcal{L}$. Defining a unimodular matrix U as an integer matrix with $|\det(U)| = 1$, every other generating matrix is of the form LU, and every such matrix is a generating matrix. However, this also shows that the determinant of a generating matrix is determined up to a sign.

DEFINITION 4.2

Let $\mathcal{L}$ be a lattice and let L be a generating matrix of $\mathcal{L}$. Then the *determinant* of $\mathcal{L}$ is defined by

$$\det(\mathcal{L}) = |\det(L)| \; .$$

In case the dimension is 1 ($n = 1$), every lattice is given as all the integer multiples of a single scalar. This scalar is unique up to a sign, and by convention one usually defines the positive scalar as *the sampling period* T (for time).

$$\mathcal{L}_T = \{nT : n \in \mathbb{Z}\} \subset \mathbb{C}, \mathbb{R}, \mathbb{Z} \tag{4.1}$$

In case the dimension is 2 ($n = 2$) it is no longer possible to single out a natural candidate as *the* generating matrix for a lattice. As an example consider the lattice $\mathcal{L}$ generated by the matrix (see also Fig. 4.1)

$$L_1 = \begin{bmatrix} \sqrt{3} & \sqrt{3} \\ -1 & 1 \end{bmatrix} .$$

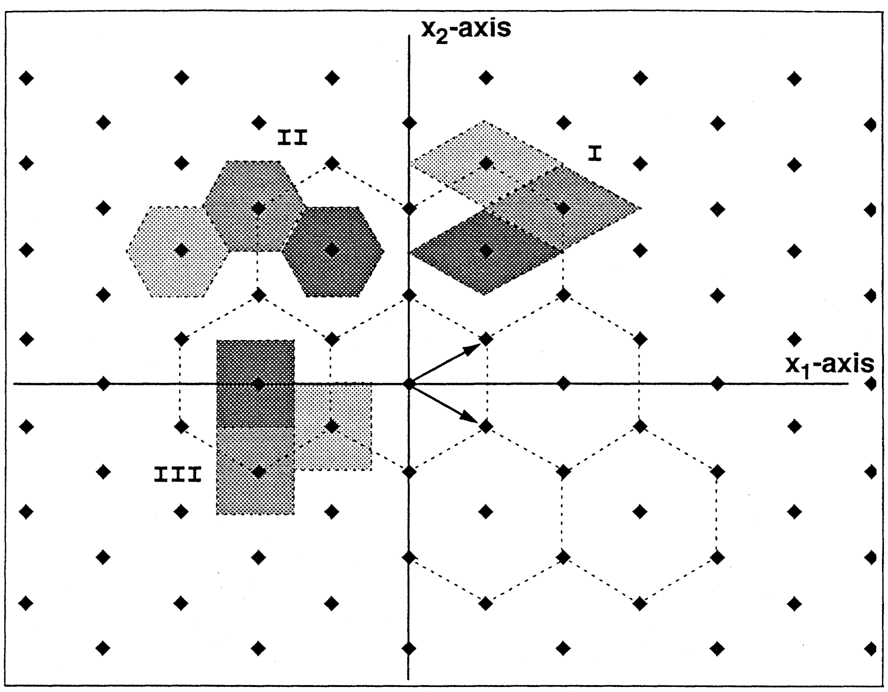


FIGURE 4.1: A hexagonal lattice in the continuous plane.

There is no reason to consider the matrix L_1 as *the* generating matrix of the lattice $\mathcal{L}$, and in fact the matrix

$$L_2 = \begin{bmatrix} \sqrt{3} & 2\sqrt{3} \\ 1 & 0 \end{bmatrix}$$

is just as valid a generating matrix as L_1.

4.2.2 Fundamental Domains and Cosets

Each lattice $\mathcal{L}$ can be used to partition its embedding space into so-called *fundamental domains*. The importance of the concept of fundamental domains lies in their ability to define $\mathcal{L}$-periodic functions, i.e., functions $f(x)$ for which $f(x) = f(x + \lambda)$ for every $\lambda \in \mathcal{L}$. Knowing a $\mathcal{L}$-periodic function $f(x)$ on a fundamental domain is sufficient to know the complete function. Periodic functions will emerge naturally when we come to speak about sampling of continuous functions.

Let $\mathcal{L} \subset \mathcal{D}$ be a lattice, where $\mathcal{D}$ is either a lattice $\mathcal{M} \subset \mathbb{R}^n$ or the space $\mathbb{R}^n$ itself. Let L be a generating matrix of $\mathcal{L}$, and let P be an arbitrary subset of $\mathcal{D}$. With every $p \in P$ we can associate a translated version or *coset* $p + \mathcal{L}$ of $\mathcal{L}$. The set of cosets is referred to as the *coset group* of $\mathcal{L}$ with respect to $\mathcal{D}$ and is denoted by the expression $\mathcal{D}/\mathcal{L}$. A fundamental domain is defined as a subset $P \subset \mathcal{D}$ which intersects every coset in exactly one point.

DEFINITION 4.3

The set P is called a fundamental domain of the lattice $\mathcal{L}$ in $\mathcal{D}$ if and only if

1. $p \neq q$ implies $p + \mathcal{L} \neq q + \mathcal{L}$, and
2. $\bigcup_{p \in P} \; p + \mathcal{L} = \mathcal{D}$.

A fundamental domain is not a uniquely defined object. For example, the shaded areas in Fig. 4.1 show three possibilities for the choice of a fundamental domain. Although the shapes may differ, their volume is defined by the lattice $\mathcal{L}$.

THEOREM 4.1 *Let P be a fundamental domain of the lattice $\mathcal{L}$ in $\mathcal{D}$, and assume that P is measurable, i.e., that its volume is defined.*

1. *If $\mathcal{D} = \mathbb{R}^n$, then the volume of P is given by*

$$vol(P) = \det(\mathcal{L}) \; .$$

2. *If $\mathcal{D} = \mathcal{M}$, and if Q is a fundamental domain of $\mathcal{L}$ in $\mathbb{R}^n$, then $Q \cap \mathcal{M}$ is a fundamental domain of $\mathcal{L}$ in $\mathcal{M}$.*
3. *If $\mathcal{D} = \mathcal{M}$, then the number of points in P is given by*

$$\#(P) = \det(\mathcal{L}) / \det(\mathcal{M}).$$

This number is referred to as the index of $\mathcal{L}$ in $\mathcal{M}$, and is denoted by the symbol $\iota(\mathcal{L}, \mathcal{M})$.

As a consequence of assertion 1 of this theorem, all the shaded areas in Fig. 4.1, being fundamental domains of the same hexagonal lattice, have a volume equal to $2\sqrt{3}$.

4.2.3 Reciprocal Lattices

For any lattice $\mathcal{L}$ there exists a *reciprocal* lattice $\mathcal{L}^*$ as defined below. Reciprocal lattices appear in the theory of Fourier transforms of sampled continuous functions (see Section 4.3).

DEFINITION 4.4 Let $\mathcal{L}$ be a lattice. Its reciprocal lattice $\mathcal{L}^*$ is defined by

$$\mathcal{L}^* = \{\lambda^* : \langle \lambda^*, \lambda \rangle \in \mathbb{Z} \ \forall \lambda \in \mathcal{L}\} ,$$

where $\langle \lambda^*, \lambda \rangle$ denotes the usual inner product $\sum_i \lambda_i^* \lambda_i$.

This notion of reciprocal lattice is made more tangible by the observation that the reciprocal lattice of $[L]$ is the lattice $[L^{-t}]$, where $[M]$ denotes the lattice generated by a matrix M. In particular $\det(\mathcal{M}^*) = \det(\mathcal{M})^{-1}$. For example, the reciprocal lattice of the lattice of Fig. 4.1 is generated by the matrix

$$\frac{1}{2\sqrt{3}} \begin{bmatrix} 1 & 1 \\ -\sqrt{3} & \sqrt{3} \end{bmatrix}$$

This lattice is very similar to the original lattice: it differs by a rotation by $\pi/2$, and a scaling factor of $1/2\sqrt{3}$. In particular, the volume of a fundamental domain of $\mathcal{L}^*$ is equal to $1/2\sqrt{3}$.

An important property of reciprocal lattices is that subset inclusions are reversed. To be precise, the inclusion $\mathcal{M} \subset \mathcal{L}$ holds if and only if $\mathcal{L}^* \subset \mathcal{M}^*$. Using some elementary math it follows that the coset groups $\mathcal{L}/\mathcal{M}$ and $\mathcal{M}^*/\mathcal{L}^*$ have the same number of elements.

4.3 Sampling of Continuous Functions

In this section we will give the main results on the theory of sampled continuous functions. It will be shown that there is a strong relationship between sampling in the spatial domain and *periodizing* in the frequency domain. In order to state this result this section starts with a short overview of multidimensional Fourier transforms. This allows us to formulate the main result (Theorem 4.3), which states very informally that sampling in the spatial domain is equivalent to periodizing in the frequency domain.

4.3.1 The Continuous Space-Time Fourier Transform

Let $f(x)$ be a *nice*[1] function defined on the continuous domain $\mathbb{R}^n$. Let its continuous space-time Fourier transform[2] (CSFT) $F(\nu)$ be defined by

$$F(\nu) = \mathcal{F}(f)(\nu) = \int_{\mathbb{R}^n} e^{-2\pi i \langle x, \nu \rangle} f(x)\, dx \tag{4.2}$$

with inverse transform given by

$$f(x) = \mathcal{F}^{-1}(F)(x) = \int_{\mathbb{R}^n} e^{2\pi i \langle x, \nu \rangle} F(\nu)\, d\nu \ . \tag{4.3}$$

Forgetting many technicalities, the CSFT has the following basic properties:

[1]Nice means in this context that all sums, integrals, Fourier transforms, etc. involving the function exist and are finite.

[2]Contrary to the conventional wisdom, we choose to exclude the factor 2π from the frequency term $\omega = 2\pi\nu$. This has the advantage that the Fourier transform is orthogonal, without any need for normalizing factors.

- The CSFT is an *isometry*, i.e., it preserves inner products.

$$\langle f, g \rangle = \langle \mathcal{F}(f), \mathcal{F}(g) \rangle \ .$$

- The CSFT of the point-wise multiplication of two functions is the convolution of the two separate CSFTs.

$$\mathcal{F}(f \cdot g) = \mathcal{F}(f) * \mathcal{F}(g) \ .$$

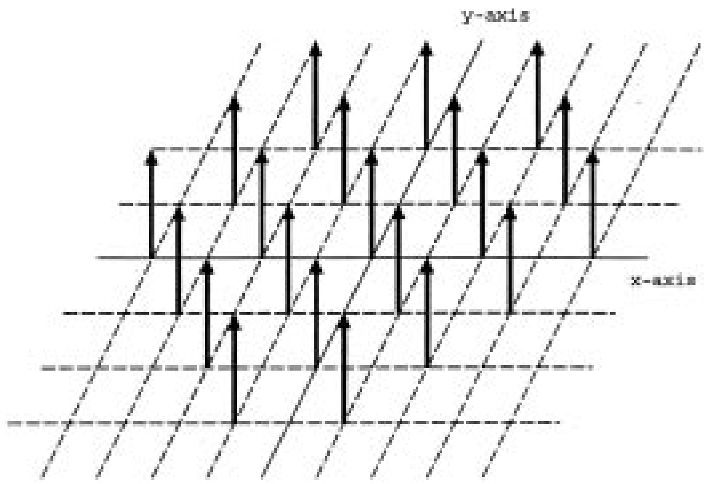


FIGURE 4.2: Lattice comb for the quincunx lattice.

A special class of functions[3] is the class of *lattice combs* (Fig. 4.2 illustrates the lattice comb of the quincunx lattice generated by the matrix $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$). If $\mathcal{L}$ is a lattice, the lattice comb $\text{Ш}_{\mathcal{L}}$ is a set of δ functions with support on $\mathcal{L}$ and is formally defined by

$$\text{Ш}_{\mathcal{L}}\ (x) = \sum_{\lambda \in \mathcal{L}} \delta_{\lambda}(x) \ . \tag{4.4}$$

The following theorem states the most important facts about lattice combs.

THEOREM 4.2 *With notations as above we have the following properties:*

$$\begin{aligned} \text{Ш}_{\mathcal{L}}\ (x) &= \frac{1}{\det(\mathcal{L})} \sum_{\lambda^* \in \mathcal{L}^*} e^{-2\pi i \langle x, \lambda^* \rangle} && (4.5) \\ \mathcal{F}(\text{Ш}_{\mathcal{L}})(\nu) &= \sum_{\lambda \in \mathcal{L}} e^{-2\pi i \langle \lambda, \nu \rangle} \\ &= \det(\mathcal{L}^*)\ \text{Ш}_{\mathcal{L}^*}\ (\nu) \ . && (4.6) \end{aligned}$$

The last equation says that the CSFT of a lattice comb is the lattice comb of the reciprocal lattice, up to a constant.

[3] Actually distributions.

4.3.2 The Discrete Space-Time Fourier Transform

The CSFT is a functional on continuous functions. We also need a similar functional on (multidimensional) sequences. This functional will be the discrete space-time Fourier transform (DSFT). In this section we will only state the definition. The properties of this functional and its relation to the CSFT will be highlighted in the next section. So let $\mathcal{L}$ be a lattice and let P^* be a fundamental domain of the reciprocal lattice $\mathcal{L}^*$. Let $\tilde{f}(x) = \Sigma_{\mathcal{L}}(f)(x)$ be the sampled version of f, and let $\tilde{F}(\nu) = \Pi_{\mathcal{L}^*}(F)(\nu)$ be the periodized version of $F(\nu)$. Then we define the forward and backward discrete space-time Fourier transform (DSFT) by

$$\tilde{\mathcal{F}}(\tilde{f})(\nu) = \sum_{x \in \mathcal{L}} e^{-2\pi i \langle x, \nu \rangle} \tilde{f}(x) \ , \tag{4.7}$$

and

$$\tilde{\mathcal{F}}^{-1}(\tilde{F})(\nu) = \det(\mathcal{L}) \int_{P^*} e^{2\pi i \langle x, \nu \rangle} \tilde{F}(\nu) d\nu \ , \tag{4.8}$$

respectively.

Note that the function $\tilde{\mathcal{F}}(\tilde{f})(\nu)$ is a $\mathcal{L}^*$-periodic function. This implies that the formula for the inverse DSFT is independent of the choice of the fundamental domain P^*.

4.3.3 Sampling and Periodizing

One of the most important issues in the sampling of functions concerns the relationship between the CSFT of the original function and the DSFT of a sampled version. In this section we will state the main theorem (Theorem 4.3) of sampling theory.

Before continuing we need two definitions. If $f(x)$ is a function and $\mathcal{L} \subset \mathbb{R}^n$ is a lattice, *sampling* $f(x)$ on $\mathcal{L}$ is defined by

$$\Sigma_{\mathcal{L}}(f)(x) = \begin{cases} f(x) & \text{if } x \in \mathcal{L} \\ 0 & \text{if } x \notin \mathcal{L} \ . \end{cases} \tag{4.9}$$

The above definition has to be read carefully: sampling a function $f(x)$ on a lattice means that we modify $f(x)$ by putting all its values outside of the lattice to 0. It *does not* mean that we forget how the lattice is embedded in the continuous domain. For example, when we sample a one-dimensional continuous function $f(x)$ on the set of even numbers, the down sampled function $f_s(k)$ is not defined by $f_s(k) = f(2k)$, but by $f_s(k) = f(k)$ when k is even, and 0 otherwise.

Closely related to the sampling operator is the periodizing operator $\Pi_{\mathcal{L}}$, which modifies a function $f(x)$ such that it becomes $\mathcal{L}$-periodic. This operator is defined by

$$\Pi_{\mathcal{L}}(f)(x) = \det(\mathcal{L}) \sum_{\lambda \in \mathcal{L}} f(x - \lambda) \tag{4.10}$$

Clearly $\Pi_{\mathcal{L}}(f)(x)$ is $\mathcal{L}$-periodic, i.e., $\Pi_{\mathcal{L}}(f)(x) = \Pi_{\mathcal{L}}(f)(x - \lambda)$ for all $\lambda \in \mathcal{L}$. With these tools at our disposal we are now in a position to formulate the main theorem of sampling theory.

THEOREM 4.3 *With definitions and notations as above, consider the following diagram:*

$$\begin{array}{ccc} f & \xrightarrow{\mathcal{F}} & F \\ \downarrow \Sigma_{\mathcal{L}} & & \downarrow \Pi_{\mathcal{L}^*} \\ \tilde{f} & \xrightarrow{\tilde{\mathcal{F}}} & \tilde{F} \end{array}$$

The following assertions hold:

1. *The above diagram commutes,*[4] *i.e., whichever way we take to go from top left to bottom right, the result is the same. Informally this can be formulated as saying that first sampling and taking the DSFT is the same as first taking the CSFT and then periodizing.*
2. $\sqrt{\det(\mathcal{L})}\,\tilde{\mathcal{F}}$ *(and, therefore,* $\sqrt{\det(\mathcal{L}^*)}\,\tilde{\mathcal{F}}^{-1}$*) is an isometry with respect to the inner products*

$$\langle \tilde{f}, \tilde{g} \rangle_{\mathcal{L}} = \sum_{\lambda \in \mathcal{L}} \tilde{f}^{\dagger}(\lambda) \tilde{g}(\lambda)$$

and

$$\langle \tilde{F}, \tilde{G} \rangle_{P^*} = \int_{P^*} \tilde{F}^{\dagger}(\nu) \tilde{G}(\nu) d\nu \; ,$$

respectively.

PROOF 4.1 The proof relies heavily on the property of lattice combs and can be found in the Appendix.

This theorem has many important consequences, the best known of which is the Shannon sampling theorem. This theorem says that a function can be retrieved from a sampled version if the support of its CSFT is contained within a fundamental domain of the reciprocal lattice. Given the above theorem this result is immediate: we only need to verify that a function $F(\nu)$ can be retrieved from $\Pi_{\mathcal{L}^*}(F)$ by restriction to a fundamental domain when $F(\nu)$ has sufficiently restricted support.

THEOREM 4.4 *(Shannon) Let $\mathcal{L}$ be a lattice, and let $f(x)$ be a continuous function with CSFT $F(\nu)$. Let $\tilde{f} = \Sigma_{\mathcal{L}}(f)$. The function $f(x)$ can be retrieved from $\tilde{f}(\lambda)$ if and only if the support of $F(\nu)$ is contained in some fundamental domain P^* of the reciprocal lattice $\mathcal{L}^*$. In that case we can retrieve $f(x)$ from $\tilde{f}(\lambda)$ with the formula*

$$f(x) = \sum_{\lambda \in \mathcal{L}} f(\lambda) \mathrm{Int}(x - \lambda) \; ,$$

where

$$\mathrm{Int}(x) = \det(\mathcal{L}) \int_{P^*} e^{2\pi i \langle x, \nu \rangle} \, d\nu \; .$$

PROOF 4.2 We only need to prove the interpolation formula.

$$\begin{aligned} f(x) &= \int_{P^*} e^{2\pi i \langle x, \nu \rangle} F(\nu) \, d\nu \\ &= \det(\mathcal{L}) \sum_{\lambda \in \mathcal{L}} f(\lambda) \int_{P^*} e^{2\pi i \langle x - \lambda, \nu \rangle} \, d\nu \\ &= \sum_{\lambda \in \mathcal{L}} f(\lambda) \mathrm{Int}(x - \lambda) \; . \end{aligned} \tag{4.11}$$

We end this section with an example showing all the aspects of Theorem 4.3.

[4] Commuting diagrams are a common mathematical tool to describe that certain sequences of function applications are equivalent.

EXAMPLE 4.1:

Let $\mathcal{L} \subset \mathbb{Z}^2$ be the *quincunx* sampling lattice generated by the matrix $L = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Let

$$f(x_1, x_2) = \mathrm{sinc}(x_1 - x_2)\mathrm{sinc}(x_1 + x_2) .$$

A simple computation shows that CSFT $F(\nu_1, \nu_2)$ of $f(x_1, x_2)$ is given by

$$F(\nu_1, \nu_2) = \frac{1}{2}\mathrm{X}_\Lambda(\nu_1, \nu_2) ,$$

where Λ is the set $\Lambda = \{(\nu_1, \nu_2) : |\nu_1| + |\nu_2| \leq 1\}$. Observing that $\mathcal{L}^*$ is generated by $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, we find that the periodized function $\Pi_{\mathcal{L}^*}(F)$ is constant with value 1.

Sampling $f(x)$ on the quincunx lattice yields the function $\tilde{f}(\lambda)$

$$\tilde{f}(\lambda_1, \lambda_2) = \begin{cases} 1 & \text{if } (\lambda_1, \lambda_2) = (0, 0) \\ 0 & \text{if } (\lambda_1, \lambda_2) \neq (0, 0) . \end{cases}$$

It is now trivial to check that $\tilde{\mathcal{F}}(\tilde{f}) = \tilde{F}$, as predicted by Theorem 4.3. Moreover, as

$$\left\|\tilde{f}\right\|_2^2 = \sum_{\lambda \in \mathcal{L}} \delta_0(\lambda)^2 = 1$$

and

$$\left\|\tilde{F}\right\|_2^2 = \int_\Lambda d\nu = 1/2 ,$$

it follows that $\left\|\tilde{\mathcal{F}}\right\|$ and $\left\|\tilde{f}\right\|$ differ by a factor of $\sqrt{2} = \sqrt{\det(\mathcal{L}^*)}$, again as predicted by Theorem 4.3.

4.4 From Infinite Sequences to Finite Sequences

In the previous section we considered sampling in the spatial domain and saw that this was equivalent to periodizing in the frequency domain. One obvious question now arises: what happens if we sample the DSFT of a (spatially) sampled function? In this section we will answer this question and show that sampling in both spatial and frequency domains simultaneously is closely related to properties of the discrete Fourier transform (DFT).

4.4.1 The Discrete Fourier Transform

The discrete Fourier transform (DFT) is a frequency transform on finite sequences. In a multidimensional context the DFT is best defined by assuming two lattices $\mathcal{L}$ and $\mathcal{M}$, $\mathcal{M} \subset \mathcal{L} \subset \mathbb{R}^n$. Let P be a fundamental domain of $\mathcal{L}$ in $\mathcal{M}$, and let P^* be a fundamental domain of $\mathcal{M}^*$ in $\mathcal{L}^*$ (recall that lattice inclusions invert when going over to the reciprocal domain [Section 4.2]). Note that both P and P^* have the same number points, viz. $\#(P) = \#(P^*) = \iota(\mathcal{L}^*, \mathcal{M}^*) = \iota(\mathcal{M}, \mathcal{L})$. Let $\hat{f}(p)$, $p \in P$ be a finite sequence over P. The DFT $\hat{\mathcal{F}}$ is now defined as functional which maps sequences $\hat{f}$ to sequences $\hat{F}$ over P^*. The formal definitions of $\hat{\mathcal{F}}$ and $\hat{\mathcal{F}}^{-1}$ are as follows.

DEFINITION 4.5

$$\hat{\mathcal{F}}(\hat{f})(p^*) = \frac{1}{\det(\mathcal{M})} \sum_{p \in P} e^{-2\pi i \langle p, p^* \rangle} \hat{f}(p) \tag{4.12}$$

$$\hat{\mathcal{F}}^{-1}(\hat{F})(p) = \frac{1}{\det(\mathcal{L}^*)} \sum_{p^* \in P^*} e^{2\pi i \langle p, p^* \rangle} \hat{F}(p^*) . \tag{4.13}$$

It is obvious that the conventional one-dimensional DFT is a special case of the more general multidimensional DFT defined above. The next example makes this more explicit.

EXAMPLE 4.2:

Let $\mathcal{M} \subset \mathcal{L} \subset \mathbb{R}$ be defined by $\mathcal{M} = \mathbb{Z}$ for some positive integer p, and let $\mathcal{L} = \frac{1}{p}\mathbb{Z}$. One easily checks that the set P and P^* can be chosen as $\{0/p, \cdots, (p-1)/p\}$ and $\{0, \cdots, p-1\}$, respectively. If x_n and X_m are the values of $\hat{f}$ on $n/p \in P$ and of $\hat{F}$ on $m \in P^*$, respectively, then the functionals $\hat{\mathcal{F}}$ and $\hat{\mathcal{F}}^{-1}$ are defined in the (x_n, X_m) domain as

$$X_m = \sum_{n=0}^{p-1} e^{-\frac{2\pi i n m}{p}} x_n, \tag{4.14}$$

$$x_n = \frac{1}{p} \sum_{m=0}^{p-1} e^{\frac{2\pi i n m}{p}} X_m . \tag{4.15}$$

This is, of course, nothing else but the usual definition of the one-dimensional DFT on finite sequences of length p.

The following example shows the general DFT at work in a two-dimensional setting.

EXAMPLE 4.3:

(Example 4.1 continued) Continuing Example 4.1, we choose the lattice $\mathcal{M} = \mathbb{Z}^2$ as the periodizing lattice. We can then choose

$$P = \{p_0, p_1\} = \left\{(0,0), \left(\frac{1}{2}, \frac{1}{2}\right)\right\}$$

and

$$P^* = \left\{p_0^*, p_1^*\right\} = \{(0,0), (1,0)\} \ .$$

The functional $\hat{\mathcal{F}}$ is then given by

$$\begin{aligned} X_0 &= x_0 e^{-2\pi i \langle p_0, p_0^* \rangle} + x_1 e^{-2\pi i \langle p_1, p_0^* \rangle} \\ &= x_0 + x_1 \\ X_1 &= x_0 e^{-2\pi i \langle p_0, p_1^* \rangle} + x_1 e^{-2\pi i \langle p_1, p_1^* \rangle} \\ &= x_0 - x_1 \ , \end{aligned}$$

and the functional $\hat{\mathcal{F}}^{-1}$ by

$$\begin{aligned} x_0 &= \frac{1}{2}\left(X_0 e^{-2\pi i \langle p_0, p_0^* \rangle} + X_1 e^{-2\pi i \langle p_0, p_1^* \rangle}\right) \\ &= \frac{1}{2}(X_0 + X_1) \\ x_1 &= \frac{1}{2}\left(X_0 e^{-2\pi i \langle p_1, p_0^* \rangle} + X_1 e^{-2\pi i \langle p_1, p_1^* \rangle}\right) \\ &= \frac{1}{2}(X_0 - X_1) \ . \end{aligned}$$

4.4.2 Combined Spatial and Frequency Sampling

We start with setting up the context of the problem. So let $f(x)$ be a nice continuous function on $\mathbb{R}^n$ and let $\mathcal{M}$ and $\mathcal{L}$ be two lattices such that $\mathcal{M} \subset \mathcal{L} \subset \mathbb{R}^n$. Sampling $f(x)$ on $\mathcal{L}$ and periodizing on $\mathcal{M}$ we construct a function $\hat{f}(x)$ that has support on $\mathcal{L}$ and is $\mathcal{M}$-periodic. In formula:

$$\hat{f}(x) = \begin{cases} \det(\mathcal{M}) \sum_{\mu \in \mathcal{M}} f(x-\mu) & \text{if } x \in \mathcal{L} \\ 0 & \text{if } x \notin \mathcal{L}\,. \end{cases}$$

A similar definition can be given for the function $\hat{F}(\nu)$, which is obtained from the CSFT $F(\nu)$ of $f(x)$ by periodizing on $\mathcal{L}^*$ and sampling on $\mathcal{M}^*$.

One easily verifies that $\hat{f}(x)$ is completely specified by its values on a (finite) fundamental domain P of $\mathcal{M}$ in $\mathcal{L}$. Similarly $\hat{F}(\nu)$ is completely specified by its values on a fundamental domain P^* of $\mathcal{L}^*$ in $\mathcal{M}^*$. Now we are in a position to extend the commutative diagram of Theorem 4.3.

THEOREM 4.5 *With notations and definitions as above, consider the following extensions of the diagram of Theorem 4.3:*

$$\begin{array}{ccc} f & \xrightarrow{\mathcal{F}} & F \\ \downarrow \Sigma_{\mathcal{L}} & & \downarrow \Pi_{\mathcal{L}^*} \\ \tilde{f} & \xrightarrow{\tilde{\mathcal{F}}} & \tilde{F} \\ \downarrow \Pi_{\mathcal{M}} & & \downarrow \Sigma_{\mathcal{M}^*} \\ \hat{f} & \xrightarrow{\hat{\mathcal{F}}} & \hat{F} \end{array}$$

The following assertions hold:

1. *The above diagram commutes;*
2. *The functionals* $\sqrt{\det(\mathcal{L})}\sqrt{\det(\mathcal{M})}\hat{\mathcal{F}}$ *and* $\sqrt{\det(\mathcal{L}^*)}\sqrt{\det(\mathcal{M}^*)}\hat{\mathcal{F}}^{-1}$ *are isometries with respect to the inner products*

$$\langle \hat{f}, \hat{g} \rangle_P = \sum_{p \in P} \hat{f}^\dagger(p)\hat{g}(p)$$

and

$$\langle \hat{F}, \hat{G} \rangle_{P^*} = \sum_{p^* \in P^*} \hat{F}^\dagger(p^*)\hat{G}(p^*)\,.$$

PROOF 4.3 See Appendix.

The theorem above says that sampling the Fourier transform of a sampled function amounts to periodizing that sampled version. In this process only a finite number of data points in both the spatial and the frequency domain are sufficient to specify the resulting functions. Moreover, the CSFT can be pushed down to a DFT to provide for a one-to-one orthogonal correspondence between the two domains.

We close this section with two examples.

EXAMPLE 4.4:

(Example 4.2 continued) The formulas for the DFT obtained in Example 4.2 are *not orthonormal*. According to Theorem 4.5 above we have to multiply the forward transform with $\sqrt{\det(\mathcal{L})\det(\mathcal{M})} = \frac{1}{\sqrt{p}}$ and the backward transform with the inverse of this number to obtain orthonormal versions of the DFT. This result in the following well-known formulas for the orthonormal one-dimensional DFT.

$$X_m = \frac{1}{\sqrt{p}} \sum_{n=0}^{p-1} e^{-\frac{2\pi i n m}{p}} x_n \,, \tag{4.16}$$

$$x_n = \frac{1}{\sqrt{p}} \sum_{m=0}^{p-1} e^{\frac{2\pi i n m}{p}} X_m \,. \tag{4.17}$$

EXAMPLE 4.5:

(Example 4.3 continued) With $\mathcal{L}, \mathcal{M}, f(x), P$ and P^* as in Example 4.3, we find that the periodized sampled function $\hat{f}$ is represented by the pair $(1, 0)$, and that the periodized sampled CSFT $\hat{F}$ of F is represented by the pair $(1, 1)$. Using the formulas for the DFT of Example 4.3 is now easy to verify that $\hat{\mathcal{F}}(\{1, 0\}) = \{1, 1\}$ and $\hat{\mathcal{F}}^{-1}(\{1, 1\}) = \{1, 0\}$, as predicted by Theorem 4.5.

4.5 Lattice Chains

In the previous section we considered the sampling of continuous functions. In this section we will consider the sampling of discrete functions. The necessity of studying this topic comes from the fact that very often the sampling of a continuous function $f(x)$ is done in steps: $f(x)$ is first sampled to a fine grid $\mathcal{L}_1$, and subsequently sampled to a coarser grid $\mathcal{L}_2$, $\mathcal{L}_2 \subset \mathcal{L}_1$. Letting $\tilde{f}^{(i)} = \Sigma_{\mathcal{L}_i}(f)$ and letting $\tilde{F}^{(i)}$ be the corresponding DFST, a natural question is whether we can obtain $\tilde{F}^{(2)}$ directly from $\tilde{F}^{(1)}$, without having to go back to CSFT of $f(x)$. This question is addressed in the following theorem and answered affirmatively.

THEOREM 4.6 *With notation as above, and letting P^* be a fundamental domain of $\mathcal{L}_1^*$ in $\mathcal{L}_2^*$, we have the following result.*

$$\tilde{F}^{(2)}(\nu) = \frac{1}{\#(P^*)} \sum_{p^* \in P^*} \tilde{F}^{(1)}(\nu - p^*) \,.$$

PROOF 4.4 See Appendix.

The above result has a natural interpretation. The function $\tilde{F}^{(1)}$ is by construction $\mathcal{L}_1^*$-periodic. The function $\tilde{F}^{(2)}$ has more symmetries as it is $\mathcal{L}_2^*$-periodic. The above theorem can be phrased as saying that $\tilde{F}^{(2)}$ is obtained from $\tilde{F}^{(1)}$ by periodizing (and thereby enlarging the set of symmetries) and averaging (dividing by $\#(P^*)$). The following example shows an application of Theorem 4.6 in the one-dimensional case.

EXAMPLE 4.6:

Let $f(x) = \text{sinc}(x/2)$. Let $\mathcal{L}_1 = \mathbb{Z}$ be the lattice of integers and let $\mathcal{L}_2 = 2\mathbb{Z}$ be the lattice of even integers. Let as before $\tilde{F}^{(i)}(x)$ denote the sampled versions of $f(x)$. Then one easily computes that

$$\begin{aligned}\tilde{F}^{(1)}(\nu) &= 2\sum_{\lambda^* \in \mathbb{Z}} \mathrm{X}_{[-1/4;1/4]}(\nu - \lambda^*) \ , \\ \tilde{F}^{(2)}(\nu) &= 1 \ ,\end{aligned}$$

where X_A denotes the characteristic function of a set A.

Using Theorem 4.6 above we can also compute $\tilde{F}^{(2)}(\nu)$ directly from $\tilde{F}^{(1)}(\nu)$. We proceed as follows. Computing the reciprocal lattices we find $\mathcal{L}_1^* = \mathbb{Z}$ and $\mathcal{L}_2^* = \frac{1}{2}\mathbb{Z}$. We find two shifted versions of $\mathcal{L}_1^*$ within $\mathcal{L}_2^*$, viz. $\mathcal{L}_1^*$ and $\frac{1}{2} + \mathcal{L}_1^*$. Picking an arbitrary point in each coset, say 0 and $\frac{1}{2}$ respectively, we find

$$\begin{aligned}\tilde{F}^{(2)}(\nu) &= \frac{1}{2}\left(\tilde{F}^{(1)}(\nu) + \tilde{F}^{(1)}\left(\nu - \frac{1}{2}\right)\right) \\ &= 1\end{aligned}$$

4.6 Change of Variables

Consider the case of a one-dimensional continuous function $f(x)$. It is not always the case that $f(x)$ has a nice form, suitable for direct mathematical treatment. In such a situation a change of variables can sometimes help out. If A is an invertible linear transformation on $\mathbb{R}^n$, it might be more convenient to work with the variable $y = Ax$. Substituting $x = A^{-1}y$ we formally define the *change of variable* functional $f(x) \to f^A(x)$ by

$$f^A(x) = f\left(A^{-1}x\right) \ .$$

A similar approach can be used for discrete functions. Instead of using a linear transform A on some continuous domain, we need in this case an *isomorphism* $A : \mathcal{L}_1 \to \mathcal{L}_2$ between two lattices $\mathcal{L}_1$ and $\mathcal{L}_2$. If $\tilde{f}(k)$ is a discrete function on $\mathcal{L}_1$, a change of variables by A yields a discrete function on $\mathcal{L}_2$ defined by

$$\tilde{f}^A(k) = \tilde{f}\left(A^{-1}k\right) \ .$$

A typical example for a change of variables on discrete functions is the following. Let the lattice $\mathcal{L}_1 = 2\mathbb{Z}$, let $\mathcal{L}_2 = \mathbb{Z}$ and define $A : \mathcal{L}_1 \to \mathcal{L}_2$ by $2k \to k$. Given a function $f(x)$ on $\mathbb{R}$, downsampling it to $\mathcal{L}_1$ and changing variables with A, yield a discrete function $\tilde{f}(k)$ on $\mathbb{Z}$ defined by $\tilde{f}(k) = f(2k)$. In many textbooks this function $\tilde{f}(k)$ is referred to as the downsampled version of $f(x)$, but our analysis shows that it is better to view the discrete function $\tilde{f}(k)$ as the result of *two* consecutive operations: downsampling and change of variables.

The following two theorems address the question of how the CSFT and DSFT behave under a change of variables for the continuous and discrete case, respectively.

THEOREM 4.7

Let A be an invertible linear transform on $\mathbb{R}^n$, and let $f(x)$ be a function on $\mathbb{R}^n$. Then the CSFT of $f^A(x)$ is given by

$$\mathcal{F}\left(f^{A}\right) = |\det(A)|\mathcal{F}(f)^{A^{-t}} .$$

PROOF 4.5 See Appendix.

THEOREM 4.8 *Let $A : \mathcal{L}_1 \to \mathcal{L}_2$ be an isomorphism of lattices, and let $\tilde{f}(k)$ be a function on $\mathcal{L}_1$. Then the DSFT of $\tilde{f}^A(k)$ is given by*

$$\tilde{\mathcal{F}}\left(\tilde{f}^{A}\right) = \tilde{\mathcal{F}}\left(\tilde{f}\right)^{A^{-t}} .$$

PROOF 4.6 See Appendix.

Note that in the assertion of Theorem 4.7 a factor $|\det(A)|$ is present, which is lacking in the assertion of Theorem 4.8. The last theorem of this section addresses the situation in which a function is extended by zero-padding to a larger domain.

THEOREM 4.9

*Let $\mathcal{L}$, $\mathcal{L} \subset \mathcal{D}$ be a lattice, where $\mathcal{D}$ is either a lattice $\mathcal{M}$ or the ambient space $\mathbb{R}^n$. Let $\tilde{f}(\lambda)$ be a function on $\mathcal{L}$. Define the $\mathcal{D}$-*extension *$\tilde{f}_{\mathcal{D}}$ of $\tilde{f}$ by*

$$\tilde{f}_{\mathcal{D}}(x) = \begin{cases} \tilde{f}(x) & \text{if } x \in \mathcal{L} \\ 0 & \text{otherwise.} \end{cases}$$

Define $\Phi(\nu)$ by

$$\Phi(\nu) = \begin{cases} \mathcal{F}\left(\tilde{f}_{\mathcal{D}}\right)(\nu) & \text{if } \mathcal{D} = \mathbb{R}^n \\ \tilde{\mathcal{F}}\left(\tilde{f}_{\mathcal{D}}\right)(\nu) & \text{if } \mathcal{D} = \mathcal{M} , \end{cases}$$

i.e., $\Phi(\nu)$ is the appropriate Fourier transform of $\tilde{f}_{\mathcal{D}}$. Then the equality $\Phi(\nu) = \tilde{\mathcal{F}}(\tilde{f})(\nu)$ holds.

Informally, the above theorem says that the Fourier transform of an extended function is equal to the Fourier transform of the function itself, i.e., extending a function does not change the Fourier transform. We will now apply the three theorems above in two examples.

EXAMPLE 4.7:

Let $A : \mathbb{Z}^n \to \mathbb{R}^n$ be a nonsingular linear mapping, and let $\mathcal{L} = [A]$ be the lattice generated by A. Let $f(x)$ be a continuous function on $\mathbb{R}^n$, and let $g = f^{A^{-1}}$. Define a discrete function $\tilde{g}(m)$ on $\mathbb{Z}^n$ by the rule[5]

$$\tilde{g}(m) = f(Am) .$$

[5]This is a common situation when we have to sample a continuous function (on points of the form An) and store it in some rectangular storage space (with addresses n).

The question is how the Fourier transforms of $f(x)$ and $\tilde{g}(k)$ are related. To answer this question we define $\tilde{f}(\lambda)$ to be the sampled version $\Sigma_{\mathcal{L}}(f)(\lambda)$ of $f(x)$. The following commutative diagram results.

$$\begin{array}{ccc} (\mathbb{R}^n, g) & \stackrel{A^{-1}}{\longleftarrow} & (\mathbb{R}^n, f) \\ \downarrow \Sigma_{\mathbb{Z}^n} & & \downarrow \Sigma_{\mathcal{L}} \\ (\mathbb{Z}^n, \tilde{g}) & \stackrel{A^{-1}}{\longleftarrow} & (\mathcal{L}, \tilde{f}) \end{array}$$

Tracing the diagram from top right to bottom right to bottom left we find

$$\begin{aligned} \tilde{\mathcal{F}}(\tilde{g})(\nu) &= (\tilde{\mathcal{F}}(\tilde{f}))^{A^t}(\nu) \\ &= \det\left(\mathcal{L}^*\right) \sum_{\lambda^* \in \mathcal{L}^*} \left(\mathcal{F}(f)^{A^t}(\nu - \lambda^*)\right) \\ &= \frac{1}{\det(A)} \sum_{\lambda^* \in \mathcal{L}^*} \mathcal{F}(f)\left(A^{-t}\nu - \lambda^*\right) , \end{aligned}$$

where we have used Theorem 4.8 and Theorem 4.3 in the first and second steps, respectively. Of course we should find the same result tracing the diagram from top right to top left to bottom left.

$$\begin{aligned} \tilde{\mathcal{F}}(\tilde{g})(\nu) &= \sum_{k \in \mathbb{Z}^n} \mathcal{F}(g)(\nu - k) \\ &= \sum_{k \in \mathbb{Z}^n} \mathcal{F}\left(f^{A^{-1}}\right)(\nu - k) \\ &= \frac{1}{\det(A)} \sum_{k \in \mathbb{Z}^n} \mathcal{F}(f)^{A^t}(\nu - k) \\ &= \frac{1}{\det(A)} \sum_{k \in \mathbb{Z}^n} \mathcal{F}(f)\left(A^{-t}\nu - A^{-t}k\right) \\ &= \frac{1}{\det(A)} \sum_{\lambda^* \in \mathcal{L}^*} \mathcal{F}(f)\left(A^{-t}\nu - \lambda^*\right) , \end{aligned}$$

where we have first applied Theorem 4.3, followed by an application of Theorem 4.8. As one sees, both calculations end up with the same result.

EXAMPLE 4.8:

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two lattices. Let $A : \mathcal{L}_1 \to \mathcal{L}_2$ be a nonsingular linear mapping, and let $\tilde{f}$ be a function on $\mathcal{L}_1$. Let $\mathcal{L}_3$ be the lattice generated by A, $\mathcal{L}_3 = [A] \subset \mathcal{L}_2$. Define $\tilde{g}$ on $\mathcal{L}_2$ by

$$\tilde{g}(\lambda_2) = \begin{cases} \tilde{f}(\lambda_1) & \text{if } \lambda_2 = A\lambda_1 \\ 0 & \text{otherwise.} \end{cases}$$

The question is to find an expression for the DSFT of $\tilde{g}$. To this end we define $\tilde{h}$ on $\mathcal{L}_3$ by $\tilde{h} = \tilde{f}^A$. The following diagram results.

$$\left(\mathcal{L}_1, \tilde{f}\right) \stackrel{A}{\longrightarrow} \left(\mathcal{L}_3, \tilde{h}\right) \stackrel{\text{extension}}{\longrightarrow} (\mathcal{L}_2, \tilde{g})$$

For the DSFT of $\tilde{g}$ we find

$$\begin{aligned}\tilde{\mathcal{F}}(\tilde{g})(\nu) &= \tilde{\mathcal{F}}\left(\tilde{h}\right)(\nu) \\ &= \tilde{\mathcal{F}}\left(\tilde{f}^A\right)(\nu) \\ &= \tilde{\mathcal{F}}\left(\tilde{f}\right)^{A^{-t}}(\nu) \\ &= \tilde{\mathcal{F}}(\tilde{f})(A^t\nu)\,,\end{aligned}$$

where we have used Theorem 4.9 and Theorem 4.8 in the first and second step, respectively.

4.7 An Extended Example: HDTV-to-SDTV Conversion

This section will introduce an application of sampling theory as it occurs in the problem of interlaced high definition television (HDTV) to interlaced standard definition television (SDTV) conversion. This problem exists because an HDTV broadcast can at present only be viewed by a minority of people. Most people can only view SDTV broadcast. As broadcasters like their programs to be viewed by as many customers as possible, they are interested in (preferably inexpensive) schemes which can convert HDTV in SDTV. In this section we present an approach to this conversion problem as has been suggested in [1].

In order to keep the notational burden low, our television signal will be one-dimensional. This leaves us with a spatial axis, referred to as the y-axis (y for vertical), and a time axis, referred to as the t-axis.

An interlaced television signal is constructed by sampling a continuous luminance signal with at times kT, but only even lines for even k and only the odd lines for odd k. Choosing T to be 1 in some unit of time, and recalling that we assume one-dimensional images, we may model an interlaced HDTV signal as a luminance signal sampled at the quincunx lattice $\mathcal{L}_2$ generated by the matrix

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

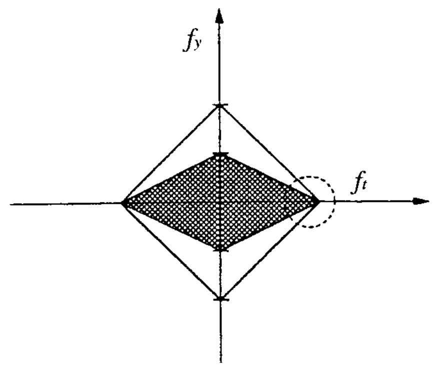
In order to prevent alias distortion, i.e., in order to prevent that frequencies overlap after sampling, the continuous luminance signal has to be sufficiently band limited. An often-used pass band region is given by the diamond in Fig. 4.3(c).

An SDTV interlaced signal has half the vertical resolution of the HDTV signal, but the same temporal resolution, and we may model this as the sampling of the continuous luminance signal on the skew quincunx lattice $\mathcal{L}_1$ generated by the matrix

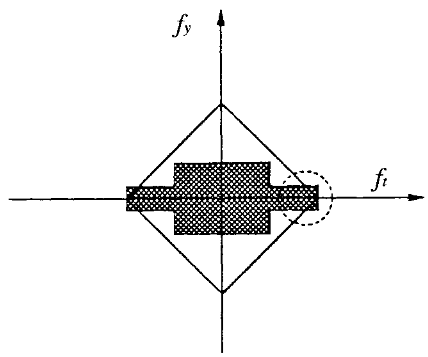
$$\begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}.$$

Note that the lattice $\mathcal{L}_1$ is not a sublattice of the $\mathcal{L}_2$. This has the consequence that the extraction of an SDTV signal from an HDTV signal is not simply a question of subsampling the HDTV signal; interpolation is needed to compute the values of the luminance signal at the missing points. In the frequency domain this is equivalent to restricting the pass band region of the HDTV signal to a smaller pass band region, such that no alias occurs when the interpolated signal is sampled to the SDTV lattice.

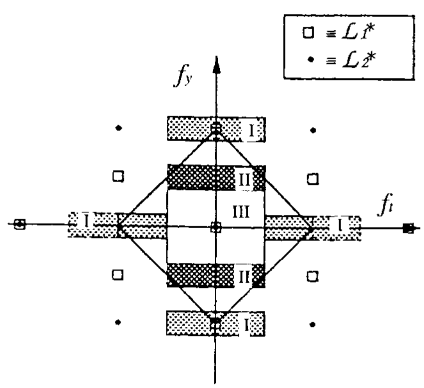
Figure 4.3(a) gives a possible solution. The SDTV pass band region is chosen as the skew diamond region within the HDTV pass band (the outer diamond). This solution has several disadvantages. One disadvantage is the fact that the realization of this diamond pass band region can only be realized



(a)



(b)



(c)

FIGURE 4.3: HDTV-to-SDTV conversion in the frequency domain.

by nonseparable filters, and, therefore, that it is expensive. A second disadvantage is the temporal attenuation at maximum temporal frequency, which may introduce visible artifacts for moving video.

As argued in [1], the best compromise between vertical resolution and temporal attenuation at maximum temporal frequency is given by a pass band of the form as given in Fig. 4.3(b). This pass band can even be realized cheaply.

Following [1] we note that the temporal information at maximum frequency (region I on the f_t-axis in Fig. 4.3(c)) is repeated at maximal vertical frequency (region I on the f_y-axis in Fig. 4.3(c)). This is simply a consequence of the fact that the DSFT of the HDTV signal is $\mathcal{L}_2^*$-periodic. We can retain this information by using an appropriately chosen *vertical high pass* filter. In a practical implementation this implies that (after temporal low-pass filtering) we extract from the HDTV signal a base-band signal using a vertical low-pass filter (the rectangle III in Fig. 4.3(c)) and a *temporal* band using a vertical high-pass filter. The temporal band is now modulated to position II in Fig. 4.3(c) by multiplying the sample at position $(2k, t)$ with $(-1)^k$.

The base band and the temporal band are now merged and sampled to the SDTV lattice. Due to this last sampling operation, region II is repeated at its original position I in frequency space: this follows immediately from computing the reciprocal SDTV quincunx lattice.

This proves (as first shown in [1]) that a high quality HDTV-to-SDTV conversion can be achieved using only separable filters.

4.8 Conclusions

We have presented the basic facts of multidimensional sampling theory. Particular attention has been paid to the interaction of the different kinds of Fourier transforms, the sampling operator, and the periodizing operator. Every basic result is accompanied by one or more examples. An application of the theory to a format conversion problem has been presented.

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Appendix

A.1 Proof of Theorem 4.3

PROOF 4.7 We first observe that

$$\begin{aligned}\Sigma_{\mathcal{L}}(f) &= f \cdot Ш_{\mathcal{L}},\\ \Pi_{\mathcal{L}}(F) &= F * Ш_{\mathcal{L}^*}.\end{aligned}$$

It follows immediately that $\mathcal{F}(\Sigma_{\mathcal{L}}(f)) = \Pi_{\mathcal{L}^*}(\mathcal{F}(f))$. To prove the first assertion of this theorem, it suffices to verify that $\tilde{\mathcal{F}}(\tilde{f}) = \tilde{F}$.

$$\begin{aligned}\tilde{F}(\nu) &= \mathcal{F}(f \cdot Ш_{\mathcal{L}})(\nu)\\ &= \int_{\mathbb{R}^n} \sum_{\lambda\in\mathcal{L}} e^{-2\pi i\langle x,\nu\rangle} f(x)\delta_\lambda(x)dx\\ &= \sum_{\lambda\in\mathcal{L}} e^{-2\pi i\langle \lambda,\nu\rangle} f(\lambda)\\ &= \tilde{\mathcal{F}}(\tilde{f}).\end{aligned}$$

The second assertion of the theorem, viz. the isometry property of the DSFT, follows from

$$\begin{aligned}\langle \tilde{F}, \tilde{G}\rangle_{P^*} &= \frac{1}{\det(\mathcal{L})^2}\int_{P^*} \langle Ш_{\mathcal{L}^*} * F, Ш_{\mathcal{L}^*} * G\rangle_{P^*}\\ &= \frac{1}{\det(\mathcal{L})^2}\int_{P^*} \left(\sum_{\lambda_1^*\in\mathcal{L}^*} F(\nu-\lambda_1^*)\right)\left(\sum_{\lambda_1^*\in\mathcal{L}^*} G(\nu-\lambda_2^*)\right)d\nu\\ &= \frac{1}{\det(\mathcal{L})^2}\int_{\mathbb{R}^n} F(\nu)\left(\sum_{\lambda^*\in\mathcal{L}^*} G(\nu-\lambda^*)\right)d\nu\\ &= \frac{1}{\det(\mathcal{L})}\langle F, \tilde{G}\rangle\\ &= \frac{1}{\det(\mathcal{L})}\langle f, \tilde{g}\rangle\\ &= \frac{1}{\det(\mathcal{L})}\langle \tilde{f}, \tilde{g}\rangle_{\mathcal{L}}.\end{aligned}$$

A.2 Proof of Theorem 4.5

PROOF 4.8 Similar to the proof of Theorem 4.3, to prove the first assertion it suffices to show that $\hat{\mathcal{F}}(\hat{f}) = \hat{F}$.

$$\tilde{\mathcal{F}}(\hat{f})(\nu) = \sum_{\lambda\in\mathcal{L}} e^{-2\pi i\langle\lambda,\nu\rangle}\hat{f}(\lambda)$$

$$
\begin{aligned}
&= \left(\sum_{\mu\in\mathcal{M}} e^{-2\pi i\langle \mu,\nu\rangle}\right)\left(\sum_{p\in P} e^{-2\pi i\langle p,\nu\rangle}\hat{f}(p)\right)\\
&= \frac{1}{\det(\mathcal{M})}\,Ш_{\mathcal{M}^*}\cdot\left(\sum_{p\in P} e^{-2\pi i\langle p,\nu\rangle}\hat{f}(p)\right)\\
&= Ш_{\mathcal{M}^*}\cdot\hat{\mathcal{F}}(\hat{f})(\nu).
\end{aligned}
$$

The isometry property of the DFT follows from

$$
\begin{aligned}
\langle \hat{f},\hat{g}\rangle_P &= \sum_{p\in P}\hat{f}^\dagger(p)\hat{g}(p)\\
&= \det(\mathcal{M})^2\sum_{p\in P}\left(\sum_{\mu_1\in\mathcal{M}}\tilde{f}^\dagger(p-\mu_1)\right)\left(\sum_{\mu_2\in\mathcal{M}}\tilde{g}(p-\mu_2)\right)\\
&= \det(\mathcal{M})^2\sum_{\lambda\in\mathcal{L}}\tilde{f}^\dagger(\lambda)\left(\sum_{\mu\in\mathcal{M}}\tilde{g}(\lambda-\mu)\right)\\
&= \det(\mathcal{M})\langle\tilde{f},\tilde{g}\rangle_{\mathcal{L}}\\
&= \det(\mathcal{M})^2\langle f, Ш_{\mathcal{L}}\cdot(Ш_{\mathcal{M}} * g)\rangle\\
&= \frac{\det(\mathcal{M})}{\det(\mathcal{L})}\langle F, Ш_{\mathcal{L}^*} * (Ш_{\mathcal{M}^*}\cdot G)\rangle\\
&= \frac{\det(\mathcal{M})}{\det(\mathcal{L})}\langle F, Ш_{\mathcal{M}^*}\cdot(Ш_{\mathcal{L}^*} * G)\rangle\\
&= \det(\mathcal{M})\det(\mathcal{L})\langle\hat{F},\hat{G}\rangle_{P^*}.
\end{aligned}
$$

The last step in this derivation follows from reversing the other steps, replacing the spatial functions f and g by their frequency domain counterparts F and G.

A.3 Proof of Theorem 4.6

PROOF 4.9

$$
\begin{aligned}
\tilde{F}^{(2)}(\nu) &= \frac{1}{\det(\mathcal{L}_2)}\sum_{\lambda_2^*\in\mathcal{L}_2^*}F(\nu-\lambda_2^*)\\
&= \frac{1}{\det(\mathcal{L}_2)}\sum_{p^*\in P^*}\sum_{\lambda_1^*\in\mathcal{L}_1^*}F(\nu-p^*-\lambda_1^*)\\
&= \frac{\det(\mathcal{L}_1)}{\det(\mathcal{L}_2)}\sum_{p^*\in P^*}\tilde{F}^{(1)}(\nu-p^*)\\
&= \frac{1}{\iota(\mathcal{L}_2,\mathcal{L}_1)}\sum_{p^*\in P^*}\tilde{F}^{(1)}(\nu-p^*)\\
&= \frac{1}{\#(P^*)}\sum_{p^*\in P^*}\tilde{F}^{(1)}(\nu-p^*).
\end{aligned}
$$

A.4 Proof of Theorem 4.7

PROOF 4.10

$$\begin{aligned}
\mathcal{F}(f^A)(\nu) &= \int_{\mathbb{R}^n} e^{-2\pi i\langle x,\nu\rangle} f^A(x)dx \\
&= \int_{\mathbb{R}^n} e^{-2\pi i\langle x,\nu\rangle} f(A^{-1}x)dx \\
&= |\det(A)| \int_{\mathbb{R}^n} e^{-2\pi i\langle Ay,\nu\rangle} f(y)dy \\
&= |\det(A)| \int_{\mathbb{R}^n} e^{-2\pi i\langle y,A^t\nu\rangle} f(y)dy \\
&= |\det(A)| F(A^t\nu) \\
&= |\det(A)| F^{A^{-t}}(\nu).
\end{aligned}$$

A.5 Proof of Theorem 4.8

PROOF 4.11

$$\begin{aligned}
\tilde{\mathcal{F}}(\tilde{f}^A)(\nu) &= \sum_{\lambda_2\in\mathcal{L}_2} e^{-2\pi i\langle\lambda_2,\nu\rangle} \tilde{f}^A(\lambda_2) \\
&= \sum_{\lambda_2\in\mathcal{L}_2} e^{-2\pi i\langle\lambda_2,\nu\rangle} \tilde{f}(A^{-1}\lambda_2) \\
&= \sum_{\lambda_1\in\mathcal{L}_1} e^{-2\pi i\langle A\lambda_1,\nu\rangle} \tilde{f}(\lambda_1) \\
&= \sum_{\lambda_1\in\mathcal{L}_1} e^{-2\pi i\langle\lambda_1,A^t\nu\rangle} \tilde{f}(\lambda_1) \\
&= \tilde{\mathcal{F}}(\tilde{f})^{A^{-t}}(\nu).
\end{aligned}$$

Glossary of Symbols and Expressions

$\mathbb{Z}^n$	n-dimensional integer space
$\mathbb{R}^n$	n-dimensional real space
$\mathbb{C}^n$	n-dimensional complex space
CSFT	Continuous space-time Fourier transform
DSFT	Discrete space-time Fourier transform
DFT	Discrete Fourier transform
$\mathcal{L}$, $\mathcal{M}$	Sampling lattice
λ, μ	Elements of lattice $\mathcal{L}$, $\mathcal{M}$
λ^*, μ^*	Elements of reciprocal lattice $\mathcal{L}^*$, $\mathcal{M}^*$
$[L]$	Lattice generated by matrix L
$\#(A)$	Number of points of set A
vol(A)	Volume (measure) of set A

$\det(\mathcal{L})$	Determinant of lattice $\mathcal{L}$
$\iota(\mathcal{M}, \mathcal{L})$	Index of lattice $\mathcal{M}$ w.r.t. lattice $\mathcal{L}$
$\mathcal{L}/\mathcal{M}$	Coset group of lattice $\mathcal{M}$ w.r.t. lattice $\mathcal{L}$
$\mathcal{L}^*$	Reciprocal lattice of $\mathcal{L}$
$Ш_{\mathcal{L}}$	Lattice comb
P	Fundamental domain
$\|\alpha\|_2$	L_2-norm of α
α^t	Hermitian transpose of α
$\langle\alpha, \beta\rangle_{\mathcal{N}}$	Inner products of α and β with respects to $\mathcal{N}$-norm
$\alpha^\dagger$	Complex conjugate of α
$\alpha \cdot \beta$	Point-wise multiplication
$\alpha * \beta$	Convolution
$f^A(x)$	Change of variables $f(A^{-1}x)$
X_A	Characteristic function of set A
$\mathcal{F}$	Continuous space-time Fourier transform
$\tilde{\mathcal{F}}$	Discrete space-time Fourier transform
$\hat{\mathcal{F}}$	Discrete Fourier transform
$\Sigma_{\mathcal{L}}$	Sampling operator
$\Pi_{\mathcal{L}}$	Periodizing operator

$$\mathrm{sinc}(x) \quad \begin{cases} \sin(\pi x)/\pi x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$