1

Bases for $L^2(\mathbb{R})$

Classical systems of orthonormal bases for $L^2([0,1))$ include the exponentials $\{e^{2\pi imx} : m \in \mathbb{Z}\}$ and various appropriate collections of trigonometric functions. (See Theorem 4.1 below.) The analogs of these bases for $L^2([\alpha,\beta))$, $-\infty < \alpha < \beta < \infty$, are obtained by appropriate translations and dilations of the ones above. To find an orthonormal basis for $L^2(\mathbb{R})$ we can cover $\mathbb{R}$ with a disjoint union of intervals

$$[\alpha_j, \alpha_{j+1}), \qquad j \in \mathbb{Z}, \quad -\infty < \cdots < \alpha_j < \alpha_{j+1} < \cdots < \infty,$$

and consider one of these bases for each space $L^2([\alpha_j, \alpha_{j+1}))$, multiply the basis elements by the characteristic function of $[\alpha_j, \alpha_{j+1})$, and take the totality of the functions so obtained. This orthonormal basis, however, produces "undesirable edge effects" at the endpoints α_j when we try to represent a function in terms of it.

In order to remedy this situation one is led to consider smooth functions that replace the characteristic function of $[\alpha_j, \alpha_{j+1})$ for $j \in \mathbb{Z}$. In the case of complex exponentials and the simple partition

$$\mathbb{R} = \bigcup_{n\in\mathbb{Z}} [n, n+1)$$

we examine systems of the form

$$\big\{g_{m,n}(x) = e^{2\pi imx} g(x-n) : m, n \in \mathbb{Z}\big\}.$$

For a system of this type (often called a **Gabor basis**) to be an orthonormal basis for $L^2(\mathbb{R})$ g cannot be "too smooth" or "very localized." This is made precise by the Balian-Low theorem presented in section 1.2. If appropriate bases of sine (or cosine) functions, however, are used, a much more general family of functions g, arbitrarily smooth and "very localized," can be used to obtain orthonormal bases of $L^2(\mathbb{R})$.

This is done in section 1.3 where we present a theory of **smooth projections**, introduced by Coifman and Meyer, that allows us to "join" appropriate bases associated with the intervals $[\alpha_j, \alpha_{j+1})$. Several examples of this construction are given, but the most relevant for our purpose are the ones that produce **orthonormal wavelets**: $\psi \in L^2(\mathbb{R})$ such that

$$\psi_{j,k}(x) = 2^{\frac{j}{2}}\psi(2^j x - k), \qquad j, k \in \mathbb{Z},$$

is an orthonormal basis for $L^2(\mathbb{R})$. It is in this way that, in section 1.4, we construct the wavelets of Lemarié and Meyer.

In section 1.5 we describe the smooth projections presented in section 1.3 in terms of certain unitary "folding operators." Some theoretical results can be obtained in an elegant manner by using these operators; however, it is, perhaps, more important that they provide some simple ways for programming the uses of these local bases. Furthermore, this approach does lend itself to extending the theory to higher dimensions.

1.1 Preliminaries

We assume that the reader is familiar with the basic notions of Lebesgue measure and integration theory, Hilbert space theory and Functional Analysis. We begin by introducing some notation and a few results that we shall assume. $\mathbb{R}$ refers to the real line; $\mathbb{T}$ will denote the unit circle in the complex plane which can be identified with the interval $[-\pi, \pi)$, though sometimes we use the interval $[-\frac{1}{2}, \frac{1}{2})$ or $[0, 1)$; and $\mathbb{Z}$ will denote the collection of integers. The inner product of functions f and g defined on either of these two spaces is

$$<f, g> = \int f\,\overline{g}\,,$$

where the integral is taken over $\mathbb{R}$ or over $\mathbb{T}$, as the case may be. We have Schwarz's inequality

$$|<f, g>| \leq \|f\|_2 \|g\|_2,$$

where

$$\|f\|_2 = \left(\int |f|^2\right)^{\frac{1}{2}}$$

is the L^2-norm of f. Schwarz's inequality allows us to prove Minkowski's inequality

$$\|f+g\|_2 \le \|f\|_2 + \|g\|_2.$$

We say that two functions f and g are **orthogonal**, and write $f \perp g$, when $<f, g> = 0$. A sequence of functions $\{f_n\}_{n\in\mathbb{Z}}$ is an **orthonormal sequence** if $<f_m, f_n> = \delta_{m,n}$, where

$$\delta_{m,n} = \begin{cases} 1, & \text{if } \; n = m, \\ 0, & \text{if } \; n \neq m. \end{cases}$$

A well known example of an orthonormal sequence on $\mathbb{T} = [-\pi, \pi)$ is $\left\{\frac{1}{\sqrt{2\pi}}\, e_n\right\}_{n\in\mathbb{Z}}$, where $e_n(x) = e^{inx}$.

Given an orthonormal system $\{f_n : n \in \mathbb{Z}\}$ and a function f, we define the Fourier coefficients of f with respect to $\{f_n : n \in \mathbb{Z}\}$ to be

$$c_k = <f, f_k>, \qquad k \in \mathbb{Z}.$$

A basic question that we shall study is to determine when, and in what sense, it is true that

$$f = \sum_{k\in\mathbb{Z}} c_k f_k. \tag{1.1}$$

When $f_k(x) = e^{ikx}$, $k \in \mathbb{Z}$, and $f \in L^2(\mathbb{T})$, the representation (1.1) is valid in the L^2-norm sense. In general, when this is the case we say that $\{f_k : k \in \mathbb{Z}\}$ is an **orthonormal basis** for $L^2(\mathbb{T})$. Equality (1.1) is a reconstruction formula and it is the basis for many applications of the theory of wavelets. Given a function f (a signal or a sound), we can encode it by means of the coefficients $\{c_k\}_{k\in\mathbb{Z}}$. Equality (1.1) allows us to reconstruct the signal from the numbers c_k and the basis used in the encoding. Some bases, in particular wavelet bases, perform this job more efficiently than others. For any orthonormal system $\{f_n : n \in \mathbb{Z}\}$ we have Bessel's inequality

$$\sum_{k\in\mathbb{Z}} |c_k|^2 \le \|f\|_2^2.$$

Moreover, if the system is a basis, we have equality. Conversely, if an orthonormal system $\{f_n : n \in \mathbb{Z}\}$ satisfies

$$\sum_{k\in\mathbb{Z}} |c_k|^2 = \|f\|_2^2 \tag{1.2}$$

for all $f \in L^2(\mathbb{R})$, the system is a basis for $L^2(\mathbb{R})$.

In $\mathbb{R}$ we have an "analogous" theory. The Fourier transform of a function $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is defined by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}\,dx.$$

We will often say that x is the "time" variable and ξ will be referred as the "frequency" variable.

The inverse Fourier transform is

$$\check{g}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\xi)e^{i\xi x}\,d\xi,$$

and if we apply it to $g = \hat{f}$ we obtain f; that is $(\hat{f}\,)^{\vee} = f$. With this definition the Plancherel theorem asserts that

$$< f, g > = \frac{1}{2\pi} < \hat{f}, \hat{g} > . \tag{1.3}$$

The Fourier transform extends to all $f \in L^2(\mathbb{R})$ and the operator $f \mapsto \frac{1}{\sqrt{2\pi}}\hat{f}$ is unitary. When f' exists in the L^2 sense, then

$$\widehat{f'}(\xi) = i\xi\hat{f}(\xi). \tag{1.4}$$

It can be proved that the integration by parts formula

$$\int_{-\infty}^{\infty} f'(x)g(x)\,dx = -\int_{-\infty}^{\infty} f(x)g'(x)\,dx \tag{1.5}$$

is valid when $f, g \in L^2(\mathbb{R})$ and $f'g, fg' \in L^1(\mathbb{R})$. In the case $f, g, f', g' \in L^2(\mathbb{R})$ this can be proved using (1.3) and (1.4).

A notion, which will be used in several proofs, is that of a Lebesgue point. Suppose f is a measurable function which is locally integrable, then a point

x_o is called a **Lebesgue point** for f whenever

$$\lim_{\delta \to 0+} \frac{1}{2\delta} \int_{x_o-\delta}^{x_o+\delta} |f(x) - f(x_o)|\, dx = 0.$$

It follows from the Lebesgue Differentiation Theorem that almost every x_o is a Lebesgue point. The reader may consult [Rud] for this particular theorem as well as for other results in measure theory.

Three simple operators on functions defined on $\mathbb{R}$ play an important role in our theory: **translation by** h, τ_h, defined by $(\tau_h f)(x) = f(x-h)$, **dilation by** r, ρ_r, defined by $(\rho_r f)(x) = f(rx)$ and **multiplication by** e^{imx}. (Sometimes referred to as a **modulation** operator.) One of our main goals is to construct orthonormal bases of $L^2(\mathbb{R})$ by applying some of these operators to a single function in $L^2(\mathbb{R})$.

Of particular interest to us are the wavelet bases for which the first two operators are applied to an appropriate function. More precisely, an **orthonormal wavelet on** $\mathbb{R}$ is a function $\psi \in L^2(\mathbb{R})$ such that $\{\psi_{j,k} : j,k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$, where

$$\psi_{j,k}(x) = 2^{\frac{j}{2}} \psi(2^j x - k), \qquad j,k \in \mathbb{Z}.$$

Observe that the $\psi_{j,k}$ are normalized so that $\|\psi_{j,k}\|_2 = \|\psi\|_2 = 1$ for all $j,k \in \mathbb{Z}$.

EXAMPLE A: If

$$\psi(x) = \begin{cases} 1, & \text{if } 0 \le x < \frac{1}{2}, \\ -1, & \text{if } \frac{1}{2} \le x < 1, \\ 0, & \text{elsewhere,} \end{cases}$$

then ψ is an orthonormal wavelet for $L^2(\mathbb{R})$. This is called the **Haar wavelet**. It is easy to prove that $\{\psi_{j,k} : j,k \in \mathbb{Z}\}$ is an orthonormal system in $L^2(\mathbb{R})$. It is also a basis for $L^2(\mathbb{R})$, a fact that will become obvious when we develop the theory of "multiresolution analysis" in Chapter 2.

EXAMPLE B: Let ψ be such that $\hat{\psi}(\xi) = \chi_I(\xi)$, where

$$I = [-2\pi, -\pi] \cup [\pi, 2\pi].$$

We shall show that ψ is an orthonormal wavelet for $L^2(\mathbb{R})$. A simple calculation shows

$$(\psi_{j,k})^{\wedge}(\xi) = 2^{-\frac{j}{2}}\hat{\psi}(2^{-j}\xi)e^{-i2^{-j}k\xi}.$$

For $j \neq \ell$ this equality shows that the intersection of the supports of $(\psi_{j,k})^{\wedge}$ and $(\psi_{\ell,m})^{\wedge}$ has measure zero; hence,

$$<\psi_{j,k}, \psi_{\ell,m}> = \frac{1}{2\pi} <(\psi_{j,k})^{\wedge}, (\psi_{\ell,m})^{\wedge}> = 0 \qquad \text{for} \;\; j \neq \ell.$$

When $j = \ell$ we can write

$$\begin{aligned} <\psi_{j,k}, \psi_{j,m}> &= \frac{1}{2\pi}2^{-j}\int_{\mathbb{R}} |\hat{\psi}(2^{-j}\xi)|^2 e^{i2^{-j}(m-k)\xi}\, d\xi \\ &= \frac{1}{2\pi}\Big\{\int_{-2\pi}^{-\pi} e^{i(m-k)\eta}\, d\eta + \int_{\pi}^{2\pi} e^{i(m-k)\eta}\, d\eta\Big\} = \delta_{k,m}. \end{aligned}$$

To prove that the system is a basis we use (1.2). The Plancherel theorem and a change of variables allow us to write

$$\begin{aligned} \sum_{j,k\in\mathbb{Z}}\sum | <f, \psi_{j,k}> |^2 &= \sum_{j,k\in\mathbb{Z}}\sum \frac{2^{-j}}{4\pi^2}\Big|\int_{\mathbb{R}} \hat{f}(\xi)\,\overline{\hat{\psi}(2^{-j}\xi)}\, e^{i2^{-j}k\xi}\, d\xi\Big|^2 \\ &= \sum_{j\in\mathbb{Z}} \frac{2^j}{2\pi} \sum_{k\in\mathbb{Z}}\Big|\int_I \hat{f}(2^j\mu)\frac{e^{ik\mu}}{\sqrt{2\pi}}\, d\mu\Big|^2. \end{aligned}$$

We now use the fact that the system $\{\frac{1}{\sqrt{2\pi}}e^{ik\mu} : k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(I)$ (a fact that is equivalent to the orthonormality of the same system on $[0, 2\pi]$) to write

$$\begin{aligned} &\sum_{j,k\in\mathbb{Z}}\sum | <f, \psi_{j,k}> |^2 = \sum_{j\in\mathbb{Z}} \frac{2^j}{2\pi}\int_I \big|\hat{f}(2^j\mu)\big|^2\, d\mu \\ &= \frac{1}{2\pi}\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}} \chi_I(2^{-j}\xi)\big|\hat{f}(\xi)\big|^2\, d\xi = \frac{1}{2\pi}\|\hat{f}\|_2^2 = \|f\|_2^2, \end{aligned}$$

since

$$\sum_{j\in\mathbb{Z}} \chi_I(2^{-j}\xi) = 1 \qquad \text{for} \;\; a.e.\ \xi \in \mathbb{R}.$$

This shows that ψ is an orthonormal wavelet for $L^2(\mathbb{R})$. This is related to the **Shannon wavelet** which will be described in Example C of Chapter 2.

1.2 Orthonormal bases generated by a single function; the Balian-Low theorem

Another way of producing an orthonormal basis from a single function involves translations and modulations. For example, a basis for $L^2(\mathbb{R})$ is the following: let $g = \chi_{[0,1]}$ and

$$g_{m,n}(x) = e^{2\pi i m x} g(x-n) \qquad \text{for} \quad m, n \in \mathbb{Z}.$$

It is not difficult to see that $\{g_{m,n} : m, n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$. D. Gabor ([Gab]) considered this type of system in 1946 and proposed its use for communication purposes. For a general $g \in L^2(\mathbb{R})$ the following theorem gives conditions that g must satisfy if the system $\{g_{m,n} : m, n \in \mathbb{Z}\}$ is an orthonormal basis.

THEOREM 2.1 **(Balian-Low)** *Suppose $g \in L^2(\mathbb{R})$ and*

$$g_{m,n}(x) = e^{2\pi i m x} g(x-n), \quad m, n \in \mathbb{Z}.$$

If $\{g_{m,n} : m, n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$, then either

$$\int_{-\infty}^{\infty} x^2 |g(x)|^2 \, dx = \infty \qquad \text{or} \qquad \int_{-\infty}^{\infty} \xi^2 |\hat{g}(\xi)|^2 \, d\xi = \infty.$$

PROOF : We introduce the operators Q and P, defined on, say, the space $\mathcal{S}'$ of tempered distributions, given by

$$(Qf)(x) = xf(x) \qquad \text{and} \qquad (Pf)(x) = -if'(x).$$

The relevance of these operators to the theorem is that

$$\int_{-\infty}^{\infty} \left|Qg(x)\right|^2 dx = \int_{-\infty}^{\infty} x^2 \left|g(x)\right|^2 dx$$

and

$$\int_{-\infty}^{\infty} \left|Pg(x)\right|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \xi^2 \left|g(\xi)\right|^2 d\xi,$$

where the last formula is a consequence of (1.3) and (1.4). Hence, we need to show that both (Qg) and (Pg) **cannot** belong to $L^2(\mathbb{R})$.

Suppose that both Qg and Pg belong to $L^2(\mathbb{R})$. We will show that this leads to a contradiction, and this proves the theorem. We claim that

$$<Qg, Pg> = \sum_{m,n\in\mathbb{Z}} <Qg, g_{m,n}><g_{m,n}, Pg>, \tag{2.2}$$

$$<Qg, g_{m,n}> = <g_{-m,-n}, Qg> \quad \text{for all} \;\; m,n \in \mathbb{Z}, \tag{2.3}$$

and

$$<Pg, g_{m,n}> = <g_{-m,-n}, Pg> \quad \text{for all} \;\; m,n \in \mathbb{Z}. \tag{2.4}$$

Equalities (2.2), (2.3) and (2.4) imply

$$<Qg, Pg> = <Pg, Qg> . \tag{2.5}$$

But (2.5) cannot hold if Pg and Qg belong to $L^2(\mathbb{R})$. If this were the case we could apply the integration by parts formula (1.5) to obtain

$$\begin{aligned} <Qg, Pg> &= \int_{-\infty}^{\infty} xg(x) \left\{\overline{-ig'(x)}\right\} dx \\ &= -i \int_{-\infty}^{\infty} \{g(x) + xg'(x)\} \, \overline{g(x)} \, dx \\ &= -i <g, g> + <Pg, Qg> . \end{aligned}$$

Since $<g, g> = \left\|g\right\|_2^2 = \left\|g_{0,0}\right\|_2^2 = 1$ we obtain

$$<Qg, Pg> = -i + <Pg, Qg>,$$

which contradicts (2.5).

Hence, the theorem is proved if we establish (2.2), (2.3) and (2.4). Since $Qg, Pg \in L^2(\mathbb{R})$ and $\{g_{m,n} : m,n \in \mathbb{Z}\}$ is an orthonormal basis we have

$$\begin{aligned} <Qg, Pg> &= <\sum\sum_{m,n\in\mathbb{Z}} <Qg, g_{m,n}> g_{m,n}, Pg > \\ &= \sum\sum_{m,n\in\mathbb{Z}} <Qg, g_{m,n}><g_{m,n}, Pg>, \end{aligned}$$

which proves (2.2). To prove (2.3) observe that $n < g, g_{m,n} >= 0$ for all $m, n \in \mathbb{Z}$; this obviously holds for $n = 0$ and if $n \neq 0$, $g = g_{0,0}$ is orthogonal to $g_{m,n}$. Thus,

$$
\begin{aligned}
&<Qg, g_{m,n}> = <Qg, g_{m,n}> - n <g, g_{m,n}> \\
&= \int_{-\infty}^{\infty} g(x)(x-n)\,\overline{g(x-n)}\, e^{-2\pi imx}\, dx \\
&= \int_{-\infty}^{\infty} g(y+n)\, y\, \overline{g(y)}\, e^{-2\pi im(y+n)}\, dy = <g_{-m,-n}, Qg>,
\end{aligned}
$$

which gives us (2.3). To prove (2.4) we use the integration by parts formula (1.5) to obtain

$$
\begin{aligned}
<Pg, g_{m,n}> &= -i \int_{-\infty}^{\infty} g'(x)\,\overline{g(x-n)}\, e^{-2\pi imx}\, dx \\
&= i \int_{-\infty}^{\infty} g(x)\{-2\pi im\,\overline{g(x-n)} + \overline{g'(x-n)}\}\, e^{-2\pi imx}\, dx \\
&= 2\pi m \delta_{m,0}\delta_{0,n} + \int_{-\infty}^{\infty} g(y+n)\{\overline{-ig'(y)}\}\, e^{-2\pi imy}\, dy \\
&= <g_{-m,-n}, Pg>.
\end{aligned}
$$

∎

Example C: For $g = \chi_{[0,1)}$, $\{g_{m,n} : m, n \in \mathbb{Z}\}$, as we have seen, is an orthonormal basis of $L^2(\mathbb{R})$; in this case the first integral in the announcement of the Balian-Low theorem is finite, but the second is infinite since

$$
\xi^2 \left|(\chi_{[0,1)})^\wedge(\xi)\right|^2 = \left[2\sin(\tfrac{\xi}{2})\right]^2.
$$

Example D: For $g(x) = \frac{\sin(\pi x)}{\pi x} \equiv \operatorname{sinc}(x)$, $\{g_{m,n} : m, n \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$; observe that

$$
(\chi_{[0,1)})^\wedge(\xi) = e^{-i\frac{\xi}{2}} \frac{\sin(\xi/2)}{(\xi/2)} = e^{-i\frac{\xi}{2}} \operatorname{sinc}(\tfrac{\xi}{2\pi}).
$$

In this case the first integral in the announcement of the Balian-Low theorem is infinite.

If $g \in L^2(\mathbb{R})$ and

$$g_{m,n}(x) = e^{imw_o x} g(x - nt_o) \tag{2.6}$$

with $w_o t_o = 2\pi$, Theorem 2.1 is still true; to see this observe that the operator U defined by $Ug(x) = (2\pi w_o^{-1})^{\frac{1}{2}} g(2\pi w_o^{-1} x)$ is unitary in $L^2(\mathbb{R})$ and

$$Ug_{m,n}(x) = e^{2\pi imx} Ug(x - n)$$

since $2\pi w_o^{-1} = t_o$. This theorem tells us that if $w_o t_o = 2\pi$, the basis given by (2.6) does not have good time and frequency localization simultaneously.

In particular, if $b(x)$ is sufficiently smooth and compactly supported the Balian-Low theorem tells us that the system

$$\{b_m(x)\}_{m\in\mathbb{Z}} = \{e^{2\pi imx} b(x)\}_{m\in\mathbb{Z}}$$

will not produce an orthonormal basis by translating the elements of the system by the integers. This is easy to see due to the decay at infinity of the Fourier transform of b, that is a consequence of the smoothness of b.

If we consider a more local situation, however, we can find a smooth and compactly supported "bell" function $b(x)$ for which

$$\{b_m(x)\}_{m\in\mathbb{Z}} = \{e^{2\pi imx} b(x)\}_{m\in\mathbb{Z}}$$

is an orthonormal system. For example, suppose that b is a function defined on $\mathbb{R}$ with $\operatorname{supp}(b) \subseteq [-\varepsilon, 1+\varepsilon']$, where $\varepsilon + \varepsilon' \leq 1$, $\varepsilon, \varepsilon' > 0$ and $b(x) \geq 0$. It is easy to find conditions on b so that $\{b_m : m \in \mathbb{Z}\}$ is an orthonormal system. The idea is to use a "folding argument" to write the orthonormal relations $< e^{2\pi im(\cdot)}b, e^{2\pi in(\cdot)}b > = \delta_{m,n}$ on the interval $[0,1]$:

$$\begin{aligned}
\delta_{m,n} &= < e^{2\pi im(\cdot)}b, e^{2\pi in(\cdot)}b > = \int_{-\varepsilon}^{1+\varepsilon'} b^2(x)\, e^{2\pi i(m-n)x}\, dx \\
&= \Big(\int_{-\varepsilon}^{0} + \int_{0}^{\varepsilon'} + \int_{\varepsilon'}^{1-\varepsilon} + \int_{1-\varepsilon}^{1} + \int_{1}^{1+\varepsilon'}\Big)\big\{b^2(x)\, e^{2\pi i(m-n)x}\, dx\big\}.
\end{aligned}$$

In the first integral we perform the change of variables $y = 1 + x$; in the last integral we use the change of variables $y = x - 1$. We therefore obtain

$$\delta_{m,n} = \int_0^{\varepsilon'} \big[b^2(x) + b^2(1+x)\big]\, e^{2\pi i(m-n)x}\, dx$$

$$+\int_{\varepsilon'}^{1-\varepsilon} b^2(x)\, e^{2\pi i(m-n)x}\, dx$$
$$+\int_{1-\varepsilon}^{1} \left[b^2(x)+b^2(x-1)\right] e^{2\pi i(m-n)x}\, dx.$$

That is, the function f having values $b^2(x)+b^2(1+x)$ on $[0,\varepsilon']$, $b^2(x)$ on $[\varepsilon', 1-\varepsilon]$ and $b^2(x)+b^2(x-1)$ on $[1-\varepsilon, 1]$ has Fourier coefficients $\hat{f}(k)=0$ if $k \neq 0$ and $\hat{f}(0)=1$. It follows easily that, if these orthonormal relations are to hold, b must satisfy

$$\left.\begin{aligned} &b^2(x)+b^2(1+x)=1 && \text{if } x\in[0,\varepsilon'],\\ &b^2(x)=1 && \text{if } x\in[\varepsilon', 1-\varepsilon],\\ &b^2(x)+b^2(x-1)=1 && \text{if } x\in[1-\varepsilon, 1]. \end{aligned}\right\} \tag{2.7}$$

It follows that (2.7) is a necessary and sufficient condition for

$$\left\{e^{2\pi i m x} b(x)\right\}_{m\in\mathbb{Z}}$$

to be an orthonormal system in $L^2(\mathbb{R})$. The Balian-Low theorem tells us that if we choose such a smooth bell function, translations by integers will not produce an orthonormal basis for $L^2(\mathbb{R})$. In the next two sections we shall show that if the exponentials $e^{2\pi i m x}$ are replaced by appropriate sines and cosines we can obtain such bases.

1.3 Smooth projections on $L^2(\mathbb{R})$

We will show that we can construct a smooth "bell" function associated with the interval $[0,1]$, in such a way that the system

$$\sqrt{2}b(x-k)\sin(\tfrac{2j+1}{2}\pi(x-k)), \qquad j,k\in\mathbb{Z},$$

is an orthonormal basis for $L^2(\mathbb{R})$. In fact, we will see that for each fixed $k\in\mathbb{Z}$, the family

$$\left\{\sqrt{2}b(x-k)\sin(\tfrac{2j+1}{2}\pi(x-k)) : j\in\mathbb{Z}\right\}$$

is an orthonormal basis for a closed subspace H_k of $L^2(\mathbb{R})$, and that $L^2(\mathbb{R})$ is the direct sum of these H_k. More generally, we shall construct smooth "bell" functions associated with a general finite interval $I = [\alpha, \beta)$ that can be multiplied by appropriate sines and cosines to obtain an orthonormal basis of a subspace H_I of $L^2(\mathbb{R})$ in such a way that if we have

$$-\infty < \cdots < \alpha_{k-1} < \alpha_k < \alpha_{k+1} < \cdots < \infty,$$

the H_{I_k}'s ($I_k = [\alpha_k, \beta_k)$) form a complete system of mutually orthogonal subspaces of $L^2(\mathbb{R})$. This does not have the form of a wavelet system, but it can be used for analyzing general functions in L^2 and, moreover, we will see how it can be used for constructing wavelets.

We start with the special case $I = [0, \infty)$ and our goal is to construct a smooth "bell" function that "approximates" $\chi_{[0,\infty)}$. Since any projection is idempotent, multiplication by a function gives a projection only if the function takes the values 0 or 1 almost everywhere on $\mathbb{R}$; this shows that the projection we are looking for cannot be given simply by multiplication by a smooth function.

Let us pose the problem of finding a non-negative bounded function $\rho \in C^\infty$ such that $\operatorname{supp}(\rho) \subseteq [-\varepsilon, \infty)$ for an $\varepsilon > 0$ and, like $\chi_{[0,\infty)}$, satisfies $\rho(x) + \rho(-x) = 1$, $x \neq 0$, and a real-valued function t so that

$$(Pf)(x) = \rho(x)f(x) + t(x)f(-x)$$

is a projection. A simple calculation, based on the fact that P has to be idempotent and self-adjoint, leads us to the equality $t(x) = \pm\sqrt{\rho(x)\rho(-x)}$. Writing $s = \sqrt{\rho}$, we are led to the formula

$$(Pf)(x) = s(x)[s(x)f(x) \pm s(-x)f(-x)].$$

In fact, more generally, no longer assuming s to be real valued, if we introduce the operator $P = P_{0,\varepsilon}$ defined by

$$(Pf)(x) \equiv (P_{0,\varepsilon}f)(x) = \overline{s(x)}\left[s(x)f(x) \pm s(-x)f(-x)\right] \tag{3.1}$$

with

$$|s(x)|^2 + |s(-x)|^2 = 1, \tag{3.2}$$

it is easy to show that it is an orthogonal projection. To see this it is enough to show that P is idempotent ($P^2 = P$) and self-adjoint ($P^* = P$). In fact,

$$(P^2f)(x) = \overline{s(x)}\left[s(x)(Pf)(x) \pm s(-x)(Pf)(-x)\right]$$

$$
\begin{aligned}
&= \overline{s(x)}\left[|s(x)|^2 s(x) f(x) \pm |s(x)|^2 s(-x) f(-x)\right. \\
&\quad \left.\pm |s(-x)|^2 s(-x) f(-x) + |s(-x)|^2 s(x) f(x)\right] \\
&= \overline{s(x)}\left[s(x) f(x) \pm s(-x) f(-x)\right] = (Pf)(x),
\end{aligned}
$$

and

$$
\begin{aligned}
<P^* f, g> = <f, Pg> &= \int_{-\infty}^{\infty} f(x) s(x)\, \overline{[s(x) g(x) \pm s(-x) g(-x)]}\, dx \\
&= \int_{-\infty}^{\infty} \left(f(x) s(x)\, \overline{s(x) g(x)} \pm f(-x) s(-x)\, \overline{s(x) g(x)}\right) dx \\
&= \int_{-\infty}^{\infty} \overline{s(x)}\left[s(x) f(x) \pm s(-x) f(-x)\right] \overline{g(x)}\, dx = <Pf, g>.
\end{aligned}
$$

For the moment we shall suppose that s is a real-valued function. Let us construct a smooth function $s(x)$ that satisfies (3.2). Choose ψ to be an even C^∞ function on $\mathbb{R}$ supported on $[-\varepsilon, \varepsilon]$, $\varepsilon > 0$, such that $\int_{-\varepsilon}^{\varepsilon} \psi(x)\, dx = \pi/2$. Let $\theta(x) = \int_{-\infty}^{x} \psi(t)\, dt$ and observe that

$$
\begin{aligned}
\theta(x) + \theta(-x) &= \int_{-\infty}^{x} \psi(t)\, dt + \int_{-\infty}^{-x} \psi(t)\, dt \\
&= \int_{-\infty}^{x} \psi(t)\, dt + \int_{x}^{\infty} \psi(-t)\, dt \\
&= \int_{-\infty}^{x} \psi(t)\, dt + \int_{x}^{\infty} \psi(t)\, dt = \frac{\pi}{2}.
\end{aligned}
$$

Putting $s(x) \equiv s_\varepsilon(x) = \sin(\theta(x))$ and $c(x) \equiv c_\varepsilon(x) = \cos(\theta(x))$ we have $s(-x) = \sin(\theta(-x)) = \sin(\frac{\pi}{2} - \theta(x)) = \cos(\theta(x)) = c(x)$. Hence,

$$
s^2(x) + s^2(-x) = \sin^2(\theta(x)) + \cos^2(\theta(x)) = 1,
$$

and (3.2) is satisfied.

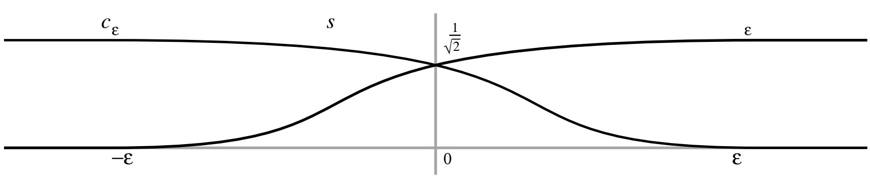


Figure 1.1: The functions s_ε and c_ε.

We have thus obtained two projections, $P^+_{0,\varepsilon}$ and $P^-_{0,\varepsilon}$, associated with the interval $[0,\infty)$ corresponding to the choice $+$ or $-$ in (3.1). We also have the analogous projections

$$(P^{0,\varepsilon'}_{+,-}f)(x) = c_{\varepsilon'}(x)\big[c_{\varepsilon'}(x)f(x) \pm c_{\varepsilon'}(-x)f(-x)\big]$$

associated with the interval $(-\infty,0]$, where $\varepsilon' > 0$.

Now we wish to construct smooth projections on a general interval

$$I = [\alpha,\beta], \qquad -\infty < \alpha < \beta < \infty.$$

We do this by using the translation operator $\tau_h f(x) = f(x-h)$, introduced at the beginning of section 1.2, and defining

$$P_\alpha = \tau_\alpha P_0 \tau_{-\alpha} \qquad \text{and} \qquad P^\beta = \tau_\beta P^0 \tau_{-\beta},$$

where we have suppressed, for the moment, the subindices and superscripts $\varepsilon, \varepsilon', +, -$. Each one of these operators is idempotent and self-adjoint since P_0 and P^0 are orthogonal projections. Thus P_α and P^β are also orthogonal projections. Using (3.1), we obtain the formulae

$$\begin{aligned}(P_\alpha f)(x) &= (\tau_\alpha P_0 \tau_{-\alpha} f)(x) = (P_0\tau_{-\alpha} f)(x-\alpha)\\ &= s_\varepsilon(x-\alpha)\big[s_\varepsilon(x-\alpha)f(x) \pm s_\varepsilon(\alpha-x)f(2\alpha-x)\big]\end{aligned} \tag{3.3}$$

and, similarly,

$$\begin{aligned}(P^\beta f)(x) &= (\tau_\beta P^0 \tau_{-\beta} f)(x) = (P^0\tau_{-\beta} f)(x-\beta)\\ &= c_{\varepsilon'}(x-\beta)\big[c_{\varepsilon'}(x-\beta)f(x) \pm c_{\varepsilon'}(\beta-x)f(2\beta-x)\big].\end{aligned} \tag{3.4}$$

Observe that $2\alpha - x$ and x are symmetric with respect to α. (That is, they lie on opposite sides and are equidistant to α.) This motivates the following definition. We say that a function g is **even with respect to** $\gamma \in \mathbb{R}$ if $g(x) = g(2\gamma - x)$ for all $x \in \mathbb{R}$.

If g is an even function with respect to α, it is easily seen that $P_\alpha(gf) = g(P_\alpha f)$ when $g \in L^\infty(\mathbb{R})$ and $f \in L^2(\mathbb{R})$; that is, multiplication by g commutes with P_α. Similarly, if g is even with respect to β, from (3.4) we see that $P^\beta(gf) = g(P^\beta f)$.

For a general interval $I = [\alpha, \beta]$ we choose $\varepsilon, \varepsilon' > 0$ such that $\alpha+\varepsilon \le \beta-\varepsilon'$ and observe that

$$\begin{aligned} P_\alpha P^\beta f &= \chi_{[\alpha-\varepsilon,\alpha+\varepsilon]} P_\alpha f + \chi_{[\alpha+\varepsilon,\beta-\varepsilon']} f + \chi_{[\beta-\varepsilon',\beta+\varepsilon']} P^\beta f \\ &= P^\beta P_\alpha f. \end{aligned} \tag{3.5}$$

To obtain this, observe that

$$\begin{aligned} P_\alpha f &= P_\alpha \chi_{[\alpha-\varepsilon,\alpha+\varepsilon]} f + P_\alpha \chi_{[\alpha+\varepsilon,\infty)} f \\ &= \chi_{[\alpha-\varepsilon,\alpha+\varepsilon]} P_\alpha f + \chi_{[\alpha+\varepsilon,\infty)} f, \end{aligned} \tag{3.6}$$

where we have used the fact that $\chi_{[\alpha-\varepsilon,\alpha+\varepsilon]}$ is even with respect to α, and, hence, commutes with P_α. Similarly, we have

$$\begin{aligned} P^\beta f &= P^\beta \chi_{(-\infty,\beta-\varepsilon']} f + P^\beta \chi_{[\beta-\varepsilon',\beta+\varepsilon']} f \\ &= \chi_{(-\infty,\beta-\varepsilon']} f + \chi_{[\beta-\varepsilon',\beta+\varepsilon']} P^\beta f. \end{aligned} \tag{3.7}$$

Now apply P^β to the first equality and P_α to the second to obtain the desired result.

Since P_α and P^β commute, the operator

$$P_I f \equiv P_{[\alpha,\beta]} f = P_\alpha P^\beta f = P^\beta P_\alpha f \tag{3.8}$$

is a bounded, orthogonal projection on $L^2(\mathbb{R})$.

Observe that $P_I \equiv P_{[\alpha,\beta]}$ depends on $\alpha, \beta, \varepsilon, \varepsilon'$ and the signs we choose at α and β. Thus, if $\alpha, \beta, \varepsilon$ and ε' are fixed, the choice of signs gives us four projections.

An expression for $P_I \equiv P_{[\alpha,\beta]}$ that is different from the one given in (3.8) is obtained by introducing the function

$$b(x) = s_\varepsilon(x-\alpha)\, c_{\varepsilon'}(x-\beta).$$

We refer to $b = b_I$ as a “bell” function associated with the interval $[\alpha, \beta]$. Observe that b depends on $\alpha, \beta, \varepsilon$ and ε'. By translating the graphs of s_ε and c_ε given in Figure 1.1 we obtain the graph of a bell function associated with $[\alpha, \beta]$:

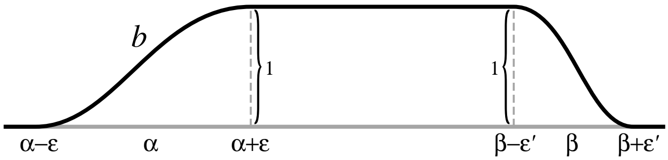


Figure 1.2: Graph of the bell function b associated with $[\alpha, \beta]$.

It is easy to prove the following basic properties of $b(x)$:

$$\left.\begin{array}{l}
\text{i)} \quad \operatorname{supp}(b) \subseteq [\alpha-\varepsilon, \beta+\varepsilon'];\\[4pt]
\text{on } [\alpha-\varepsilon, \alpha+\varepsilon]\\
\qquad \text{ii)} \ \ b(x) = s_\varepsilon(x-\alpha),\\
\qquad \text{iii)} \ \ b(2\alpha - x) = s_\varepsilon(\alpha - x) = c_\varepsilon(x-\alpha),\\
\qquad \text{iv)} \ \ b^2(x) + b^2(2\alpha - x) = 1;\\[4pt]
\text{v)} \quad \operatorname{supp}(b(\cdot)b(2\alpha-\cdot)) \subseteq [\alpha-\varepsilon, \alpha+\varepsilon];\\[4pt]
\text{vi)} \ \ \text{on } \ [\alpha+\varepsilon, \beta-\varepsilon'], \ b(x) = 1;\\[4pt]
\text{on } [\beta-\varepsilon', \beta+\varepsilon']\\
\qquad \text{vii)} \ \ b(x) = c_{\varepsilon'}(x-\beta),\\
\qquad \text{viii)} \ \ b(2\beta - x) = c_{\varepsilon'}(\beta - x) = s_{\varepsilon'}(x-\beta),\\
\qquad \text{ix)} \ \ b^2(x) + b^2(2\beta - x) = 1;\\[4pt]
\text{x)} \quad \operatorname{supp}(b(\cdot)b(2\beta-\cdot)) \subseteq [\beta-\varepsilon', \beta+\varepsilon'];\\[4pt]
\text{xi)} \ \ b^2(x) + b^2(2\alpha - x) + b^2(2\beta - x) = 1 \quad \text{on} \ \operatorname{supp}(b).
\end{array}\right\} \qquad (3.9)$$

Not all these properties are independent. For example, iv) follows from ii) and iii). The reader will find it instructive to compare these conditions with (2.7).

Using (3.5), the definition of P_α and P^β given in (3.3) and (3.4) and these properties, we can easily derive the following new formula for P_I in terms of the bell function b:

$$(P_I f)(x) = b(x)\{b(x)f(x) \pm b(2\alpha - x)f(2\alpha - x) \pm b(2\beta - x)f(2\beta - x)\}. \quad (3.10)$$

Observe that we have four choices for such a projection. The choice of $\pm$ associated with α is referred to as the **polarity** of $P_{[\alpha,\beta]}$ at α, and the choice of $\pm$ associated with β is referred to as the **polarity** of $P_{[\alpha,\beta]}$ at

β. Thus, if we choose "+" before the second summand in the bracket in (3.10), we say that the projection has **positive polarity at** α.

DEFINITION 3.11 *Suppose $I = [\alpha, \beta]$ and $J = [\beta, \gamma]$ are adjacent; we say that they have* **compatible bell functions** b_I **and** b_J *if*

$$\alpha - \varepsilon < \alpha < \alpha + \varepsilon \le \beta - \varepsilon' < \beta < \beta + \varepsilon' \le \gamma - \varepsilon'' < \gamma < \gamma + \varepsilon''$$

and

$$b_I = s_\varepsilon(x - \alpha)\, c_{\varepsilon'}(x - \beta), \qquad b_J = s_{\varepsilon'}(x - \beta)\, c_{\varepsilon''}(x - \gamma).$$

If $I = [\alpha, \beta]$ and $J = [\beta, \gamma]$ are intervals with compatible bell functions, we have

$$b_I(x) = b_J(2\beta - x), \qquad \text{if } x \in [\beta - \varepsilon', \beta + \varepsilon']; \tag{3.12}$$

$$b_I^2(x) + b_J^2(x) = 1, \qquad \text{if } x \in [\beta - \varepsilon', \beta + \varepsilon']; \tag{3.13}$$

$$b_I^2(x) + b_J^2(x) = b_{I\cup J}^2(x) \qquad \text{for all } x \in \mathbb{R}. \tag{3.14}$$

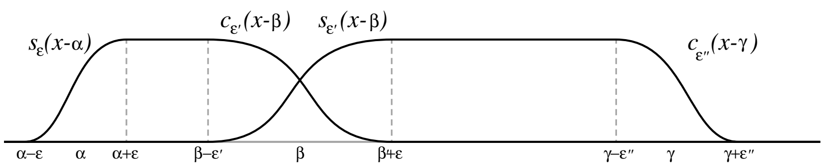


Figure 1.3: Compatible bell functions on $[\alpha, \beta]$ and $[\beta, \gamma]$.

These properties follow easily from (3.9). The next result establishes the main property of these projections that will allow us to decompose $L^2(\mathbb{R})$ as a direct sum of orthogonal subspaces.

THEOREM 3.15 *Let $I = [\alpha, \beta]$ and $J = [\beta, \gamma]$ be adjacent intervals with compatible bell functions and suppose P_I and P_J have opposite polarities at β. Then*

$$P_I + P_J = P_{I\cup J}\,, \tag{3.16}$$

$$P_I P_J = 0 = P_J P_I\,. \tag{3.17}$$

PROOF: According to (3.5), letting I also denote the identity operator,

$$\begin{aligned}P_I + P_J = \chi_{[\alpha-\varepsilon,\alpha+\varepsilon]} P_\alpha + \chi_{[\alpha+\varepsilon,\beta-\varepsilon']} I + \chi_{[\beta-\varepsilon',\beta+\varepsilon']} P^\beta \\ + \chi_{[\beta-\varepsilon',\beta+\varepsilon']} P_\beta + \chi_{[\beta+\varepsilon',\gamma-\varepsilon'']} I + \chi_{[\gamma-\varepsilon'',\gamma+\varepsilon'']} P^\gamma\end{aligned}$$

with P^β and P_β chosen with opposite polarity at β. This last property allows us to prove that the two middle terms in the above formula add up to $\chi_{[\beta-\varepsilon',\beta+\varepsilon']} I$, and, hence, the result equals $P_{I\cup J}$ according to (3.5). Thus, $P_I + P_J = P_{I\cup J}$.

Formula (3.17) is a consequence of a general result about projections. In fact, if P and Q are orthogonal projections on a Hilbert space such that $P+Q$ is an orthogonal projection, then $PQ = QP = 0$. To see this, observe that $(P+Q)^2 = P+Q$ implies $PQ = -QP$; from this we deduce $PQ = P^2Q = P(PQ) = -P(QP) = QP^2 = QP$; these two results give us $PQ = QP = 0$.

∎

If H is a Hilbert space and $\{H_k : k \in \mathbb{Z}\}$ is a sequence of mutually orthogonal closed subspaces we let

$$V = \bigoplus_{k=-\infty}^{\infty} H_k$$

denote the closed subspace consisting of all $f = \sum_{k\in\mathbb{Z}} f_k$ with $f_k \in H_k$ and $\sum_{k\in\mathbb{Z}} \|f_k\|^2 < \infty$. We call V the **orthogonal direct sum of the spaces** H_k. If the family consist of two spaces H_1 and H_2 we write $H_1 \bigoplus H_2$.

The above theorem allows us to decompose $L^2(\mathbb{R})$ as an orthogonal direct sum

$$L^2(\mathbb{R}) = \bigoplus_{k=-\infty}^{\infty} H_k, \tag{3.18}$$

where $H_k = P_k(L^2(\mathbb{R}))$, $P_k = P_{[\alpha_k,\alpha_{k+1}]}$ with

$$-\infty < \cdots < \alpha_{k-1} < \alpha_k < \alpha_{k+1} < \cdots < \infty;$$

moreover, adjacent intervals, $[\alpha_k, \alpha_{k+1}]$ and $[\alpha_{k+1}, \alpha_{k+2}]$, have compatible bell functions, and P_k and P_{k+1} have opposite polarity at α_{k+1}. The orthogonality of the H_k's follows from (3.17); formula (3.16) gives us the decomposition of $L^2(\mathbb{R})$.

Another orthogonal decomposition of $L^2(\mathbb{R})$, that we shall show is pertinent to wavelets, can be achieved as follows. Let $I = [\pi, 2\pi]$ and $J = -I = [-2\pi, -\pi]$. Choose $\varepsilon > 0$ such that $0 < \varepsilon \leq \frac{1}{3}\pi$ and $\varepsilon' = 2\varepsilon$; put $I_k = 2^k I$, $J_k = 2^k J$ for $k \in \mathbb{Z}$. Then, associating $\varepsilon_k = 2^k\varepsilon$, $\varepsilon_{k+1} = 2\varepsilon_k$ with I_k, the adjacent intervals I_k and I_{k+1} have compatible bell functions (similarly for J_k and J_{k+1}) and we have

$$L^2(\mathbb{R}) = \Big\{ \bigoplus_{k=-\infty}^{\infty} H_{J_k} \Big\} \bigoplus \Big\{ \bigoplus_{k=-\infty}^{\infty} H_{I_k} \Big\}, \tag{3.19}$$

if we choose the appropriate polarities for P_{I_k}, P_{J_k} and denote the images $P_{I_k}(L^2(\mathbb{R}))$ and $P_{J_k}(L^2(\mathbb{R}))$ by H_{I_k} and H_{J_k}.

Let us now characterize the subspace $H_I = P_I(L^2(\mathbb{R}))$. We say that f is **even** with respect to α on $[\alpha-\varepsilon, \alpha+\varepsilon]$ if $f(2\alpha - x) = f(x)$ on this interval. Similarly, a function g is said to be **odd** with respect to α on $[\alpha-\varepsilon, \alpha+\varepsilon]$ if $g(2\alpha - x) = -g(x)$ on this interval.

By (3.10) we can write

$$(P_I f)(x) = b_I(x)\, S(x),$$

where $S(x) = b_I(x)f(x) \pm b_I(2\alpha - x)f(2\alpha - x) \pm b_I(2\beta - x)f(2\beta - x)$. Observe that there are four choices for $S(x)$ depending on the signs considered, which give us the four functions $S_+^+(x)$, $S_-^+(x)$, $S_+^-(x)$, and $S_-^-(x)$. S_+^+ is even with respect to α on $[\alpha-\varepsilon, \alpha+\varepsilon]$ and even with respect to β on $[\beta-\varepsilon', \beta+\varepsilon']$; S_-^+ is odd with respect to α on $[\alpha-\varepsilon, \alpha+\varepsilon]$ and even with respect to β on $[\beta-\varepsilon', \beta+\varepsilon']$. The obvious similar statements apply to S_+^- and S_-^-.

THEOREM 3.20 *Let $I = [\alpha, \beta]$; then $f \in H_I = P_I(L^2(\mathbb{R}))$ if and only if $f = b_I S$, where $S \in L^2(\mathbb{R})$, b_I is the bell function associated with I, and S is even or odd on $[\alpha-\varepsilon, \alpha+\varepsilon]$ according to the choice of polarity at α, and even or odd on $[\beta-\varepsilon', \beta+\varepsilon']$ according to the choice of polarity at β.*

PROOF: If $f \in H_I$, there exists $g \in L^2(\mathbb{R})$ such that $f = P_I g$; then $f = P_I g = b_I S$ by (3.10) where, clearly, S has the same polarity at α and β as P_I. Observe that $S \in L^2(\mathbb{R})$.

Suppose now that $f = bS$ with $S \in L^2(\mathbb{R})$, locally even at α and locally odd at β. (The other cases are handled similarly.) It is enough to show that if P_I has the same polarity as S, then $P_I(bS) = bS$, since then $P_I(f) = P_I(bS) = bS = f$. To show $P_I(bS) = bS$ we use (3.10), and the properties of b_I given in iv), vi) and ix) of (3.9). We leave the details to the reader. ∎

1.4 Local sine and cosine bases and the construction of some wavelets

In this section we shall introduce orthonormal bases for the subspaces $H_I = P_I(L^2(\mathbb{R}))$, where P_I are the projection operators defined in the previous section. As we shall see, these bases are closely allied to certain trigonometric systems and consistent with the polarity of P_I. That is, if P_I is chosen with negative polarity at the left endpoint of the interval I and with positive polarity at the right endpoint of I, the elements of the basis will be locally odd at the left endpoint and locally even at the right endpoint. In addition, the bases for these subspaces will be expressed in terms of trigonometric functions and the associated bell function. (As explained at the beginning of the last section.)

Let us first consider I to be the interval $[0, 1]$, and suppose that P_I has polarities $-$ and $+$ at 0 and 1, respectively. (We tacitly assume that P_I is associated with positive ε and ε' such that $\varepsilon + \varepsilon' \le 1$. As we did on several occasions in the previous section, we do not indicate the dependence of P_I on ε and ε'.) Let $f \in L^2([0, 1])$ and extend f to a function F on $[-2, 2]$ so that F is even with respect to 1 and odd with respect to 0; this is consistent with the choice of the polarities for P_I. (See Figure 1.4 below.)

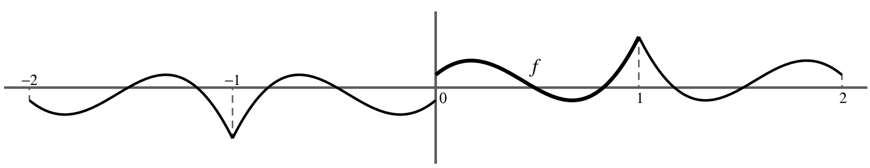


Figure 1.4: Extension F of f to $[-2, 2]$.

On $[-2, 2]$ we have the usual cosine and sine basis

$$\left\{\frac{1}{2},\ \frac{1}{\sqrt{2}}\sin\frac{\pi k x}{2},\ \frac{1}{\sqrt{2}}\cos\frac{\pi \ell x}{2}\right\}, \quad k, \ell = 1, 2, \cdots$$

Since F is odd on $[-2,2]$, the cosines are not involved in the Fourier expansion of F. Moreover, the functions $\sin(\frac{2k+1}{2}\pi x)$, $k = 0,1,2,\cdots$, are even with respect to 1 and the functions $\sin(k\pi x)$, $k = 1,2,3,\cdots$, are odd with respect to 1. Therefore, we only need $\sin(\frac{2k+1}{2}\pi x)$, $k = 0,1,2,\cdots$, to represent F. That is,

$$F(x) = \sum_{k=0}^{\infty} c_k \sin\left(\tfrac{2k+1}{2}\pi x\right),$$

where

$$c_k = \frac{1}{2}\int_{-2}^{2} F(x)\sin\left(\tfrac{2k+1}{2}\pi x\right) dx,$$

and the above series converges in the norm of $L^2([-2,2])$. Observe that the convergence is also true in the pointwise almost everywhere sense by a deep theorem of L. Carleson concerning almost everywhere convergence of Fourier series. (See [Car1].)

If we restrict ourselves to $[0,1]$, and use the appropriate normalization, we find that $\{\sqrt{2}\sin(\frac{2k+1}{2}\pi x),\ k = 0,1,2,\cdots\}$ is an orthonormal basis for $L^2([0,1])$ with polarities of its elements at 0 and 1 that match the ones of P_I. This provides us with the proof of the first part of the following result.

THEOREM 4.1 *Each one of the systems*

i) $\{\sqrt{2}\sin(\frac{2k+1}{2}\pi x)\},\ k = 0,1,2,\cdots$

ii) $\{\sqrt{2}\sin(k\pi x)\},\ k = 1,2,3,\cdots$

iii) $\{\sqrt{2}\cos(\frac{2k+1}{2}\pi x)\},\ k = 0,1,2,\cdots$

iv) $\{1,\ \sqrt{2}\cos(k\pi x)\},\ k = 1,2,3,\cdots$

is an orthonormal basis of $L^2([0,1])$ *and the polarities are* $(-,+)$ *for* i), $(-,-)$ *for* ii), $(+,-)$ *for* iii) *and* $(+,+)$ *for* iv).

We have already seen how to obtain i); the other three statements are obtained in a similar way.

We use this result to obtain the desired orthonormal bases for $H_I = P_I(L^2(\mathbb{R}))$ when $I = [0,1]$. Let $\varepsilon, \varepsilon' > 0$ with $\varepsilon + \varepsilon' \le 1$ and consider the associated bell function $b(x) = s_\varepsilon(x)c_{\varepsilon'}(x-1)$. Suppose, as before, that the polarities of P_I are $-$ at 0 and $+$ at 1. Thus, (3.10) in this case becomes

$$P_I f(x) = b(x)\{b(x)f(x) - b(-x)f(-x) + b(2-x)f(2-x)\} = b(x)S(x).$$

The function $S(x)$ is odd with respect to 0 and even with respect to 1 because of the properties of b (see (3.9)); hence, S has the right polarity to be represented by the orthonormal basis i) in Theorem (4.1). Therefore, we can write

$$S(x) = \sqrt{2}\sum_{k=0}^{\infty} c_k \sin(\tfrac{2k+1}{2}\pi x),$$

where

$$c_k = \sqrt{2}\int_0^1 S(x)\sin(\tfrac{2k+1}{2}\pi x)\,dx,$$

where the convergence is in $L^2([0,1])$ and, by Carleson's theorem, almost everywhere. Since S and the sine functions we are using have the same polarities at 0 and 1, the expansion is valid on $[-\varepsilon, 1+\varepsilon']$ in the L^2-sense and almost everywhere. Multiplying by $b(x)$ we obtain

$$(P_I f)(x) = b(x)S(x) = \sum_{k=0}^{\infty} c_k\sqrt{2}\,b(x)\sin(\tfrac{2k+1}{2}\pi x)$$

and the convergence is valid in $L^2([-\varepsilon, 1+\varepsilon'])$ and almost everywhere, since b is bounded. This shows that the system

$$\{\sqrt{2}\,b(x)\sin(\tfrac{2k+1}{2}\pi x)\}, \quad k = 0, 1, 2, \cdots \tag{4.2}$$

is complete in $H_I = P_I(L^2(\mathbb{R}))$ when P_I has the polarities $(-,+)$. To show that this system is an orthonormal basis, we need to prove the orthonormality relations; if $e_k = \sin(\frac{2k+1}{2}\pi x)$, $k = 0, 1, 2, \cdots$, we have to show

$$2\int_{-\varepsilon}^{1+\varepsilon'} b^2(x)e_k(x)e_\ell(x)\,dx = \delta_{k\ell}\,, \quad k, \ell = 0, 1, 2, \cdots.$$

Since e_k is locally odd with respect to 0, a change of variables together with property iv) of (3.9) gives us

$$\int_{-\varepsilon}^{\varepsilon} b^2(x)e_k(x)e_\ell(x)\,dx = \int_0^{\varepsilon} e_k(x)e_\ell(x)\,dx.$$

Similarly, using that e_k is locally even at 1 and property ix) of (3.9), a change of variables gives us

$$\int_{1-\varepsilon'}^{1+\varepsilon'} b^2(x)e_k(x)e_\ell(x)\,dx = \int_{1-\varepsilon'}^{1} e_k(x)e_\ell(x)\,dx.$$

Finally, since $b \equiv 1$ on $[\varepsilon, 1-\varepsilon']$, the orthonormality of (4.2) on $[-\varepsilon, 1+\varepsilon']$ is equivalent to the orthonormality of the system i) on the interval $[0,1]$ given in Theorem (4.1). Since we know that this is true, we have proved the desired result.

Performing the appropriate translations and dilations and taking into account the different types of polarity, we obtain the following result for the spaces $H_I = P_I(L^2(\mathbb{R}))$ when $I = [\alpha, \beta]$ is an arbitrary finite interval:

THEOREM 4.3 *If $P_I = P_{[\alpha,\beta]}$ has negative polarity at α and positive polarity at β, then*

i) $$\left\{\sqrt{\tfrac{2}{|I|}}\, b_I(x) \sin\left(\tfrac{2k+1}{2}\tfrac{\pi}{|I|}(x-\alpha)\right)\right\}, \quad k = 0,1,2,\cdots,$$

is an orthonormal basis for $H_I = P_I(L^2(\mathbb{R}))$. If the polarities are $(-,-)$, $(+,-)$ and $(+,+)$ at (α,β) the same is true, respectively, for

ii) $$\left\{\sqrt{\tfrac{2}{|I|}}\, b_I(x) \sin\left(k\tfrac{\pi}{|I|}(x-\alpha)\right)\right\}, \quad k = 1,2,3,\cdots,$$

iii) $$\left\{\sqrt{\tfrac{2}{|I|}}\, b_I(x) \cos\left(\tfrac{2k+1}{2}\tfrac{\pi}{|I|}(x-\alpha)\right)\right\}, \quad k = 0,1,2,\cdots,$$

iv) $$\left\{\sqrt{\tfrac{1}{|I|}}\, b_I(x),\ \sqrt{\tfrac{2}{|I|}}\, b_I(x) \cos\left(k\tfrac{\pi}{|I|}(x-\alpha)\right)\right\}, \quad k = 1,2,3,\cdots.$$

This theorem, together with the orthogonal decomposition (3.18), can be used to obtain bases for $L^2(\mathbb{R})$. Choose a strictly increasing sequence of real numbers $\{\alpha_j\}_{j\in\mathbb{Z}}$ such that $\lim_{j\to\infty}\alpha_j = \infty$ and $\lim_{j\to-\infty}\alpha_j = -\infty$; let $\{\varepsilon_j\}_{j\in\mathbb{Z}}$ be a sequence of positive real numbers such that

$$\varepsilon_j + \varepsilon_{j+1} \le \alpha_{j+1} - \alpha_j \equiv \ell_j \qquad \text{for all } \ j \in \mathbb{Z}.$$

If we choose the polarities $(-,+)$ for each $P_j = P_{[\alpha_j,\alpha_{j+1}]}$ we obtain that the system

$$\theta_{k,j} = \sqrt{\tfrac{2}{\ell_j}}\, b_{[\alpha_j,\alpha_{j+1}]}(x) \sin\left(\tfrac{2k+1}{2}\tfrac{\pi}{\ell_j}(x-\alpha_j)\right), \quad k = 0,1,2,..., \ j \in \mathbb{Z}, \quad (4.4)$$

is an orthonormal basis for $L^2(\mathbb{R})$. The convergence of the series expansion of a function $f \in L^2(\mathbb{R})$ with respect to the basis given in (4.4) is valid in $L^2(\mathbb{R})$. A much deeper result is the almost everywhere convergence, which is a consequence of the celebrated theorem of Carleson. More precisely, we have

$$\lim_{N\to\infty} \sum_{|j|\le N} \sum_{k=0}^{\infty} < f, \theta_{k,j} > \theta_{k,j}(x) = f(x)$$

for almost every $x \in \mathbb{R}$, where the second sum indicates the a.e. convergence of the partial sums

$$\sum_{k=0}^{M} < f, \theta_{k,j} > \theta_{k,j}(x)$$

as $M \to \infty$, for each $j \in \mathbb{Z}$.

Combining appropriately the polarities for different intervals $[\alpha_j, \alpha_{j+1}]$ we can obtain, in a similar manner, other bases for $L^2(\mathbb{R})$. Observe that we obtain, by using appropriate sine and cosine functions, a result which is not true in general if we use modulations, that is multiplications by exponentials. (See the Balian-Low theorem, Theorem 2.1.)

The orthogonal decomposition of $L^2(\mathbb{R})$ given in (3.19) can be used to obtain a new orthonormal basis of this space. The elements of this basis are the Fourier transforms of the wavelet basis introduced by Lemarié and Meyer in [LM].

THEOREM 4.5 *The system*

$$\gamma_{j,k}(\xi) = \frac{2^{\frac{j}{2}}}{\sqrt{2\pi}}\, b(2^j\xi)\, e^{i\frac{2k+1}{2}2^j\xi}, \qquad j,k \in \mathbb{Z},$$

is an orthonormal basis for $L^2(\mathbb{R})$, where b restricted to $[0,\infty)$ is a bell function for $[\pi, 2\pi]$ associated with $0 < \varepsilon \le \frac{\pi}{3}$, $\varepsilon' = 2\varepsilon$, and b is even on $\mathbb{R}$.

PROOF: Let

$$\left.\begin{aligned} C_{j,k}(\xi) &= \frac{2^{\frac{j}{2}}}{\sqrt{2\pi}}\, b(2^j\xi)\cos\left(\tfrac{2k+1}{2}2^j\xi\right), \\ S_{j,k}(\xi) &= \frac{2^{\frac{j}{2}}}{\sqrt{2\pi}}\, b(2^j\xi)\sin\left(\tfrac{2k+1}{2}2^j\xi\right), \end{aligned}\right\} \qquad k \ge 0, \quad j \in \mathbb{Z},$$

so that $\gamma_{j,k}(\xi) = C_{j,k}(\xi) + iS_{j,k}(\xi)$, $k \geq 0$, $j \in \mathbb{Z}$. Observe that $C_{j,k}$ is an even function on $\mathbb{R}$ and $S_{j,k}$ is an odd function on $\mathbb{R}$.

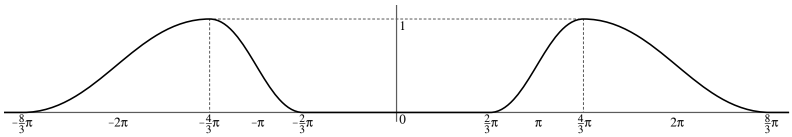


Figure 1.5: The graph of b with $\varepsilon = \frac{\pi}{3}$.

We shall use the trigonometric formulas

$$\left.\begin{array}{ll} \text{i)} & \sin\left(\frac{2k+1}{2}(\xi - \pi)\right) = (-1)^{k+1}\cos\left(\frac{2k+1}{2}\xi\right), \\ \text{ii)} & \cos\left(\frac{2k+1}{2}(\xi + 2\pi)\right) = -\cos\left(\frac{2k+1}{2}\xi\right), \\ \text{iii)} & \cos\left(\frac{2k+1}{2}(\xi - \pi)\right) = (-1)^{k}\sin\left(\frac{2k+1}{2}\xi\right), \\ \text{iv)} & \sin\left(\frac{2k+1}{2}(\xi + 2\pi)\right) = -\sin\left(\frac{2k+1}{2}\xi\right). \end{array}\right\} \tag{4.6}$$

Let $b^-(\xi) = \chi_{(-\infty,0]}(\xi)b(\xi)$ and $b^+(\xi) = \chi_{[0,\infty)}(\xi)b(\xi)$ and define $C^+_{j,k}$, $C^-_{j,k}$, $S^+_{j,k}$ and $S^-_{j,k}$ as at the beginning of the proof, replacing b with b^+ and b^-. Observe that $C_{j,k} = C^+_{j,k} + C^-_{j,k}$ and $S_{j,k} = S^+_{j,k} + S^-_{j,k}$.

Using formula i) in (4.6) and the basis i) in Theorem 4.3, we deduce that $\{2C^+_{j,k} : k \geq 0\}$ is an orthonormal basis for the projection spaces $P^{-,+}_{I_j}(L^2(\mathbb{R}))$, where $I_j = 2^{-j}[\pi, 2\pi]$. In fact, using i) of (4.6), we obtain

$$\begin{aligned} 2C^+_{j,k}(\xi) &= \sqrt{\tfrac{2}{|I_j|}}\, b^+(2^j\xi)\cos\left(\tfrac{2k+1}{2}2^j\xi\right) \\ &= (-1)^{k+1}\sqrt{\tfrac{2}{|I_j|}}\, b^+(2^j\xi)\sin\left(\tfrac{2k+1}{2}(2^j\xi - \pi)\right) \\ &= (-1)^{k+1}\sqrt{\tfrac{2}{|I_j|}}\, b^+(2^j\xi)\sin\left(\tfrac{2k+1}{2}\tfrac{\pi}{|I_j|}(\xi - 2^{-j}\pi)\right), \end{aligned}$$

which is the basis i) of Theorem 4.3, except for the factor $(-1)^{k+1}$, which does not affect the orthonormality.

Using formula iii) of (4.6) and the basis iii) of Theorem 4.3, an analogous argument shows that $\{2S^+_{j,k} : k \geq 0\}$ is an orthonormal basis for the projection spaces $P^{+,-}_{I_j}(L^2(\mathbb{R}))$.

Similarly, it can be proved that $\{2C^-_{j,k} : k \geq 0\}$ and $\{2S^-_{j,k} : k \geq 0\}$ are orthonormal bases for $P^{+,-}_{-I_j}(L^2(\mathbb{R}))$ and $P^{-,+}_{-I_j}(L^2(\mathbb{R}))$, respectively, where $-I_j = 2^{-j}[-2\pi, -\pi]$.

Hence, each one of the systems

$$\{2C^+_{j,k} : k \geq 0,\ j \in \mathbb{Z}\} \quad \text{and} \quad \{2S^+_{j,k} : k \geq 0,\ j \in \mathbb{Z}\}$$

is an orthonormal basis for $L^2((0,\infty))$, and each one of the systems

$$\{2C^-_{j,k} : k \geq 0,\ j \in \mathbb{Z}\} \quad \text{and} \quad \{2S^-_{j,k} : k \geq 0,\ j \in \mathbb{Z}\}$$

is an orthonormal basis for $L^2((-\infty,0))$.

For $k \geq 0$ and $j \in \mathbb{Z}$, define

$$\alpha_{j,k}(\xi) = C_{j,k}(\xi) + iS_{j,k}(\xi) = \frac{2^{\frac{j}{2}}}{\sqrt{2\pi}}\, b(2^j\xi)\, e^{i\frac{2k+1}{2}2^j\xi}$$

and

$$\beta_{j,k}(\xi) = C_{j,k}(\xi) - iS_{j,k}(\xi) = \frac{2^{\frac{j}{2}}}{\sqrt{2\pi}}\, b(2^j\xi)\, e^{-i\frac{2k+1}{2}2^j\xi}.$$

If $m \leq -1,\ \beta_{j,-(m+1)}(\xi) = \gamma_{j,m}(\xi)$ and if $k \geq 0,\ \alpha_{j,k}(\xi) = \gamma_{j,k}(\xi)$. Hence, the theorem is proved if we show that the system

$$\{\alpha_{j,k} : j \in \mathbb{Z},\ k \geq 0\} \cup \{\beta_{j,k} : j \in \mathbb{Z},\ k \geq 0\}$$

is an orthonormal basis of $L^2(\mathbb{R})$.

We start by showing the orthonormality of the system.

$$\begin{aligned} 4 < \alpha_{j,n}, \alpha_{k,\ell} > &= 4 < C_{j,n}, C_{k,\ell} > + 4 < S_{j,n}, S_{k,\ell} > \\ &= < 2C^+_{j,n}, 2C^+_{k,\ell} > + < 2C^-_{j,n}, 2C^-_{k,\ell} > \\ &\quad + < 2S^+_{j,n}, 2S^+_{k,\ell} > + < 2S^-_{j,n}, 2S^-_{k,\ell} > \\ &= 4\delta_{j,k}\delta_{n,\ell}. \end{aligned}$$

Similarly, $< \beta_{j,n}, \beta_{k,\ell} > = \delta_{j,k}\delta_{n,\ell}$. Finally, using the evenness of $C_{j,n}$ and the oddness of $S_{k,\ell}$ we obtain

$$\begin{aligned} 4 < \alpha_{j,n}, \beta_{k,\ell} > &= 4 < C_{j,n}, C_{k,\ell} > + 4i < C_{j,n}, S_{k,\ell} > \\ &\quad + 4i < S_{j,n}, C_{k,\ell} > - 4 < S_{j,n}, S_{k,\ell} > \end{aligned}$$

$$= <2C^+_{j,n}, 2C^+_{k,\ell}> + <2C^-_{j,n}, 2C^-_{k,\ell}>$$
$$- <2S^+_{j,n}, 2S^+_{k,\ell}> - <2S^-_{j,n}, 2S^-_{k,\ell}>$$
$$= 2\delta_{j,k}\delta_{n,\ell} - 2\delta_{j,k}\delta_{n,\ell} = 0.$$

Now we must show completeness. Given $f \in L^2(\mathbb{R})$, let $f^{(e)}$ be the even function $[f(x) + f(-x)]/2$ and $f^{(o)}$ be the odd function $[f(x) - f(-x)]/2$, so that $f = f^{(e)} + f^{(o)}$. Using the evenness of $C_{j,k}$ and the oddness of $S_{j,k}$ we obtain

$$\sum_{j\in\mathbb{Z}}\sum_{k\geq 0} <f, \alpha_{j,k}> \alpha_{j,k} + <f, \beta_{j,k}> \beta_{j,k}$$
$$= 2\sum_{j\in\mathbb{Z}}\sum_{k\geq 0} <f^{(e)}, C_{j,k}> C_{j,k} + <f^{(o)}, S_{j,k}> S_{j,k}$$
$$= 4\sum_{j\in\mathbb{Z}}\sum_{k\geq 0}\{<f^{(e)}, C^+_{j,k}> C^+_{j,k} + <f^{(e)}, C^-_{j,k}> C^-_{j,k}$$
$$+ <f^{(o)}, S^+_{j,k}> S^+_{j,k} + <f^{(o)}, S^-_{j,k}> S^-_{j,k}\}$$
$$= f^{(e)}\chi_{(0,\infty)} + f^{(e)}\chi_{(-\infty,0)} + f^{(o)}\chi_{(0,\infty)} + f^{(o)}\chi_{(-\infty,0)} = f,$$

where we have used the already observed fact that the systems $\{2C^{+,-}_{j,k}\}$ and $\{2S^{+,-}_{j,k}\}$, $k \geq 0$, $j \in \mathbb{Z}$, form an orthonormal basis of $L^2((0,\infty))$ an $L^2((-\infty,0))$ for the appropriate choice of $+$ and $-$.

∎

Corollary 4.7 *Let* $\gamma(\xi) = \frac{1}{\sqrt{2\pi}} e^{i\frac{\xi}{2}} b(\xi)$ *be the function* $\gamma_{0,0}$ *of* Theorem 4.5 *and define* ψ *by*

$$\hat{\psi}(\xi) = \sqrt{2\pi}\,\gamma(\xi) = e^{i\frac{\xi}{2}} b(\xi).$$

Then, ψ *is an orthonormal wavelet.*

Proof: By the Plancherel theorem $\|\psi\|_2^2 = \frac{1}{2\pi}\|\hat{\psi}\|_2^2 = \|\gamma\|_2^2 = 1$. Moreover,

$$(\psi_{j,k})^\wedge(\xi) = 2^{-\frac{j}{2}} e^{-i2^{-j}k\xi}\hat{\psi}(2^{-j}\xi) = 2^{-\frac{j}{2}} e^{-i2^{-j}k\xi} b(2^{-j}\xi) e^{i2^{-j}\frac{\xi}{2}}$$
$$= 2^{-\frac{j}{2}} b(2^{-j}\xi) e^{i2^{-j}\frac{1-2k}{2}\xi} = \sqrt{2\pi}\,\gamma_{-j,-k}(\xi).$$

By Theorem 4.5, $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$. ∎

The orthonormal wavelets obtained in Corollary 4.7 are the ones described by P.G. Lemarié and Y. Meyer in [LM] (see also [Me5]), and will be called the **Lemarié-Meyer wavelets**.

In Figure 1.6 we give the graph of a wavelet ψ whose Fourier tranform is of the form $\hat{\psi}(\xi) = b(\xi)e^{i\frac{\xi}{2}}$ with

$$b(\xi) = \begin{cases} \sin\left(\frac{3}{4}(|\xi| - \frac{2}{3}\pi)\right), & \text{if } \frac{2}{3}\pi < |\xi| \le \frac{4}{3}\pi, \\ \sin\left(\frac{3}{8}(\frac{8}{3}\pi - |\xi|)\right), & \text{if } \frac{4}{3}\pi < |\xi| \le \frac{8}{3}\pi, \\ 0 & \text{otherwise.} \end{cases}$$

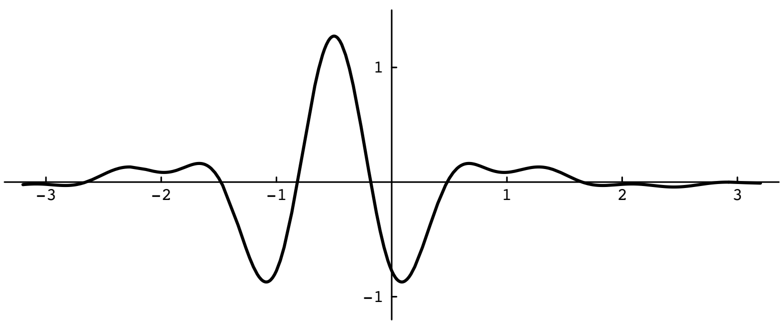


Figure 1.6: The graph of a Lemarié-Meyer wavelet.

Theorem 4.3 together with the orthogonal decomposition (3.18) can be used to obtain other bases for $L^2(\mathbb{R})$. Let

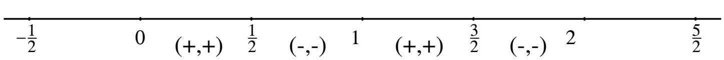
$$\alpha_j = \tfrac{j}{2}, \quad I_j = [\alpha_j, \alpha_{j+1}] \quad \text{and} \quad \ell_j = |I_j| = \alpha_{j+1} - \alpha_j = \tfrac{1}{2} \quad \text{for} \quad j \in \mathbb{Z},$$

and choose $0 < \varepsilon \le \frac{1}{4}$. Let b be the "bell" function associated with $[0, \frac{1}{2}]$ and ε at each endpoint. Observe that

$$b_j \equiv b_{I_j} = b(x - \tfrac{j}{2})$$

if we use the same ε at each endpoint of the interval I_j.

For the interval $I_j = [\frac{j}{2}, \frac{j+1}{2}]$ we choose the polarities indicated in the figure below:



We now construct an orthonormal basis for $L^2(\mathbb{R})$. If j is even we use the local cosine basis given in iv) of Theorem 4.3 to obtain

$$\left.\begin{aligned} &\sqrt{2}\, b(x-\tfrac{j}{2}),\\ &2b(x-\tfrac{j}{2})\cos\bigl(2\pi k(x-\tfrac{j}{2})\bigr), \quad k=1,2,\cdots. \end{aligned}\right\} \tag{4.8}$$

If j is odd we use the local sine basis given in ii) of Theorem 4.3 to obtain

$$2b(x-\tfrac{j}{2})\sin\bigl(2\pi k(x-\tfrac{j}{2})\bigr), \quad k=1,2,\cdots. \tag{4.9}$$

For j even we have

$$\begin{aligned} \cos\bigl(2\pi k(x-\tfrac{j}{2})\bigr) &= \cos(2\pi kx)\cos\bigl(2\pi k\tfrac{j}{2}\bigr)+\sin(2\pi kx)\sin\bigl(2\pi k\tfrac{j}{2}\bigr)\\ &= \cos(2\pi kx); \end{aligned}$$

and for j odd,

$$\begin{aligned} \sin\bigl(2\pi k(x-\tfrac{j}{2})\bigr) &= \sin(2\pi kx)\cos\bigl(2\pi k\tfrac{j}{2}\bigr)-\cos(2\pi kx)\sin\bigl(2\pi k\tfrac{j}{2}\bigr)\\ &= (-1)^k\sin(2\pi kx). \end{aligned}$$

Thus,

$$\left.\begin{aligned} &\sqrt{2}\, b(x-\tfrac{j}{2}) && \text{if } j\in 2\mathbb{Z},\\ &2b(x-\tfrac{j}{2})\cos(2\pi kx) && \text{if } j\in 2\mathbb{Z},\ k=1,2,\cdots,\\ &(-1)^k 2b(x-\tfrac{j}{2})\sin(2\pi kx) && \text{if } j\in 2\mathbb{Z}+1,\ k=1,2,\cdots, \end{aligned}\right\} \tag{4.10}$$

is an orthonormal basis for $L^2(\mathbb{R})$. Observe that we have "defeated" the Balian-Low phenomenon by using cosines and sines instead of exponentials, but the product of the "translation step" and the "frequency step" is still 2π (see (2.6)). Observe that if $g_{m,n}(x)=e^{2\pi imx}g(x-\frac{n}{2})$, the family

$$\left.\begin{aligned} &\sqrt{2}\, g_{0,j} && \text{if } j\in 2\mathbb{Z},\\ &\bigl[g_{k,j}+(-1)^j g_{-k,j}\bigr] && \text{if } j\in \mathbb{Z},\ k=1,2,3,\cdots, \end{aligned}\right\} \tag{4.11}$$

coincides with (4.10) when $g=b$, except that the factor $(-1)^k$ in the third family of (4.10) is replaced by $2i$.

A basis similar to the one described in (4.10) arises in the work of K. Wilson in quantum mechanics ([Wil]). He observed that for the study of his operators one does not need basis functions that distinguish between positive and negative frequencies of the same order. Instead of having a "peak" function localized around $x = \frac{n}{2}$, he uses functions that arise from the combination of two functions having peaks symmetrically distributed about the origin; this produces a system similar to the one in (4.11). We shall call the basis he uses a **Wilson basis**; more explicitly, using the notation of (4.11), we have

$$\left.\begin{array}{ll} \sqrt{2}\, g_{0,j} & \text{if } \ j \in 2\mathbb{Z}, \\ \left[g_{k,j} + (-1)^{k+j} g_{-k,j}\right] & \text{if } \ j \in \mathbb{Z}, \ \ k = 1, 2, 3, \cdots, \end{array}\right\} \tag{4.12}$$

(observe the difference between the powers of -1 in (4.11) and (4.12)). This family can be written in the following way:

$$\left.\begin{array}{ll} \sqrt{2}\, g(x-j) & \text{if } \ k = 0, \ \ j \in \mathbb{Z}, \\ 2g(x-\frac{j}{2})\cos(2\pi kx) & \text{if } \ k > 0, \ \ j+k \text{ even}, \\ 2g(x-\frac{j}{2})\sin(2\pi kx) & \text{if } \ k > 0, \ \ j+k \text{ odd}. \end{array}\right\} \tag{4.13}$$

The proof that (4.13) is an orthonormal basis for some function g was simplified in [DJJ]. Here we can give a very simple proof as a consequence of our results on smooth projections and local sine and cosine basis. This was observed independently by P. Auscher ([Au1]) and E. Laeng ([Lae]). What is needed is a simple modification of the scheme developed to obtain (4.10).

Take $\alpha_j = \frac{2j-1}{4}$ for $j \in \mathbb{Z}$ and $0 < \varepsilon < \frac{1}{4}$, $\varepsilon' = \varepsilon$, and use the polarities described below:

$-\frac{3}{4}$ $\quad -\frac{1}{4}$ (+,+) $\frac{1}{4}$ (-,-) $\frac{3}{4}$ (+,+) $\frac{5}{4}$ (-,-) $\frac{7}{4}$ $\quad \frac{9}{4}$

By using simple trigonometric identities, it is not hard to show that the family

$$\left.\begin{array}{ll} \sqrt{2}\, b(x-\frac{j}{2}) & \text{if } \ j \ \text{ is even and } \ k = 0, \\ 2b(x-\frac{j}{2})\cos\big(2\pi k(x+\frac{1}{4})\big) & \text{if } \ j \ \text{ is even and } \ k > 0, \\ 2b(x-\frac{j}{2})\sin\big(2\pi k(x-\frac{1}{4})\big) & \text{if } \ j \ \text{ is odd and } \ k > 0, \end{array}\right\} \tag{4.14}$$

coincides with (4.13) when $b = g$, except for some factors of -1 which do not change the orthonormality of the system.

1.5 The unitary folding operators and the smooth projections

In this section we present another way of defining the projections P_I of section 1.3 and a new proof of Theorem 3.15, which allowed us to obtain orthonormal bases for $L^2(\mathbb{R})$. This section is not necessary to understand the chapters that follow, so that the reader who is interested in the concept of multiresolution analysis can proceed directly to Chapter 2.

We begin with the projections associated with the interval $[0, \infty)$. Recall the definition of $P \equiv P^{+,-}_{0,\varepsilon}$ given in (3.1). Motivated by this definition, we introduce the operator U defined by

$$Uf(x) = \begin{cases} s(x)f(x) + s(-x)f(-x), & x > 0, \\ \overline{s(-x)}f(x) - \overline{s(x)}f(-x), & x < 0, \end{cases} \tag{5.1}$$

where $\operatorname{supp}(s) \subseteq [-\varepsilon, \infty)$ and satisfies

$$|s(x)|^2 + |s(-x)|^2 = 1 \qquad \text{for all} \quad x. \tag{5.2}$$

The condition $\operatorname{supp}(s) \subseteq [-\varepsilon, \infty)$ is not necessary for the first result we shall prove. In Figure 1.7 we show the graph of Uf for $f(x) = \frac{1}{x^2+1}$ and $\varepsilon = \frac{1}{2}$.

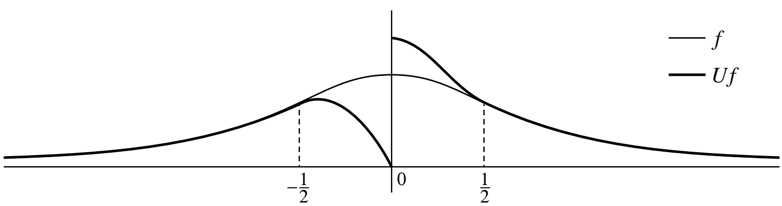


Figure 1.7: The graph of Uf for $f(x) = \frac{1}{x^2+1}$ and $\varepsilon = \frac{1}{2}$.

We consider the space $L^2(\mathbb{R}^+, \mathbb{C}^2)$ of all functions

$$F(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix},$$

where f_1 and f_2 are complex-valued functions defined on $\mathbb{R}^+$ belonging to $L^2((0,\infty))$. On this space the inner product is defined by

$$[F,G] = \int_0^\infty \left[f_1(t)\overline{g_1(t)} + f_2(t)\overline{g_2(t)}\,\right]dt.$$

The **folding operator** $\mathcal{F} : L^2(\mathbb{R},\mathbb{C}) \longrightarrow L^2(\mathbb{R}^+,\mathbb{C}^2)$ is defined by

$$(\mathcal{F}f)(t) = \begin{bmatrix} f(t) \\ f(-t) \end{bmatrix}, \quad t > 0.$$

It is easy to see that $\mathcal{F}$ has an inverse, $\mathcal{F}^{-1}$, given by

$$\left(\mathcal{F}^{-1}\begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}\right)(t) = \begin{cases} f_1(t), & \text{if } \ t > 0, \\ f_2(-t), & \text{if } \ t < 0. \end{cases}$$

A simple computation shows that $[\mathcal{F}f, \mathcal{F}g] = < f, g >$, so that $\mathcal{F}$ is a unitary operator.

We can "transfer" U to $L^2(\mathbb{R}^+,\mathbb{C}^2)$ by using the following matrix:

$$A(t) = \begin{pmatrix} s(t) & s(-t) \\ -\overline{s(-t)} & \overline{s(t)} \end{pmatrix}, \quad t > 0.$$

This matrix is unitary due to (5.2); thus, $A(t)A(t)^* = I = A(t)^*A(t)$. We now define the operators $\mathcal{A}, \mathcal{A}^* : L^2(\mathbb{R}^+,\mathbb{C}^2) \longrightarrow L^2(\mathbb{R}^+,\mathbb{C}^2)$ by letting

$$(\mathcal{A}F)(t) = A(t)F(t) \quad \text{and} \quad (\mathcal{A}^*F)(t) = A(t)^*F(t).$$

THEOREM 5.3 *$U = \mathcal{F}^{-1}\mathcal{A}\mathcal{F}$ and $U^* = \mathcal{F}^{-1}\mathcal{A}^*\mathcal{F}$, so that U is unitary. Moreover,*

$$(U^*f)(x) = \begin{cases} \overline{s(x)}f(x) - s(-x)f(-x), & x > 0, \\ s(-x)f(x) + \overline{s(x)}f(-x), & x < 0, \end{cases}$$

and $U^*\chi_{[0,\infty)}U = P^+_{0,\varepsilon}$, where $P^+_{0,\varepsilon}$ is defined by (3.1).

PROOF: The equality $U = \mathcal{F}^{-1}\mathcal{A}\mathcal{F}$ is easy to check by using the above definitions. That $U^* = \mathcal{F}^{-1}\mathcal{A}^*\mathcal{F}$ follows immediately from the fact that $\mathcal{F}$

is also a unitary operator. The expression for $(U^* f)$ in terms of f follows readily from this last equality.

Finally, let us prove that $U^*\chi_{[0,\infty)}U = P^+_{0,\varepsilon}$: if $x > 0$,

$$\begin{aligned}(U^*\chi_{[0,\infty)}Uf)(x) &= \overline{s(x)}(\chi_{[0,\infty)}Uf)(x) - s(-x)(\chi_{[0,\infty)}Uf)(-x)\\ &= \overline{s(x)}(Uf)(x) = \overline{s(x)}\,[s(x)f(x) + s(-x)f(-x)];\end{aligned}$$

if $x < 0$,

$$\begin{aligned}(U^*\chi_{[0,\infty)}Uf)(x) &= \overline{s(x)}(\chi_{[0,\infty)}Uf)(-x)\\ &= \overline{s(x)}\,[s(-x)f(-x) + s(x)f(x)].\end{aligned}$$

These two formulae coincide with the definition of $P^+_{0,\varepsilon}$ given in (3.1). ∎

The graph of U^* is shown in Figure 1.8 for $f(x) = \frac{1}{x^2+1}$ and $\varepsilon = \frac{1}{2}$.

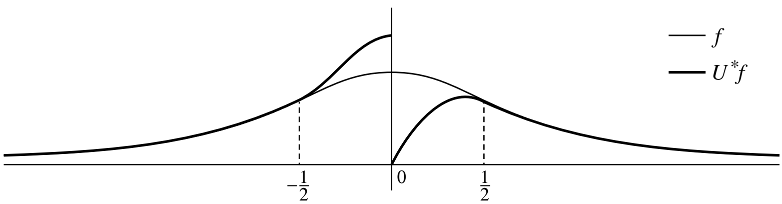


Figure 1.8: The graph of U^*f for $f(x) = \frac{1}{x^2+1}$ and $\varepsilon = \frac{1}{2}$.

Observe that f is unchanged under the actions of U^* and U outside the interval $(-\varepsilon, \varepsilon)$. (See Figures 1.7 and 1.8.)

We can translate the point 0 to α as we did in section 1.3. As before, let $\tau_\alpha f(x) = f(x-\alpha)$ be the translation by α operator; we then define

$$U_\alpha = \tau_\alpha U \tau^*_\alpha \quad \text{and} \quad U^*_\alpha = \tau_\alpha U^* \tau^*_\alpha.$$

Observe that $U = U_0$ and $U^* = U^*_0$. With these definitions we have the general formulas:

$$(U_\alpha f)(x) = \begin{cases} s(x-\alpha)f(x) + s(\alpha-x)f(2\alpha-x), & x > \alpha,\\ -\overline{s(x-\alpha)}f(2\alpha-x) + \overline{s(\alpha-x)}f(x), & x < \alpha,\end{cases}$$

and

$$(U_\alpha^* f)(x) = \begin{cases} \overline{s(x-\alpha)}\,f(x) - \overline{s(\alpha - x)}\,f(2\alpha - x), & x > \alpha, \\ \overline{s(x-\alpha)}\,f(2\alpha - x) + s(\alpha - x)f(x), & x < \alpha. \end{cases}$$

PROPOSITION 5.4 *Let $E = E_\alpha = [\alpha - \varepsilon, \alpha + \varepsilon]$,*

$$L^2(E) = \{f \in L^2(\mathbb{R}) : \mathit{supp}\,(f) \subseteq E\},$$

and suppose that s satisfies (5.2) *and, also, $s(x) = 1$ if $x > \varepsilon$. Then U_α and U_α^* satisfy:*

(i) $U_\alpha, U_\alpha^* : L^2(E) \longrightarrow L^2(E)$, *and, hence, are unitary on $L^2(E)$,*

(ii) $U_\alpha\big|_{L^2(E)^\perp} = I = U_\alpha^*\big|_{L^2(E)^\perp}$, *where $L^2(E)^\perp = L^2(E^c)$, and*

(iii) *U_α and U_α^* commute with multiplication by χ_E.*

PROOF: Since $2\alpha - x$ is the point symmetric to x with respect to α, $\mathrm{supp}\,(f(2\alpha - \cdot)) \subseteq [\alpha - \varepsilon, \alpha + \varepsilon]$; this proves that U_α and U_α^* take $L^2(E)$ into $L^2(E)$. (i) is now immediate.

To prove (ii), look at the general formulas for U_α, U_α^* and use the equalities $s(x - \alpha) = 1$, $s(\alpha - x) = 0$ if $x > \alpha + \varepsilon$. Again, examine the general formulas for U_α and U_α^* and observe that χ_E is symmetric with respect to α; this proves (iii).

∎

THEOREM 5.5 *Let s satisfy* (5.2) *with support on $[\varepsilon, \infty)$, and suppose that $s \in C^d$, where C^d is the space of all functions with continuous derivatives up to order d. Then*

$$U_\alpha : C^d \cap L^2(\mathbb{R}) \longrightarrow \mathcal{S}_\alpha \quad \textit{and} \quad U_\alpha^* : \mathcal{S}_\alpha \longrightarrow C^d \cap L^2(\mathbb{R}),$$

and both operators are one-to-one and onto, where

$$\begin{aligned} \mathcal{S}_\alpha = \Big\{ f \in C^d(\mathbb{R} - \{\alpha\}) \cap L^2(\mathbb{R}) : f^{(n)}(\alpha\pm) \ \ &\textit{exist for} \ \ 0 \le n \le d, \\ \lim_{x\to\alpha+} f^{(n)}(x) = 0 \ \ &\textit{if} \ \ n \ \ \textit{is odd}, \\ \textit{and} \ \ \lim_{x\to\alpha-} f^{(n)}(x) = 0 \ \ &\textit{if} \ \ n \ \ \textit{is even} \Big\}. \end{aligned}$$

PROOF: Since U_α and U_α^* are unitary (see part (i) of Proposition 5.4), it suffices to show $U_\alpha(C^d \cap L^2(\mathbb{R})) \subseteq \mathcal{S}_\alpha$ and $U_\alpha^*(\mathcal{S}_\alpha) \subseteq C^d \cap L^2(\mathbb{R})$. Moreover, these inclusions are proved if we can show that $U_0(C^d \cap L^2(\mathbb{R})) \subseteq \mathcal{S}_0$ and $U_0^*(\mathcal{S}_0) \subseteq C^d \cap L^2(\mathbb{R})$, where $U_0 = U$.

To show the first inclusion, let $h(x) = s(x)f(x)$ so that

$$(Uf)(x) = h(x) + h(-x) \qquad \text{when} \;\; x > 0.$$

If $f \in C^d \cap L^2(\mathbb{R})$ we have

$$(Uf)^{(n)}(x) = h^{(n)}(x) + (-1)^n h^{(n)}(-x) \quad \text{for} \;\; x > 0,$$

which shows that $(Uf)^{(n)}(0+)$ exists and is zero if n is odd and $0 \le n \le d$. Let $g(x) = \overline{s(-x)}f(x)$ so that $(Uf)(x) = g(x) - g(-x)$ for $x < 0$. If $f \in C^d \cap L^2(\mathbb{R})$,

$$(Uf)^{(n)}(x) = g^{(n)}(x) - (-1)^n g^{(n)}(-x) \quad \text{for} \;\; x < 0,$$

which shows that $(Uf)^{(n)}(0-)$ exists and is zero if n is even and $0 \le n \le d$.

We now show the second inclusion, which is a little more complicated. If $f \in \mathcal{S}_0$ it is clear from the formula satisfied by U^*f that $(Uf)^{(n)}(0\pm)$ exists when $0 \le n \le d$. It is enough to show

$$\lim_{x \to 0+} \left\{ (U^*f)^{(n)}(x) - (U^*f)^{(n)}(-x) \right\} = 0. \tag{5.6}$$

Let $H(x) = (U^*f)(x) - (-1)^n (U^*f)(-x)$ so that we have to show

$$\lim_{x \to 0+} H^{(n)}(x) = 0.$$

A simple computation using the formula for U^*f shows

$$H(x) = \left[\overline{s(x)} - (-1)^n \overline{s(-x)}\right] f(x) - \left[s(-x) + (-1)^n s(x)\right] f(-x), \;\; x > 0.$$

Taking derivatives we obtain

$$H^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x) \left[\overline{s^{(k)}(x)} - (-1)^{n+k} \overline{s^{(k)}(-x)}\right]$$

$$- \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f^{(n-k)}(-x) \left[(-1)^k s^{(k)}(-x) + (-1)^n s^{(k)}(x)\right]$$

$$\equiv \sum_{k=0}^{n} \binom{n}{k} \{G_k(f,s)(x)\}.$$

If $n-k$ is odd, $\lim_{x\to 0+} f^{(n-k)}(x) = 0$ so that

$$\begin{aligned}\lim_{x\to 0+} G_k(f,s)(x) &= -(-1)^{n-k} f^{(n-k)}(0-)\big[(-1)^k s^{(k)}(0) + (-1)^n s^{(k)}(0)\big] \\ &= 0,\end{aligned}$$

since $(-1)^k = -(-1)^n$. If $n-k$ is even, $\lim_{x\to 0+} f^{(n-k)}(-x) = 0$ so that

$$\lim_{x\to 0+} G_k(f,s)(x) = f^{(n-k)}(0+)\big[\,\overline{s^{(k)}(0)} - (-1)^{n+k}\,\overline{s^{(k)}(0)}\,\big] = 0.$$

This finishes the proof of (5.6) and, hence, the theorem. ∎

We shall examine the smooth projection operators defined in section 1.3 in terms of the unitary operators we have defined in this section.

In Theorem 5.3 we have shown that $U^*\chi_{(0,\infty)}U = P^+_{0,\varepsilon} \equiv P^+_0$ where $(P^+_0 f)(x) = \overline{s(x)}\,\big[s(x)f(x) \pm s(-x)f(-x)\big]$ was given in (3.1); we already knew that P^+_0 is an orthogonal projection, but this follows immediately from this equality since U is a unitary operator and the multiplication by $\chi_{(0,\infty)}$ operator is self-adjoint: in fact,

$$(P^+_0)^* = U^*\chi^*_{(0,\infty)}U = U^*\chi_{(0,\infty)}U = P^+_0$$

and

$$(P^+_0)^2 = U^*\chi_{(0,\infty)}UU^*\chi_{(0,\infty)}U = U^*\chi_{(0,\infty)}U = P^+_0.$$

To find the projection P^+_α corresponding to the interval (α,∞), we recall the definition $P^+_\alpha = \tau_\alpha P^+_0 \tau^*_\alpha$ and observe that $\tau_\alpha\chi_{(0,\infty)}\tau^*_\alpha = \chi_{(\alpha,\infty)}$ to obtain

$$P^+_\alpha = U^*_\alpha \chi_{(\alpha,\infty)} U_\alpha. \tag{5.7}$$

The following, in fact, establishes (5.7):

$$\begin{aligned}P^+_\alpha &= \tau_\alpha P^+_0 \tau^*_\alpha = \tau_\alpha U^*\chi_{(0,\infty)}U\tau^*_\alpha = \tau_\alpha U^*(\tau^*_\alpha\tau_\alpha\chi_{(0,\infty)}\tau^*_\alpha\tau_\alpha)U\tau^*_\alpha \\ &= U^*_\alpha(\tau_\alpha\chi_{(0,\infty)}\tau^*_\alpha)U_\alpha = U^*_\alpha\chi_{(\alpha,\infty)}U_\alpha.\end{aligned}$$

Observe that equality (5.7) shows immediately that P_α^+ is an orthogonal projection, since U_α is unitary.

We can also define, as we did in section 1.3, the projections associated with the interval $(-\infty,0)$. Just define $P_-^0 = U^*\chi_{(-\infty,0)}U$ and it is immediate that P_-^0 is an orthogonal projection, since U is unitary. Moreover,

$$(P_-^0 f)(x) = s(-x)\big[\,\overline{s(-x)}f(x) - \overline{s(x)}f(-x)\big].$$

(Observe that this projection has negative polarity at 0.) We can translate this projection to the interval $(-\infty,\beta)$ to obtain

$$P_-^\beta = U_\beta^*\chi_{(-\infty,\beta)}U_\beta$$

as we did when we obtained (5.7).

Let us now show how these unitary operators can be used to obtain the smooth projections P_I associated with the interval $I=[\alpha,\beta]$. Choose the real numbers α, β, ε and ε' with $\varepsilon,\varepsilon'>0$ and

$$-\infty < \alpha-\varepsilon < \alpha < \alpha+\varepsilon < \beta-\varepsilon' < \beta < \beta+\varepsilon' < \infty.$$

Observe that $U_\alpha f$ and $U_\alpha^* f$ have the same values as f outside the interval $(\alpha-\varepsilon,\alpha+\varepsilon)$, and both $U_\beta f$ and $U_\beta^* f$ coincide with f outside the interval $(\beta-\varepsilon',\beta+\varepsilon')$. As a consequence, we have several commutativity relations; some of them are:

$$\left.\begin{aligned}
&\text{i)} && U_\alpha U_\beta^* = U_\beta^* U_\alpha \quad\text{and}\quad U_\alpha U_\beta = U_\beta U_\alpha,\\
&\text{ii)} && \chi_{(\alpha,\infty)}U_\beta^* = U_\beta^*\chi_{(\alpha,\infty)},\\
&\text{iii)} && U_\alpha\chi_{(-\infty,\beta)} = \chi_{(-\infty,\beta)}U_\alpha.
\end{aligned}\right\}\tag{5.8}$$

These commutativity relations allow us to show $P_\alpha^+P_-^\beta = P_-^\beta P_\alpha^+$. In fact, using (5.8), we have

$$P_\alpha^+P_-^\beta = U_\alpha^*\chi_{(\alpha,\infty)}U_\alpha U_\beta^*\chi_{(-\infty,\beta)}U_\beta = U_\alpha^*U_\beta^*\chi_{(\alpha,\beta)}U_\alpha U_\beta,$$

and, similarly,

$$P_-^\beta P_\alpha^+ = U_\alpha^*U_\beta^*\chi_{(\alpha,\beta)}U_\alpha U_\beta.$$

Since $P_\alpha^+ P_-^\beta = P_-^\beta P_\alpha^+$, the operator $P^{+,-}_{(\alpha,\beta)} = U_\alpha^* U_\beta^* \chi_{(\alpha,\beta)} U_\alpha U_\beta$ is an orthogonal projection. Observe that this projection has polarity $+$ at α and polarity $-$ at β. Again, observe that this equality giving us $P^{+,-}_{(\alpha,\beta)}$ also immediately implies that it is a projection.

It is now easy to obtain the version of Theorem 3.15 for $P^{+,-}_{(\alpha,\beta)}$ by using the definition of this projection we have just given in terms of the folding operators. In fact, suppose

$$-\infty < \alpha - \varepsilon < \alpha < \alpha + \varepsilon < \beta - \varepsilon' < \beta < \beta + \varepsilon' < \gamma - \varepsilon'' < \gamma < \gamma + \varepsilon'' < \infty$$

so that the intervals $I = [\alpha, \beta]$ and $J = [\beta, \gamma]$, with these choices of $\varepsilon, \varepsilon', \varepsilon''$, are compatible. Then, if we write

$$P_I = P^{+,-}_{(\alpha,\beta)}, \quad P_J = P^{+,-}_{(\beta,\gamma)} \quad \text{and} \quad P_{I\cup J} = P^{+,-}_{(\alpha,\gamma)},$$

we have

$$\text{i)} \quad P_I + P_J = P_{I\cup J}$$

and

$$\text{ii)} \quad P_I P_J = P_J P_I.$$

Equality i) follows easily from (5.8) once we express P_I, P_J and $P_{I\cup J}$ in terms of the associated folding operators and the characteristic functions χ_I, χ_J and $\chi_{I\cup J}$.

Equality ii) can be proved as we did for Theorem 3.15. But it can also be easily obtained directly by using the fact that $\chi_I \chi_J = 0 = \chi_J \chi_I$ and the commutativity relations (5.8) that the folding operators satisfy.

There are some remarks that should be made about this approach. We have only obtained the special case of Theorem 3.15 for projections with polarities $(+,-)$. To obtain the full statement of this theorem we need to define folding operators related to projections having the other polarities. In addition to the operators U and U^*, we also define V and V^* by

$$(Vf)(x) = \begin{cases} s(x)f(x) - s(-x)f(-x), & x > 0, \\ \overline{s(-x)}f(x) + s(x)f(-x), & x < 0, \end{cases}$$

and

$$(V^*f)(x) = \begin{cases} \overline{s(x)}f(x) + s(-x)f(-x), & x > 0, \\ s(-x)f(x) - \overline{s(x)}f(-x), & x < 0. \end{cases}$$

Thus,

$$U^*\chi_{(0,\infty)}U = P_0^+, \qquad U^*\chi_{(-\infty,0)}U = P_-^0,$$
$$V^*\chi_{(0,\infty)}V = P_0^-, \qquad V^*\chi_{(-\infty,0)}V = P_+^0,$$

where

$$(P_0^{+,-}f)(x) = \overline{s(x)}\left[s(x)f(x) \pm s(-x)f(-x)\right],$$
$$(P_{+,-}^0 f)(x) = s(-x)\left[\overline{s(-x)}f(x) \pm \overline{s(x)}f(-x)\right].$$

We can now construct the four projections associated with an interval $I = (\alpha, \beta)$ having chosen appropriate $\varepsilon, \varepsilon' > 0$. They are

$$\left.\begin{aligned} P_I^{+,-} &= U_\alpha^* U_\beta^* \chi_I U_\alpha U_\beta, \\ P_I^{+,+} &= U_\alpha^* V_\beta^* \chi_I U_\alpha V_\beta, \\ P_I^{-,+} &= V_\alpha^* V_\beta^* \chi_I V_\alpha V_\beta, \\ P_I^{-,-} &= V_\alpha^* U_\beta^* \chi_I V_\alpha U_\beta. \end{aligned}\right\} \tag{5.9}$$

The full statement of Theorem 3.15 can then be obtained by using these equalities, as long as we choose compatible projections for adjacent intervals, and provided they have opposite polarities at the common end point.

It is illustrative to present the graphs of P_0^+f, P_0^-f, P_+^0f and P_-^0f (see Figure 1.9 below) for $f(x) = \frac{1}{(x+1)^2+1}$ and $\varepsilon = \frac{1}{2}$.

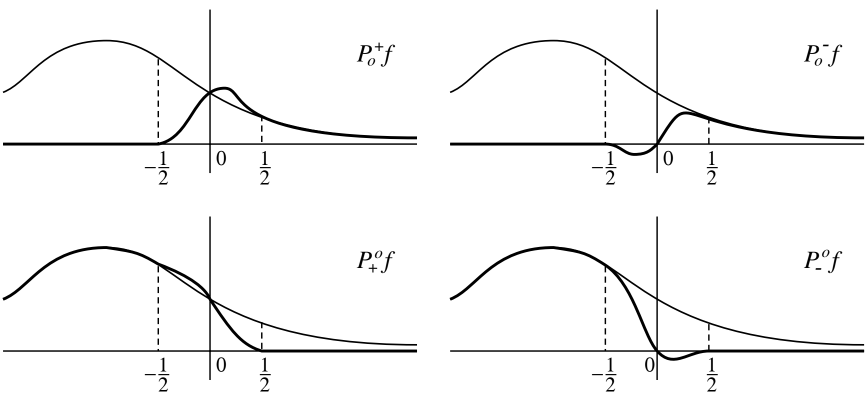


Figure 1.9: Graphs of P_0^+f, P_0^-f, P_+^0f and P_-^0f.

This method for obtaining the projections associated with the interval I by the factorizations presented in (5.9) is particularly useful in applications. The factors used are very simple operators that can be easily represented in a computer program. More generally, the bases we have constructed in terms of a specific function to which we apply certain translation operators, dilation operators and/or modulations are also well suited for applications. The fact that an elementary function is used and the fact that these operators applied to it are particularly simple lead to relatively elegant expressions for the partial sums of the series representing general functions.

1.6 Notes and references

1. Appropriate general assumptions that guarantee the validity of most of the formulas in section 1.1 can be found in [SW]. The Haar function presented in Example A was discovered by A. Haar (see [Haa]) in 1910. More information about Gabor bases can be obtained from [Gab]. Those readers who are not familiar with the space $\mathcal{S}'$ of tempered distributions involved in the proof of Theorem 2.1 can find the definition and relevant properties of $\mathcal{S}'$ in [SW]. Theorem 2.1, referred to as the Balian-Low theorem in this book, was originally proved independently by R. Balian [Bal] and F. Low [Low] in the early 1980s. (See also item 1 in section 8.5.) The proof we presented is due to G. Battle [Bat2]. A good source for the basic properties of orthogonal projections used in section 1.3 is the book written by P. Halmos [Hal]. The almost everywhere convergence of the trigonometric series considered in section 1.4 follows from the Carleson-Hunt theorem (see [Car1] and [Hun]). The detailed construction of the Lemarié-Meyer wavelets introduced in section 1.4 can be found in [LM]. While the local sine and cosine series in Theorem 4.5 were first described by R. Coifman and Y. Meyer in [CM2], they were also introduced by H. Malvar ([Malv]) in connection with the theory of signal processes. A complete account of these facts is also discussed in [AWW]. The Wilson basis mentioned in the same section was introduced by K. Wilson in an unpublished manuscript [Wil]. The proof that the family (4.13) is an orthonormal basis for some function g was presented in [DJJ]. The unitary folding operators of section 1.5 and their application to obtaining smooth localized orthonormal bases were developed in [Wi1] and are also described in [Wi2].

2. Dilation factors other than 2 can be considered to decompose $L^2(\mathbb{R})$ as an orthogonal direct sum in a way similar to (3.19). Consider the intervals $I = [\pi, \lambda\pi]$ and $J = [-\lambda\pi, -\pi]$ for some $\lambda > 1$, and let $I_k = \lambda^{-k} I$ and $J_k = \lambda^{-k} J$ for all $k \in \mathbb{Z}$. Since

$$(0, \infty) = \bigcup_{k \in \mathbb{Z}} \lambda^k I \qquad \text{and} \qquad (-\infty, 0) = \bigcup_{k \in \mathbb{Z}} \lambda^k J,$$

it follows from the theory of smooth projections developed in section 1.3 that

$$L^2(\mathbb{R}) = \Big\{ \bigoplus_{k \in \mathbb{Z}} P_{J_k}(L^2(\mathbb{R})) \Big\} \bigoplus \Big\{ \bigoplus_{k \in \mathbb{Z}} P_{I_k}(L^2(\mathbb{R})) \Big\}$$

when we choose compatible bell functions for adjacent intervals and appropriate polarities. It then follows, as in the proof that led to Theorem 4.5, that the collection of functions

$$\alpha^{\lambda}_{j,k}(\xi) = c^{\lambda}_{j,k}(\xi) + i s^{\lambda}_{j,k}(\xi) \qquad \text{and} \qquad \beta^{\lambda}_{j,k}(\xi) = c^{\lambda}_{j,k}(\xi) - i s^{\lambda}_{j,k}(\xi),$$

where

$$c^{\lambda}_{j,k}(\xi) \equiv \frac{\lambda^{\frac{j}{2}}}{\sqrt{2(\lambda - 1)\pi}}\, b(\lambda^j \xi) \cos\Big(\frac{2k+1}{2} \frac{1}{\lambda - 1} (\lambda^j \xi - \pi) \Big)$$

and

$$s^{\lambda}_{j,k}(\xi) \equiv \frac{\lambda^{\frac{j}{2}}}{\sqrt{2(\lambda - 1)\pi}}\, b(\lambda^j \xi) \sin\Big(\frac{2k+1}{2} \frac{1}{\lambda - 1} (\lambda^j \xi - \pi) \Big),$$

is an orthonormal basis for $L^2(\mathbb{R})$. It is shown in [AWW] that in order for the functions $\alpha^{\lambda}_{j,k}$ and $\beta^{\lambda}_{j,k}$ to be generated by a single function, $\alpha^{\lambda}_{0,0}$, via dilations by λ and multiplications by $e^{in\frac{\xi}{\lambda-1}}$, $n \in \mathbb{Z}$, as was the case for the basis of Theorem 4.5, we must have

$$\lambda = 1 + \frac{1}{m}$$

for some $m \in \mathbb{Z}^+$. With this value of λ one can obtain a wavelet basis of the form $\big\{ \lambda^{\frac{j}{2}} \psi(\lambda^j x - k\frac{1}{\lambda - 1}) : j, k \in \mathbb{Z} \big\}$, where

$$\hat{\psi}(\xi) = \frac{1}{\sqrt{2\pi(\lambda - 1)}}\, e^{i\frac{\xi}{2(\lambda-1)}} b(\xi).$$

The function b we need to use in the above considerations is an even function on $\mathbb{R}$ that, when restricted to $(0, \infty)$, is a bell function associated with the interval $[\pi, \lambda\pi]$, $\varepsilon = \frac{\lambda-1}{\lambda+1}\pi$ and $\varepsilon' = \frac{\lambda(\lambda-1)}{\lambda+1}\pi = \lambda\varepsilon$. This result is due to G. David (see [Dav]).

3. For information on the theory of wavelet-like bases with more general dilation factors see [Au3] and note 2 in section 2.5.