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A First Course on WAVELETS

Eugenio Hernández
Universidad Autónoma de Madrid

Guido Weiss
Washington University in St. Louis

CRC

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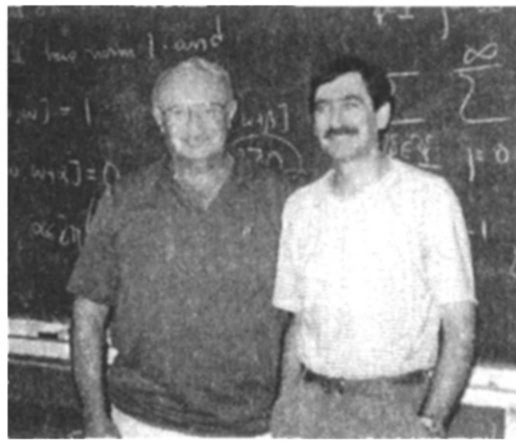
About the Authors



Guido Weiss (left) and Eugenio Hernandez

Eugenio Hernandez graduated from the Universidad Complutense de Madrid in 1977 and obtained his Ph.D. degree from Washington University in St. Louis in 1981. He has been a member of the faculty of the Universidad Autonoma de Madrid since 1982 and is now Professor of Mathematics. During the academic year 1987-88 Dr. Hernandez was a Fulbright Fellow and a member of the Mathematical Sciences Research Institute in Berkeley, California. He was a visiting professor at Washington University in St. Louis in 1994-95. His research interests lie in the areas of the theory of interpolation of operators, weighted inequalities, and most recently, in the theory of wavelets.

Guido Weiss obtained his undergraduate and graduate degrees from the University of Chicago, receiving his Ph.D. degree in 1956. He served on the faculty at DePaul University from 1955 to 1960, and joined the faculty of Washington University in 1960 where he is now the Elinor Anheuser Professor of Mathematics. During the past 35 years he has had leaves of absence that have allowed him to be visiting professor in several different institutions: the Sorbonne, the University of Geneva, the University of Paris in Orsay, the Mathematical Sciences Research Institute in Berkeley, California (in each case for an academic year). He also was visiting professor during semester academic leave at the Universidad de Buenos Aires, Peking University, Beijing Normal University, and the Universidad Autonoma de Madrid.

His research involves a broad area of mathematical analysis, particularly harmonic analysis. Some of his work, especially his contributions to the atomic and molecular characterizations of certain function spaces (particularly the Hardy spaces), is closely related to the theory of wavelets, a subject that has commanded his attention during the last few years. He has been awarded several honors-among them the Chauvenet Prize and honorary degrees from Beijing Normal University, the University of Milano, and the University of Barcelona.

To **Barbara** and **Jody**

Contents

Foreword

by

Yves Meyer

Membre de l'Institut (Académie des Sciences)

Foreign Honorary Member of the American Academy of Arts and Sciences

Wavelet analysis can be defined as an alternative to the classical windowed Fourier analysis. In the latter case the goal is to measure the local frequency content of a signal, while in the wavelet case one is comparing several magnifications of this signal, with distinct resolutions. The building blocks of a windowed Fourier analysis are sines and cosines (waves) multiplied by a sliding window. They are usually referred to as time-frequency atoms. In a wavelet analysis, the window is already oscillating and is called a mother wavelet. This mother wavelet is no longer multiplied by sines or cosines. Instead it is translated and dilated by arbitrary translations and dilations. That is the way the mother wavelet generates the other wavelets which are the building blocks of a wavelet analysis. These dilations are precisely the magnifications we alluded to, and the building blocks are called time-scale atoms.

Fourier analysis, windowed Fourier analysis, and wavelet analysis are based on an identical recipe. In the three cases, the analysis of a function amounts to computing all the correlations between this function and the time-frequency or time-scale atoms which are being used. The synthesis is obtained exactly as if these building blocks were an orthonormal basis.

A common wisdom among numerical analysts and image processing people is that the inverse of a scale is a frequency: small scales correspond to large frequencies and large scales to small frequencies. Moreover, very distinct scales should provide independent (i.e., non-redundant) information. Wavelet analysis could be defined as an attempt to give a very precise meaning to this folk belief.

Wavelets were implicit in mathematics, physics, signal or image processing, and numerical analysis long before they were given the status of a unified scientific field.

In pure mathematics, three algorithms have been created to overcome some drawbacks of standard Fourier series expansions. These difficulties appear when one is facing the problem of measuring the size or the smoothness of a function. For example, the simplest norms, based on quadratic estimates, can easily be extracted from Fourier coefficients. But as soon as L^p or H^p estimates are addressed, Fourier coefficients do not answer the problem, while the algorithms that do answer it involve the **Haar basis** (1909), the **Franklin orthonormal system** (1927), or the **Littlewood-Paley theory** (1930); these have, in the past, proven to be the correct tools.

Later, **Calderón's reproducing identity** (1960) and **atomic decompositions** (1972) were widely used in other functional settings (Hardy spaces, for example). Both the Littlewood-Paley theory and atomic decompositions play a key role in a branch of operator theory created by Calderón, Zygmund, and their school which is known as the Calderón-Zygmund Theory. Just before wavelets became popular, J.O. Strömberg used this precise tool for solving a celebrated problem in the geometry of Banach spaces: the existence of a specific unconditional basis for the Hardy space $H^1(\mathbb{R})$.

In signal or image processing a similar and parallel evolution started from the standard windowed Fourier analysis and culminated in some discrete versions of Calderón's reproducing identity. Indeed, D. Gabor (1946) introduced **time-frequency atoms** in speech signal processing; Croisier, Esteban, and Galand developed **subband coding** in signal processing (1975); and only a little later Burt and Adelson described **pyramidal algorithms** in image processing (1982). D. Marr was convinced that both human vision and computer vision were based on similar algorithms which should be, in some sense, independent of the "wires" used in their realizations. These specific algorithms involve the **zero-crossings** of the wavelet transform of a two-dimensional signal (1982). In numerical analysis, wavelets are related to spline approximation. Before wavelets became fashionable, V. Rokhlin created the so-called **multipole algorithms**: refinement schemes that play a key role in computer graphics.

Finally, let us turn to mathematical physics. **Coherent states** are fundamental in quantum mechanics. **Renormalization in quantum field theory** is needed for extracting finite numbers from divergent integrals. It is based on some variants of Littlewood-Paley techniques which were mainly developed by K. Wilson, K. Gawedzki and A. Kupiainen, J. Glimm and A. Jaffe, G. Battle and P. Federbush.

Therefore wavelets were implicit in several scientific fields but nobody knew that, for instance, Littlewood-Paley theory and the Burt & Adelson pyramidal algorithms were telling the same story. The great unification was a shock, and many people still do not accept it. This unification was a fairy tale come true, which explains why the subject became immediately popular. The great unification meant a scientific status incorporating the heuristics and the wisdom of the distinct fields where protowavelets were already used. This unification was made possible through the efforts of several people. Let me especially mention Alex Grossmann and Stephane Mallat.

I have a vivid and nostalgic memory of many discussions with Antoni Zygmund. He used to test me on whatever problem he was dreaming about. He silently waited for my answer. Then he listened with a smile to my often stupid comments. Finally he often tried to correct my erroneous viewpoints. This happened when R.R. Coifman and G. Weiss and their collaborators launched the so-called **atomic decompositions** program. Zygmund asked my opinion about what Guido Weiss was doing. Zygmund immediately recognized the relevance of this endeavour, while it took me a slightly longer time.

But it is hard to believe that Zygmund would have guessed that atomic decompositions are also relevant in signal processing. He would have been surprised to learn that the celebrated composer and conductor Pierre Boulez and his collaborators decided to find a compact atomic decomposition for an aria by Mozart interpreted by Rita Streich. P. Boulez and his collaborators were indeed using (time-frequency) waveforms instead of (time-scale) wavelets.

We now come to the present book. It is not just one more book about wavelets. This unique book is distinct, since it is co-authored by one of the pioneers of atomic decompositions. Who else is more appropriate to talk about wavelets? Indeed atomic decompositions are at the heart of signal and image processing.

The careful writing of the authors, Eugenio Hernández and Guido Weiss, is well known and this book reflects their desire to make this subject most accessible. It will be applauded by all lovers of the precise, powerful, and elegant mathematics which Guido Weiss and his school have promoted.

This book contains many new and impressive results. Nowadays, there is a tendency to derive wavelets from the multiresolution analysis construction. By this method one cannot address basic issues like the ones that are discussed in this book and are, indeed, crucial. For example, the Fourier localization of a wavelet is discussed in full detail. This has been neglected by other authors. I hope the reader will enjoy this remarkable contribution as much as I did, and I thank the authors for letting me read the manuscript.

Preface

Introduction

Wavelets were introduced relatively recently, in the beginning of the 1980s. They attracted considerable interest from the mathematical community and from members of many diverse disciplines in which wavelets had promising applications. A consequence of this interest is the appearance of several books on this subject and a large volume of research articles. In order to explain why we have written this book, describe where it might play a useful role in this field and to whom it is addressed, we find it necessary to state what we mean by the word "wavelet" and mention some of its properties. Let us do this for wavelets defined on the real line $\mathbb{R}$.

The real line is endowed with two basic algebraic operations, addition and multiplication. From these two operations we obtain two families of operators acting on functions defined on $\mathbb{R}$: the **translations** and the **dilations**. More precisely, translation by $h \in \mathbb{R}$ is the operator τ_h that maps a function f into the function whose value at $x \in \mathbb{R}$ is $(\tau_h f)(x) = f(x-h)$. The dilation ρ_r, $r > 0$, is defined by the equality $(\rho_r f)(x) = f(rx)$. Many of the important linear operators acting on functions defined on $\mathbb{R}$ have simple relations with these two families. For example, differentiation commutes with the translations. More generally, in the setting of tempered distributions, the class of convolution operators are characterized by this property of commuting with translations (differentiation is obtained by convolving with the distribution that is the derivative of the "Dirac-delta function"). Similar observations can be made about the family of dilations. A most important operator acting on functions (or, more generally, on tempered distributions) is the **Fourier Transform**, which maps f into $\hat{f}$, where

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x)\, dx.$$

It is well known that convolution operators are converted, via the Fourier transform, into multiplication operators. This is a consequence of the formula $(f * g)^{\wedge} = \hat{f}\,\hat{g}$. In particular, $(\tau_h f)^{\wedge}(\xi) = e^{-ih\xi}\hat{f}(x)$; that is, translation by h corresponds to multiplication by the exponential $e^{-ih\xi}$. All these properties are particularly natural if we consider them in the context of $L^2(\mathbb{R})$: the Fourier transform can then be expressed in terms of a unitary operator, and this allows one to study many convolution operators in terms of particularly simple multiplier operators.

In view of these observations, it is only natural to look for bases of $L^2(\mathbb{R})$ having properties that reflect the importance of translations, dilations and the Fourier transform. For example, in the analogous periodic case, the "trigonometric" system, $\{\frac{1}{\sqrt{2\pi}} e^{inx} : n \in \mathbb{Z}\}$, is an orthonormal basis for $L^2(0, 2\pi)$ that simultaneously diagonalizes all the bounded operators on this space that commute with translations. This property makes this system a most important basis for $L^2(0, 2\pi)$ and is of fundamental importance to the study of Fourier series. The various wavelets provide us with orthonormal bases for $L^2(\mathbb{R})$ that are particularly natural when dealing with the analysis that involves the action of translations, dilations and the Fourier transform (that is, Harmonic Analysis). We see that this is most plausible from their definition: a function $\psi \in L^2(\mathbb{R})$ is an **orthonormal wavelet** provided the system $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$, where

$$\psi_{j,k}(x) = 2^{\frac{j}{2}} \psi(2^j x - k) \qquad \text{for all} \quad j, k \in \mathbb{Z}.$$

That is, this system is generated from one function, ψ, by translating it by the integers and applying the dyadic dilations ρ_r, where $r = 2^j$, to these translates. The multiplication by the factor $2^{j/2}$ is forced upon us if we require each member of this system to have L^2-norm equal to one; moreover, it renders the action of the Fourier transform on this system particularly simple: if $\gamma = \hat{\psi}$, then the Fourier transform of $\psi_{j,k}$ is

$$(\psi_{j,k})^\wedge(\xi) = e^{-i2^{-j}k\xi} 2^{-\frac{j}{2}} \gamma(2^{-j}\xi).$$

That is, we still have dyadic dilations and the translations are converted into "modulations" (which is a term that means multiplication by exponentials).

The philosophy of the book

The purpose of this book is to show how such wavelets can be constructed, illustrate why they provide us with a particularly powerful tool in mathematical analysis, and indicate how they can be used in applications. The title of the book reflects our hope that it can be read by those who are familiar with the Fourier transform and its basic properties; we feel that this amount of knowledge suffices for the understanding of the material presented. Let us explain in more detail what we mean by this. We shall show that wavelets can be applied to a large variety of mathematical subjects. For example, they can be used to characterize several function spaces: the Lebesgue, Hardy, Sobolev, Besov and Triebel-Lizorkin spaces are some of these. The Lebesgue spaces are easily defined, and some of their basic properties are not hard to explain. This is not the case for all these spaces. For example, the Hardy spaces have many different, but equivalent, definitions. Originally they were introduced as spaces of holomorphic functions in the domain in the complex plane that lies above the x-axis. About twenty-five years ago it was discovered that they can be identified as functions (really, distributions) on $\mathbb{R}$ having an appropriate maximal function. A few years later their "atomic" characterization was discovered. This approach involves certain "building blocks" called atoms, which are particularly simple functions, that can be used to express the general element of the Hardy space. It would carry us way beyond the scope of this book if, before discussing these spaces, we were to present all the material that is necessary to establish the equivalence of these various versions. It is not difficult, however, to present clear statements of those properties that are most relevant to the use of wavelets; when we do this, we do give appropriate references. In this sense this book is not "self-contained," but this does not mean that more is demanded from the reader in order to appreciate the roles that are played by wavelets in these applications.

Wavelets can be defined on other domains. For example, we can introduce a natural extension of the definition of the function $\psi_{j,k}$ by considering ψ to be defined in $\mathbb{R}^n$, n-dimensional Euclidean space, by letting $k = (k_1, \cdots, k_n) \in \mathbb{Z}^n$ be an n-tuple of integers and replacing $2^{j/2}$ by $2^{nj/2}$ (so that $\|\psi_{j,k}\|_2 = \|\psi\|_2$). The situation, in this case, is more complicated: if one makes certain natural assumptions, it can be shown that one cannot obtain an orthonormal basis of $L^2(\mathbb{R}^n)$ by such a construction; in fact, $2^n - 1$ such generating functions are needed if one wants to obtain such a basis. Other domains can be considered where the roles played by the translations and dilations need to be played by different actions on the domain. We decided in this "first course on wavelets" not to present the theory of wavelets in these more complicated settings and to concentrate on the one-dimensional case. We felt that a good understanding of the one-dimensional theory provides a good background for its extensions to other domains.

Let us make a few comments about some of the other books on wavelets. Perhaps the two most important treatises on the subject are Y. Meyer's three-volume set ([Me1], [Me2], and [CM1] – the third one is co-authored with R. Coifman) and I. Daubechies "Ten lectures on wavelets" ([Da1]). Both are excellent presentations, and we recommend them with enthusiasm. They are more advanced than this book and cover much more material. Since they were written, however, the theory has advanced considerably (partly due to their contributions). Some of the original constructions have been simplified and extended. We hope that this book can serve as an introduction to these two treatises. The book by C. Chui ([Chu]) should also be mentioned. It is a good complement to the ones by Daubechies and Meyer (as, we hope, is ours). We cite it often, particularly when we discuss spline wavelets.

Description of the book

It is, perhaps, useful to describe this book in more detail and give some advice about how to read it. The first four chapters, together with Chapter 7, make up a "natural" inter-related group. They are devoted to the construction of wavelets. We feel that Chapter 7 is the most important one in the book. There are two simple equations that completely characterize all orthonormal wavelets. They are

$$\sum_{j\in\mathbb{Z}} \left|\hat{\psi}(2^j\xi)\right|^2 = 1 \qquad \text{for } \ a.e.\ \xi\in\mathbb{R}, \tag{1}$$

and for every odd integer m,

$$\sum_{j=0}^{\infty} \hat{\psi}(2^j\xi)\,\overline{\hat{\psi}(2^j(\xi+2m\pi))} = 0 \qquad \text{for } \ a.e.\ \xi\in\mathbb{R}. \tag{2}$$

More precisely, $\psi\in L^2(\mathbb{R})$ is an orthonormal wavelet if and only if ψ satisfies (1) and (2), provided $\|\psi\|_2 = 1$. The proof of this is elementary but it is not simple, and we present it in the seventh chapter. These equations are known and have been used by many investigators working with wavelets. The proof of this characterization in full generality, however, did not appear in the published literature until recently. It can be found in a paper by G. Gripenberg ([Gri1]), the Ph.D. thesis of one of our students, X. Wang ([Wan]), and will appear in an expository article we wrote with him ([HWW3]). It has been one of our goals to study the properties of wavelets by examining their Fourier transforms. One of the principal features of this book, in fact, is the important role played by the Fourier transform.

The first four chapters are devoted to different ways of constructing wavelets. Chapter 1 deals with the local sine and cosine bases that were discovered by R. Coifman and Y. Meyer. We show how they lead us to bases for $L^2(\mathbb{R})$ that have the important features described in the beginning of this introduction; that is, they enjoy particularly simple relations with the basic operators: translations, dilations and the Fourier transform. We use these bases to construct the wavelets of Lemarié and Meyer, the first class of orthonormal wavelets that were introduced and that includes ones such that they and their Fourier transform are smooth.

In the second chapter we develop a general method that was introduced by Mallat and Meyer for constructing wavelets: the **multiresolution analysis** (MRA). We apply this method to obtain the compactly supported wavelets introduced by Daubechies. The third chapter is devoted to the "band-limited" wavelets (the ones having compactly supported Fourier transforms). We show that the elements of this class have some surprising properties; for example, their Fourier transforms vanish in a neighborhood of the origin. Perhaps one of the best reasons for studying this class separately is that the basic equations (1) and (2) are particularly easy to study. Among other things, the series involved have only a finite number of non-zero terms and we do not need to worry about their convergence. This allows us to pave the way for the technically more difficult analysis involved in the seventh chapter. The fourth chapter introduces the reader to the "spline wavelets." This class appears to be particularly important in the various applications of wavelet theory to signal and image analyses. We also explain in this fourth chapter how one can construct periodic wavelets.

By the end of the first four chapters we have enough examples and have obtained sufficiently many properties of wavelets to introduce the reader to some of the uses of wavelets and their connection to other parts of Analysis. We therefore interrupt our program of characterizing all wavelets in terms of their Fourier transform and show how they provide us with tools for the study of the important scales of function spaces we mentioned above. In addition to providing us with orthonormal bases for the Hilbert space $L^2(\mathbb{R})$, some wavelets give us natural bases for these other topological linear spaces as well. Let us illustrate this with the Lebesgue spaces $L^p(\mathbb{R})$, $1 < p < \infty$, of all those measurable functions f such that

$$\|f\|_p = \Big(\int_{-\infty}^{\infty} |f(x)|^p \, dx\Big)^{\frac{1}{p}} < \infty. \tag{3}$$

When $p = 2$ the finiteness of this norm $\|f\|_2$ is equivalent to the finiteness of the norm

$$\|\mathfrak{c}\|_2 = \Big(\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |c_{j,k}|^2\Big)^{\frac{1}{2}}$$

of the coefficient sequence $\mathfrak{c} = \{c_{j,k}\} = \big\{\langle f, \psi_{j,k}\rangle\big\}$ that provides us with the representation

$$f = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} c_{j,k}\,\psi_{j,k}\,.$$

Thus, $L^2(\mathbb{R})$ can be represented as the space $\ell^2(\mathbb{Z}\times\mathbb{Z})$ of all sequences $\mathfrak{c}$ such that $\|\mathfrak{c}\|_2 < \infty$. Appropriate wavelet bases provide us with the characterization of $L^p(\mathbb{R})$ in terms of a sequence space for the other indices $p \in (1,\infty)$. It can be shown that f belongs to $L^p(\mathbb{R})$ if and only if

$$\|\mathfrak{c}\|^{(p)} = \big\|\{c_{j,k}\}\big\|^{(p)} = \Big(\int_{-\infty}^{\infty}\Big\{\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} 2^j|c_{j,k}|^2\chi_{j,k}(x)\Big\}^{\frac{p}{2}} dx\Big)^{\frac{1}{p}} < \infty,$$

where $\chi_{j,k}$ is the characteristic function of the interval $[2^{-j}k, 2^{-j}(k+1)]$ and $\mathfrak{c}$ is the sequence of coefficients of f associated with $\{\psi_{j,k}\}$. Observe that the finiteness of $\|\mathfrak{c}\|^{(p)}$ is a condition on the size (or absolute value) of the coefficients $c_{j,k}$. This provides us with the ability to study $L^p(\mathbb{R})$ in terms of a corresponding sequence space in a way that is analogous to the reduction of properties of $L^2(\mathbb{R})$ to properties of $\ell^2(\mathbb{Z}\times\mathbb{Z})$. Note that $\|\mathfrak{c}\|^{(2)} = \|\mathfrak{c}\|_2$. Thus, an operator that is diagonalized by the basis $\{\psi_{j,k}\}$ can be analyzed in terms of its proper values, as is the case in Hilbert space theory. Many important operators are "essentially" diagonalized by wavelet bases. It is this circle of ideas that is presented in Chapter 5 and Chapter 6. More specifically, we present a brief treatment of bases in Banach spaces, with an emphasis on the notion of unconditionality, in Chapter 5. In Chapter 6 we give the characterizations described above. This treatment uses properties of **Calderón-Zygmund operators**; consequently, we have an opportunity to see how wavelets are associated with the study of these important operators.

In Chapter 7 we resume the study of wavelets in $L^2(\mathbb{R})$. We not only give a characterization of all wavelets, as described above, but we also characterize all wavelets that arise from an MRA and the basic functions (the scaling functions and low-pass filters) involved in this method. These characterizations allow us to construct several other classes of wavelets as well.

Though most of the bases discussed in the first seven chapters are orthonormal, we do mention some other types of bases. In Chapter 8 we present a more thorough treatment of systems that are more general, with particular attention to **frames** and their importance to wavelets. We pay special attention to the way they can be used to analyze and reconstruct functions; we also extend the Balian-Low theorem to frames.

The last chapter is devoted to certain topics that are important and relevant to the applications of the theory of wavelets. We indicate how the mathematical theory is transformed when it is applied to "discrete" signals. We develop the Discrete Fourier (and Cosine) Transform in what is, probably, a manner that is different from the usual treatment but has some features that are adaptable to programming for computers. We also describe the decomposition and reconstruction algorithms for wavelets and we end the chapter with a treatment of "wavelet packets."

One of our colleagues, M.V. Wickerhauser, has recently written a book, "Adapted wavelet analysis from theory to software" ([Wi2]), that treats the subject we just mentioned, and many more applications, in great detail. We believe that his book will prove to be most useful. We found no need, therefore, to go further than we did in this direction. As we stated about the books by Daubechies and Meyer, we hope that our book makes a good companion to, and complements, the book by Wickerhauser.

Some advice to the reader

The background we assume of the reader is a "good undergraduate" preparation in mathematics. We do use the language of measure theory; for example, we talk about "measurable functions." One should not be discouraged if he/she only knows the ordinary Riemann integral. Substituting the Lebesgue integral for the latter will, in general, not affect the meaning or the validity of most statements. Some notions in elementary functional analysis are used; again, the results or statements involving these notions can almost always be understood by ignoring unfamiliar language.

It is our hope that graduate students in mathematics, the sciences and engineering can profit from our presentation. We advise the reader not to be discouraged by the **few** somewhat technical notions we introduce at times (distributions, maximal functions, vector-valued inequalities, etc). If it's "too much," just skip it at first; there is enough material that can be understood with the background mentioned in the previous paragraph. It is our experience that even those whose main interest is in the applications can profit by learning about the theory we present.

In each section we number results consecutively; that is, we do **not** form separate lists of theorems, propositions, corollaries, lemmas, formulae and inequalities. These items are listed as ordered pairs n.m, where "n" denotes the section (in the chapter) and "m" the m^{th} item so numbered in the section. If we need to refer to something in another chapter, we mention the chapter and the relevant ordered pair. The sections in each chapter are also assigned an ordered pair, n.m; in this case, "n" denotes the number of the chapter and "m" the m^{th} section.

We do not present a list of exercises at the end of each chapter. In many cases we leave certain calculations to be worked out by the reader. This is particularly true of the comments made in the last section of each chapter, which is labelled "Notes and references."

We also feel that we should state quite clearly that, though the bibliography we include is quite large, it is far from a list that comes anywhere close to exhausting what has been published in the theory of wavelets during the relatively short period of its existence. We have tried to give proper credits; however, since some of the material we discuss is quite new, we realize that it is very likely we omitted some references that should have been included.

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