## Chapter 14

## FFTs for Real Input

In Chapter 3, the discrete Fourier transform on $N$ discrete samples from a complex time series is defined by formula (3.1):

$$
\begin{align*}
X_{r} & =\sum_{\ell=0}^{N-1} x_{\ell} \omega_{N}^{r \ell}, \quad r=0,1, \ldots, N-1  \tag{14.1}\\
& =\sum_{\ell=0}^{N-1}\left(\operatorname{Re}\left(x_{\ell}\right)+j \operatorname{Im}\left(x_{\ell}\right)\right) \omega_{N}^{r \ell}
\end{align*}
$$

When the samples come from a real time series, they can be treated as complex numbers with zero-valued imaginary part, i.e., $\operatorname{Im}\left(x_{\ell}\right)=0$ for $0 \leq \ell \leq N-1$. In other words, real data represent a special case when approximately one half of the arithmetic operations are redundantly performed on zeros. Since many FFTs are performed on real-valued time series, it is worthwhile to handle real input more efficiently. Two such algorithms are described below. The first algorithm allows one to compute two real FFTs of size $N$ by computing one complex FFT of size $N$; and the second algorithm allows one to compute a real FFT of size $N$ by computing a complex FFT of size $N / 2$.

### 14.1 Computing Two Real FFTs Simultaneously

In this section, a method which computes two real FFTs of size $N$ by computing one complex FFT of size $N$ is introduced. The two sets of real numbers are denoted by $f_{\ell}$ and $g_{\ell}$ for $0 \leq \ell \leq N-1$. By setting $\operatorname{Re}\left(x_{\ell}\right)=f_{\ell}$, and $\operatorname{Im}\left(x_{\ell}\right)=g_{\ell}$, one obtains a set of $N$ complex numbers $x_{\ell}=f_{\ell}+j g_{\ell}$ for $0 \leq \ell \leq N-1$.

The definition of the DFT implies that

$$
\begin{equation*}
F_{r}=\sum_{\ell=0}^{N-1} f_{\ell} \omega_{N}^{r \ell} \quad \text { and } \quad G_{r}=\sum_{\ell=0}^{N-1} g_{\ell} \omega_{N}^{r \ell}, \quad 0 \leq r \leq N-1 \tag{14.2}
\end{equation*}
$$

and

$$
\begin{align*}
X_{r} & =\sum_{\ell=0}^{N-1} x_{\ell} \omega_{N}^{r \ell} \\
& =\sum_{\ell=0}^{N-1}\left(f_{\ell}+j g_{\ell}\right) \omega_{N}^{r \ell}  \tag{14.3}\\
& =\sum_{\ell=0}^{N-1} f_{\ell} \omega_{N}^{r \ell}+j \sum_{\ell=0}^{N-1} g_{\ell} \omega_{N}^{r \ell} \\
& =F_{r}+j G_{r} .
\end{align*}
$$

Thus, one complex FFT on the $x_{\ell}$ 's can be computed to obtain $X_{r}$ 's, and almost half of the arithmetic operations can be saved if the $F_{r}$ 's and the $G_{r}$ 's can be recovered efficiently from the computed $X_{r}$ 's. This can be done by using the symmetry property for the DFT of a real-valued series, which was established in Chapter 1. For convenience, the result is rederived here.

The symmetry property ensures that the complex conjugate of $F_{N-r}$ is equal to $F_{r}$. This property is derived using the DFT definition and the fact that the complex conjugate of a real-valued $f_{\ell}$ is equal to itself, $\omega_{N}^{N}=1$, and the complex conjugate of $\omega_{N}^{-r \ell}$ is equal to $\omega_{N}^{r \ell}$.

$$
\begin{equation*}
\bar{F}_{N-r}=\sum_{\ell=0}^{N-1} \bar{f}_{\ell}\left(\bar{\omega}_{N}^{N \ell}\right) \bar{\omega}_{N}^{-r \ell}=\sum_{\ell=0}^{N-1} f_{\ell} \omega_{N}^{r \ell}=F_{r} . \tag{14.4}
\end{equation*}
$$

Since the $g_{\ell}$ 's are also real, $\bar{G}_{N-r}=G_{r}$. Now the complex conjugate of $X_{N-r}$ can be expressed in terms of $F_{r}$ and $G_{r}$ as shown below.

$$
\begin{equation*}
\bar{X}_{N-r}=\bar{F}_{N-r}-j \bar{G}_{N-r}=F_{r}-j G_{r} . \tag{14.5}
\end{equation*}
$$

Combining (14.3) and (14.5), one immediately obtains

$$
\begin{equation*}
F_{r}=\frac{1}{2}\left(X_{r}+\bar{X}_{N-r}\right), \quad G_{r}=\frac{j}{2}\left(\bar{X}_{N-r}-X_{r}\right) . \tag{14.6}
\end{equation*}
$$

Therefore, only $2 N$ extra complex additions/subtractions are required to recover the two real FFTs after one complex FFT is performed, which requires $\Theta\left(N \log _{2} N\right)$ arithmetic operations as usual.

### 14.2 Computing a Real FFT

To apply the results in the previous section to transform a single series, the latter is first split into two real series of half the size. The derivation is similar to the work in
deriving the DIT FFT algorithm in Section 3.1, namely,

$$
\begin{align*}
X_{r} & =\sum_{\ell=0}^{N-1} x_{\ell} \omega_{N}^{r \ell}, \quad r=0,1, \ldots, N-1, \\
& =\sum_{\ell=0}^{\frac{N}{2}-1} x_{2 \ell} \omega_{N}^{r(2 \ell)}+\omega_{N}^{r} \sum_{\ell=0}^{\frac{N}{2}-1} x_{2 \ell+1} \omega_{N}^{r(2 \ell)}  \tag{14.7}\\
& =\sum_{\ell=0}^{\frac{N}{2}-1} x_{2 \ell} \omega_{\frac{N}{2}}^{r \ell}+\omega_{N}^{r} \sum_{\ell=0}^{\frac{N}{2}-1} x_{2 \ell+1} \omega_{\frac{N}{2}}^{r \ell} .
\end{align*}
$$

By setting $f_{\ell}=x_{2 \ell}, g_{\ell}=x_{2 \ell+1}$ for $0 \leq \ell \leq N / 2-1$, the DFT of two real series and a DFT of $N / 2$ complex numbers $y_{\ell}=f_{\ell}+j g_{\ell}$ are defined below.

$$
\begin{equation*}
F_{r}=\sum_{\ell=0}^{\frac{N}{2}-1} f_{\ell} \omega_{\frac{N}{2}}^{r \ell}, \quad G_{r}=\sum_{\ell=0}^{\frac{N}{2}-1} g_{\ell} \omega_{\frac{N}{2}}^{r \ell}, \tag{14.8}
\end{equation*}
$$

and

$$
\begin{equation*}
Y_{r}=\sum_{\ell=0}^{\frac{N}{2}-1} y_{\ell} \omega_{\frac{N}{2}}^{r \ell}=\sum_{\ell=0}^{\frac{N}{2}-1}\left(f_{\ell}+j g_{\ell}\right) \omega_{\frac{N}{2}}^{r \ell}=\sum_{\ell=0}^{\frac{N}{2}-1} f_{\ell} \omega_{\frac{N}{2}}^{r \ell}+j \sum_{\ell=0}^{\frac{N}{2}-1} g_{\ell} \omega_{\frac{N}{2}}^{r \ell}=F_{r}+j G_{r} \tag{14.9}
\end{equation*}
$$

Using the results from the previous section, one complex FFT on the ye's can be computed to obtain the $Y_{r}$ 's, and the $F_{r}$ 's and $G_{r}$ 's ( for $0 \leq r \leq N / 2-1$ ) can be recovered using the following equations.

$$
\begin{equation*}
F_{r}=\frac{1}{2}\left(Y_{r}+\bar{Y}_{\frac{N}{2}-r}\right), \quad G_{r}=\frac{j}{2}\left(\bar{Y}_{\frac{N}{2}-r}-Y_{r}\right) . \tag{14.10}
\end{equation*}
$$

However, it is no longer sufficient to have successfully recovered the $F_{r}$ 's and $G_{r}$ 's, because the goal is to compute the $X_{r}$ 's defined by equation (14.7), which can now be obtained from the available $F_{r}$ 's and $G_{r}$ 's as shown below.

$$
\begin{align*}
X_{r} & =\sum_{\ell=0}^{\frac{N}{2}-1} x_{2 \ell} \omega_{\frac{N}{2}}^{r \ell}+\omega_{N}^{r} \sum_{\ell=0}^{\frac{N}{2}-1} x_{2 \ell+1} \omega_{\frac{N}{2}}^{r \ell} \\
& =\sum_{\ell=0}^{\frac{N}{2}-1} f_{\ell} \omega_{\frac{N}{2}}^{r \ell}+\omega_{N}^{r} \sum_{\ell=0}^{r-1} g_{\ell} \omega_{\frac{N}{2}}^{r \ell}  \tag{14.11}\\
& =F_{r}+\omega_{N}^{r} G_{r}, \quad r=0,1,2, \ldots, N / 2-1 .
\end{align*}
$$

Since the $x_{\ell}$ 's are real, the $X_{r}$ 's have the symmetry property derived in (14.4); thus, $X_{\frac{N}{2}+1}, X_{\frac{N}{2}+2}, \ldots, X_{N-1}$ can be obtained by taking the complex conjugate of the previously computed $X_{r}$ 's.

$$
\begin{equation*}
X_{N-r}=\bar{X}_{r}, \quad r=1,2, \ldots, N / 2-1 . \tag{14.12}
\end{equation*}
$$

Using equations (14.11) and (14.12), all the $X_{r}$ 's can be obtained except for $X_{\frac{N}{2}}$. To compute $X_{r+\frac{N}{2}}=F_{r+\frac{N}{2}}+\omega_{N}^{r+\frac{N}{2}} G_{r+\frac{N}{2}}$, recall that $F_{r+\frac{N}{2}}=F_{r}, G_{r+\frac{N}{2}}=G_{r}$, and
$\omega_{N}^{r+\frac{N}{2}}=-\omega_{N}^{r}$. Using these properties with $r=0, X_{r+\frac{N}{2}}=X_{\frac{N}{2}}$ can now be computed by

$$
\begin{equation*}
X_{\frac{N}{2}}=F_{0}-G_{0} . \tag{14.13}
\end{equation*}
$$

In total, $N$ extra complex additions/subtractions are needed to recover the $F_{r}$ 's and $G_{r}$ 's after one complex FFT is performed on $N / 2$ complex numbers, and an additional $N / 2$ complex multiplications and $N / 2+1$ complex additions/subtractions are needed to compute the $X_{r}$ 's using equations (14.11), (14.12), and (14.13). Therefore, for large $N$, almost half of the arithmetic operations can be saved by performing the FFT on $N / 2$ complex numbers instead of treating the real-valued series as consisting of $N$ complex numbers.

### 14.3 Notes and References

According to Bergland [4], there are two basic approaches to the evaluation of realvalued time series. The approach which makes use of the conventional complex FFT algorithm and depends on forming an artificial $N / 2$-term complex record from each $N$-term real record was due to Cooley, Lewis, and Welch [31]; an alternative approach was proposed in [4]. The former approach was used by Brigham [17] and Walker [106]. The algorithms developed in this chapter are based on Walker's approach [106]. Other algorithms for computing real-valued series may be found in [4, 8, 88]. Implementation of split-radix FFT algorithms for real and real-symmetric data is described in [38].

