

Functional analysis lecture notes

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COURSE OBJECTIVES

In order to reach the more interesting and useful ideas, we shall adopt a fairly brutal approach to some early material. Lengthy proofs will sometimes be left out, though full versions will be made available. By the end of the course, you should have a good understanding of *normed vector spaces*, *Hilbert and Banach spaces*, *fixed point theorems* and examples of *function spaces*. These ideas will be illustrated with applications to differential equations.

BOOKS

You do not need to buy a book for this course, but the following may be useful for background reading. If you do buy something, the starred books are recommended [1] *Functional Analysis*, W. Rudin, McGraw–Hill (1973). This book is thorough, sophisticated and demanding.

[2] *Functional Analysis*, F. Riesz and B. Sz.-Nagy, Dover (1990). This is a classic text, also much more sophisticated than the course.

[3]* *Foundations of Modern Analysis*, A. Friedman, Dover (1982). Cheap and cheerful, includes a useful few sections on background.

[4]* *Essential Results of Functional Analysis*, R.J. Zimmer, University of Chicago Press (1990). Lots of good problems and a useful chapter on background.

[5]* *Functional Analysis in Modern Applied Mathematics*, R.F. Curtain and A.J. Pritchard, Academic Press (1977). This book is closest to the course.

[6]* *Linear Analysis*, B. Bollobas, Cambridge University Press (1995). This book is excellent but makes heavy demands on the reader.

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Normed Linear Spaces

A linear space is simply an abstract version of the familiar vector spaces \mathbb{R} , \mathbb{R}^2 , \mathbb{R}^3 and so on. Recall that vector spaces have certain *algebraic* properties: vectors may be added, multiplied by scalars, and vector spaces have bases and subspaces. Linear maps between vector spaces may be described in terms of matrices. Using the Euclidean norm or distance, vector spaces have other *analytic* properties (though you may not have called them that): for example, certain functions from \mathbb{R} to \mathbb{R} are continuous, differentiable, Riemann integrable and so on.

We need to make three steps of generalization.

BASES: The first is familiar: instead of, for example, \mathbb{R}^3 , we shall sometimes want to talk about an abstract three-dimensional vector space V over the field \mathbb{R} . This distinction amounts to having a specific basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in mind, in which case every element of V corresponds to a triple $(a, b, c) = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3$ of reals – or choosing not to think of a specific basis, in which case the elements of V are just abstract vectors \mathbf{v} . In the abstract language we talk about linear maps or operators between vector spaces; after choosing a basis linear maps become matrices – though in an infinite dimensional setting it is rarely useful to think in terms of matrices.

GROUND FIELDS: The second is fairly trivial and is also familiar: the ground field can be any field. We shall only be interested in \mathbb{R} (real vector spaces) and \mathbb{C} (complex vector spaces). Notice that \mathbb{C} is itself a two-dimensional vector space over \mathbb{R} with additional structure (multiplication). Choosing a basis $\{1, i\}$ for \mathbb{C} over \mathbb{R} we may identify $z \in \mathbb{C}$ with the vector $(\Re(z), \Im(z)) \in \mathbb{R}^2$.

DIMENSION: In linear algebra courses, you deal with finite dimensional vector spaces. Such spaces (over a fixed ground field) are determined up to isomorphism by their dimension. We shall be mainly looking at linear spaces that are not finite-dimensional, and several new features appear. All of these features may be summed up in one line: the algebra of infinite dimensional linear spaces is intimately connected to the topology. For example, linear maps between \mathbb{R}^2 and \mathbb{R}^2 are automatically continuous. For infinite dimensional spaces, some linear maps are not continuous.

1. Linear (vector) spaces

DEFINITION 1.1. A linear space over a field k is a set V equipped with maps $\oplus : V \times V \rightarrow V$ and $\cdot : k \times V \rightarrow V$ with the properties

- (1) $x \oplus y = y \oplus x$ for all $x, y \in V$ (addition is commutative);
- (2) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ for all $x, y, z \in V$ (addition is associative);
- (3) there is an element $0 \in V$ such that $x \oplus 0 = 0 \oplus x = x$ for all $x \in V$ (a zero element);
- (4) for each $x \in V$ there is a unique element $-x \in V$ with $x \oplus (-x) = 0$ (additive inverses);

- (notice that $(V, +)$ therefore forms an abelian group)
- (5) $\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x$ for all $\alpha, \beta \in k$ and $x \in V$;
- (6) $(\alpha + \beta) \cdot x = \alpha \cdot x \oplus \beta \cdot x$ for all $\alpha, \beta \in k$ and $x \in V$ (scalar multiplication distributes over scalar addition);
- (7) $\alpha \cdot (x \oplus y) = \alpha \cdot x \oplus \alpha \cdot y$ for all $\alpha \in k$ and $x, y \in V$ (scalar multiplication distributes over vector addition);
- (8) $1 \cdot x = x$ for all $x \in V$ where 1 is the multiplicative identity in the field k .

EXAMPLE 1.1. [1] Let $V = \mathbb{R}^n = \{x = (x_1, \dots, x_n) \mid x_i \in \mathbb{R}\}$ with the usual vector addition and scalar multiplication.

[2] Let V be the set of all polynomials with coefficients in \mathbb{R} of degree $\leq n$ with usual addition of polynomials and scalar multiplication.

[3] Let V be the set $M_{(m,n)}(\mathbb{C})$ of complex-valued $m \times n$ matrices, with usual addition of matrices and scalar multiplication.

[4] Let ℓ_∞ denote the set of infinite sequences (x_1, x_2, x_3, \dots) that are *bounded*: $\sup\{|x_n|\} < \infty$. Then ℓ_∞ is linear space, since $\sup\{|x_n + y_n|\} \leq \sup\{|x_n|\} + \sup\{|y_n|\} < \infty$ and $\sup\{|\alpha x_n|\} = |\alpha| \sup\{|x_n|\}$.

[5] Let $C(S)$ be the set of continuous functions $f : S \rightarrow \mathbb{R}$ with addition $(f \oplus g)(x) = f(x) + g(x)$ and scalar multiplication $(\alpha \cdot f)(x) = \alpha f(x)$. Here S is, for example, any subset of \mathbb{R} . The dimension of $C(S)$ is infinite if S is an infinite set, and is exactly $|S|$ if not¹.

[6] Let V be the set of Riemann-integrable functions $f : (0, 1) \rightarrow \mathbb{R}$ which are *square-integrable*: that is, with the property that $\int_0^1 |f(x)|^2 dx < \infty$. We need to check that this is a linear space. Closure under scalar multiplication is clear: if $\int_0^1 |f(x)|^2 dx < \infty$ and $\alpha \in \mathbb{R}$ then $\int_0^1 |\alpha f(x)|^2 dx = |\alpha|^2 \int_0^1 |f(x)|^2 dx < \infty$. For vector addition we need the Cauchy-Schwartz inequality:

$$\begin{aligned} \int_0^1 |f(x) + g(x)|^2 dx &\leq \int_0^1 (|f(x)|^2 + 2|f(x)||g(x)| + |g(x)|^2) dx \\ &\leq \int_0^1 |f(x)|^2 dx + \left(\int_0^1 |f(x)|^2 dx \right)^{1/2} \left(\int_0^1 |g(x)|^2 dx \right)^{1/2} \\ &\quad + \int_0^1 |g(x)|^2 dx < \infty. \end{aligned}$$

[7] Let $C^\infty[a, b]$ be the space of infinitely differentiable functions on $[a, b]$.

[8] Let Ω be a subset of \mathbb{R}^n , and $C^k(\Omega)$ the space of k times continuously differentiable functions. This means that if $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$ has $|\mathbf{a}| = a_1 + \dots + a_n \leq k$, then the partial derivatives

$$D^{\mathbf{a}} f = \frac{\partial^{|\mathbf{a}|} f}{\partial x_1^{a_1} \dots \partial x_n^{a_n}}$$

exist and are continuous.

From now on we will drop the special notation \oplus, \cdot for vector addition and scalar multiplication. We will also (normally) use plain letters x, y and so on for elements of linear spaces.

¹This may be seen as follows. If $S = \{s_1, \dots, s_n\}$ is finite, then the map that sends a function $f \in C(S)$ to the vector $(f(s_1), \dots, f(s_n)) \in \mathbb{R}^n$ is an isomorphism of linear spaces. If S is infinite, then the map that sends a polynomial $f \in \mathbb{R}[x]$ to the function $f \in C(S)$ is injective (since two polynomials that agree on infinitely many values must be identical). This shows that $C(S)$ contains an isomorphic copy of an infinite-dimensional space, so must be infinite-dimensional.

As in the linear algebra of finite-dimensional vector spaces, subsets of linear spaces that are themselves linear spaces are called *linear subspaces*.

2. Linear subspaces

DEFINITION 1.2. Let V be a linear space over the field k . A subset $W \subset V$ is a linear subspace of V if for all $x, y \in W$ and $\alpha, \beta \in k$, the linear combination $\alpha x + \beta y \in W$.

EXAMPLE 1.2. [1] The set of vectors in \mathbb{R}^n of the form $(x_1, x_2, x_3, 0, \dots, 0)$ forms a three-dimensional linear subspace.

[2] The set of polynomials of degree $\leq r$ forms a linear subspace of the set of polynomials of degree $\leq n$ for any $r \leq n$.

[3] (cf. Example 1.1(8)) The space $C^{k+1}(\Omega)$ is a linear subspace of $C^k(\Omega)$.

3. Linear independence

Let V be a linear space. Elements x_1, x_2, \dots, x_n of V are *linearly dependent* if there are scalars $\alpha_1, \dots, \alpha_n$ (not all zero) such that

$$\alpha_1 x_1 + \dots + \alpha_n x_n = 0.$$

If there is no such set of scalars, then they are *linearly independent*.

The linear span of the vectors x_1, x_2, \dots, x_n is the linear subspace

$$\text{span}\{x_1, \dots, x_n\} = \left\{ x = \sum_{j=1}^n \alpha_j x_j \mid \alpha_j \in k \right\}.$$

DEFINITION 1.3. If the linear space V is equal to the span of a linearly independent set of n vectors, then V is said to have dimension n . If there is no such set of vectors, then V is infinite-dimensional.

A linearly independent set of vectors that spans V is called a basis for V .

EXAMPLE 1.3. [1] (cf. Example 1.1(1)) The space \mathbb{R}^n has dimension n ; the standard basis is given by the vectors $e_1 = (1, 0, \dots, 0), e_2 = (0, 1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1)$.

[2] (cf. Example 1.1[2]) A basis is given by $\{1, t, t^2, \dots, t^n\}$, showing the space to have dimension $(n + 1)$.

[3] Examples 1.1 [4], [5], [6], [7], [8] are all infinite-dimensional.

4. Norms

A norm on a vector space is a way of measuring distance between vectors.

DEFINITION 1.4. A norm on a linear space V over k is a non-negative function $\|\cdot\| : V \rightarrow \mathbb{R}$ with the properties that

- (1) $\|x\| = 0$ if and only if $x = 0$ (positive definite);
- (2) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$ (triangle inequality);
- (3) $\|\alpha x\| = |\alpha| \|x\|$ for all $x \in V$ and $\alpha \in k$.

In Definition 1.4(3) we are assuming that k is \mathbb{R} or \mathbb{C} and $|\cdot|$ denotes the usual absolute value. If $\|\cdot\|$ is a function with properties (2) and (3) only it is called a *semi-norm*.

DEFINITION 1.5. A normed linear space is a linear space V with a norm $\|\cdot\|$ (sometimes we write $\|\cdot\|_V$).

DEFINITION 1.6. A set C in a linear space is *convex* if for any two points $x, y \in C$, $tx + (1-t)y \in C$ for all $t \in [0, 1]$.

DEFINITION 1.7. A norm $\|\cdot\|$ is *strictly convex* if $\|x\| = 1$, $\|y\| = 1$, $\|x+y\| = 2$ together imply that $x = y$.

We won't be using convexity methods much, but for each of the examples try to work out whether or not the norm is strictly convex. Strict convexity is automatic for Hilbert spaces.

EXAMPLE 1.4. [1] Let $V = \mathbb{R}^n$ with the usual Euclidean norm

$$\|x\| = \|x\|_2 = \left(\sum_{j=1}^n |x_j|^2 \right)^{1/2}.$$

To check this is a norm the only difficulty is the triangle inequality: for this we use the Cauchy-Schwartz inequality.

[2] There are many other norms on \mathbb{R}^n , called the p -norms. For $1 \leq p < \infty$ defined

$$\|x\|_p = \left(\sum_{j=1}^n |x_j|^p \right)^{1/p}.$$

Then $\|\cdot\|_p$ is a norm on V : to check the triangle inequality use Minkowski's Inequality

$$\left(\sum_{j=1}^n |x_j + y_j|^p \right)^{1/p} \leq \left(\sum_{j=1}^n |x_j|^p \right)^{1/p} + \left(\sum_{j=1}^n |y_j|^p \right)^{1/p}.$$

There is another norm corresponding to $p = \infty$,

$$\|x\|_\infty = \max_{1 \leq j \leq n} \{|x_j|\}.$$

It is conventional to write ℓ_p^n for these spaces. Notice that the linear spaces ℓ_p^n and ℓ_q^n have exactly the same elements.

[3] Let $X = \ell_\infty$ be the linear space of bounded infinite sequences (cf. Example 1.1[4]). Consider the function $\|\cdot\|_p : \ell_\infty \rightarrow \mathbb{R} \cup \{\infty\}$ given by

$$\|x\|_p = \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p}.$$

If we restrict attention to the linear subspace on which $\|\cdot\|_p$ is finite, then $\|\cdot\|_p$ is a norm (to check this use the infinite version of Minkowski's inequality). This gives an infinite family of normed linear spaces,

$$\ell_p = \{x = (x_1, x_2, \dots) \mid \|x\|_p < \infty\}.$$

Notice that for $p < \infty$ there is a strict inclusion $\ell_p \subset \ell_\infty$. Indeed, for any $p < q$ there is a strict inclusion $\ell_p \subset \ell_q$ so ℓ_p is a linear subspace of ℓ_q . That is, the sets ℓ_p and ℓ_q for $p \neq q$ do not contain the same elements.

[4] Let $X = C[a, b]$, and put $\|f\| = \sup_{t \in [a, b]} |f(t)|$. This is called the *uniform or supremum* norm. Why is it finite?

[5] Let $X = C[a, b]$, and choose $1 \leq p < \infty$. Then (using the integral form of Minkowski's inequality) we have the p -norm

$$\|f\|_p = \left(\int_a^b |f(t)|^p dt \right)^{1/p}.$$

[6] (cf. Example 1.1[6]). Let V be the set of Riemann-integrable functions $f : (0, 1) \rightarrow \mathbb{R}$ which are square-integrable. Let $\|f\|_2 = \int_0^1 |f(x)|^2 dx < \infty$. Then V is a normed linear space.

5. Isomorphism of normed linear spaces

Recall from linear algebra that linear spaces V and W are (algebraically) isomorphic if there is a bijection $T : V \rightarrow W$ that is *linear*:

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

for all $\alpha, \beta \in k$ and $x, y \in V$.

A pair $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ of normed linear spaces are (topologically) isomorphic if there is a linear bijection $T : X \rightarrow Y$ with the property that there are positive constants a, b with

$$a\|x\|_X \leq \|T(x)\|_Y \leq b\|x\|_X. \quad (1)$$

We shall usually denote topological isomorphism by $X \cong Y$.

LEMMA 1.1. *If X and Y are n -dimensional normed linear spaces over \mathbb{R} (or \mathbb{C}) then X and Y are topologically isomorphic.*

If the constants a and b in equation (1) may both be taken as 1, so $\|T(x)\|_Y = \|x\|_X$, then T is called an *isometry* and the normed spaces X and Y are called *isometric*.

EXAMPLE 1.5. The real linear spaces $(\mathbb{C}, |\cdot|)$ and $(\mathbb{R}^2, \|\cdot\|_2)$ are isometric.

If Y is a subspace of a linear normed space $(X, \|\cdot\|_X)$ then $\|\cdot\|_X$ restricted to Y makes Y into a normed subspace.

EXAMPLE 1.6. Let Y denote the space of infinite real sequences with only finitely many non-zero terms. Then Y is a linear subspace of ℓ_p for any $1 \leq p \leq \infty$ so the p -norm makes Y into a normed space.

6. Products of normed spaces

If $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed linear spaces, then the product

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

is a linear space which may be made into a normed space in many different ways, a few of which follow.

EXAMPLE 1.7. [1] $\|(x, y)\| = (\|x\|_X + \|y\|_Y)^{1/p}$;
[2] $\|(x, y)\| = \max\{\|x\|_X, \|y\|_Y\}$.

7. Continuous maps between normed spaces

We have seen continuous maps between \mathbb{R} and \mathbb{R} in first year analysis. To make this definition we used the *distance function* $|x - y|$ on \mathbb{R} : a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous if

$$\forall a \in \mathbb{R}, \forall \epsilon > 0, \exists \delta > 0 \text{ such that } |x - a| < \delta \implies |f(x) - f(a)| < \epsilon. \quad (2)$$

Looking at (2), we see that exactly the same definition can be made for maps between linear normed spaces, which in view of Example 1.4 will give us the possibility of talking about continuous maps between spaces of functions. Thus, on suitably defined spaces, questions like “is the map $f \mapsto f'$ continuous?” or “is the map $f \mapsto \int_0^x f$ continuous?” can be asked.

DEFINITION 1.8. A map $f : X \rightarrow Y$ between normed linear spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ is *continuous at* $a \in X$ if

$$\forall \epsilon > 0 \exists \delta = \delta(\epsilon, a) > 0 \text{ such that } \|x - a\|_X < \delta \implies \|f(x) - f(a)\|_Y < \epsilon.$$

If f is continuous at every $a \in X$ then we simply say f is *continuous*.

Finally, f is *uniformly continuous* if

$$\forall \epsilon > 0 \exists \delta = \delta(\epsilon) > 0 \text{ such that } \|x - y\|_X < \delta \implies \|f(x) - f(y)\|_Y < \epsilon \forall x, y \in X.$$

EXAMPLE 1.8. [1] The map $x \mapsto x^2$ from $(\mathbb{R}, |\cdot|)$ to itself is continuous but not uniformly continuous.

[2] Let $f(x) = Ax$ be the non-trivial linear map from \mathbb{R}^n to \mathbb{R}^m (with Euclidean norms) defined by the $m \times n$ matrix $A = (a_{ij})$. Using the Cauchy-Schwartz inequality, we see that f is uniformly continuous: fix $a \in \mathbb{R}^n$ and $b = Aa$. Then for any $x \in \mathbb{R}^n$ we have

$$\begin{aligned} \|Ax - Aa\|^2 &= \sum_{i=1}^m \left| \sum_{j=1}^n a_{ij}(x_j - a_j) \right|^2 \\ &\leq \sum_{i=1}^m \left(\sum_{j=1}^n |a_{ij}^2| \right) \left(\sum_{j=1}^n |x_j - a_j|^2 \right) \\ &= C^2 \|x - a\|^2 \end{aligned}$$

where $C^2 = \sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2 > 0$. It follows that f is uniformly continuous, and we may take $\delta = \epsilon/C$.

[3] Let X be the space of continuous functions $[-1, 1] \rightarrow \mathbb{R}$ with the sup norm (cf. Example 1.4[4]). Define a map $F : X \rightarrow X$ by $F(u) = v$, where

$$v(t) = 1 + \int_0^t (\sin u(s) + \tan s) ds.$$

The map F is uniformly continuous on X . Notice that F is intimately connected to a certain differential equation: a *fixed point* for F (that is, an element $u \in X$ for which $F(u) = u$) is a continuous solution to the ordinary differential equation

$$\frac{du}{dt} = \sin(u) + \tan(t); \quad u(0) = 1,$$

in the region $t \in [-1, 1]$. We shall see later that F does indeed have a fixed point – knowing that F is uniformly continuous is a step towards this. To see that F is

continuous, calculate

$$\begin{aligned}
\|F(u) - F(v)\| &= \sup_{t \in [-1, 1]} |F(u)(t) - F(v)(t)| \\
&= \sup_{t \in [-1, 1]} \left| \left(1 + \int_0^t (\sin u(s) + \tan s) ds \right) - \right. \\
&\quad \left. \left(1 + \int_0^t (\sin v(s) + \tan s) ds \right) \right| \\
&= \sup_{t \in [-1, 1]} \left| \int_0^t (\sin u(s) - \sin v(s)) ds \right| \\
&\leq \sup_{t \in [-1, 1]} \left| \int_0^t |\sin u(s) - \sin v(s)| ds \right| \\
&\leq \|u - v\|.
\end{aligned}$$

Notice we have used the inequality $|\sin u - \sin v| \leq |u - v|$, an easy consequence of the Mean Value Theorem.

[4] Let X be the space of complex-valued square-integrable Riemann integrable functions on $[0, 1]$ with 2-norm (cf. Example 1.4[6]). Define a map $F : X \rightarrow X$ by $F(u) = v$, with

$$v(t) = \int_0^t u^2(s) ds.$$

Then F is continuous (but not uniformly continuous):

$$\begin{aligned}
|Fu(t) - Fv(t)| &= \left| \int_0^t (u^2(s) - v^2(s)) ds \right| \\
&\leq \int_0^1 (|u(s)| + |v(s)|) (|u(s)| - |v(s)|) ds \\
&\leq \left(\int_0^1 (|u(s)|^2 + |v(s)|^2) ds \right)^{1/2} \left(\int_0^1 (|u(s) - v(s)|^2) ds \right)^{1/2},
\end{aligned}$$

so that

$$\|Fu - Fv\|_2 \leq \sup_{t \in [0, 1]} |u(t) - v(t)| \leq (\|u\| + \|v\|_2) \|u - v\|.$$

[5] The same map as in [4] applied to square-integrable Riemann integrable functions on $[0, \infty)$ is not continuous. To see this, let $a, b \in \mathbb{R}$ and define

$$u(t) = \begin{cases} a, & 0 \leq t \leq 2b^2 \\ ia, & 2b^2 \leq t \leq 4b^2 \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|u - 0\|_2 = 2ab$. On the other hand,

$$F(u)(t) = \begin{cases} a^2 t, & 0 \leq t \leq 2b^2 \\ 4b^2 a^2 - a^2 t, & 2b^2 \leq t \leq 4b^2 \\ 0, & \text{otherwise.} \end{cases}$$

Then $\|F(u) - F(0)\|_2 = \frac{16}{3} a^4 b^6$. Now, given any $\delta > 0$ we may choose constants a, b with $2ab < \delta$ but $\frac{16}{3} a^4 b^6 = 1$. That is, given any $\delta > 0$ there is a function u

with the property that $\|u - 0\| < \delta$ but $\|F(u) - F(0)\| = 1$, showing that F is not continuous.

The moral is that the topological properties of infinite spaces are a little counter-intuitive.

8. Sequences and completeness in normed spaces

Just as for continuity, we can use the norm on a normed linear space to define convergence for sequences and series in a normed space using the corresponding notion for \mathbb{R} .

Let $X = (X, \|\cdot\|_X)$ be a normed linear space. A sequence (x_n) in X is said to converge to $a \in X$ if

$$\|x_n - a\| \longrightarrow 0$$

as $n \rightarrow \infty$.

Similarly, a series $\sum_{n=1}^{\infty} x_n$ converges if the sequence of partial sums (s_N) defined by $s_N = \sum_{n=1}^N x_n$ is a convergent sequence.

EXAMPLE 1.9. [1] If (x_j) is a sequence in \mathbb{R}^n , with $x_j = (x_j^{(1)}, \dots, x_j^{(n)})$, then check that

$$\|x_j\|_p \rightarrow 0$$

(that is, (x_j) converges to 0 in the space ℓ_p^n) if and only if $x_j^{(k)} \rightarrow 0$ in \mathbb{R} for each $k = 1, \dots, n$.

[2] For infinite-dimensional spaces, it is not enough to check convergence on each component using a basis. Let (x_j) be the sequence in ℓ_p defined by

$$x_j = (0, 0, \dots, 1, \dots)$$

(where the 1 appears in the j th position. Then if we write $x_j = (x_j^{(1)}, x_j^{(2)}, \dots)$ we certainly have $x_j^{(k)} \rightarrow 0$ as $j \rightarrow \infty$ for each k . However, we also have $\|x_j\|_p = 1$ for all j , so the sequence is certainly not converging to 0. Indeed, it is not converging to anything.

LEMMA 1.2. A map $F : X \rightarrow Y$ between normed linear spaces is continuous at $a \in X$ if and only if

$$\lim_{n \rightarrow \infty} F(x_n) = F(a)$$

for every sequence (x_n) converging to a .

PROOF. Replace $|\cdot|$ with $\|\cdot\|$ in the proof of this statement for functions $\mathbb{R} \rightarrow \mathbb{R}$. \square

DEFINITION 1.9. A sequence (x_n) is a *Cauchy sequence* if

$$\forall \epsilon > 0 \exists N \text{ such that } n, m > N \implies \|x_n - x_m\| < \epsilon.$$

It is clear that a convergent sequence is a Cauchy sequence. We know that in the normed linear space $(\mathbb{R}, |\cdot|)$ the converse also holds, and it is a simple matter to check that in \mathbb{R}^n the converse holds. In many reasonable infinite-dimensional normed linear spaces however there are Cauchy sequences that do not converge.

DEFINITION 1.10. A normed linear space is said to be *complete* if all Cauchy sequences are convergent.

EXAMPLE 1.10. [1] The sequence $3, \frac{31}{100}, \frac{314}{1000}, \frac{31415}{10000}, \dots$ is a Cauchy sequence of rationals converging to the real number π .

[2] Consider the space $C[0, 1]$ of continuous functions under the sup norm (cf. Example 1.4[4]). This is complete.

[3] The space $C[0, 1]$ under the 2-norm (cf. Example 1.4[5]) is not complete. To see this, consider the sequence of functions

$$u_n(t) = \begin{cases} 0, & 0 \leq t \leq \frac{1}{2} - \frac{1}{n} \\ \frac{nt}{2} - \frac{n}{4} + \frac{1}{2}, & \frac{1}{2} - \frac{1}{n} \leq t \leq \frac{1}{2} + \frac{1}{n} \\ 1 & \frac{1}{2} + \frac{1}{n} \leq t \leq 1. \end{cases}$$

Then (u_n) is a Cauchy sequence, since

$$\begin{aligned} \|u_m - u_n\|_2^2 &= \int_0^1 |u_m(t) - u_n(t)|^2 dt \\ &= \int_{1/2-1/m}^{1/2} |u_m(t) - u_n(t)|^2 dt + \int_{1/2}^{1/2+1/m} |u_m(t) - u_n(t)|^2 dt \\ &\rightarrow 0 \text{ as } m > n \rightarrow \infty. \end{aligned}$$

We claim that the sequence (u_n) is not convergent in $C[0, 1]$ under the 2-norm. To see this, let g be the function defined by $g(t) = 0$ for $0 \leq t \leq \frac{1}{2}$ and $g(t) = 1$ for $\frac{1}{2} < t \leq 1$, and assume that there is a continuous function f with $\|u_n - f\|_2 \rightarrow 0$ as $n \rightarrow \infty$. It is clear that $\|u_n - g\|_2 \rightarrow 0$ as $n \rightarrow \infty$ also, so we must have

$$\|f - g\|_2 = 0. \quad (3)$$

Now examine $f(\frac{1}{2})$. If $f(\frac{1}{2}) \neq g(\frac{1}{2}) = 0$ then $|f - g|$ must be positive on $(\frac{1}{2} - \delta, \frac{1}{2})$ for some $\delta > 0$, which contradicts (3). We must therefore have $f(\frac{1}{2}) = 0$; but in this case $|f - g|$ must be positive on $(\frac{1}{2}, \frac{1}{2} + \delta)$ for some $\delta > 0$, again contradicting (3). We conclude that there is no continuous function f that is the 2-norm limit of the sequence (u_n) . Thus the normed space $(C[0, 1], \|\cdot\|_2)$ is not complete.

9. Topological language

There are certain properties of subsets of normed linear spaces (and other more general spaces) that we use very often. Topology is a subject that begins by attaching names to these properties and then develops a shorthand for talking about such things.

DEFINITION 1.11. Let X be a normed linear space.

A set $C \subset X$ is *closed* if whenever (c_n) is a sequence in C that is a convergent sequence in X , the limit $\lim_{n \rightarrow \infty} c_n$ also lies in C .

A set $U \subset X$ is *open* if for every $u \in U$ there exists $\epsilon > 0$ such that $\|x - u\| < \epsilon \implies x \in U$.

A set $S \subset X$ is *bounded* if there is an $R < \infty$ with the property that $x \in S \implies \|x\| < R$.

A set $S \subset X$ is *connected* if there do not exist open sets A, B in X with $S \subset A \cup B$, $S \cap A \neq \emptyset$, $S \cap B \neq \emptyset$ and $S \cap A \cap B = \emptyset$.

Associated to any set $S \subset X$ in a normed space there are sets $S^o \subset S \subset \bar{S}$ defined as follows: the *interior* of S is the set

$$S^o = \{x \in X \mid \exists \epsilon > 0 \text{ such that } \|x - y\| < \epsilon \implies y \in S\}, \quad (4)$$

and the *closure* of S ,

$$\bar{S} = \{x \in X \mid \forall \epsilon > 0 \exists s \in S \text{ such that } \|s - x\| < \epsilon\}. \quad (5)$$

EXERCISE 1.1. [1] Prove that a map $f : X \rightarrow Y$ is continuous (cf. Definition 1.8) if and only if for every open set $U \subset Y$, the pre-image $f^{-1}(U) \subset X$ is also open.

[2] Show by example that for a continuous map $f : \mathbb{R} \rightarrow \mathbb{R}$ there may be open sets U for which $f(U)$ is not open.

It is clear from first year analysis that closed bounded sets (closed intervals, for example) have special properties. For example, recall the theorem of Bolzano–Weierstrass.

THEOREM 1.1. *Let S be a closed and bounded subset of \mathbb{R} . Then a continuous function $f : S \rightarrow \mathbb{R}$ attains its bounds: there exist $\xi, \eta \in S$ with the property that*

$$f(\xi) = \sup_{s \in S} f(s), \quad f(\eta) = \inf_{s \in S} f(s).$$

DEFINITION 1.12. A subset S of a normed linear space is (sequentially) *compact* if and only if every sequence (s_n) in S has a subsequence $(s_{n_j}) = (s_{n_1}, s_{n_2}, \dots)$ that converges in S .

Recall the following theorem (the Heine–Borel theorem) – which is really the same one as Theorem 1.1.

THEOREM 1.2. *A subset of \mathbb{R}^n is compact if and only if it is closed and bounded.*

By now you should be used to the idea that any such result does not extend to infinite-dimensional normed linear spaces: Example 1.9[2] is a bounded sequence with no convergent subsequences. Thus the result Theorem 1.2 does not extend to infinite-dimensional normed spaces. However the analogue of Theorem 1.1 does hold in great generality. This is also a version of the Bolzano–Weierstrass theorem.

THEOREM 1.3. *If A is a compact subset of a normed linear space X , and $f : X \rightarrow Y$ is a continuous map between normed linear spaces, then $f(A)$ is a compact subset of Y .*

As an exercise, convince yourself that Theorem 1.3 implies Theorem 1.1. Some standard sets are used so often that we give them special names.

DEFINITION 1.13. Let X be a normed space. Then the *open ball* of radius $\epsilon > 0$ and centre x_0 is the set

$$B_\epsilon(x_0) = \{x \in X \mid \|x - x_0\| < \epsilon\}.$$

The *closed ball* of radius $\epsilon > 0$ and center x_0 is the set

$$\bar{B}_\epsilon(x_0) = \{x \in X \mid \|x - x_0\| \leq \epsilon\}.$$

EXERCISE 1.2. Open and closed balls in normed spaces are convex (cf. Definition 1.6).

DEFINITION 1.14. A subset S of a normed space X is *dense* if every open ball in X has non-empty intersection with S . A normed space is said to be *separable* if there is a countable set $S = \{x_1, x_2, \dots\}$ that is dense in X .

10. Quotient spaces

As an application of Section 9, quotients of normed spaces may be formed. Notice that we need both the algebraic structure (subspace of a linear space) and a topological property (closed) to make it all work.

Recall from Definition 1.11 and Definition 1.2 that a *closed linear subspace* Y of a normed linear space X is a subset $Y \subset X$ that is itself a linear space, with the property that any sequence (y_n) of elements of Y that converges in X has the limit in Y .

The linear space X/Y (the *quotient* or *factor* space) is formed as follows. The elements of X/Y are *cosets* of Y – sets of the form $x + Y$ for $x \in X$. The set of cosets is a linear space under the operations

$$(x_1 + Y) \oplus (x_2 + Y) = (x_1 + x_2) + Y, \quad \lambda \cdot (x + Y) = \lambda x + Y.$$

Notice that this makes sense precisely because Y is itself a linear space, so for example $Y + Y = Y$ and $\lambda Y = Y$ for $\lambda \neq 0$. Two cosets $x_1 + Y$ and $x_2 + Y$ are equal if as sets $x_1 + Y = x_2 + Y$, which is true if and only if $x_1 + x_2 \in Y$.

EXAMPLE 1.11. [1] Let $X = \mathbb{R}^3$, and let Y be the subspace spanned by $(1, 1, 0)$. Then X/Y is a two-dimensional real vector space. There are many pairs of elements that generate X/Y , for example

$$(1, 0, 1) + Y \quad \text{and} \quad (0, 0, 1) + Y.$$

[2] The linear space Y of finitely supported sequences in ℓ_1 is a linear subspace. The quotient space ℓ_1/Y is very hard to visualize: its elements are equivalence classes under the relation $(x_n) \sim (y_n)$ if the sequences (x_n) and (y_n) differ in finitely many positions.

[3] The linear space Y of ℓ_1 sequences of the form $(0, \dots, 0, x_{n+1}, \dots)$ (first n are zero) is a linear subspace of ℓ_1 . Here the quotient space ℓ_1/Y is quite reasonable: in fact it is isomorphic to \mathbb{R}^n .

[4] We know that for $p, q \in [1, \infty]$, $p < q \implies \ell_p \subset \ell_q$. This means that for any $p < q$ there is a linear quotient space ℓ_q/ℓ_p . These quotient spaces are very pathological.

[5] The linear space $Y = C[0, 1]$ is a linear subspace of the space X of square-Riemann-integrable functions on $[0, 1]$. The quotient X/Y is again a linear space that is impossible to work with.

[6] Let $X = C[0, 1]$, and let $Y = \{f \in X \mid f(0)\}$. Then X/Y is isomorphic to \mathbb{R} .

It is clear from these examples that not all linear subspaces are equally good: Examples 1.11 [1], [3], and [6] are quite reasonable, whereas [2], [4] and [5] are examples of linear spaces unlike any we have seen. The reason is the following: the space X/Y is guaranteed to be a normed space with a norm related to the original norm on X only when the subspace Y is itself closed. Notice that Examples 1.11 [1], [3], and [6] are precisely the ones in which the subspace is closed.

THEOREM 1.4. *If X is a normed space, and Y is a normed linear subspace, then X/Y is a normed space under the norm*

$$\|x + Y\| = \inf_{z \in x+Y} \|z\|. \tag{6}$$

Before proving this theorem, try to convince yourself that the norm (6) is the obvious candidate: if $X = \mathbb{R}^2$ and $Y = (1, 0)\mathbb{R}$, then the space X/Y consists of

lines in X of the form $(s, t) + Y$. Notice that each such line may be written uniquely in the form $(0, t) + Y$, and this choice minimizes the norm of the element of X that represents the line.

PROOF. Let $x + Y$ be any coset of X , and let $(x_n) \subset z + Y$ be a convergent sequence with $x_n \rightarrow x$. Then for any fixed n , $x_n - x_m \rightarrow x_n - x$ is a sequence in Y converging in X . Since Y is closed, we must have $x_n - x \in Y$, so $x + Y = x_n + Y = z + Y$. That is, the limit of the sequence defines the same coset as does the sequence – the set $z + Y$ is a closed set.

Assume now that $\|x + Y\| = 0$. Then there is a sequence $(x_n) \subset x + Y$ with $\|x_n\| \rightarrow 0$. Since $x + Y$ is closed and $x_n \rightarrow 0$, we must have $0 \in x + Y$, so $x + Y = Y$, the zero element in X/Y .

Homogeneity is clear:

$$\|\lambda(x + Y)\| = \inf_{z \in x + Y} \|\lambda z\| = |\lambda| \inf_{z \in x + Y} \|z\| = |\lambda| \|x + Y\|.$$

Finally, the triangle inequality:

$$\begin{aligned} \|(x_1 + Y) + (x_2 + Y)\| &= \inf_{z_1 \in x_1 + Y; z_2 \in x_2 + Y} \|z_1 + z_2\| \\ &\leq \inf_{z_1 \in x_1 + Y} \|z_1\| + \inf_{z_2 \in x_2 + Y} \|z_2\| \\ &= \|x_1 + Y\| + \|x_2 + Y\|. \end{aligned}$$

□

EXAMPLE 1.12. Even if the subspace is closed, the quotient space may be a little odd. For example, let c denote the space of all sequences (x_n) with the property that $\lim_n x_n$ exists. This is a closed subspace of ℓ_∞ . What is the quotient ℓ_∞/c ?

CHAPTER 2

Banach spaces

It turns out to be very important and natural to work in complete spaces – trying to do functional analysis in non-complete spaces is a little like trying to do elementary analysis over the rationals.

DEFINITION 2.1. A complete normed linear space is called¹ a *Banach space*.

EXAMPLE 2.1. [1] We are already familiar with a large class of Banach spaces: any finite-dimensional normed linear space is a Banach space. In our notation, this means that ℓ_p^n is a Banach space for all $1 \leq p \leq \infty$ and all n .

[2] The space of continuous functions with the sup norm is a Banach space (cf. Example 1.4[4] and Example 1.10[2]).

[3] The sequence space ℓ_p is a Banach space. To see this, assume that (x_n) is a Cauchy sequence in ℓ_p , and write

$$x_n = (x_n^{(1)}, x_n^{(2)}, \dots).$$

Recall that $\|\cdot\|_p \geq \|\cdot\|_\infty$ for all p (cf. Example 1.4[3]). So, given $\epsilon > 0$ we may find N with the property that

$$m, n > N \implies \|x_n - x_m\|_p < \epsilon$$

which in turn implies that $\|x_n - x_m\|_\infty < \epsilon$, so for each k , $|x_n^{(k)} - x_m^{(k)}| < \epsilon$. That is, if (x_n) is a Cauchy sequence in ℓ_p , then for each k $(x_n^{(k)})$ is a Cauchy sequence in \mathbb{R} . Since \mathbb{R} is complete, we deduce that for each k we have $x_n^{(k)} \rightarrow y^{(k)}$. Notice that this does not imply by itself that $x_n \rightarrow y$ (cf. Example 1.9[2]). However, if we know (as we do) that (x_n) is Cauchy, then it does: we prove this for $p < \infty$ but the $p = \infty$ case is similar. Fix $\epsilon > 0$, and use the Cauchy criterion to find N such that $n, m > N$ implies that

$$\sum_{k=1}^{\infty} |x_n^{(k)} - x_m^{(k)}|^p < \epsilon.$$

Now fix n and let $m \rightarrow \infty$ to see that

$$\sum_{k=1}^{\infty} |x_n^{(k)} - y^{(k)}|^p \leq \epsilon$$

(notice that $<$ has become \leq). This last inequality means that

$$\|x_n - y\|_p \leq \epsilon^{1/p},$$

¹After the Polish mathematician Stefan Banach (1892–1945) who gave the first abstract treatment of complete normed spaces in his 1920 thesis (*Fundamenta Math.*, **3**, 133–181, 1922). His later book (*Théorie des opérations linéaires*, Warsaw, 1932) laid the foundations of functional analysis.

showing that in ℓ_p , $x_n \rightarrow y = (y^{(1)}, y^{(2)}, \dots)$.

LEMMA 2.1. *Let (x_n) be a sequence in a Banach space. If the series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent, then it is convergent.*

Recall that absolutely convergent means that the numerical series $\sum_{n=1}^{\infty} \|x_n\|$ is convergent. The lemma is clearly not true for general normed spaces: take, for example, a sequence of functions in $C[0, 1]$ each with $\|f_n\|_2 = \frac{1}{n^2}$ with the property that $\sum_{n=1}^{\infty} f_n$ is not continuous.

PROOF. Consider the sequence of partial sums $s_m = \sum_{n=1}^m x_n$:

$$\|s_m - s_k\| \leq \sum_{n=k+1}^m \|x_n\| \rightarrow 0$$

as $m > k \rightarrow \infty$. It follows that the sequence (s_m) is Cauchy; since X is complete this sequence converges, so the series $\sum_{n=1}^{\infty} x_n$ converges. \square

1. Completions

Completeness is so important that in many applications we deal with non-complete normed spaces by completing them. This is analogous to the process of passing from \mathbb{Q} to \mathbb{R} by working with Cauchy sequences of rationals. In this section we simply outline what is done. In later sections we will see more details about what the completions look like.

Let X be a normed linear space. Let $\mathcal{C}(X)$ denote the set of all Cauchy sequences in X . An element of $\mathcal{C}(X)$ is then a Cauchy sequence (x_n) . The linear space structure of X extends to $\mathcal{C}(X)$ by defining $\alpha \cdot (x_n) + (y_n) = (\alpha x_n + y_n)$. The norm $\|\cdot\|$ on X extends to a *semi-norm* on $\mathcal{C}(X)$, defined by

$$\|(x_n)\| = \lim_{n \rightarrow \infty} \|x_n\|.$$

Finally, define an equivalence relation \sim on $\mathcal{C}(X)$ by $(x_n) \sim (y_n)$ if and only if $x_n - y_n \rightarrow 0$. Then the linear space operations and the semi-norm are well-defined on the space of equivalence classes $\mathcal{C}(X)/\sim$, giving a complete normed linear space \bar{X} called the completion of X .

EXERCISE 2.1. [1] Apply the process outlined above to the rationals \mathbb{Q} . Try to see why the obvious extension of the norm to the space of Cauchy sequences only gives a semi-norm.

[2] Construct a Cauchy sequence (f_n) in $(C[0, 1], \|\cdot\|_2)$ with the property that $f_n \neq 0$ for any n but $\|f_n\|_2 \rightarrow 0$. This means that the Cauchy sequence (f_n) and the Cauchy sequence (0) are not separated by the semi-norm $\|\cdot\|_2$, showing it is not a norm.

[3] Show that if X is already a Banach space, then there is a bijective isomorphism between X and \bar{X} .

It should be clear from the above that it is going to be difficult to work with elements of the completions in this formal way, where an element of \bar{X} is an equivalence class of Cauchy sequences. However all we will ever need is the simple statement: *for any normed linear space X , there is a Banach space \hat{X} such that X is isomorphic to a dense subspace $\iota(X)$ of \hat{X} ; the map ι from X into \hat{X} preserves all the linear space operations.*

EXAMPLE 2.2. [1] We have seen that $C[0, 1]$ under the 2–norm is not complete (cf. Example 1.10[2]). Similar examples will show that $C[0, 1]$ is not complete under any of the p –norms. Let X denote the non–complete space $(C[0, 1], \|\cdot\|_p)$. A reasonable guess for \bar{X} might be the space of Riemann–integrable functions with finite p –norm, but this is still not complete. It is easy to construct a Cauchy sequence of Riemann–integrable functions that does not converge to a Riemann–integrable function in the p –norm. However, if you use Lebesgue integration, you do get a complete space, called $L^p[0, 1]$. For now, think of this space as consisting of all Riemann–integrable functions with finite p –norm together with extra functions obtained as limits of sequences of Riemann–integrable functions. Then L_p provides a further example of a Banach space.

[2] A function $f : X \rightarrow Y$ is said to have *compact support* if it is zero outside some compact subset of X ; the *support* of f is the smallest closed set containing $\{x \in X \mid f(x) \neq 0\}$. This example is of importance in distribution theory and the study of partial differential equations. Let $C_0^\infty(\Omega)$ be the space of infinitely differentiable functions of compact support on Ω , an open subset of \mathbb{R}^n . Recall the definition of higher–order derivatives $D^{\mathbf{a}}$ from Example 1.1(8). For each $k \in \mathbb{N}$ and $1 \leq p \leq \infty$ define a norm

$$\|f\|_{k,p} = \left(\int_{\Omega} \sum_{|\mathbf{a}| \leq k} |D^{\mathbf{a}} f(x)|^p dx \right)^{1/p}.$$

This gives an infinite family of (different) normed space structures on the linear space $C_0^\infty(\Omega)$. None of these spaces are complete because there are sequences of C^∞ functions whose (n, p) –limit is not even continuous. The completions of these spaces are the *Sobolev spaces*.

2. Contraction mapping theorem

In this section we prove the simplest of the many fixed–point theorems. Such theorems are useful for solving equations, and with the formalism of function spaces one uniform treatment may be given for numerical equations like $x = \cos(x)$ and differential equations like $\frac{dy}{dx} = x + \tan(xy), y(0) = y_0$.

EXERCISE 2.2. If you have an electronic calculator, put it in “radians” mode. Starting with any initial value, press the cos button repeatedly. What happens? Can you explain why this happens? (Draw a graph) How does this relate to the equation $x = \cos(x)$.

DEFINITION 2.2. A map $F : X \rightarrow Y$ between normed linear spaces is called a *contraction* if there is a constant $K < 1$ for which

$$\|F(x) - F(y)\|_Y \leq K \cdot \|x - y\|_X \tag{7}$$

for all $x, y \in X$.

EXERCISE 2.3. [1] Any contraction is uniformly continuous.

[2] If $f : [a, b] \rightarrow [a, b]$ has the property that $|f(x) - f(y)| < |x - y|$ then f is a contraction.

[3] Find an example of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ that has the property $|f(x) - f(y)| < |x - y|$ for all $x, y \in \mathbb{R}$, but f is not a contraction.

THEOREM 2.1. *If $F : X \rightarrow X$ is a contraction and X is a Banach space, then there is a unique point $x^* \in X$ which is fixed by F (that is, $F(x) = x$). Moreover, if x_0 is any point in X , then the sequence defined by $x_1 = F(x_0), x_2 = F(x_1), \dots$ converges to x^* .*

COROLLARY 2.1. *If S is a closed subset of the Banach space X , and $F : S \rightarrow S$ is a contraction, then F has a unique fixed point in S .*

PROOF. Simply notice that S is itself complete (since it is a closed subset of a complete space), and the proof of Theorem 2.1 does not use the linear space structure of X . \square

COROLLARY 2.2. *If S is a closed subset of a Banach space, and $F : S \rightarrow S$ has the property that for some n there is a $K < 1$ such that*

$$\|F^n(x) - F^n(y)\|_Y \leq K \cdot \|x - y\|_X$$

for all $x, y \in S$, then F has a unique fixed point.

PROOF. Choose any point $x_0 \in S$. Then by Corollary 2.1 we have

$$x = \lim_{k \rightarrow \infty} F^{kn} x_0,$$

where x is the unique fixed point of F^n . By the continuity of F ,

$$Fx = \lim_{k \rightarrow \infty} FF^{kn} x_0.$$

On the other hand, F^n is a contraction, so

$$\|F^{kn} Fx_0 - F^{kn} x_0\| \leq K \|F^{(k-1)n} Fx_0, F^{(k-1)n} x_0\| \leq \dots \leq K^k \|F(x_0) - x_0\|,$$

so

$$\|F(x) - x\| = \lim_{k \rightarrow \infty} \|FF^{kn} x_0 - F^{kn} x_0\| = 0.$$

It follows that $F(x) = x$ so x is a fixed point for F . This fixed point is automatically unique: if F has more than one fixed point, then so does F^n which is impossible by Corollary 2.1. \square

EXERCISE 2.4. [1] Give an example of a map $f : \mathbb{R} \rightarrow \mathbb{R}$ which has the property that $|f(x) - f(y)| < |x - y|$ for all $x, y \in \mathbb{R}$ but f has no fixed point.

[2] Let f be a function from $[0, 1]$ to $[0, 1]$. Check that the contraction condition (7) holds if f has a continuous derivative f' on $[0, 1]$ with the property that

$$|f'(x)| \leq K < 1$$

for all $x \in [0, 1]$. As an exercise, draw graphs to illustrate convergence of the iterates² of f to a fixed point for examples with $0 < f'(x) < 1$ and $-1 < f'(x) < 0$.

EXAMPLE 2.3. A basic linear problem is the following: let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the affine map defined by

$$F(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$$

²Iteration of continuous functions on the interval may be used to illustrate many of the features of dynamical systems, including frequency locking, sensitive dependence on initial conditions, period doubling, the Feigenbaum phenomena and so on. An excellent starting point is the article and demonstration "One-dimensional iteration" at the web site <http://www.geom.umn.edu/java/>.

where $A = (a_{ij})$ is an $n \times n$ matrix. Equivalently, $F(\mathbf{x}) = \mathbf{y}$, where

$$y_i = \sum_{j=1}^n a_{ij}x_j + b_i$$

for $i = 1, \dots, n$. If F is a contraction, then we can apply Theorem 2.1 to solve³ the equation $F(\mathbf{x}) = \mathbf{x}$. The conditions under which F is a contraction depend on the choice of norm for \mathbb{R}^n . Three examples follow.

[1] Using the max norm, $\|\mathbf{x}\|_\infty = \max\{|x_i|\}$. In this case,

$$\begin{aligned} \|F(\mathbf{x}) - F(\tilde{\mathbf{x}})\|_\infty &= \max_i \left| \sum_j a_{ij}(x_j - \tilde{x}_j) \right| \\ &\leq \max_i \sum_j |a_{ij}| |x_j - \tilde{x}_j| \\ &\leq \max_i \sum_j |a_{ij}| \max_j |x_j - \tilde{x}_j| \\ &= \left(\max_i \sum_j |a_{ij}| \right) \|\mathbf{x} - \tilde{\mathbf{x}}\|_\infty. \end{aligned}$$

Thus the contraction condition is

$$\sum_j |a_{ij}| \leq K < 1 \quad \text{for } i = 1, \dots, n. \quad (8)$$

[2] Using the 1-norm, $\|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i|$. In this case,

$$\begin{aligned} \|F(\mathbf{x}) - F(\tilde{\mathbf{x}})\|_1 &= \sum_i \left| \sum_j a_{ij}(x_j - \tilde{x}_j) \right| \\ &\leq \sum_i \sum_j |a_{ij}| |x_j - \tilde{x}_j| \\ &\leq \left(\max_j \sum_i |a_{ij}| \right) \|\mathbf{x} - \tilde{\mathbf{x}}\|_1. \end{aligned}$$

The contraction condition is now

$$\sum_i |a_{ij}| \leq K < 1 \quad \text{for } j = 1, \dots, n. \quad (9)$$

[3] Using the 2-norm, $\|\mathbf{x}\|_2 = (\sum_{i=1}^n |x_i|^2)^{1/2}$. In this case,

$$\begin{aligned} \|F(\mathbf{x}) - F(\tilde{\mathbf{x}})\|_2^2 &= \sum_i \left(\sum_j a_{ij}(x_j - \tilde{x}_j) \right)^2 \\ &\leq \left(\sum_i \sum_j a_{ij}^2 \right) \|\mathbf{x} - \tilde{\mathbf{x}}\|_2^2. \end{aligned}$$

³Of course the equation is in one sense trivial. However, it is sometimes of importance computationally to avoid inverting matrices, and more importantly to have an iterative scheme that converges to a solution in some predictable fashion.

The contraction condition is now

$$\sum_i \sum_j |a_{ij}^2| \leq K < 1. \quad (10)$$

It follows that if any one of the conditions (8), (9), or (10) holds, then there exists a unique solution in \mathbb{R}^n to the affine equation $A\mathbf{x} + \mathbf{b} = \mathbf{x}$. Moreover, the solution may be approximated using the iterative scheme $\mathbf{x}_1 = F(\mathbf{x}_0)$, $\mathbf{x}_2 = F(\mathbf{x}_1)$, \dots .

Notice that each of the conditions (8), (9), (10) is sufficient for the method to work, but none of them are necessary. In fact there are examples in which exactly one of the condition holds for each of them conditions.

It remains only to prove the contraction mapping theorem.

PROOF OF THEOREM 2.1. Given any point $x_0 \in X$, define a sequence (x_n) by

$$x_1 = F(x_0), x_2 = F(x_1), \dots$$

Then, for any $n \leq m$ we have by the contraction condition (7)

$$\begin{aligned} \|x_n - x_m\| &= \|F^n x_0 - F^m x_0\| \\ &\leq K^n \|x_0 - F^{m-n} x_0\| \\ &\leq K^n (\|x_0 - x_1\| + \|x_1 - x_2\| + \dots + \|x_{m-n-1} - x_{m-n}\|) \\ &\leq K^n \|x_0 - x_1\| (1 + K + K^2 + \dots + K^{m-n-1}) \\ &< \frac{K^n}{1 - K} \|x_0 - x_1\|. \end{aligned}$$

Now for fixed x_0 , the last expression converges to zero as n goes to infinity, so (cf. Definition 1.9) the sequence (x_n) is a Cauchy sequence.

Since the linear space X is complete (cf. Definition 1.10), the sequence x_n therefore converges, say

$$x^* = \lim_{x \rightarrow \infty} X_N.$$

Since F is continuous,

$$F(x^*) = F\left(\lim_{n \rightarrow \infty} x_n\right) = \lim_{n \rightarrow \infty} F(x_n) = \lim_{n \rightarrow \infty} x_{n+1} = x^*,$$

so F has a fixed point x^* . To prove that x^* is the only fixed point for F , notice that if $F(y) = y$ say, then

$$\|x^* - y\| = \|F(x^*) - F(y)\| \leq K \|x^* - y\|,$$

which requires that $x^* = y$ since $K < 1$. \square

3. Applications to differential equations

As mentioned before, the most important applications of the contraction mapping method are to function spaces. We have seen already in Example 1.8[3] that fixed points for certain integral operators on function spaces are solutions of ordinary differential equations. The first result in this direction is due to Picard⁴.

⁴(Charles) Emile Picard (1856–1941), who was Professor of higher analysis at the Sorbonne and became permanent secretary of the Paris Academy of Sciences. Some of his deepest results lie in complex analysis: 1) a non-constant entire function can omit at most one finite value, 2) a non-polynomial entire function takes on every value (except the possible exceptional one), an infinite number of times.

THEOREM 2.2. *Let $f : G \rightarrow \mathbb{R}$ be a continuous function defined on a set G containing a neighbourhood $\{(x, y) \mid \|(x, y) - (x_0, y_0)\|_\infty < e\}$ of (x_0, y_0) for some $e > 0$. Suppose that f satisfies a Lipschitz condition of the form*

$$|f(x, y) - f(x, \tilde{y})| \leq M|y - \tilde{y}| \quad (11)$$

in the variable y on G . Then there is an interval $(x_0 - \delta, x_0 + \delta)$ on which the ordinary differential equation

$$\frac{dy}{dx} = f(x, y) \quad (12)$$

has a unique solution $y = \phi(x)$ satisfying the initial condition

$$\phi(x_0) = y_0. \quad (13)$$

PROOF. The differential equation (12) with initial condition (13) is equivalent to the integral equation

$$\phi(x) = y_0 + \int_{x_0}^x f(t, \phi(t)) dt. \quad (14)$$

Since f is continuous there is a bound

$$|f(x, y)| \leq R \quad (15)$$

for all (x, y) with $\|(x, y) - (x_0, y_0)\|_\infty < e'$ for some $e' > 0$. Choose $\delta > 0$ such that

- (1) $|x - x_0| \leq \delta, |y - y_0| \leq R\delta$ together imply that $\|(x, y) - (x_0, y_0)\|_\infty < e'$;
- (2) $M\delta < 1$ where M is the Lipschitz constant in (11).

Let S be the set of continuous functions ϕ defined on the interval $|x - x_0| \leq \delta$ with the property that $|\phi(x) - y_0| \leq R\delta$, equipped with the sup metric. The set S is complete, since it is a closed subset of a complete space. Define a mapping $F : S \rightarrow S$ by the equation

$$(F(\phi))(x) = y_0 + \int_{x_0}^x f(t, \phi(t)) dt. \quad (16)$$

First check that F does indeed map S into S : if $\phi \in S$, then

$$\begin{aligned} |F\phi(x) - y_0| &= \left| \int_{x_0}^x f(t, \phi(t)) dt \right| \\ &\leq \int_{x_0}^x |f(t, \phi(t))| dt \\ &\leq R|x - x_0| \leq R\delta \end{aligned}$$

by (15), so $F(\phi) \in S$. Moreover,

$$\begin{aligned} |F\phi(x) - F\tilde{\phi}(x)| &\leq \int_{x_0}^x |f(t, \phi(t)) - f(t, \tilde{\phi}(t))| dt \\ &\leq M\delta \max_x |\phi(x) - \tilde{\phi}(x)|, \end{aligned}$$

so that

$$\|F(\phi) - F(\tilde{\phi})\| \leq M\delta \|\phi - \tilde{\phi}\|,$$

after taking sups over x . By construction, $M\delta < 1$, so that F is a contraction mapping. It follows from Corollary 2.2 that the operator F has a unique fixed point in S , so the differential equation (12) with initial condition (13) has a unique solution. \square

The conditions on the set G used in Theorem 2.2 arise very often so it is useful to have a short description for them. A *domain* in a normed linear space X is an open connected set (cf. Definition 1.11). An example of a domain in \mathbb{R} containing the point a is an interval $(a - \delta, a + \delta)$ for some $\delta > 0$. Notice that if G is a domain in $(X, \|\cdot\|_X)$, and $a \in G$ then for some $\epsilon > 0$ the *open ball* $B_\epsilon(a) = \{x \in X \mid \|x - a\|_X < \epsilon\}$ lies in G (cf. Definition 1.13).

Picard's theorem easily generalises to systems of simultaneous differential equations, and we shall see in the next section that the contraction mapping method also applies to certain integral equations.

THEOREM 2.3. *Let G be a domain in \mathbb{R}^{n+1} containing the point $(x_0, y_{01}, \dots, y_{0n})$, and let f_1, \dots, f_n be continuous functions from G to \mathbb{R} each satisfying a Lipschitz condition*

$$|f_i(x, y_1, \dots, y_n) - f_i(x, \tilde{y}_1, \dots, \tilde{y}_n)| \leq M \max_{1 \leq i \leq n} |y_i - \tilde{y}_i| \quad (17)$$

in the variables y_1, \dots, y_n . Then there is an interval $(x_0 - \delta, x_0 + \delta)$ on which the system of simultaneous ordinary differential equations

$$\frac{dy_i}{dx} = f_i(x, y_1, \dots, y_n) \quad \text{for } i = 1, \dots, n \quad (18)$$

has a unique solution

$$y_1 = \phi_1(x), \dots, y_n = \phi_n(x)$$

satisfying the initial conditions

$$\phi_1(x_0) = y_{01}, \dots, \phi_n(x_0) = y_{0n}. \quad (19)$$

PROOF. As in the proof of Theorem 2.2, write the system defined by (18) and (19) in integral form

$$\phi_i(x) = y_{0i} + \int_{x_0}^x f_i(t, \phi_1(t), \dots, \phi_n(t)) dt \quad \text{for } i = 1, \dots, n. \quad (20)$$

Since each of the functions f_i is continuous on G , there is a bound

$$|f_i(x, y_1, \dots, y_n)| \leq R \quad (21)$$

in some domain $G' \subset G$ with $G' \ni (x_0, y_{01}, \dots, y_{0n})$. Choose $\delta > 0$ with the properties that

- (1) $|x - x_0| \leq \delta$ and $\max_i |y_i - y_{0i}| \leq R\delta$ together imply that $(x, y_1, \dots, y_n) \in G'$;
- (2) $M\delta < 1$.

Now define the set S to be the set of n -tuples (ϕ_1, \dots, ϕ_n) of continuous functions defined on the interval $[x_0 - \delta, x_0 + \delta]$ and such that $|\phi_i(x) - y_{0i}| \leq R\delta$ for all $i = 1, \dots, n$. The set S may be equipped with the norm

$$\|\phi - \tilde{\phi}\| = \max_{x,i} |\phi_i(x) - \tilde{\phi}_i(x)|.$$

It is easy to check that S is complete. The mapping F defined by the set of integral operators

$$(F(\phi))_i(x) = y_{0i} + \int_{x_0}^x f_i(t, \phi_1(t), \dots, \phi_n(t)) dt \quad \text{for } |x - x_0| \leq \delta, i = 1, \dots, n$$

is a contraction from S to itself. To see this, first notice that if

$$\phi = (\phi_1, \dots, \phi_n) \in S, \quad \text{and } |x - x_0| \leq \delta$$

then

$$|\phi_i(x) - y_{0i}| = \left| \int_{x_0}^x f_i(t, \phi_1(t), \dots, \phi_n(t)) dt \right| \leq R\delta \quad \text{for } i = 1, \dots, n$$

by (21), so that $F(\phi) = (F(\phi)_1, \dots, F(\phi)_n)$ lies in S . It remains to check that F is a contraction:

$$\begin{aligned} |(F(\phi))_i(x) - (F(\tilde{\phi}))_i(x)| &\leq \int_{x_0}^x |f_i(t, \phi_1(t), \dots, \phi_n(t)) - f_i(t, \tilde{\phi}_1(t), \dots, \tilde{\phi}_n(t))| dt \\ &\leq M\delta \max_i |\phi_i(x) - \tilde{\phi}_i(x)|; \end{aligned}$$

after maximising over x and i we have

$$\|F(\phi) - F(\tilde{\phi})\| \leq M\delta \|\phi - \tilde{\phi}\|,$$

so $F : S \rightarrow S$ is a contraction. It follows that the equation (20) has a unique solution, so the system of differential equations (18) and (19) has a unique solution. \square

4. Applications to integral equations

Integral equations may be a little less familiar than differential equations (though we have seen already that the two are intimately connected), so we begin with some important examples. The theory of integral equations is largely modern (twentieth-century) mathematics, but several specific instances of integral equations had appeared earlier.

Certain problems in physics led to the need to “invert” the integral equation

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy} f(y) dy \quad (22)$$

for functions f and g of specific kinds. This was solved – formally at least – by Fourier⁵ in 1811, who noted that (22) requires that

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixy} g(y) dy.$$

We shall see later that this is really due to properties of particularly good Banach spaces called Hilbert spaces.

⁵Jean Baptiste Joseph Fourier (1768–1830), who pursued interests in mathematics and mathematical physics. He became famous for his *Theorie analytique de la Chaleur* (1822), a mathematical treatment of the theory of heat. He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions. Though these series had been used before, Fourier investigated them in much greater detail. His research, initially criticized for its lack of rigour, was later shown to be valid. It provided the impetus for later work on trigonometric series and the theory of functions of a real variable.

Abel⁶ studied generalizations of the tautochrone⁷ problem, and was led to the integral equation

$$g(x) = \int_a^x \frac{f(y)}{(x-y)^b} dy, \quad b \in (0, 1), g(a) = 0$$

for which he found the solution

$$f(y) = \frac{\sin \pi b}{\pi} \int_a^y \frac{g'(x)}{(y-x)^{1-b}} dx.$$

This equation is an example of a Volterra⁸ equation.

We shall briefly study two kinds of integral equation (though the second is formally a special case of the first).

EXAMPLE 2.4. A *Fredholm equation*⁹ is an integral equation of the form

$$f(x) = \lambda \int_a^b K(x, y) f(y) dy + \phi(x), \quad (23)$$

where K and ϕ are two given functions, and we seek a solution f in terms of the arbitrary (constant) parameter λ . The function K is called the *kernel* of the equation, and the equation is called *homogeneous* if $\phi = 0$.

We assume that $K(x, y)$ and $\phi(x)$ are continuous on the square $\{(x, y) \mid a \leq x \leq b, a \leq y \leq b\}$. It follows in particular (see Section 1.9) that there is a bound M so that

$$|K(x, y)| \leq M \quad \text{for all } a \leq x \leq b, a \leq y \leq b.$$

Define a mapping $F : C[a, b] \rightarrow C[a, b]$ by

$$(F(f))(x) = \lambda \int_a^b K(x, y) f(y) dy + \phi(x) \quad (24)$$

Now

$$\begin{aligned} \|F(f_1) - F(f_2)\| &= \max_x |F(f_1)(x) - F(f_2)(x)| \\ &\leq |\lambda| M(b-a) \max_x |f_1(x) - f_2(x)| \\ &= |\lambda| M(b-a) \|f_1 - f_2\|, \end{aligned}$$

⁶Niels Henrik Abel (1802–1829), was a brilliant Norwegian mathematician. He earned wide recognition at the age of 18 with his first paper, in which he proved that the general polynomial equation of the fifth degree is insolvable by algebraic procedures (problems of this sort are studied in Galois Theory). Abel was instrumental in establishing mathematical analysis on a rigorous basis. In his major work, *Recherches sur les fonctions elliptiques* (Investigations on Elliptic Functions, 1827), he revolutionized the understanding of elliptic functions by studying the inverse of these functions.

⁷Also called an *isochrone*: a curve along which a pendulum takes the same time to make a complete oscillation independent of the amplitude of the oscillation. The resulting differential equation was solved by James Bernoulli in May 1690, who showed that the result is a cycloid.

⁸Vito Volterra (1860–1940) succeeded Beltrami as professor of Mathematical Physics at Rome. His method for solving the equations that carry his name is exactly the one we shall use. He worked widely in analysis and integral equations, and helped drive Lebesgue to produce a more sophisticated integration by giving an example of a function with bounded derivative whose derivative is not Riemann integrable.

⁹This is really a Fredholm equation “of the second kind”, named after the Swedish geometer Erik Ivar Fredholm (1866–1927).

so that F is a contraction mapping if

$$|\lambda| < \frac{1}{M(b-a)}.$$

It follows by Theorem 2.1 that the equation (23) has a unique continuous solution f for small enough values of λ , and the solution may be obtained by starting with any continuous function f_0 and then iterating the scheme

$$f_{n+1}(x) = \lambda \int_a^b K(x, y) f_n(y) dy + \phi(x).$$

EXAMPLE 2.5. Now consider the *Volterra equation*,

$$f(x) = \lambda \int_a^x K(x, y) f(y) dy + \phi(x), \quad (25)$$

which only differs¹⁰ from the Fredholm equation (23) in that the variable x appears as the upper limit of integration. As before, define a function $F : C[a, b] \rightarrow C[a, b]$ by

$$(F(f))(x) = \lambda \int_a^x K(x, y) f(y) dy + \phi(x).$$

Then for $f_1, f_2 \in C[a, b]$ we have

$$\begin{aligned} |F(f_1)(x) - F(f_2)(x)| &= \left| \lambda \int_a^x K(x, y) [f_1(y) - f_2(y)] dy \right| \\ &\leq |\lambda| M(x-a) \max_x |f_1(x) - f_2(x)|, \end{aligned}$$

where $M = \max_{x,y} |K(x, y)| < \infty$. It follows that

$$\begin{aligned} |F^2(f_1)(x) - F^2(f_2)(x)| &= \left| \lambda \int_a^x K(x, y) [F(f_1)(y) - F(f_2)(y)] dy \right| \\ &\leq |\lambda| M \int_a^x |F(f_1)(y) - F(f_2)(y)| dy \\ &\leq |\lambda|^2 M^2 \max_x |f_1(x) - f_2(x)| \int_a^x |y-a| dy \\ &= |\lambda|^2 M^2 \frac{(x-a)^2}{2} \max_x |f_1(x) - f_2(x)|, \end{aligned}$$

and in general

$$\begin{aligned} |F^n(f_1)(x) - F^n(f_2)(x)| &\leq |\lambda|^n M^n \frac{(x-a)^n}{n!} \max_x |f_1(x) - f_2(x)| \\ &\leq |\lambda|^n M^n \frac{(b-a)^n}{n!} \max_x |f_1(x) - f_2(x)|. \end{aligned}$$

It follows that

$$\|F^n f_1 - F^n f_2\| \leq |\lambda|^n M^n \frac{(b-a)^n}{n!} \|f_1 - f_2\|,$$

so that F^n is a contraction mapping if n is chosen large enough to ensure that

$$|\lambda|^n M^n \frac{(b-a)^n}{n!} < 1.$$

¹⁰If we extend the definition of the kernel $K(x, y)$ appearing in (25) by setting $K(x, y) = 0$ for all $y > x$ then (25) becomes an instance of the Fredholm equation (23). This is not done because the contraction mapping method applied to the Volterra equation directly gives a better result in that the condition on λ can be avoided.

It follows by Corollary 2.2 that the equation (25) has a unique solution for all λ .

Linear Transformations

Let X and Y be linear spaces, and T a function from the set $D_T \subset X$ into Y . Sometimes such functions will be called *operators*, *mappings* or *transformations*. The set D_T is the *domain* of T , and $T(D_T) \subset Y$ is the *range* of T . If the set D_T is a linear subspace of X and T is a linear map,

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \quad \text{for all } \alpha, \beta \in \mathbb{R} \text{ or } \mathbb{C}, x, y \in X. \quad (26)$$

Notice that a linear operator is injective if and only if the kernel $\{x \in X \mid Tx = 0\}$ is trivial.

LEMMA 3.1. *A linear transformation $T : X \rightarrow Y$ is continuous if and only if it is continuous at one point.*

PROOF. Assume that T is continuous at a point a . Then for any sequence $x_n \rightarrow a$, $T(x_n) \rightarrow T(a)$. Let z be any point in X , and y_n a sequence with $y_n \rightarrow z$. Then $y_n - z + a$ is a sequence converging to a , so $T(y_n - z + a) = T(y_n) - T(z) + T(a) \rightarrow T(a)$. It follows that $T(y_n) \rightarrow T(z)$. \square

A simple observation that is useful in differential equations, where it is called the *principle of superposition*: if $\sum_{n=1}^{\infty} \alpha_n x_n$ is convergent, and T is a continuous linear map, then $T(\sum_{n=1}^{\infty} \alpha_n x_n) = \sum_{n=1}^{\infty} \alpha_n T x_n$.

1. Bounded operators

EXAMPLE 3.1. Consider a voltage $v(t)$ applied to a resistor R , capacitor C , and inductor L arranged in series (an “LCR” circuit). The charge $u = u(t)$ on the capacitor satisfies the equation

$$L \frac{d^2 u}{dt^2} + R \frac{du}{dt} + \frac{1}{C} u = v, \quad (27)$$

with some initial conditions say $u(0) = 0$, $\frac{du}{dt}(0) = 0$. Assuming that $R^2 > 4L/C$, then the solution of (27) is

$$u(t) = \int_0^t k(t-s)v(s)ds, \quad (28)$$

where

$$k(t) = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{L(\lambda_1 - \lambda_2)}$$

and λ_1, λ_2 are the (distinct) roots of $L\lambda^2 + R\lambda + \frac{1}{C} = 0$.

This problem may be phrased in terms of linear operators. Let $X = C[0, \infty)$; then the transformation defined by $T(v) = u$ in (28) is a linear operator from X to X .

Similarly, (27) can be written in the form $S(u) = v$ for some linear operator S . However, S cannot be defined on all of X – only on the dense linear subspace of twice-differentiable functions. The transformations T and S are closely related, and we would like to develop a framework for viewing them as inverse to each other.

DEFINITION 3.1. A linear transformation $T : X \rightarrow Y$ is (algebraically) *invertible* if there is a linear transformation $S : Y \rightarrow X$ with the property that $TS = 1_Y$ and $ST = 1_X$.

For example, in Example 3.1, if we take $X = C[0, \infty)$ and $Y = C^2[0, \infty)$, then T is algebraically invertible with $T^{-1} = S$.

DEFINITION 3.2. A linear operator $T : X \rightarrow Y$ is *bounded* if there is a constant K such that

$$\|Tx\|_Y \leq K\|x\|_X \text{ for all } x \in X.$$

The *norm* of the bounded linear operator T is

$$\|T\| = \sup_{x \neq 0} \left\{ \frac{\|Tx\|_Y}{\|x\|_X} \right\}. \quad (29)$$

EXAMPLE 3.2. In Example 3.1, the operator T is bounded when restricted to any $C[0, a]$ for any a , since

$$|Tv(s)| \leq \int_0^t |k(t-s)| \cdot |v(s)| ds,$$

which shows that

$$\|Tv\|_\infty \leq a \sup_{0 \leq t \leq a} |k(t)| \|v\|_\infty < \frac{a\|v\|_\infty}{L|\lambda_1 - \lambda_2|}.$$

The operator S is not bounded of course – think about what differentiation does.

EXERCISE 3.1 (1). Show that $\|T\| = \sup_{\|x\|=1} \{\|Tx\|_Y\}$.
 [2] Prove the following useful inequality:

$$\|Tx\|_Y \leq \|T\| \cdot \|x\|_X \text{ for all } x \in X. \quad (30)$$

THEOREM 3.1. A linear transformation $T : X \rightarrow Y$ is continuous if and only if it is bounded.

PROOF. If T is bounded and $x_n \rightarrow 0$, then by Definition 3.2, $Tx_n \rightarrow 0$ also. It follows that T is continuous at 0, so by Lemma 3.1 T is continuous everywhere.

Conversely, suppose that T is continuous but unbounded. Then for any $n \in \mathbb{N}$ there is a point x_n with $\|Tx_n\| > n\|x_n\|$. Let $y_n = \frac{x_n}{n\|x_n\|}$, so that $y_n \rightarrow 0$ as $n \rightarrow \infty$. On the other hand, $\|Ty_n\| > 1$ and $T(0) = 0$, contradicting the assumption that T is continuous at 0. \square

2. The space of linear operators

The set of all linear transformations $X \rightarrow Y$ is itself a linear space with the operations

$$(T + S)(x) = Tx + Sx, \quad (\lambda T)(x) = \lambda Tx.$$

Denote this linear space by $\mathcal{L}(X, Y)$. If X and Y are normed spaces, denote by $\mathcal{B}(X, Y)$ the subspace of continuous linear transformations. If $X = Y$, then write $\mathcal{L}(X, X) = \mathcal{L}(X)$ and $\mathcal{B}(X, X) = \mathcal{B}(X)$.

LEMMA 3.2. *Let X and Y be normed spaces. Then $\mathcal{B}(X, Y)$ is a normed linear space with the norm (29). If in addition Y is a Banach space, then $\mathcal{B}(X, Y)$ is a Banach space.*

PROOF. We have to show that the function $T \mapsto \|T\|$ satisfies the conditions of Definition 1.4.

(1) It is clear that $\|T\| \geq 0$ since it is defined as the supremum of a set of non-negative numbers. If $\|T\| = 0$ then $\|Tx\|_Y = 0$ for all x , so $Tx = 0$ for all x – that is, $T = 0$.

(2) The triangle inequality is also clear:

$$\|T + S\| = \sup_{\|x\|=1} \|(T + S)x\| \leq \sup_{\|x\|=1} \|Tx\| + \sup_{\|x\|=1} \|Sx\| = \|T\| + \|S\|.$$

(3) $\|\lambda T\| = \sup_{\|x\|=1} \|(\lambda T)x\| = |\lambda| \sup_{\|x\|=1} \|Tx\| = |\lambda| \|T\|$.

Finally, assume that Y is a Banach space and let (T_n) be a Cauchy sequence in $\mathcal{B}(X, Y)$. Then the sequence is bounded: there is a constant K with $\|T_n x\| \leq K\|x\|$ for all $x \in X$ and $n \geq 1$. Since $\|T_n x - T_m x\| \leq \|T_n - T_m\| \|x\| \rightarrow 0$ as $n \geq m \rightarrow \infty$, the sequence $(T_n x)$ is a Cauchy sequence in Y for each $x \in X$. Since Y is complete, for each $x \in X$ the sequence $(T_n x)$ converges; define T by

$$Tx = \lim_{n \rightarrow \infty} T_n x.$$

It is clear that T is linear, and $\|Tx\| \leq K\|x\|$ for all x , so $T \in \mathcal{B}(X, Y)$.

We have not yet established that $T_n \rightarrow T$ in the norm of $\mathcal{B}(X, Y)$ (cf. 29). Since (T_n) is Cauchy, for any $\epsilon > 0$ there is an N such that

$$\|T_m - T_n\| \leq \epsilon \text{ for all } m \geq n \geq N.$$

For any $x \in X$ we therefore have

$$\|T_m x - T_n x\|_Y \leq \epsilon \|x\|_X.$$

Take the limit as $m \rightarrow \infty$ to see that

$$\|Tx - T_n x\| \leq \epsilon \|x\|,$$

so that $\|T - T_n\| \leq \epsilon$ if $n \geq N$. This proves that $\|T - T_n\| \rightarrow 0$ as $n \rightarrow \infty$. \square

EXAMPLE 3.3. Once the space of linear operators is known to be complete, we can do analysis on the operators themselves. For example, if X is a Banach space and $A \in \mathcal{B}(X)$, then we may define an operator

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots,$$

which makes sense since

$$\begin{aligned} \|e^A\| &\leq 1 + \|A\| + \frac{1}{2!}\|A\|^2 + \dots \\ &\leq e^{\|A\|}. \end{aligned}$$

This is particularly useful in linear systems theory and control theory; if $x(t) \in \mathbb{R}^n$ then the linear differential equation $\frac{dx}{dt} = Ax(t)$, $x(0) = x_0$, where A is an $n \times n$ matrix, has as solution $x(t) = e^{At}x_0$.

3. Banach algebras

In many situations it makes sense to multiply elements of a normed linear space together.

DEFINITION 3.3. Let X be a Banach space, and assume there is a multiplication $(x, y) \mapsto xy$ from $X \times X \rightarrow X$ such that addition and multiplication make X into a ring, and

$$\|xy\| \leq \|x\|\|y\|.$$

Then X is called a *Banach algebra*.

Recall that a ring does not need to have a unit; if X has a unit then it is called *unital*.

EXAMPLE 3.4. [1] The continuous functions $C[0, 1]$ with sup norm form a Banach algebra with $(fg)(x) = f(x)g(x)$.

[2] If X is any Banach space, then $\mathcal{B}(X)$ is a Banach algebra:

$$\|ST\| = \sup_{\|x\|=1} \|(ST)x\| = \sup_{\|x\|=1} \|S(Tx)\| \leq \|S\| \sup_{\|x\|=1} \|Tx\| = \|S\|\|T\|.$$

The algebra has an identity, namely $I(x) = x$.

[3] A special case of [2] is the case $X = \mathbb{R}^n$. By choosing a basis for \mathbb{R}^n we may identify $\mathcal{B}(\mathbb{R}^n)$ with the space of $n \times n$ real matrices.

In the next few sections we will prove the more technical results about linear transformations that provide the basic tools of functional analysis.

4. Uniform boundedness

The first theorem is the *principle of uniform boundedness* or the *Banach–Steinhaus theorem*.

THEOREM 3.2. Let X be a Banach space and let Y be a normed linear space. Let $\{T_\alpha\}$ be a family of bounded linear operators from X into Y . If, for each $x \in X$, the set $\{T_\alpha x\}$ is a bounded subset of Y , then the set $\{\|T_\alpha\|\}$ is bounded.

PROOF. Assume first that there is a ball $B_\epsilon(x_0)$ on which $\{T_\alpha x\}$ (a set of functions) is uniformly bounded: that is, there is a constant K such that

$$\|T_\alpha x\| \leq K \text{ if } \|x - x_0\| < \epsilon. \quad (31)$$

Then it is possible to find a uniform bound on the whole family $\{\|T_\alpha\|\}$. For any $y \neq 0$ define

$$z = \frac{\epsilon}{\|y\|}y + x_0.$$

Then $z \in B_\epsilon(x_0)$ by construction, so (31) implies that $\|T_\alpha z\| \leq K$.

Now by linearity of T_α this shows that

$$\frac{\epsilon}{\|y\|}\|T_\alpha y\| - \|T_\alpha x_0\| \leq \left\| \frac{\epsilon}{\|y\|}T_\alpha y + T_\alpha x_0 \right\| = \|T_\alpha z\| \leq K,$$

which can be solved for $\|T_\alpha y\|$:

$$\|T_\alpha y\| \leq \frac{K + \|T_\alpha x_0\|}{\epsilon} \|y\| \leq \frac{K + K'}{\epsilon} \|y\|$$

where $K' = \sup_{\alpha} \|T_{\alpha}x_0\| < \infty$. It follows that

$$\|T_{\alpha}\| \leq \frac{K + K'}{\epsilon}$$

as required.

To finish the proof we have to show that there is a ball on which property (31) holds. This is proved by a contradiction argument: assume for not that there is no ball on which (31) holds. Fix an arbitrary ball B_0 . By assumption there is a point $x_1 \in B_0$ such that

$$\|T_{\alpha_1}x_1\| > 1$$

for some index α_1 say. Since each T_{α} is continuous, there is a ball $B_{\epsilon_1}(x_1)$ in which $\|T_{\alpha_1}(x_1)\| > 1$. Assume without loss of generality that $\epsilon_1 < 1$. By assumption, in this new ball the family $\{T_{\alpha}x\}$ is not bounded, so there is a point $x_2 \in B_{\epsilon_1}(x_1)$ with

$$\|T_{\alpha_2}x_2\| > 2$$

for some index $\alpha_2 \neq \alpha_1$. Continue in the same way: by continuity of α_2 there is a ball $B_{\epsilon_2}(x_2) \subset B_{\epsilon_1}(x_1)$ on which $\|T_{\alpha_2}x\| > 2$. Assume without loss of generality that $\epsilon_2 < \frac{1}{2}$.

Repeating this process produces points x_3, x_4, x_5, \dots , indices $\alpha_3, \alpha_4, \alpha_5, \dots$, and positive numbers $\epsilon_3, \epsilon_4, \epsilon_5, \dots$ such that $B_{\epsilon_n}(x_n) \subset B_{\epsilon_{n-1}}(x_{n-1})$, $\epsilon_n < \frac{1}{n}$, all the α_j 's are distinct, and

$$\|T_{\alpha_n}x\| > n \text{ for all } x \in B_{\epsilon_n}(x_n).$$

Now the sequence (x_n) is clearly Cauchy and therefore converges to $z \in X$ say (equivalently, prove that $\bigcap_{n=1}^{\infty} \bar{B}_{\epsilon_n}(x_n)$ contains the single point z).

By construction, $\|T_{\alpha_n}z\| \geq n$ for all $n \geq 1$, which contradicts the hypothesis that the set $\{T_{\alpha}z\}$ is bounded. \square

Recall the operator norm in Definition 3.2. Corresponding to this norm there is a notion of convergence in $\mathcal{B}(X, Y)$: we say that a sequence (T_n) is *uniformly convergent* if there is $T \in \mathcal{B}(X, Y)$ with $\|T_n - T\| \rightarrow 0$ as $n \rightarrow \infty$ (so uniform convergence of a sequence of operators is simply convergence in the operator norm).

DEFINITION 3.4. A sequence (T_n) in $\mathcal{B}(X, Y)$ is *strongly convergent* if, for any $x \in X$, the sequence (T_nx) converges in Y . If there is a $T \in \mathcal{B}(X, Y)$ with $\lim_n T_nx = Tx$ for all $x \in X$, then (T_n) is *strongly convergent to T* .

EXERCISE 3.2 (1). Prove that uniform convergence implies strong convergence. [2] Show by example that strong convergence does not imply uniform convergence.

THEOREM 3.3. *Let X be a Banach space, and Y any normed linear space. If a sequence (T_n) in $\mathcal{B}(X, Y)$ is strongly convergent, then there exists $T \in \mathcal{B}(X, Y)$ such that (T_n) is strongly convergent to T .*

PROOF. For each $x \in X$ the sequence (T_nx) is bounded since it is convergent. By the uniform boundedness principle (Theorem 3.2), there is a constant K such that $\|T_n\| \leq K$ for all n . Hence

$$\|T_nx\| \leq K\|x\| \text{ for all } x \in X. \quad (32)$$

Define T by requiring that $Tx = \lim_{n \rightarrow \infty} T_nx$ for all $x \in X$. It is clear that T is linear, and (32) shows that $\|Tx\| \leq K\|x\|$ for all $x \in X$, showing that T is bounded. The construction of T means that (T_n) converges strongly to T . \square

5. An application of uniform boundedness to Fourier series

This section is an application of Theorem 3.2 to Fourier analysis. We will encounter Fourier analysis again, in the context of Hilbert spaces and L^2 functions. For now we take a naive view of Fourier analysis: the functions will all be continuous periodic functions, and we compute Fourier coefficients using Riemann integration.

LEMMA 3.3.

$$\int_0^{2\pi} \left| \frac{\sin(n + \frac{1}{2})x}{\sin \frac{1}{2}x} \right| dx \longrightarrow \infty \text{ as } n \rightarrow \infty.$$

PROOF. Recall that $|\sin(x)| \leq |x|$ for all x . It follows that

$$\int_0^{2\pi} \left| \frac{\sin(n + \frac{1}{2})x}{\sin \frac{1}{2}x} \right| dx \geq \int_0^{2\pi} \frac{2}{x} |\sin(n + \frac{1}{2})x| dx.$$

Now $|\sin(n + \frac{1}{2})x| \geq \frac{1}{2}$ for all x with $(n + \frac{1}{2})x$ between $k\pi + \frac{1}{6}\pi$ and $k\pi + \frac{1}{3}\pi$ for $k = 1, 2, \dots$. It follows (by thinking of the Riemann approximation to the integral) that

$$\int_0^{2\pi} \frac{2}{x} |\sin(n + \frac{1}{2})x| dx \geq \sum_{k=0}^{2n} \left(\frac{\pi(k + \frac{1}{3})}{n + \frac{1}{2}} \right)^{-1} = \frac{1}{\pi} \left(n + \frac{1}{2} \right) \sum_{k=0}^{2n} \frac{1}{k + \frac{1}{3}} \rightarrow \infty$$

as $n \rightarrow \infty$. □

DEFINITION 3.5. If $f : (0, 2\pi) \rightarrow \mathbb{R}$ is Riemann-integrable, then the *Fourier series* of f is the series

$$s(x) = \sum_{m=-\infty}^{\infty} a_m e^{imx}, \text{ where } a_m = \frac{1}{2\pi} \int_0^{2\pi} f(\xi) e^{-im\xi} d\xi.$$

Extend the definition of f to make it 2π -periodic, so $f(x + 2\pi) = f(x)$ for all x . Define the n th partial sum of the Fourier series to be

$$s_n(x) = \sum_{m=-n}^n a_m e^{-imx}.$$

The basic questions of Fourier analysis are then the following: is there any relation between $s(x)$ and $f(x)$? Does the function $s_n(x)$ approximate $f(x)$ for large n in some sense?

LEMMA 3.4. Let¹ $D_n(x) = \frac{\sin(n + \frac{1}{2})x}{\sin \frac{1}{2}x}$. Then

$$s_n(y) = \frac{1}{2\pi} \int_0^{2\pi} f(y+x) D_n(x) dx.$$

PROOF. Exercise. □

Now let X be the Banach space of continuous functions $f : [0, 2\pi] \rightarrow \mathbb{R}$ with $f(0) = f(2\pi)$, with the uniform norm.

¹This function is called the Dirichlet kernel. For the lemma, it is helpful to notice that $D_n(x) = \sum_{j=-n}^n e^{ijx}$. If you read up on Fourier analysis, it will be helpful to note that the Dirichlet kernel is not a “summability kernel”.

LEMMA 3.5. *The linear operator $T_n : X \rightarrow \mathbb{R}$ defined by*

$$T_n(f) = \frac{1}{2\pi} \int_0^{2\pi} f(x)D_n(x)dx$$

is bounded, and

$$\|T_n\| = \frac{1}{2\pi} \int_0^{2\pi} |D_n(x)|dx.$$

PROOF. For any $f \in X$,

$$|T_n(f)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(x)||D_n(x)|dx \leq \frac{1}{2\pi} \|f\| \int_0^{2\pi} |D_n(x)|dx,$$

so

$$\|T_n\| \leq \frac{1}{2\pi} \int_0^{2\pi} |D_n(x)|dx.$$

Assume that for some $\delta > 0$ we have

$$\|T_n\| = \frac{1}{2\pi} \int_0^{2\pi} |D_n(x)|dx - \delta. \quad (33)$$

Then since for fixed n $|D_n(x)| \leq M_n$ is bounded, we may find a continuous function f_n that differs from $\text{sign}(D_n(x))$ on a finite union of intervals whose total length does not exceed $\frac{1}{M_n}\delta$. Then (don't think about this – just draw a picture)

$$\left| \frac{1}{2\pi} \int_0^{2\pi} f_n(x)D_n(x)dx \right| > \frac{1}{2\pi} \int_0^{2\pi} |D_n(x)|dx - \delta,$$

which contradicts the assumption (33). We conclude that

$$\|T_n\| = \frac{1}{2\pi} \int_0^{2\pi} |D_n(x)|dx.$$

□

We are now ready to see a genuinely non-trivial and important observation about the basic theorems of Fourier analysis.

THEOREM 3.4. *There exists a continuous function $f : [0, 2\pi] \rightarrow \mathbb{R}$, with $f(0) = f(2\pi)$, such that its Fourier series diverges at $x = 0$.*

PROOF. By Lemma 3.4, we have

$$T_n(f) = s_n(0)$$

for all $f \in X$. Moreover, for fixed $f \in X$, if the Fourier series of f converges at 0, then the family $\{T_n f\}$ is bounded as n varies (since each element is just a partial sum of a convergent series). Thus if the Fourier series of f converges at 0 for all $f \in X$, then for each $f \in X$ the set $\{T_n f\}$ is bounded. By Theorem 3.2, this implies that the set $\{\|T_n\|\}$ is bounded, which contradicts Lemma 3.5.

The conclusion is that there must be some $f \in X$ whose Fourier series does not converge at 0. □

EXERCISE 3.3. The problem of deciding whether or not the Fourier series of a given function converges at a specific point (or everywhere) is difficult and usually requires some degree of smoothness (differentiability). You can read about various results in many books – a good starting point is *Fourier Analysis*, Tom Körner, Cambridge University Press (1988).

It is more natural in functional analysis to ask for an appropriate semi-norm in which $\|s(x) - f(x)\| = 0$ for some class of functions f .

6. Open mapping theorem

Recall that a continuous map between normed spaces has the property that the pre-image of any open set is open, but in general the image of an open set is not open (Exercise 1.1). Bounded linear maps between Banach spaces cannot do this.

THEOREM 3.5. *Let X and Y be Banach spaces, and let T be a bounded linear map from X onto Y . Then T maps open sets in X onto open sets in Y .*

Of course the assumption that X maps onto Y is crucial: think of the projection $(x, y) \mapsto (x, 0)$ from $\mathbb{R}^2 \rightarrow \mathbb{R}^2$. This is bounded and linear, but not onto, and certainly cannot send open sets to open sets.

The proof of the Open-Mapping theorem is long and requires the Baire category theorem, so it will be omitted from the lectures. For completeness it is given here in the next three lemmas.

Some notation: use B_r^X and B_r^Y to denote the open balls of radius r centre 0 in X and Y respectively.

LEMMA 3.6. *For any $\epsilon > 0$, there is a $\delta > 0$ such that*

$$T\bar{B}_{2\epsilon}^X \supset B_\delta^Y. \quad (34)$$

PROOF. Since $X = \bigcup_{n=1}^{\infty} nB_\epsilon^X$, and T is onto, we have $Y = T(X) = \bigcup_{n=1}^{\infty} nTB_\epsilon^X$. By the Baire category theorem (Theorem A.4) it follows that, for some n , the set $nT\bar{B}_\epsilon^X$ contains some ball $B_r^Y(z)$ in Y . Then $T\bar{B}_\epsilon^X$ must contain the ball $B_\delta^Y(y_0)$, where $y_0 = \frac{1}{n}z$ and $\delta = \frac{1}{n}r$. It follows that the set

$$P = \{y_1 - y_2 \mid y_1 \in B_\delta^Y(y_0), y_2 \in B_\delta^Y(y_0)\}$$

is contained in the closure of the set TQ , where

$$Q = \{x_1 - x_2 \mid x_1 \in B_\epsilon^X, x_2 \in B_\epsilon^X\} \subset B_{2\epsilon}^X.$$

Thus, $T\bar{B}_{2\epsilon}^X \subset P$. Any point $y \in B_\delta^Y$ can be written in the form $y = (y + y_0) - y_0$, so $B_\delta^Y \subset P$. and (34) follows. \square

LEMMA 3.7. *For any $\epsilon_0 > 0$ there is a $\delta_0 > 0$ such that*

$$TB_{2\epsilon_0}^X \supset B_{\delta_0}^Y. \quad (35)$$

PROOF. Choose a sequence (ϵ_n) with each $\epsilon_n > 0$ and $\sum_{n=1}^{\infty} \epsilon_n < \epsilon_0$. By Lemma 3.6 there is a sequence (δ_n) of positive numbers such that

$$T\bar{B}_{\epsilon_n}^X \supset B_{\delta_n}^Y \quad (36)$$

for all $n \geq 1$. Without loss of generality, assume that $\delta_n \rightarrow 0$ as $n \rightarrow \infty$.

Let y be any point in $B_{\delta_0}^Y$. By (36) with $n = 0$ there is a point $x_0 \in B_{\epsilon_0}^X$ with $\|y - Tx_0\| < \delta_1$. Since $(y - Tx_0) \in B_{\delta_1}^Y$, (36) with $n = 1$ implies that there exists a point $x_1 \in B_{\epsilon_1}^X$ such that $\|y - Tx_0 - Tx_1\| < \delta_2$. Continuing, we obtain a sequence (x_n) such that $x_n \in B_{\epsilon_n}^X$ for all n , and

$$\left\| y - T \left(\sum_{k=0}^n x_k \right) \right\| < \delta_{n+1}. \quad (37)$$

Since $\|x_n\| < \epsilon_n$, the series $\sum_n x_n$ is absolutely convergent, so by Lemma 2.1 it is convergent; write $x = \sum_n x_n$. Then

$$\|x\| \leq \sum_{n=0}^{\infty} \|x_n\| \leq \sum_{n=0}^{\infty} \epsilon_n < 2\epsilon_0.$$

The map T is continuous, so (37) shows that $y = Tx$ since $\delta_n \rightarrow 0$.

That is, for any $y \in B_{\delta_0}^Y$ we have found a point $x \in B_{2\epsilon_0}^X$ such that $Tx = y$, proving (35). \square

LEMMA 3.8. *For any open set $G \subset X$ and for any point $\bar{y} = T\bar{x}$, $\bar{x} \in G$, there is an open ball B_η^Y such that $\bar{y} + B_\eta^Y \subset T(G)$.*

Notice that Lemma 3.8 proves Theorem 3.5 since it implies that $T(G)$ is open.

PROOF. Since G is open, there is a ball B_ϵ^X such that $\bar{x} + B_\epsilon^X \subset G$. By Lemma 3.7, $T(B_\epsilon^X) \supset B_\eta^Y$ for some $\eta > 0$. Hence

$$T(G) \supset T(\bar{x} + B_\epsilon^X) = T(\bar{x}) + T(B_\epsilon^X) \supset \bar{y} + B_\eta^Y.$$

\square

As an application of Theorem 3.5, we establish a general property of inverse maps. Generalizing Definition 3.1 slightly, we have the following.

DEFINITION 3.6. Let $T : X \rightarrow Y$ be an injective linear operator. Define the inverse of T , T^{-1} by requiring that

$$T^{-1}y = x \text{ if and only if } Tx = y.$$

Then the domain of T^{-1} is a linear subspace of Y , and T^{-1} is a linear operator.

It is easy to check that $T^{-1}Tx = x$ for all $x \in X$, and $TT^{-1}y = y$ for all y in the domain of T^{-1} .

LEMMA 3.9. *Let X and Y be Banach spaces, and let T be an injective bounded linear map from X to Y . Then T^{-1} is a bounded linear map.*

PROOF. Since T^{-1} is a linear operator, we only need to show it is continuous by Theorem 3.1. By Theorem 3.5 $(T^{-1})^{-1}$ maps open sets onto open sets. By Exercise 1.1[1], this means that T^{-1} is continuous. \square

COROLLARY 3.1. *If X is a Banach space with respect to two norms $\|\cdot\|^{(1)}$ and $\|\cdot\|^{(2)}$ and there is a constant K such that*

$$\|x\|^{(1)} \leq K\|x\|^{(2)},$$

then the two norms are equivalent: there is another constant K' with

$$\|x\|^{(2)} \leq K'\|x\|^{(1)}$$

for all $x \in X$.

PROOF. Consider the map $T : x \mapsto x$ from $(X, \|\cdot\|^{(1)})$ to $(X, \|\cdot\|^{(2)})$. By assumption, T is bounded, so by Lemma 3.9, T^{-1} is also bounded, giving the bound in the other direction. \square

DEFINITION 3.7. Let $T : X \rightarrow Y$ be a linear operator from a normed linear space X into a normed linear space Y , with domain D_T . The *graph* of T is the set

$$G_T = \{(x, Tx) \mid x \in D_T\} \subset X \times Y.$$

If G_T is a closed set in $X \times Y$ (see Example 1.7) then T is a *closed operator*.

Notice as usual that this notion becomes trivial in finite dimensions: if X and Y are finite-dimensional, then the graph of T is simply some linear subspace, which is automatically closed. The next theorem is called the *closed-graph theorem*.

THEOREM 3.6. *Let X and Y be Banach spaces, and $T : X \rightarrow Y$ a linear operator (notice that the notation means $D_T = X$). If T is closed, then it is continuous.*

PROOF. Fix the norm $\|(x, y)\| = \|x\|_X + \|y\|_Y$ on $X \times Y$. The graph G_T is, by linearity of T , a closed linear subspace in $X \times Y$, so G_T is itself a Banach space. Consider the projection $P : G_T \rightarrow X$ defined by $P(x, Tx) = x$. Then P is clearly bounded, linear, and bijective. It follows by Lemma 3.9 that P^{-1} is a bounded linear operator from X into G_T , so

$$\|(x, Tx)\| = \|P^{-1}x\| \leq K\|x\|_X \quad \text{for all } x \in X,$$

for some constant K . It follows that $\|x\|_X + \|Tx\|_Y \leq K\|x\|_X$ for all $x \in X$, so T is bounded – and therefore T is continuous by Theorem 3.1. \square

7. Hahn–Banach theorem

Let X be a normed linear space. A bounded linear operator from X into the normed space \mathbb{R} is a (*real*) *continuous linear functional* on X . The space of all continuous linear functionals is denoted $\mathcal{B}(X, \mathbb{R}) = X^*$, and it is called the *dual* or *conjugate* space of X . All the material here may be done again with \mathbb{C} instead of \mathbb{R} without significant changes.

Notice that Lemma 3.2 shows that X^* is itself a Banach space independently of X .

One of the most important questions one may ask of X^* is the following: are there “enough” elements in X^* ? (to do what we need: for example, to separate points). This is answered in great generality using the Hahn–Banach theorem (Theorem 3.7 below); see Corollary 3.4. First we prove the Hahn–Banach lemma.

LEMMA 3.10. *Let X be a real linear space, and $p : X \rightarrow \mathbb{R}$ a continuous function with*

$$p(x + y) \leq p(x) + p(y), \quad p(\lambda x) = \lambda p(x) \quad \text{for all } \lambda \geq 0, x, y \in X.$$

Let Y be a subspace of X , and $f \in Y^$ with*

$$f(x) \leq p(x) \quad \text{for all } x \in Y.$$

Then there exists a functional $F \in X^$ such that*

$$F(x) = f(x) \quad \text{for } x \in Y; \quad F(x) \leq p(x) \quad \text{for all } x \in X.$$

PROOF. Let \mathcal{K} be the set of all pairs (Y_α, g_α) in which Y_α is a linear subspace of X containing Y , and g_α is a real linear functional on Y_α with the properties that

$$g_\alpha(x) = f(x) \quad \text{for all } x \in Y, \quad g_\alpha(x) \leq p(x) \quad \text{for all } x \in Y_\alpha.$$

Make \mathcal{K} into a partially ordered set by defining the relation $(Y_\alpha, g_\alpha) \leq (Y_\beta, g_\beta)$ if $Y_\alpha \subset Y_\beta$ and $g_\alpha = g_\beta$ on Y_α . It is clear that any totally ordered subset $\{(Y_\lambda, g_\lambda)\}$ has an upper bound given by the subspace $\bigcup_\lambda Y_\lambda$ and the functional defined to be g_λ on each Y_λ .

By Theorem A.1, there is a maximal element (Y_0, g_0) in \mathcal{K} . All that remains is to check that Y_0 is all of X (so we may take F to be g_0).

Assume that $y_1 \in X \setminus Y_0$. Let Y_1 be the linear space spanned by Y_0 and y_1 : each element $x \in Y_1$ may be expressed uniquely in the form

$$x = y + \lambda y_1, \quad y \in Y_0, \lambda \in \mathbb{R},$$

because y_1 is assumed not to be in the linear space Y_0 . Define a linear functional $g_1 \in Y_1^*$ by $g_1(y + \lambda y_1) = g_0(y) + \lambda c$.

Now we choose the constant c carefully. Note that if $x \neq y$ are in Y_0 , then

$$g_0(y) - g_0(x) = g_0(y - x) \leq p(y - x) \leq p(y + y_1) + p(-y_1 - x),$$

so

$$-p(-y_1 - x) - g_0(x) \leq p(y + y_1) - g_0(y).$$

It follows that

$$A = \sup_{x \in Y_0} \{-p(-y_1 - x) - g_0(x)\} \leq \inf_{y \in Y_0} \{p(y + y_1) - g_0(y)\} = B.$$

Choose c to be any number in the interval $[A, B]$. Then by construction of A and B ,

$$c \leq p(y + y_1) - g_0(y) \quad \text{for all } y \in Y_0, \quad (38)$$

$$-p(-y_1 - y) - g_0(y) \leq c \quad \text{for all } y \in Y_0. \quad (39)$$

Multiply (38) by $\lambda > 0$ and substitute $\frac{y}{\lambda}$ for y to obtain

$$\lambda c \leq p(y + \lambda y_1) - g_0(y). \quad (40)$$

Now multiply (39) by $\lambda < 0$, substitute $\frac{y}{\lambda}$ for y and use the homogeneity assumption on p to obtain (40) again. Since (40) is clear for $\lambda = 0$, we deduce that

$$g_1(y + \lambda y_1) = g_0(y) + \lambda c \leq p(y + \lambda y_1)$$

for all $\lambda \in \mathbb{R}$ and $y \in Y_0$. That is, $(Y_1, g_1) \in \mathcal{K}$ and $(Y_0, g_0) \leq (Y_1, g_1)$ with $Y_0 \neq Y_1$. This contradicts the maximality of (Y_0, g_0) . \square

For real linear spaces, the Hahn–Banach theorem follows at once (for complex spaces a little more work is needed).

THEOREM 3.7. *Let X be a real normed space, and Y a linear subspace. Then for any $y^* \in Y^*$ there corresponds an $x^* \in X^*$ such that*

$$\|x^*\| = \|y^*\|, \quad \text{and } x^*(y) = y^*(y) \text{ for all } y \in Y.$$

That is, any linear functional defined on a subspace may be extended to a linear functional on the whole space with the same norm.

PROOF. Let $p(x) = \|y^*\| \|x\|$, $f(x) = y^*(x)$, and $x^* = F$. Apply the Hahn–Banach Lemma 3.10. To check that $\|x^*\| \leq \|y^*\|$, write $x^*(x) = \theta |x^*(x)|$ for $\theta = \pm 1$. Then

$$|x^*(x)| = \theta x^*(x) = x^*(\theta x) \leq p(\theta x) = \|y^*\| \|\theta x\| = \|y^*\| \|x\|.$$

The reverse inequality is clear, so $\|x^*\| = \|y^*\|$. \square

Many useful results follow from the Hahn–Banach theorem.

COROLLARY 3.2. *Let Y be a linear subspace of a normed linear space X , and let $x_0 \in X$ have the property that*

$$\inf_{y \in Y} \|y - x_0\| = d > 0. \quad (41)$$

Then there exists a point $x^ \in X^*$ such that*

$$x^*(x_0) = 1, \|x^*\| = \frac{1}{d}, x^*(y) = 0 \text{ for all } y \in Y_0.$$

PROOF. Let Y_1 be the linear space spanned by Y and x_0 . Since $x_0 \notin Y$, every point x in Y_1 may be represented uniquely in the form $x = y + \lambda x_0$, with $y \in Y$, $\lambda \in \mathbb{R}$. Define a linear functional $z^* \in Y_1^*$ by $z^*(y + \lambda x_0) = \lambda$. If $\lambda \neq 0$, then

$$\|y + \lambda x_0\| = |\lambda| \left\| \frac{y}{\lambda} + x_0 \right\| \geq |\lambda|d.$$

It follows that $|z^*(x)| \leq \|x\|/d$ for all $x \in Y_1$, so $\|z^*\| \leq \frac{1}{d}$. Choose a sequence $(y_n) \subset Y$ with $\|x_0 - y_n\| \rightarrow d$ as $n \rightarrow \infty$. Then

$$1 = z^*(x_0 - y_n) \leq \|z^*\| \|x_0 - y_n\| \rightarrow \|z^*\|d,$$

so $\|z^*\| = \frac{1}{d}$. Apply Theorem 3.7 to z^* . □

COROLLARY 3.3. *Let X be a normed linear space. Then, for any $x \neq 0$ in X there is a functional $x^* \in X^*$ with $\|x^*\| = 1$ and $x^*(x) = \|x\|$.*

PROOF. Apply Corollary 3.2 with $Y = \{0\}$ to find $z^* \in X^*$ such that $\|z^*\| = 1/\|x\|$, $z^*(x) = 1$. We may therefore take x^* to be $\|x\|z^*$. □

COROLLARY 3.4. *If $z \neq y$ in a normed linear space X , then there exists $x^* \in X^*$ such that $x^*(y) \neq x^*(z)$.*

PROOF. Apply Corollary 3.3 with $x = y - z$. □

COROLLARY 3.5. *If X is a normed linear space, then*

$$\|x\| = \sup_{x^* \neq 0} \frac{|x^*(x)|}{\|x^*\|} = \sup_{\|x^*\|=1} |x^*(x)|.$$

PROOF. The last two expressions are clearly equal. It is also clear that

$$\sup_{\|x^*\|=1} |x^*(x)| \leq \|x\|.$$

By Corollary 3.3, there exists x_0^* such that $x_0^*(x) = \|x\|$ and $\|x_0^*\| = 1$, so

$$\sup_{\|x^*\|=1} |x^*(x)| \geq \|x\|.$$

□

COROLLARY 3.6. *Let Y be a linear subspace of the normed linear space X . If Y is not dense in X , then there exists a functional $x^* \neq 0$ such that $x^*(y) = 0$ for all $y \in Y$.*

PROOF. Notice that if there is no point $x_0 \in X$ satisfying (41) then Y must be dense in X . So we may choose x_0 with (41) and apply Corollary 3.2). □

Notice finally that linear functionals allow us to decompose a linear space: let X be a normed linear space, and $x^* \in X^*$. The *null space* or *kernel* of x^* is the linear subspace $N_{x^*} = \{x \in X \mid x^*(x) = 0\}$. If $x^* \neq 0$, then there is a point $x_0 \neq 0$ such that $x^*(x_0) = 1$. Any element $x \in X$ can then be written $x = z + \lambda x_0$, with $\lambda = x^*(x)$ and $z = x - \lambda x_0 \in N_{x^*}$. Thus, $X = N_{x^*} \oplus Y$, where Y is the one-dimensional space spanned by x_0 .

Integration

We have seen in Examples 1.10[3] that the space $C[0, 1]$ of continuous functions with the p -norm

$$\|f\|_p = \left(\int_0^1 |f(t)|^p dt \right)^{1/p}$$

is not complete, even if we extend the space to Riemann-integrable functions.

As discussed in the section on completions, we can think of the completion of the space in terms of all limit points of (equivalence classes) of Cauchy sequences. This does not give any real sense of what kind of functions are in the completion. In this chapter we construct the completions L_p for $1 \leq p \leq \infty$ by describing (without proofs) the *Lebesgue¹ integral*.

1. Lebesgue measure

DEFINITION 4.1. Let \mathcal{B} denote the smallest collection of subsets of \mathbb{R} that includes all the open sets and is closed under countable unions, countable intersections and complements. These sets are called the *Borel sets*.

In fact the Borel sets form a σ -algebra: $\mathbb{R}, \emptyset \in \mathcal{B}$, and \mathcal{B} is closed under countable unions and intersections. We will call Borel sets *measurable*. Many subsets of \mathbb{R} are not measurable, but all the ones you can write down or that might arise in a practical setting are measurable.

The *Lebesgue measure* on \mathbb{R} is a map $\mu : \mathcal{B} \rightarrow \mathbb{R} \cup \{\infty\}$ with the properties that

- (i) $\mu[a, b] = \mu(a, b) = b - a$;
- (ii) $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$.

Notice that the Lebesgue measure attaches a measure to all measurable sets. Sets of measure zero are called *null sets*, and something that happens everywhere except on a set of measure zero is said to happen *almost everywhere*, often written simply a.e. For technical reasons, allow any subset of a null set to also be regarded as “measurable”, with measure zero.

EXERCISE 4.1. [1] Prove that $\mu(\mathbb{Q}) = 0$. Thus a.e. real number is irrational.

¹Henri Leon Lebesgue (1875–1941), was a French mathematician who revolutionized the field of integration by his generalization of the Riemann integral. Up to the end of the 19th century, mathematical analysis was limited to continuous functions, based largely on the Riemann method of integration. Building on the work of others, including that of the French mathematicians Emile Borel and Camille Jordan, Lebesgue developed (in 1901) his theory of measure. A year later, Lebesgue extended the usefulness of the definite integral by defining the Lebesgue integral: a method of extending the concept of area below a curve to include many discontinuous functions. Lebesgue served on the faculty of several French universities. He made major contributions in other areas of mathematics, including topology, potential theory, and Fourier analysis.

[2] More can be said: call a real number *algebraic* if it is a zero of some polynomial with rational coefficients, and *transcendental* if not. Then a.e. real number is transcendental.

[3] Prove that for any measurable sets A, B , $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$.

[4] Can you construct² a set that is *not* a member of \mathcal{B} ?

DEFINITION 4.2. A function $f : \mathbb{R} \rightarrow \mathbb{R} \cup \{\pm\infty\}$ is a *Lebesgue measurable function* if $f^{-1}(A) \in \mathcal{B}$ for every $A \in \mathcal{A}$.

EXAMPLE 4.1. [1] The characteristic function $\chi_{\mathbb{Q}}$, defined by $\chi_{\mathbb{Q}}(x) = 1$ if $x \in \mathbb{Q}$, and $= 0$ if $x \notin \mathbb{Q}$ is an example of a measurable function that is not Riemann integrable.

[2] All continuous functions are measurable (by Exercise 1.1[1]).

The basic idea in Riemann integration is to approximate functions by step functions, whose “integrals” are easy to find. These give the upper and lower estimates. In the Lebesgue theory, we do something similar, using *simple* functions instead of step functions.

A *simple function* is a map $f : \mathbb{R} \rightarrow \mathbb{R}$ of the form

$$f(x) = \sum_{i=1}^n c_i \chi_{E_i}(x), \quad (42)$$

where the c_i are non-zero constants and the E_i are disjoint measurable sets with $\mu(E_i) < \infty$.

The integral of the simple function (42) is defined to be

$$\int_E f d\mu = \sum_{i=1}^n c_i \mu(E \cap E_i)$$

for any measurable set E .

The basic approximation fact in the Lebesgue integral is the following: if $f : \mathbb{R} \rightarrow \mathbb{R} \cup \{\pm\infty\}$ is measurable and non-negative, then there is an increasing sequence (f_n) of simple functions with the property that $f_n(t) \rightarrow f(t)$ a.e. We write this as $f_n \uparrow f$ a.e., and define the integral of f to be

$$\int_E f d\mu = \lim_{n \rightarrow \infty} \int_E f_n d\mu.$$

Notice that (once we allow the value ∞), the limit is guaranteed to exist since the sequence is increasing.

²This “construction” requires the use of the Axiom of Choice and is closely related to the existence of a Hamel basis for \mathbb{R} as a vector space over \mathbb{Q} . The question really has two faces: 1) using the usual axioms of set theory (including the Axiom of Choice), can you exhibit a non-measurable subset of \mathbb{R} ? 2) using the usual axioms of set theory *without* the Axiom of Choice, is it still possible to exhibit a non-measurable subset of \mathbb{R} ?

The first question is easily answered. The second question is much deeper because the answer is “no”. This is part of a subject called Model Theory. Solovay showed that there is a model of set theory (excluding the Axiom of Choice but including a further axiom) in which every subset of \mathbb{R} is measurable. Shelah tried to remove Solovay’s additional axiom, and answered a related question by exhibiting a model of set theory (excluding the Axiom of Choice but otherwise as usual) in which every subset of \mathbb{R} has the Baire property. The references are R.M. Solovay, “A model of set-theory in which every set of reals is Lebesgue measurable”, *Annals of Math.* **92** (1970), 1–56, and S. Shelah, “Can you take Solovay’s inaccessible away?”, *Israel Journal of Math.* **48** (1984), 1–47 but both of them require extensive additional background to read.

For a general measurable function f , write $f = f^+ - f^-$ where both f^+ and f^- are non-negative and measurable, then define

$$\int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu.$$

EXAMPLE 4.2. Let $f(x) = \chi_{\mathbb{Q} \cap [0,1]}(x)$. Then f is itself a simple function, so

$$\int_0^1 f d\mu = \mu(\mathbb{Q} \cap [0,1]) = 0.$$

A measurable function f on $[a, b]$ is *essentially bounded* if there is a constant K such that $|f(x)| \leq K$ a.e. on $[a, b]$. The *essential supremum* of such a function is the infimum of all such essential bounds K , written

$$\|f\|_\infty = \text{ess.sup.}_{[a,b]} |f|.$$

DEFINITION 4.3. Define $\mathcal{L}_p[a, b]$ to be the linear space of measurable functions f on $[a, b]$ for which

$$\|f\|_p = \left(\int_a^b |f|^p d\mu \right)^{1/p} < \infty$$

for $p \in [1, \infty)$ and $\mathcal{L}_\infty[a, b]$ to be the linear space of essentially bounded functions. Notice that $\|\cdot\|_p$ on \mathcal{L}_p is only a semi-norm, since many functions will for example have $\|f\|_p = 0$. Define an equivalence relation on \mathcal{L}_p by $f \sim g$ if $\{x \in \mathbb{R} \mid f(x) \neq g(x)\}$ is a null set. Then define

$$L_p[a, b] = \mathcal{L}_p / \sim,$$

the space of L_p functions.

In practice we will not think of elements of L_p as equivalence classes of functions, but as functions defined a.e. A similar definition may be made of p -integrable functions on \mathbb{R} , giving the linear space $L_p(\mathbb{R})$.

The following theorems are proved in any book on measure theory or modern analysis or may be found in any of the references. Theorem 4.1 is sometimes called the *Riesz–Fischer theorem*; Theorem 4.2 is *Hölder’s inequality*.

THEOREM 4.1. *The normed spaces $L_p[a, b]$ and $L_p(\mathbb{R})$ are (separable) Banach spaces under the norm $\|\cdot\|_p$.*

THEOREM 4.2. *If $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, then*

$$\|fg\|_r \leq \|f\|_p \|g\|_q$$

for any $f \in L_p[a, b]$, $g \in L_q[a, b]$. It follows that for any measurable f on $[a, b]$,

$$\|f\|_1 \leq \|f\|_2 \leq \|f\|_3 \leq \cdots \leq \|f\|_\infty.$$

Hence

$$L_1[a, b] \supset L_2[a, b] \supset \cdots \supset L_\infty[a, b].$$

In the theorem we allow p and q to be anything in $[1, \infty]$ with the obvious interpretation of $\frac{1}{\infty}$.

Note the “opposite” behaviour to the sequence spaces ℓ_p in Example 1.4[3], where we saw that

$$\ell_1 \subset \ell_2 \subset \cdots \subset \ell_\infty.$$

Two easy consequences of Hölder's inequality are the Cauchy–Schwartz inequality,

$$\|fg\|_1 \leq \|f\|_2 \|g\|_2$$

and Minkowski's inequality,

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

The most useful general result about Lebesgue integration is Lebesgue's *dominated convergence theorem*.

THEOREM 4.3. *Let (f_n) be a sequence of measurable functions on a measurable set E such that $f_n(t) \rightarrow f(t)$ a.e. and there exists an integrable function g such that $|f_n(t)| \leq g(t)$ a.e. Then*

$$\int_E f d\mu = \lim_{n \rightarrow \infty} \int_E f_n d\mu.$$

EXERCISE 4.2. [1] Prove that the L_p -norm is strictly convex for $1 < p < \infty$ but is not strictly convex if $p = 1$ or ∞ .

2. Product spaces and Fubini's theorem

Let X and Y be two subsets of \mathbb{R} . Let \mathcal{A} , \mathcal{B} denote the σ -algebra of Borel sets in X and Y respectively.

Subsets of $X \times Y$ (Cartesian product) of the form

$$A \times B = \{(x, y) : x \in A, y \in B\}$$

with $A \in \mathcal{A}$, $B \in \mathcal{B}$ are called (*measurable*) *rectangles*. Let $\mathcal{A} \times \mathcal{B}$ denote the smallest σ -algebra on $X \times Y$ containing all the measurable rectangles. Notice that, despite the notation, this is much larger than the set of all measurable rectangles. The measure space $(X \times Y, \mathcal{A} \times \mathcal{B})$ is the *Cartesian product* of (X, \mathcal{A}) and (Y, \mathcal{B}) . Let μ_X and μ_Y denote Lebesgue measure on X and Y . Then there is a unique measure λ on $X \times Y$ with the property that

$$\lambda(A \times B) = \mu_X(A) \times \mu_Y(B)$$

for all measurable rectangles $A \times B$. This measure is called the *product measure* of μ_X and μ_Y and we write $\lambda = \mu_X \times \mu_Y$.

The most important result on product measures is Fubini's theorem.

THEOREM 4.4. *If h is an integrable function on $X \times Y$, then $x \mapsto h(x, y)$ is an integrable function of X for a.e. y , $y \mapsto h(x, y)$ is an integrable function of Y for a.e. x , and*

$$\int h d(\mu_X \times \mu_Y) = \int \int h d\mu_X d\mu_Y = \int \int h d\mu_Y d\mu_X.$$

Hilbert spaces

We have seen how useful the property of completeness is in our applications of Banach-space methods to certain differential and integral equations. However, some obvious ideas for use in differential equations (like Fourier analysis) seem to go wrong in the obvious Banach space setting (cf. Theorem 3.4). It turns out that not all Banach spaces are equally good – there are distinguished ones in which the *parallelogram law* (equation (43) below) holds, and this has enormous consequences. It makes more sense in this section to deal with complex linear spaces, so from now on assume that the ground field is \mathbb{C} .

1. Hilbert spaces

DEFINITION 5.1. A complex linear space H is called a *Hilbert¹ space* if there is a complex-valued function $(\cdot, \cdot) : H \times H \rightarrow \mathbb{C}$ with the properties

- (i) $(x, x) \geq 0$, and $(x, x) = 0$ if and only if $x = 0$;
- (ii) $(x + y, z) = (x, z) + (y, z)$ for all $x, y, z \in H$;
- (iii) $(\lambda x, y) = \lambda(x, y)$ for all $x, y \in H$ and $\lambda \in \mathbb{C}$;
- (iv) $(x, y) = \overline{(y, x)}$ for all $x, y \in H$;
- (v) the norm defined by $\|x\| = (x, x)^{1/2}$ makes H into a Banach space.

If only properties (i), (ii), (iii), (iv) hold then $(H, (\cdot, \cdot))$ is called an *inner-product space*.

Notice that property (v) makes sense since by (i) $(x, x) \geq 0$, and we shall see below (Lemma 5.2) that $\|\cdot\|$ is indeed a norm.

The function (\cdot, \cdot) is called an *inner* or *scalar* product, and so a Hilbert space is a *complete inner product space*.

If the scalar product is real-valued on a real linear space, then the properties determine a *real Hilbert space*; all the results below apply to these.

Notice that (iii) and (iv) imply that $(x, \lambda y) = \overline{\lambda}(x, y)$, and $(x, 0) = (0, x) = 0$.

EXAMPLE 5.1. [1] If $X = \mathbb{C}^n$, then $(x, y) = \sum_{i=1}^n x_i \overline{y_i}$ makes \mathbb{C}^n into an n -dimensional Hilbert space.

¹David Hilbert (1862–1943) was a German mathematician whose work in geometry had the greatest influence on the field since Euclid. After making a systematic study of the axioms of Euclidean geometry, Hilbert proposed a set of 21 such axioms and analyzed their significance. Hilbert received his Ph.D. from the University of Königsberg and served on its faculty from 1886 to 1895. He became (1895) professor of mathematics at the University of Göttingen, where he remained for the rest of his life. Between 1900 and 1914, many mathematicians from the United States and elsewhere who later played an important role in the development of mathematics went to Göttingen to study under him. Hilbert contributed to several branches of mathematics, including algebraic number theory, functional analysis, mathematical physics, and the calculus of variations. He also enumerated 23 unsolved problems of mathematics that he considered worthy of further investigation. Since Hilbert's time, nearly all of these problems have been solved.

[2] Let $X = C[a, b]$ (complex-valued continuous functions). Then the inner-product $(f, g) = \int_a^b f(t)g(\bar{t})dt$ makes X into an inner-product space that is not a Hilbert space.

[3] Let $X = \ell_2$ (square-summable sequences; see Example 1.4[3]) with the inner-product $((x_n), (y_n)) = \sum_{n=1}^{\infty} x_n y_n$. This is well-defined by the Schwartz inequality Lemma 5.1, and it is a Hilbert space by Example 2.1[3]. We shall see later that ℓ_2 is the only ℓ_p space that is a Hilbert space.

[4] Let $X = L_2[a, b]$ with inner-product $(f, g) = \int_a^b f(t)g(\bar{t})dt$. Then X is a Hilbert space (by the Cauchy-Schwartz inequality and Theorem 4.1).

LEMMA 5.1. *In a Hilbert space,*

$$|(x, y)| \leq \|x\| \|y\|.$$

PROOF. Assume that x, y are non-zero (the result is clear if x or y is zero), and let $\lambda \in \mathbb{C}$. Then

$$\begin{aligned} 0 &\leq (x + \lambda y, x + \lambda y) \\ &= \|x\|^2 + |\lambda|^2 \|y\|^2 + \lambda(y, x) + \bar{\lambda}(x, y) \\ &= \|x\|^2 + |\lambda|^2 \|y\|^2 + 2\Re[\lambda(x, y)]. \end{aligned}$$

Let $\lambda = -re^{i\theta}$ for some $r > 0$, and choose θ such that $\theta = -\arg(x, y)$ if $(x, y) \neq 0$. Then

$$\|x\|^2 + r^2 \|y\|^2 \geq 2r|(x, y)|.$$

Take $r = \|x\|/\|y\|$ to obtain the result. \square

LEMMA 5.2. *The function defined by $\|x\| = (x, x)^{1/2}$ is a norm on a Hilbert space.*

PROOF. All the properties are clear except the triangle inequality. Since

$$(x, y) + (y, x) = 2\Re(x, y) \leq 2\|x\| \|y\|,$$

we have

$$\begin{aligned} \|x + y\|^2 &= \|x\|^2 + \|y\|^2 + (x, y) + (y, x) \\ &\leq \|x\|^2 + \|y\|^2 + 2\|x\| \|y\| = (\|x\| + \|y\|)^2, \end{aligned}$$

so $\|x + y\| \leq \|x\| + \|y\|$. \square

LEMMA 5.3. *The norm on a Hilbert space is strictly convex (cf. Definition 1.7).*

PROOF. From the proof of Lemma 5.1, if $|(x, y)| = \|x\| \|y\|$, then $x = -\lambda y$. From the proof of Lemma 5.2 it follows that if $\|x\| + \|y\| = \|x + y\|$ and $y \neq 0$ then $x = -\lambda y$. Hence if $\|x\| = \|y\| = 1$ and $\|x + y\| = 2$, then $|\lambda| = 1$ and $|1 - \lambda| = 2$, so $\lambda = -1$ and $x = y$. \square

Next there is the peculiar *parallelogram law*.

THEOREM 5.1. *If H is a Hilbert space, then*

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad (43)$$

for all $x, y \in H$.

Conversely, if H is a complex Banach space with norm $\|\cdot\|$ satisfying (43), then H is a Hilbert space with scalar product (\cdot, \cdot) satisfying $\|x\| = (x, x)^{1/2}$.

PROOF. The forward direction is easy: simply expand the expression

$$(x + y, x + y) + (x - y, x - y).$$

For the reverse direction, define

$$(x, y) = \frac{1}{4} ([\|x + y\|^2 - \|x - y\|^2] + i [\|x + iy\|^2 - \|x - iy\|^2]) \quad (44)$$

(in the real case, with the second expression simply omitted). Since

$$(x, x) = \|x\|^2 + \frac{i}{4} \|x\|^2 |1 + i|^2 - \frac{i}{4} \|x\|^2 |1 - i|^2 = \|x\|^2,$$

the inner-product norm $(x, x)^{1/2}$ coincides with the norm $\|x\|$.

To prove that (\cdot, \cdot) satisfies condition (ii) in Definition 5.1, use (43) to show that

$$\begin{aligned} \|u + v + w\|^2 + \|u + v - w\|^2 &= 2\|u + v\|^2 + 2\|w\|^2, \\ \|u - v + w\|^2 + \|u - v - w\|^2 &= 2\|u - v\|^2 + 2\|w\|^2. \end{aligned}$$

It follows that

$$\begin{aligned} (\|u + v - w\|^2 - \|u - v + w\|^2) &+ (\|u + v - w\|^2 - \|u - v - w\|^2) \\ &= 2\|u + v\|^2 - 2\|u - v\|^2, \end{aligned}$$

showing that

$$\Re(u + w, v) + \Re(u - w, v) = 2\Re(u, v).$$

A similar argument shows that

$$\Im(u + w, v) + \Im(u - w, v) = 2\Im(u, v),$$

so

$$(u + w, v) + (u - w, v) = 2(u, v).$$

Taking $w = u$ shows that $(2u, v) = 2(u, v)$. Taking $u + w = x$, $u - w = y$, $v = z$ then gives

$$(x, z) + (y, z) = 2\left(\frac{x + y}{2}, z\right) = (x + y, z).$$

To prove condition (iii) in Definition 5.1, use (ii) to show that

$$\begin{aligned} (mx, y) &= ((m - 1)x + x, y) = ((m - 1)x, y) + (x, y) \\ &= ((m - 2)x, y) + 2(x, y) \\ &= \dots \\ &= m(x, y). \end{aligned}$$

The same argument in reverse shows that $n(x/n, y) = (x, y)$, so $(x/n, y) = (1/n)(x, y)$. If $r = m/n$ ($m, n \in \mathbb{N}$) then

$$r(x, y) = \frac{m}{n}(x, y) = m\left(\frac{x}{n}, y\right) = \left(\frac{m}{n}x, y\right) = (rx, y).$$

Now (x, y) is a continuous function in x (by (44)); we deduce that $\lambda(x, y) = (\lambda x, y)$ for all $\lambda > 0$. For $\lambda < 0$,

$$\begin{aligned} \lambda(x, y) - (\lambda x, y) &= \lambda(x, y) - (|\lambda|(-x), y) = \lambda(x, y) - |\lambda|(-x, y) \\ &= \lambda(x, y) + \lambda(-x, y) = \lambda(0, y) = 0, \end{aligned}$$

so (iii) holds for all $\lambda \in \mathbb{R}$. For $\lambda = i$, (iii) is clear, so if $\lambda = \mu + i\nu$,

$$\begin{aligned}\lambda(x, y) &= \mu(x, y) + i\nu(x, y) = (\mu x, y) + i(\nu x, y) \\ &= (\mu x, y) + (i\nu x, y) = (\lambda x, y).\end{aligned}$$

Condition (iv) is clear, and (v) follows from the assumption that H is Banach space. \square

2. Projection theorem

Let H be a Hilbert space. A point $x \in H$ is *orthogonal* to a point $y \in H$, written $x \perp y$, if $(x, y) = 0$. For sets N, M in H , x is *orthogonal to N* , written $x \perp N$, if $(x, y) = 0$ for all $y \in N$. The sets N and M are orthogonal (written $N \perp M$) if $x \perp M$ for all $x \in N$. The *orthogonal complement* of M is defined as

$$M^\perp = \{x \in H \mid x \perp M\}.$$

Notice that for any M , M^\perp is a closed linear subspace of H .

LEMMA 5.4. *Let M be a closed convex set in a Hilbert space H . For every point $x_0 \in H$ there is a unique point $y_0 \in M$ such that*

$$\|x_0 - y_0\| = \inf_{y \in M} \|x_0 - y\|. \quad (45)$$

That is, it makes sense in a Hilbert space to talk about the point in a closed convex set that is “closest” to a given point.

PROOF. Let $d = \inf_{y \in M} \|x_0 - y\|$ and choose a sequence (y_n) in M such that $\|x_0 - y_n\| \rightarrow d$ as $n \rightarrow \infty$. By the parallelogram law (43),

$$\begin{aligned}4\|x_0 - \tfrac{1}{2}(y_m + y_n)\|^2 + \|y_m - y_n\|^2 &= 2\|x_0 - y_m\|^2 + 2\|x_0 - y_n\|^2 \\ &\rightarrow 4d^2\end{aligned}$$

as $m, n \rightarrow \infty$. By convexity (Definition 1.6), $\frac{1}{2}(y_m + y_n) \in M$, so

$$4\|x_0 - \tfrac{1}{2}(y_m + y_n)\|^2 \geq 4d^2.$$

It follows that $\|y_m - y_n\| \rightarrow 0$ as $m, n \rightarrow \infty$. Now H is complete and M is a closed subset, so $\lim_{n \rightarrow \infty} y_n = y_0$ exists and lies in M . Now $\|x_0 - y_0\| = \lim_{n \rightarrow \infty} \|x_0 - y_n\| = d$, showing (45).

It remains to check that the point y_0 is the only point with property (45). Let y_1 be another point in M with

$$\|x_0 - y_1\| = \inf_{y \in M} \|x_0 - y\|.$$

Then

$$\begin{aligned}2\left\|x_0 - \frac{y_0 + y_1}{2}\right\| &\leq \|x_0 - y_0\| + \|x_0 - y_1\| \\ &\leq 2 \inf_{y \in M} \|x_0 - y\| \leq 2\left\|x_0 - \frac{y_0 + y_1}{2}\right\|,\end{aligned}$$

since $(y_0 + y_1)/2$ lies in M . It follows that

$$2\left\|x_0 - \frac{y_0 + y_1}{2}\right\| = \|x_0 - y_0\| + \|x_0 - y_1\|.$$

Since the Hilbert norm is strictly convex (Lemma 5.3), we deduce that $x_0 - y_0 = x_0 - y_1$, so $y_1 = y_0$. \square

This gives us the Orthogonal Projection Theorem.

THEOREM 5.2. *Let M be a closed linear subspace of a Hilbert space H . Then any $x_0 \in H$ can be written $x_0 = y_0 + z_0$, $y_0 \in M$, $z_0 \in M^\perp$. The elements y_0, z_0 are determined uniquely by x_0 .*

PROOF. If $x_0 \in M$ then $y_0 = x_0$ and $z_0 = 0$. If $x_0 \notin M$, then let y_0 be the point in M with

$$\|x_0 - y_0\| = \inf_{y \in M} \|x_0 - y\|$$

(this point exists by Lemma 5.4). Now for any $y \in M$ and $\lambda \in \mathbb{C}$, $y_0 + \lambda y \in M$ so

$$\|x_0 - y_0\|^2 \leq \|x_0 - y_0 - \lambda y\|^2 = \|x_0 - y_0\|^2 - 2\Re\lambda(y, x_0 - y_0) + |\lambda|^2\|y\|^2.$$

Hence

$$-2\Re\lambda(y, x_0 - y_0) + |\lambda|^2\|y\|^2 \geq 0.$$

Assume now that $\lambda = \epsilon > 0$ and divide by ϵ . As $\epsilon \rightarrow 0$ we deduce that

$$\Re(y, x_0 - y_0) \leq 0. \quad (46)$$

Assume next that $\lambda = -i\epsilon$ and divide by ϵ . As $\epsilon \rightarrow 0$, we get

$$\Im(y, x_0 - y_0) \leq 0. \quad (47)$$

Exactly the same argument may be applied to $-y$ since $-y \in M$, showing that (46) and (47) hold with y replaced by $-y$. Thus $(y, x_0 - y_0) = 0$ for all $y \in M$. It follows that the point $z_0 = x_0 - y_0$ lies in M^\perp .

Finally, we check that the decomposition is unique. Suppose that $x_0 = y_1 + z_1$ with $y_1 \in M$ and $z_1 \in M^\perp$. Then $y_0 - y_1 = z_1 - z_0 \in M \cap M^\perp = \{0\}$. \square

COROLLARY 5.1. *If M is a closed linear subspace and $M \neq H$, then there exists an element $z_0 \neq 0$ such that $z_0 \perp M$.*

PROOF. Apply the projection theorem (Theorem 5.2) to any $x_0 \in H \setminus M$. \square

It follows that all linear functionals on a Hilbert space are given by taking inner products – the *Riesz theorem*.

THEOREM 5.3. *For every bounded linear functional x^* on a Hilbert space H there exists a unique element $z \in H$ such that $x^*(x) = (x, z)$ for all $x \in H$. The norm of the functional is given by $\|x^*\| = \|z\|$.*

PROOF. Let N be the null space of x^* ; N is a closed linear subspace of H . If $N = H$, then $x^* = 0$ and we may take $x^*(x) = (x, 0)$. If $N \neq H$, then by Corollary 5.1 there is a point $z_0 \in N^\perp$, $z_0 \neq 0$. By construction, $\alpha = x^*(z_0) \neq 0$. For any $x \in H$, the point $x - x^*(x)z_0/\alpha$ lies in N , so

$$(x - x^*(x)z_0/\alpha, z_0) = 0.$$

It follows that

$$x^*(x) \left(\frac{z_0}{\alpha}, z_0 \right) = (x, z_0).$$

If we substitute $z = \frac{\bar{\alpha}}{(z_0, z_0)} z_0$, we get $x^*(x) = (x, z)$ for all $x \in H$.

To check uniqueness, assume that $x^*(x) = (x, z')$ for all $x \in H$. Then $(x, z - z') = 0$ for all $x \in H$, so (taking $x = z - z'$), $\|z - z'\| = 0$ and therefore $z = z'$.

Finally,

$$\|x^*\| = \sup_{\|x\|=1} |x^*(x)| = \sup_{\|x\|=1} |(x, z)| \leq \sup_{\|x\|=1} (\|x\| \|z\|) = \|z\|.$$

On the other hand,

$$\|z\|^2 = (z, z) = |x^*(z)| \leq \|x^*\| \|z\|,$$

so $\|z\| \leq \|x^*\|$. \square

COROLLARY 5.2. *If H is a Hilbert space, then the space H^* is also a Hilbert space. The map $\sigma : H \rightarrow H^*$ given by $(\sigma x)(y) = (y, x)$ is an isometric embedding of H onto H^* .*

DEFINITION 5.2. Let M and N be linear subspaces of a Hilbert space H . If every element in the linear space $M + N$ has a unique representation in the form $x + y$, $x \in M$, $y \in N$, then we say $M + N$ is a *direct sum*. If $M \perp N$, then we write $M \oplus N$ – and this sum is automatically a direct one. If $Y = M \oplus N$, then we also write $N = Y \ominus M$ and call N the *orthogonal complement* of M in Y .

Notice that the projection theorem says that if M is a closed linear space in H , then $H = M \oplus M^\perp$.

3. Projection and self-adjoint operators

DEFINITION 5.3. Let M be a closed linear subspace of the Hilbert space H . By the projection theorem, every $x \in H$ can be written uniquely in the form $x = y + z$ with $y \in M$, $z \in M^\perp$. Call y the *projection* of x in M , and the operator $P = P_M$ defined by $Px = y$ is the *projection on M* . The space M is called the *subspace of the projection P* .

DEFINITION 5.4. Let $T : H \rightarrow H$ be a bounded linear operator. The *adjoint* T^* of T is defined by the relation $(Tx, y) = (x, T^*y)$ for all $x, y \in H$. An operator with $T = T^*$ is called *self-adjoint*.

Notice that if T is self-adjoint, then for every $x \in H$, $(Tx, x) \in \mathbb{R}$.

EXERCISE 5.1. Let T and S be bounded linear operators in Hilbert space H , and $\lambda \in \mathbb{C}$. Prove the following: $(T + S)^* = T^* + S^*$; $(TS)^* = S^*T^*$; $(\lambda T)^* = \bar{\lambda}T^*$; $I^* = I$; $T^{**} = T$; $\|T^*\| = \|T\|$. If T^{-1} is also a bounded linear operator with domain H , then $(T^*)^{-1}$ is a bounded linear map with domain H and $(T^{-1})^* = (T^*)^{-1}$.

THEOREM 5.4. [1] *If P is a projection, then P is self-adjoint, $P^2 = P$ and $\|P\| = 1$ if $P \neq 0$.*

[2] *If P is a self-adjoint operator with $P^2 = P$, then P is a projection.*

PROOF. [1] Let $P = P_M$, and $x_i = y_i + z_i$ for $i = 1, 2$ where $y_i \in M$ and $z_i \in M^\perp$. Then $\lambda_1 x_1 + \lambda_2 x_2 = (\lambda_1 y_1 + \lambda_2 y_2) + (\lambda_1 z_1 + \lambda_2 z_2)$ and

$$(\lambda_1 y_1 + \lambda_2 y_2) \in M, \quad (\lambda_1 z_1 + \lambda_2 z_2) \in M^\perp.$$

It follows that P is linear. To see that $P^2 = P$, notice that $P^2 x_1 = P(Px_1) = P(y_1) = y_1 = Px_1$ since $y_1 \in M$. Notice that $\|x_1\|^2 = \|y_1\|^2 + \|z_1\|^2 \geq \|y_1\|^2 = \|Px_1\|^2$ so $\|P\| \leq 1$. If $P \neq 0$ then for any $x \in M \setminus \{0\}$ we have $Px = x$ so $\|P\| \geq 1$.

Self-adjointness is clear:

$$(Px_1, x_2) = (y_1, x_2) = (y_1, y_2) = (x_1, y_2) = (x_1, Px_2).$$

[2] Let $M = P(H)$; then M is a linear subspace of H . If $y_n = P(x_n)$, with $y_n \rightarrow z$, then $Py_n = P^2 x_n = Px_n = Py_n$, so $z = \lim_n y_n = \lim_n Py_n = Pz \in M$ so M is closed. Since P is self-adjoint and $P^2 = P$,

$$(x - Px, Py) = (Px - P^2 x, y) = 0$$

for all $y \in H$ so $x - Px \in M^\perp$. This means that $x = Px + (x - Px)$ is the unique decomposition of x as a sum $y + z$ with $y \in M$ and $z \in M^\perp$. That is, P is the projection P_M . \square

We collect all the elementary properties of projections into the next theorem.

Projections P_1 and P_2 are *orthogonal* if $P_1P_2 = 0$. Since projections are self-adjoint, $P_1P_2 = 0$ if and only if $P_2P_1 = 0$.

The projection P_L is *part of* the projection P_M if and only if $L \subset M$.

- THEOREM 5.5. [1] *Projections P_M and P_N are orthogonal if and only if $M \perp N$.*
 [2] *The sum of two projections P_M and P_N is a projection if and only if $P_MP_N = 0$. In that case, $P_M + P_N = P_{M \oplus N}$.*
 [3] *The product of two projections P_M and P_N is another projection if and only if $P_MP_N = P_NP_M$. In that case, $P_MP_N = P_{M \cap N}$.*
 [4] *P_L is part of $P_M \iff P_MP_L = P_L \iff P_LP_M = P_L \iff \|P_Lx\| \leq \|P_Mx\| \forall x \in H$.*
 [5] *If P is a projection, then $I - P$ is a projection.*
 [6] *More generally, $P = P_M - P_L$ is a projection if and only if P_L is a part of P_M . If so, then $P = P_{M \ominus L}$.*

PROOF. [1] Let $P_MP_N = 0$ and $x \in M, y \in N$. Then

$$(x, y) = (P_Mx, P_Ny) = (P_NP_Mx, y) = 0,$$

so $M \perp N$. Conversely, if $M \perp N$ then for any $x \in H$, $P_Nx \perp M$ so $P_M(P_Nx) = 0$.

[2] If $P = P_M + P_N$ is a projection, then $P^2 = P$, so $P_MP_N + P_NP_M = 0$. Hence

$$P_MP_N + P_MP_NP_M = 0,$$

after multiplying by P_M on the left. Multiplying on the right by P_M then gives $2P_MP_NP_M = 0$ so $P_MP_N = 0$.

Conversely, if $P_MP_N = 0$ then $P_NP_M = 0$ also, so $P^2 = P$. Since P is self-adjoint, it is a projection.

Finally, it is clear that $(P_M + P_N)(H) = M \oplus N$ so $P = P_{M \oplus N}$.

[3] If $P = P_MP_N$ is a projection, then $P^* = P$, so $P_MP_N = (P_MP_N)^* = P_N^*P_M^* = P_NP_M$.

Conversely, let $P_MP_N = P_NP_M = P$. Then $P^* = P$, so P is self-adjoint. Also $P^2 = P_MP_NP_MP_N = P_M^2P_N^2 = P_MP_N = P$, so P is a projection. Moreover, $Px = P_M(P_Nx) = P_N(P_Mx)$ so $Px \in M \cap N$. On the other hand, if $x \in M \cap N$ then $Px = P_M(P_Nx) = P_Mx = x$ so $P = P_{M \cap N}$.

[4] Assume that P_L is part of P_M , so $L \subset M$. Then $P_Lx \in M$ for all $x \in H$. Hence $P_MP_Lx = P_Lx$, and $P_MP_L = P_L$.

If $P_MP_L = P_L$, then

$$P_L = P_L^* = (P_MP_L)^* = P_L^*P_M^* = P_LP_M,$$

so $P_LP_M = P_L$.

If $P_LP_M = P_L$, then for any $x \in H$,

$$\|P_Lx\| = \|P_LP_Mx\| \leq \|P_L\| \|P_Mx\| \leq \|P_Mx\|,$$

so that $\|P_Lx\| \leq \|P_Mx\|$.

Finally, assume that $\|P_Lx\| \leq \|P_Mx\|$. If there is a point $x_0 \in L \setminus M$ then let $x_0 = y_0 + z_0$, $y_0 \in M$, $z_0 \perp M$, and $z_0 \neq 0$. Then

$$\|P_Lx_0\|^2 = \|y_0\|^2 + \|z_0\|^2 > \|y_0\|^2 = \|P_Mx_0\|^2,$$

so there can be no such point. It follows that $L \subset M$ so P_L is a part of P_M .

[5] $I - P$ is self-adjoint, and $(I - P)^2 = I - P - P + P^2 = I - P$.

[6] If P is a projection, then by [5] so is $I - P = (I - P_M) + P_L$. Also by [5], $I - P_M$ is a projection, so by [2] we must have $(I - P_M)P_L = 0$. That is, $P_L = P_M P_L$. Hence, by [4], P_L is a part of P_M .

Conversely, if P_L is part of P_M , then $P_M - P_L$ and P_L are orthogonal. By [2], the subspace Y of $P_M - P_L$ must therefore satisfy $Y \oplus L = M$, so $Y = M \ominus L$. \square

4. Orthonormal sets

A subset K in a Hilbert space H is *orthonormal* if each element of K has norm 1, and if any two elements of K are orthogonal. An orthonormal set K is *complete* if $K^\perp = 0$.

THEOREM 5.6. *Let $\{x_n\}$ be an orthonormal sequence in H . Then for any $x \in H$,*

$$\sum_{n=1}^{\infty} |(x, x_n)|^2 \leq \|x\|^2. \quad (48)$$

The inequality (48) is *Bessel's inequality*. The scalar coefficients (x, x_n) are called the *Fourier coefficients* of x with respect to $\{x_n\}$.

PROOF. We have

$$\begin{aligned} \left\| x - \sum_{n=1}^m (x, x_n) x_n \right\|^2 &= \|x\|^2 - \left(x, \sum_{n=1}^m (x, x_n) x_n \right) \\ &\quad - \left(\sum_{n=1}^m (x, x_n) x_n, x \right) + \sum_{n=1}^m (x, x_n) (x_n, x), \end{aligned}$$

so

$$\left\| x - \sum_{n=1}^m (x, x_n) x_n \right\|^2 = \|x\|^2 - \sum_{n=1}^m |(x, x_n)|^2. \quad (49)$$

It follows that

$$\sum_{n=1}^m |(x, x_n)|^2 \leq \|x\|^2,$$

and Bessel's inequality follows by taking $m \rightarrow \infty$. \square

The next result shows that the Fourier series of Theorem 5.6 is the best possible approximation of fixed length.

THEOREM 5.7. *Let $\{x_n\}$ be an orthonormal sequence in a Hilbert space H and let $\{\lambda_n\}$ be any sequence of scalars. Then, for any $n \geq 1$,*

$$\left\| x - \sum_{n=1}^m \lambda_n x_n \right\| \geq \left\| x - \sum_{n=1}^m (x, x_n) x_n \right\|.$$

PROOF. Write $c_n = (x, x_n)$. Then

$$\begin{aligned} \left\| x - \sum_{n=1}^m \lambda_n x_n \right\|^2 &= \|x\|^2 - \sum_{n=1}^m \bar{\lambda}_n c_n - \sum_{n=1}^m \lambda_n \bar{c}_n + \sum_{n=1}^m |\lambda_n|^2 \\ &= \|x\|^2 - \sum_{n=1}^m |c_n|^2 + \sum_{n=1}^m |c_n - \lambda_n|^2 \\ &\geq \|x\|^2 - \sum_{n=1}^m |c_n|^2. \end{aligned}$$

Now apply equation (49). \square

THEOREM 5.8. *Let $\{x_n\}$ be an orthonormal sequence in a Hilbert space H , and let $\{\alpha_n\}$ be any sequence of scalars. Then the series $\sum \alpha_n x_n$ is convergent if and only if $\sum |\alpha_n|^2 < \infty$, and if so*

$$\left\| \sum_{n=1}^{\infty} \alpha_n x_n \right\| = \left(\sum_{n=1}^{\infty} |\alpha_n|^2 \right)^{1/2}. \quad (50)$$

Moreover, the sum $\sum \alpha_n x_n$ is independent of the order in which the terms are arranged.

PROOF. For $m > n$ we have (by orthonormality)

$$\left\| \sum_{j=n}^m \alpha_j x_j \right\|^2 = \sum_{j=n}^m |\alpha_j|^2. \quad (51)$$

Since H is complete, (51) shows the first part of the theorem. Take $n = 1$ and $m \rightarrow \infty$ in (51) to get (50)

Assume that $\sum |\alpha_j|^2 < \infty$ and let $z = \sum \alpha_{j_n} x_{j_n}$ be a rearrangement of the series $x = \sum \alpha_j x_j$. Then

$$\|x - z\|^2 = (x, x) + (z, z) - (x, z) - (z, x), \quad (52)$$

and $(x, x) = (z, z) = \sum |\alpha_j|^2$. Write

$$s_m = \sum_{j=1}^m \alpha_j x_j, \quad t_m = \sum_{n=1}^m \alpha_{j_n} x_{j_n}.$$

Then

$$(x, z) = \lim_m (s_m, t_m) = \sum_{j=1}^{\infty} |\alpha_j|^2.$$

Also, $(z, x) = (\bar{x}, z) = (x, z)$ so (52) shows that $\|x - z\|^2 = 0$ and hence $x = z$. \square

THEOREM 5.9. *Let K be any orthonormal set in a Hilbert space H , and for each $x \in H$ let $K_x = \{y \in K, (x, y) \neq 0\}$. Then:*

- (i) for any $x \in H$, K_x is countable;
- (ii) the sum $Ex = \sum_{y \in K_x} (x, y)y$ converges independently of the order in which the terms are arranged;
- (iii) E is the projection operator onto the closed linear space spanned by K .

PROOF. From Bessel's inequality (48), for any $\epsilon > 0$ there are no more than $\|x\|^2/\epsilon^2$ points y in K with $|(x, y)| > \epsilon$. Taking $\epsilon = \frac{1}{2}, \frac{1}{3}, \dots$ we see that K_x is countable for any x .

Bessel's inequality and Theorem 5.8 show (ii).

Let $\langle \bar{K} \rangle$ denote the closed linear subspace spanned by K . If $x \perp \langle \bar{K} \rangle$ then $Ex = 0$. If $x \in \langle \bar{K} \rangle$ then for any $\epsilon > 0$ there are scalars $\lambda_1, \dots, \lambda_n$ and elements $y_1, \dots, y_n \in K$ such that

$$\left\| x - \sum_{j=1}^n \lambda_j y_j \right\| < \epsilon.$$

Then, by Theorem 5.7,

$$\left\| x - \sum_{j=1}^n (x, y_j) y_j \right\| < \epsilon. \quad (53)$$

Without loss of generality, all of the y_j lie in K_x . Arrange the set K_x in a sequence $\{y_j\}$. From (49) notice that the left-hand side of (53) does not increase with n . Taking $n \rightarrow \infty$, we get $\|x - Ex\| < \epsilon$. Since $\epsilon > 0$ is arbitrary, we deduce that $Ex = x$ for all $x \in \langle \bar{K} \rangle$. This proves that $E = P_{\langle \bar{K} \rangle}$. \square

DEFINITION 5.5. A set K is an *orthonormal basis* of H if K is orthonormal and for every $x \in H$,

$$x = \sum_{y \in K_x} (x, y) y. \quad (54)$$

THEOREM 5.10. Let K be an orthonormal set in a Hilbert space H . Then the following properties are equivalent.

- (i) K is complete;
- (ii) $\langle \bar{K} \rangle = H$;
- (iii) K is an orthonormal basis for H ;
- (iv) for any $x \in H$, $\|x\|^2 = \sum_{y \in K_x} |(x, y)|^2$.

The equality in (iv) is called *Parseval's formula*.

PROOF. That (i) implies (ii) follows from Corollary 5.1. Assume (ii). Then by Theorem 5.9, $Ex = x$ for all $x \in H$, so K is an orthonormal basis. Now assume (iii). Arrange the elements of K_x in a sequence $\{x_n\}$, and take $n \rightarrow \infty$ in (49) to obtain Parseval's formula (iv). Finally, assume (iv). If $x \perp K$, then $\|x\|^2 = \sum |(x, y)|^2 = 0$, so $x = 0$. This means that (iv) implies (i). \square

THEOREM 5.11. Every Hilbert space has an orthonormal basis. Any orthonormal basis in a separable Hilbert space is countable.

EXAMPLE 5.2. Classical Fourier analysis comes about using the orthonormal basis $\{e^{2\pi i n t}\}_{n \in \mathbb{Z}}$ for $L_2[0, 1]$.

PROOF. Let H be a Hilbert space, and consider the classes of orthonormal sets in H with the partial order of inclusion. By Lemma A.1 there exists a maximal orthonormal set K . Since K is maximal, it is complete and is therefore an orthonormal basis.

Now let H be separable, and suppose that $\{x_\alpha\}$ is an uncountable orthonormal basis. Since, for any $\alpha \neq \beta$,

$$\|x_\alpha - x_\beta\|^2 = \|x_\alpha\|^2 + \|x_\beta\|^2 = 2,$$

the balls $B_{1/2}(x_\alpha)$ are mutually disjoint. If $\{y_n\}$ is a dense sequence in H , then there is a ball $B_{1/2}(x_{\alpha_0})$ that does not contain any of the points y_n . Hence x_{α_0} is not in the closure of $\{y_n\}$, a contradiction. \square

COROLLARY 5.3. *Any two infinite-dimensional separable Hilbert spaces are isometrically isomorphic.*

PROOF. Let H_1 and H_2 be two such spaces. By Theorem 5.11 there are sequences $\{x_n\}$ and $\{y_n\}$ that form orthonormal bases for H_1 and H_2 respectively. Given any points $x \in H_1$ and $y \in H_2$, we may write

$$x = \sum_{n=1}^{\infty} c_n x_n, \quad y = \sum_{n=1}^{\infty} d_n x_n, \quad (55)$$

where $c_n = (x, x_n)$ and $d_n = (y, y_n)$ for all $n \geq 1$. Define a map $T : H_1 \rightarrow H_2$ by $Tx = y$ if $c_n = d_n$ for all n in (55). It is clear that T is linear and it maps H_1 onto H_2 since the sequences (c_n) and (d_n) run through all of ℓ_2 . Also,

$$\|Tx\|^2 = \sum_{n=1}^{\infty} |d_n|^2 = \sum_{n=1}^{\infty} |c_n|^2 = \|x\|^2,$$

so T is an isometry. \square

5. Gram-Schmidt orthonormalization

Starting with any linearly independent set $\{x_1, x_2, \dots\}$ in a Hilbert space H , we can inductively construct an orthonormal set that spans the same subspace by the *Gram-Schmidt Orthonormalization process* (Theorem 5.12). The idea is simple: first, any vector v can be reduced to unit length simply by dividing by the length $\|v\|$. Second, if x_1 is a fixed unit vector and x_2 is another unit vector with $\{x_1, x_2\}$ linearly independent, then $x_2 - (x_2, x_1)x_1$ is a non-zero vector (since x_1 and x_2 are independent), is orthogonal to x_1 (since $(x_1, x_2 - (x_2, x_1)x_1) = (x_1, x_2) - (x_2, x_1)(x_1, x_1) = (x_1, x_2) - (x_2, x_1) = 0$), and $\{x_1, x_2 - (x_2, x_1)x_1\}$ spans the same space as $\{x_1, x_2\}$. This idea can be extended as follows – the notational complexity comes about because of the need to renormalize (make the new vector unit length).

We will only need this for sets whose linear span is dense.

THEOREM 5.12. *If $\{x_1, x_2, \dots\}$ is a linearly independent set whose linear span is dense in H , then the set $\{\phi_1, \phi_2, \dots\}$ defined below is an orthonormal basis for H :*

$$\begin{aligned} \phi_1 &= \frac{x_1}{\|x_1\|}, \\ \phi_2 &= \frac{x_2 - (x_2, \phi_1)\phi_1}{\|x_2 - (x_2, \phi_1)\phi_1\|}, \end{aligned}$$

and in general for any $n \geq 1$,

$$\phi_n = \frac{x_n - (x_n, \phi_1)\phi_1 - (x_n, \phi_2)\phi_2 - \dots - (x_n, \phi_{n-1})\phi_{n-1}}{\|x_n - (x_n, \phi_1)\phi_1 - (x_n, \phi_2)\phi_2 - \dots - (x_n, \phi_{n-1})\phi_{n-1}\|}.$$

The proof is obvious unless you try to write it down: the idea is that at each stage the piece of the next vector x_n that is *not* orthogonal to the space spanned by $\{x_1, \dots, x_{n-1}\}$ is subtracted. The vector ϕ_n so constructed cannot be zero by linear independence.

The most important situation in which this is used is to find orthonormal bases for certain weighted function spaces.

Given $a < b$, $a, b \in [-\infty, \infty]$ and a function $M : (a, b) \rightarrow (0, \infty)$ with the property that $\int_a^b t^n M(t) dt < \infty$ for all $n \geq 1$, define the Hilbert space $L_P^M[a, b]$ to be the linear space of measurable functions f with $\|f\|_M = (f, f)_M^{1/2} < \infty$ where

$$(f, g)_M = \int_a^b M(t) f(t) \bar{g}(t) dt.$$

It may be shown that the linearly independent set $\{1, t, t^2, t^3, \dots\}$ has a linear span dense in L_P^M . The Gram-Schmidt orthonormalization process may be applied to this set to produce various families of classical orthonormal functions.

EXAMPLE 5.3. [1] If $M(t) = 1$ for all t , $a = -1$, $b = 1$, then the process generates the *Legendre polynomials*.

[2] If $M(t) = \frac{1}{\sqrt{1-t^2}}$, $a = -1$, $b = 1$, then the process generates the *Tchebychev polynomials*.

[3] If $M(t) = t^{q-1}(1-t)^{p-q}$, $a = 0$, $b = 1$ (with $q > 0$ and $p - q > -1$), then the process generates the *Jacobi polynomials*.

[4] If $M(t) = e^{-t^2}$, $a = -\infty$, $b = \infty$, then the process generates the *Hermite polynomials*.

[5] If $M(t) = e^{-t}$, $a = 0$, $b = \infty$, then the process generates the *Laguerre polynomials*.

Fourier analysis

In the last chapter we saw some very general methods of “Fourier analysis” in Hilbert space. Of course the methods started with the classical setting on periodic complex-valued functions on the real line, and in this chapter we describe the elementary theory of classical Fourier analysis using summability kernels. The classical theory of Fourier series is a huge subject: the introduction below comes mostly from Katznelson¹ and from Körner²; both are highly recommended for further study.

1. Fourier series of L_1 functions

Denote by $L_1(\mathbb{T})$ the Banach space of complex-valued, Lebesgue integrable functions on $\mathbb{T} = [0, 2\pi)/0 \sim 2\pi$ (this just means periodic functions).

Modify the L_1 -norm on this space so that

$$\|f\|_1 = \frac{1}{2\pi} \int_0^{2\pi} |f(t)| dt.$$

What is going on here is simply this: to avoid writing “ 2π ” hundreds of times, we make the unit circle have “length” 2π . To recover the useful normalization that the L_1 -norm of the constant function 1 is 1, the usual L_1 -norm is divided by 2π .

Notice that the translate f_x of a function has the same norm, where $f_x(t) = f(t - x)$.

DEFINITION 6.1. A *trigonometric polynomial* on \mathbb{T} is an expression of the form

$$P(t) = \sum_{n=-N}^N a_n e^{int},$$

with $a_n \in \mathbb{C}$.

LEMMA 6.1. *The functions $\{e^{int}\}_{n \in \mathbb{Z}}$ are pairwise orthogonal in L_2 . That is,*

$$\frac{1}{2\pi} \int_0^{2\pi} e^{int} e^{-imt} dt = \frac{1}{2\pi} \int_0^{2\pi} e^{i(n-m)t} dt = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$$

It follows that if the function $P(t)$ is given, we can recover the coefficients a_n by computing

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} P(t) e^{-int} dt.$$

¹An *introduction to Harmonic Analysis*, Y. Katznelson, Dover Publications, New York (1976).

²*Fourier Analysis*, T. Körner, Cambridge University Press, Cambridge.

It will be useful later to write things like

$$P \sim \sum_{n=-N}^N a_n e^{int}$$

which means that P is identified with the *formal sum* on the right hand side. The expression $P(t) = \dots$ is a function defined by the value of the right hand side for each value of t .

DEFINITION 6.2. A *trigonometric series* on \mathbb{T} is an expression

$$S \sim \sum_{n=-\infty}^{\infty} a_n e^{int}. \quad (56)$$

The *conjugate* of S is the series

$$\tilde{S} \sim \sum_{n=-\infty}^{\infty} -i \operatorname{sign}(n) a_n e^{int} \quad (57)$$

where $\operatorname{sign}(n) = 0$ if $n = 0$ and $= n/|n|$ if not.

Notice that there is no assumption about convergence, so in general S is not related to a function at all.

DEFINITION 6.3. Let $f \in L_1(\mathbb{T})$. Define the n th (classical) *Fourier coefficient* of f to be

$$\hat{f}(n) = \frac{1}{2\pi} \int f(t) e^{-int} dt \quad (58)$$

(the integration is from 0 to 2π as usual). Associate to f the *Fourier series* $S[f]$, which is defined to be the formal trigonometric series

$$S[f] \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{int}. \quad (59)$$

We say that a given trigonometric series (56) is a Fourier series if it is of the form (59) for some $f \in L_1(\mathbb{T})$.

THEOREM 6.1. Let $f, g \in L_1(\mathbb{T})$. Then

[1] $\widehat{(f+g)}(n) = \hat{f}(n) + \hat{g}(n)$.

[2] For $\lambda \in \mathbb{C}$, $\widehat{(\lambda f)}(n) = \lambda \hat{f}(n)$.

[3] If $\bar{f}(t) = \overline{f(t)}$ is the complex conjugate of f then $\widehat{\bar{f}}(n) = \overline{\hat{f}(-n)}$.

[4] If $f_x(t) = f(t-x)$ is the translate of f , then $\hat{f}_x(n) = e^{-inx} \hat{f}(n)$.

[5] $|\hat{f}(n)| \leq \frac{1}{2\pi} \int |f(t)| dt = \|f\|_1$.

Prove these as an exercise.

Notice that $f \mapsto \hat{f}$ sends a function in $L_1(\mathbb{T})$ to a function in $C(\mathbb{Z})$, the continuous functions on \mathbb{Z} with the sup norm. This map is continuous in the following sense.

COROLLARY 6.1. Assume (f_j) is a sequence in $L_1(\mathbb{T})$ with $\|f_j - f\|_1 \rightarrow 0$. Then $\hat{f}_j \rightarrow \hat{f}$ uniformly.

PROOF. This follows at once from Theorem 6.1[5]. \square

THEOREM 6.2. Let $f \in L_1(\mathbb{T})$ have $\hat{f}(0) = 0$. Define

$$F(t) = \int_0^t f(s) ds.$$

Then F is continuous, 2π periodic, and

$$\hat{F}(n) = \frac{1}{in} \hat{f}(n)$$

for all $n \neq 0$.

PROOF. It is clear that F is continuous since it is the integral of an L_1 function. Also,

$$F(t + 2\pi) - F(t) = \int_t^{t+2\pi} f(s) ds = 2\pi \hat{f}(0) = 0.$$

Finally, using integration by parts

$$\hat{F}(n) = \frac{1}{2\pi} \int_0^{2\pi} F(t) e^{-int} dt = -\frac{1}{2\pi} \int_0^{2\pi} F'(t) \frac{-1}{in} e^{-int} dt = \frac{1}{in} \hat{f}(n).$$

□

Notice that we have used the symbol F' – the function F is differentiable because of the way it was defined.

2. Convolution in L_1

In this section we introduce a form of “multiplication” on $L_1(\mathbb{T})$ that makes it into a Banach algebra (see Definition 3.3). Notice that the only real properties we will use is that the circle \mathbb{T} is a group on which the measure ds is translation invariant:

$$\int f_x(s) ds = \int f ds.$$

THEOREM 6.3. Assume that f, g are in $L_1(\mathbb{T})$. Then, for almost every s , the function $f(t - s)g(s)$ is integrable as a function of s . Define the convolution of f and g to be

$$(F * g)(t) = \frac{1}{2\pi} \int f(t - s)g(s) ds. \quad (60)$$

Then $f * g \in L_1(\mathbb{T})$, with norm

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$

Moreover

$$\widehat{(f * g)}(n) = \hat{f}(n) \hat{g}(n).$$

PROOF. It is clear that $F(t, s) = f(t - s)g(s)$ is a measurable function of the variable (s, t) . For almost all s , $F(t, s)$ is a constant multiple of f_s , so is integrable. Moreover

$$\frac{1}{2\pi} \int \left(\frac{1}{2\pi} \int |F(t, s)| dt \right) ds = \frac{1}{2\pi} \int |g(s)| \|f\|_1 ds = \|f\|_1 \|g\|_1.$$

So, by Fubini's Theorem 4.4, $f(t - s)g(s)$ is integrable as a function of s for almost all t , and

$$\frac{1}{2\pi} \int |(f * g)(t)| dt = \frac{1}{2\pi} \int \left| \frac{1}{2\pi} \int F(t, s) ds \right| dt \leq \frac{1}{4\pi^2} \int \int |F(t, s)| dt ds = \|f\|_1 \|g\|_1,$$

showing that $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$. Finally, using Fubini again to justify a change in the order of integration,

$$\begin{aligned} \widehat{(f * g)}(n) &= \frac{1}{2\pi} \int (f * g)(t) e^{-int} dt = \frac{1}{4\pi^2} \int \int f(t-s) e^{-in(t-s)} g(s) dt ds \\ &= \frac{1}{2\pi} \int f(t) e^{-int} dt \cdot \frac{1}{2\pi} \int g(s) e^{-ins} ds \\ &= \hat{f}(n) \hat{g}(n). \end{aligned}$$

□

LEMMA 6.2. *The operation $(f, g) \mapsto f * g$ is commutative, associative, and distributive over addition.*

Prove this as an exercise.

LEMMA 6.3. *If $f \in L_1(\mathbb{T})$ and $k(t) = \sum_{n=-N}^N a_n e^{int}$ then*

$$(k * f)(t) = \sum_{n=-N}^N a_n \hat{f}(n) e^{int}.$$

Thus convolving with the function e^{int} picks out the n th Fourier coefficient.

PROOF. Simply check this one term at a time: if $\chi_n(t) = e^{int}$, then

$$(\chi_n * f)(t) = \frac{1}{2\pi} \int e^{in(t-s)} f(s) ds = e^{int} \frac{1}{2\pi} \int f(s) e^{-ins} ds.$$

□

3. Summability kernels and homogeneous Banach algebras

Two properties of the Banach space $L_1(\mathbb{T})$ are particularly important for Fourier analysis.

THEOREM 6.4. *If $f \in L_1(\mathbb{T})$ and $x \in \mathbb{T}$, then*

$$f_x(t) = f(t-x) \in L_1(\mathbb{T}) \text{ and } \|f_x\|_1 = \|f\|_1.$$

Also, the function $x \mapsto f_x$ is continuous on \mathbb{T} for each $f \in L_1(\mathbb{T})$.

PROOF. The translation invariance is clear.

In order to prove the continuity we must show that

$$\lim_{x \rightarrow x_0} \|f_x - f_{x_0}\|_1 = 0. \quad (61)$$

Now (61) is clear if f is continuous. On the other hand, the continuous functions are dense in $L_1(\mathbb{T})$, so given $f \in L_1(\mathbb{T})$ and $\epsilon > 0$ we may choose $g \in C(\mathbb{T})$ such that

$$\|g - f\|_1 < \epsilon.$$

Then

$$\begin{aligned} \|f_x - f_{x_0}\|_1 &\leq \|f_x - g_x\|_1 + \|g_x - f_{x_0}\|_1 + \|g_{x_0} - f_{x_0}\|_1 \\ &= \|(f-g)_x\|_1 + \|g_x - g_{x_0}\|_1 + \|(g-f)_{x_0}\|_1 \\ &< 2\epsilon + \|g_x - g_{x_0}\|_1. \end{aligned}$$

It follows that

$$\limsup_{x \rightarrow x_0} \|f_x - f_{x_0}\|_1 < 2\epsilon,$$

so the theorem is proved. \square

DEFINITION 6.4. A *summability kernel* is a sequence (k_n) of continuous 2π -periodic functions with the following properties:

$$\frac{1}{2\pi} \int k_n(t) dt = 1 \text{ for all } n. \quad (62)$$

$$\text{There is an } R \text{ such that } \frac{1}{2\pi} \int |k_n(t)| dt \leq R \text{ for all } n. \quad (63)$$

$$\text{For all } \delta > 0, \lim_{n \rightarrow \infty} \int_{\delta}^{2\pi-\delta} |k_n(t)| dt = 0. \quad (64)$$

If in addition $k_n(t) \geq 0$ for all n and t then (k_n) is called a *positive summability kernel*.

THEOREM 6.5. *Let $f \in L_1(\mathbb{T})$ and let (k_n) be a summability kernel. Then*

$$f = \lim_{n \rightarrow \infty} \frac{1}{2\pi} \int k_n(s) f_s ds$$

in the L_1 norm.

PROOF. Write $\phi(s) = f_s(t) = f(t-s)$ for fixed t . By Theorem 6.4 ϕ is a continuous $L_1(\mathbb{T})$ -valued function on \mathbb{T} , and $\phi(0) = f$. We will be integrating $L_1(\mathbb{T})$ -valued functions – see the Appendix for a brief definition of what this means.

Then for any $0 < \delta < \pi$, by (62) we have

$$\begin{aligned} \frac{1}{2\pi} \int k_n(s) \phi(s) ds - \phi(0) &= \frac{1}{2\pi} \int k_n(s) (\phi(s) - \phi(0)) ds \\ &= \frac{1}{2\pi} \int_{-\delta}^{\delta} k_n(s) (\phi(s) - \phi(0)) ds \\ &\quad + \frac{1}{2\pi} \int_{\delta}^{2\pi-\delta} k_n(s) (\phi(s) - \phi(0)) ds. \end{aligned}$$

The two parts may be estimated separately:

$$\left\| \frac{1}{2\pi} \int_{-\delta}^{\delta} k_n(s) (\phi(s) - \phi(0)) ds \right\|_1 \leq \max_{|s| \leq \delta} \|\phi(s) - \phi(0)\|_1 \|k_n\|_1, \quad (65)$$

and

$$\left\| \frac{1}{2\pi} \int_{\delta}^{2\pi-\delta} k_n(s) (\phi(s) - \phi(0)) ds \right\|_1 \leq \max \|\phi(s) - \phi(0)\|_1 \frac{1}{2\pi} \int_{\delta}^{2\pi-\delta} |k_n(s)| ds. \quad (66)$$

Using (63) and the fact that ϕ is continuous at $s = 0$, given any $\epsilon > 0$ there is a $\delta > 0$ such that (65) is bounded by ϵ . With the same δ , (64) implies that (66) converges to 0 as $n \rightarrow \infty$, so that $\frac{1}{2\pi} \int k_n(s) \phi(s) ds - \phi(0)$ is bounded by 2ϵ for large n . \square

The integral appearing in Theorem 6.5 looks a bit like a convolution of $L_1(\mathbb{T})$ -valued functions. This is not a problem for us. Consider first the following lemma.

LEMMA 6.4. *Let k be a continuous function on \mathbb{T} , and $f \in L_1(\mathbb{T})$. Then*

$$\frac{1}{2\pi} \int k(s) f_s ds = k * f. \quad (67)$$

PROOF. Assume first that f is continuous on \mathbb{T} . Then, making the obvious definition for the integral,

$$\frac{1}{2\pi} \int k(s) f_s ds = \frac{1}{2\pi} \lim \sum_j (s_{j+1} - s_j) k(s_j) f_{s_j},$$

with the limit taken in the $L_1(\mathbb{T})$ norm as the partition of \mathbb{T} defined by $\{s_1, \dots, s_j, \dots\}$ becomes finer. On the other hand,

$$\frac{1}{2\pi} \lim \sum_j (s_{j+1} - s_j) k(s_j) f(t - s_j) = (k * f)(t)$$

uniformly, proving the lemma for continuous f .

For arbitrary $f \in L_1(\mathbb{T})$, fix $\epsilon > 0$ and choose a continuous function g with $\|f - g\|_1 < \epsilon$. Then

$$\frac{1}{2\pi} \int k(s) f_s ds - k * f = \frac{1}{2\pi} \int k(s) (f - g)_s ds + k * (g - f),$$

so

$$\left\| \frac{1}{2\pi} \int k(s) f_s ds - k * f \right\|_1 \leq 2\|k\|_1 \epsilon.$$

□

Lemma 6.4 means that Theorem 6.5 can be written in the form

$$f = \lim_{n \rightarrow \infty} k_n * f \text{ in } L_1. \quad (68)$$

4. Fejér's kernel

Define a sequence of functions

$$K_n(t) = \sum_{j=-n}^n \left(1 - \frac{|j|}{n+1}\right) e^{ijt}.$$

LEMMA 6.5. *The sequence (K_n) is a summability kernel.*

PROOF. Property (62) is clear.

Now notice that

$$\begin{aligned} & \left(-\frac{1}{4}e^{-it} + \frac{1}{2} - \frac{1}{4}e^{it}\right) \sum_{j=-n}^n \left(1 - \frac{|j|}{n+1}\right) e^{ijt} \\ &= \frac{1}{n+1} \left(-\frac{1}{4}e^{-i(n+1)t} + \frac{1}{2} - \frac{1}{4}e^{i(n+1)t}\right). \end{aligned}$$

On the other hand,

$$\sin^2 \frac{t}{2} = \frac{1}{2}(1 - \cos t) = -\frac{1}{4}e^{-it} + \frac{1}{2} - \frac{1}{4}e^{it},$$

so

$$K_n(t) = \frac{1}{n+1} \left\{ \frac{\sin \frac{n+1}{2}t}{\sin \frac{1}{2}t} \right\}^2. \quad (69)$$

Property (64) follows, and this also shows that $K_n(t) \geq 0$ for all n and t .

Prove property (63) as an exercise. □

The following graph is the Fejér kernel K_{11} .

DEFINITION 6.5. Write $\sigma_n(f) = K_n * f$.

Using Lemma 6.3, it follows that

$$\sigma_n(f)(t) = \sum_{j=-n}^n \left(1 - \frac{|j|}{n+1}\right) \hat{f}(j) e^{ijt}, \quad (70)$$

and (68) means that

$$\sigma_n(f) \rightarrow f$$

in the L_1 norm for every $f \in L_1(\mathbb{T})$. It follows at once that the trigonometric polynomials are dense in $L_1(\mathbb{T})$. The most important consequences are however more general statements about Fourier series.

THEOREM 6.6. *If $f, g \in L_1(\mathbb{T})$ have $\hat{f}(n) = \hat{g}(n)$ for all $n \in \mathbb{Z}$, then $f = g$.*

PROOF. It is enough to show that $\hat{f}(n) = 0$ for all n implies that $f = 0$. Using (70), we see that if $\hat{f}(n) = 0$ for all n , then $\sigma_n(f) = 0$ for all n ; since $\sigma_n(f) \rightarrow f$, it follows that $f = 0$. \square

COROLLARY 6.2. *The family of functions $\{e^{int}\}_{n \in \mathbb{Z}}$ form a complete orthonormal system in $L_2(\mathbb{T})$.*

PROOF. It is enough to notice that

$$(f, e^{int}) = \hat{f}(n).$$

Then for all $f \in L_2(\mathbb{T})$, the function f and its Fourier series have identical Fourier coefficients, so must agree. \square

We also find a very general statement about the decay of Fourier coefficients: the *Riemann–Lebesgue Lemma*.

THEOREM 6.7. *Let $f \in L_1(\mathbb{T})$. Then $\lim_{|n| \rightarrow \infty} \hat{f}(n) = 0$.*

PROOF. Fix an $\epsilon > 0$, and choose a trigonometric polynomial P with the property that

$$\|f - P\|_1 < \epsilon.$$

If $|n|$ exceeds the degree of P , then

$$|\hat{f}(n)| = |(\widehat{f - P})(n)| \leq \|f - P\|_1 < \epsilon.$$

□

Recall that for $f \in L_1(\mathbb{T})$, the Fourier series was defined (formally) to be

$$S[f] \sim \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{int},$$

and the n th partial sum corresponds to the function

$$S_n(f)(t) = \sum_{j=-n}^n \hat{f}(j)e^{ijt}. \quad (71)$$

Looking at equations (71) and (70), we see that $\sigma_n(f)$ is the arithmetic mean of $S_0(f), S_1(f), \dots, S_n(f)$:

$$\sigma_n(f) = \frac{1}{n+1} (S_0(f) + S_1(f) + \dots + S_n(f)). \quad (72)$$

It follows that if $S_n(f)$ converges in $L_1(\mathbb{T})$, then it must converge to the same thing as σ_n , that is to f (if this is not clear to you, look at Corollary 6.3 below).

The partial sums $S_n(f)$ also have a convolution form: using (70) we have that $S_n(f) = D_n * f$ where (D_n) is the *Dirichlet kernel* defined by

$$D_n(t) = \sum_{j=-n}^n e^{ijt} = \frac{\sin(n + \frac{1}{2})t}{\sin \frac{1}{2}t}.$$

Notice that (D_n) is not a summability kernel: it has property (62) but does not have (63) (as we saw in Lemma 3.3) nor does it have (64). This explains why the question of convergence for Fourier series is so much more subtle than the problem of summability. The following graph is the Dirichlet kernel D_{11} .

DEFINITION 6.6. The *de la Vallée Poussin kernel* is defined by

$$V_n(t) = 2K_{2n+1}(t) - K_n(t).$$

Properties (62), (63) and (64) are clear.

The next picture is the de la Vallée Poussin kernel with $n = 11$.

This kernel is useful because V_n is a polynomial of degree $2n+1$ with $\widehat{V}_n(j) = 1$ for $|j| \leq n+1$, so it may be used to construct approximations to a function f by trigonometric polynomials having the same Fourier coefficients as f for small frequencies.

5. Pointwise convergence

Recall that a sequence of elements (x_n) in a normed space $(X, \|\cdot\|)$ converges to x if $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$. If the space X is a space of complex-valued functions on some set Z (for example, $L_1(\mathbb{T})$, $C(\mathbb{T})$), then there is another notion of convergence: x_n converges to x *pointwise* if for every $z \in Z$, $x_n(z) \rightarrow x(z)$ as a sequence of complex numbers. The question addressed in this section is the following: does the Fourier series of a function converge pointwise to the original function?

In the last section, we showed that for L_1 functions on the circle, $\sigma_n(f)$ converges to f with respect to the norm of any homogeneous Banach algebra containing f . Applying this to the Banach algebra of continuous functions with the sup norm, we have that $\sigma_n(f) \rightarrow f$ uniformly for all $f \in C(\mathbb{T})$.

If the function f is not continuous on \mathbb{T} , then the convergence in norm of $\sigma_n(f)$ does not tell us anything about the pointwise convergence. In addition, if $\sigma_n(f, t)$ converges for some t , there is no real reason for the limit to be $f(t)$.

THEOREM 6.8. *Let f be a function in $L_1(\mathbb{T})$.*

(a) *If*

$$\lim_{h \rightarrow 0} (f(t+h) + f(t-h))$$

exists (the possibility that the limit is $\pm\infty$ is allowed), then

$$\sigma_n(f, t) \longrightarrow \frac{1}{2} \lim_{h \rightarrow 0} (f(t+h) + f(t-h)).$$

(b) *If f is continuous at t , then $\sigma_n(f, t) \longrightarrow f(t)$.*

(c) *If there is a closed interval $I \subset \mathbb{T}$ on which f is continuous, then $\sigma_n(f, \cdot)$ converges uniformly to f on I .*

COROLLARY 6.3. *If f is continuous at t , and if the Fourier series of f converges at t , then it must converge to $f(t)$.*

PROOF. Recall equation (72):

$$\sigma_n(f) = \frac{1}{n+1} (S_0(f) + S_1(f) + \cdots + S_n(f)).$$

By assumption and (b), $\sigma_n(f, t) \rightarrow f(t)$ and $S_n(f, t) \rightarrow S(t)$ say. Write the right hand side as

$$\frac{1}{n+1} (S_0(t) + S_1(t) + \cdots + S_{\sqrt{n}}(t)) + \frac{1}{n+1} (S_{\sqrt{n}+1}(t) + \cdots + S_n(t)).$$

The first term converges to zero as $n \rightarrow \infty$ (since the convergent sequence $(S_n(t))$ is bounded). For the second term, choose and fix ϵ and choose n so large that

$$|S_k(t) - S(t)| < \epsilon$$

for all $k \geq \sqrt{n}$. Then the whole second term is within $\left(\frac{n-\sqrt{n}}{n+1}\right)\epsilon$ of $S(t)$. It follows that

$$\frac{1}{n+1} (S_0(f) + S_1(f) + \cdots + S_n(f)) \rightarrow S(t)$$

as $n \rightarrow \infty$, so $S(t)$ must coincide with $\lim_{n \rightarrow \infty} \sigma_n(f, t) = f(t)$. \square

Turning to the proof of Theorem 6.8, recall that the Fejér kernel (K_n) (see Lemma 6.5) is a positive summability kernel with the following properties:

$$\lim_{n \rightarrow \infty} \left(\sup_{\theta < t < 2\pi - \theta} K_n(t) \right) = 0 \text{ for any } \theta \in (0, \pi), \quad (73)$$

and

$$K_n(t) = K_n(-t). \quad (74)$$

PROOF OF THEOREM 6.8. Define

$$\check{f}(t) = \lim_{h \rightarrow 0} \frac{1}{2} (f(t+h) + f(t-h)),$$

and assume that this limit is finite (a similar argument works for the infinite cases). We wish to show that $\sigma_n(f, t) - \check{f}(t)$ is small for large n . Evaluate the difference,

$$\begin{aligned} \sigma_n(f, t) - \check{f}(t) &= \frac{1}{2\pi} \int_{\mathbb{T}} K_n(\tau) (f(t-\tau) - \check{f}(t)) d\tau \\ &= \frac{1}{2\pi} \int_{-\theta}^{\theta} K_n(\tau) (f(t-\tau) - \check{f}(t)) d\tau \\ &\quad + \int_{\theta}^{2\pi-\theta} K_n(\tau) (f(t-\tau) - \check{f}(t)) d\tau. \end{aligned}$$

Applying (74) this may be written

$$\sigma_n(f, t) - \check{f}(t) = \frac{1}{\pi} \left(\int_0^{\theta} + \int_{\theta}^{\pi} \right) K_n(\tau) \left(\frac{f(t-\tau) + f(t+\tau)}{2} - \check{f}(t) \right) d\tau. \quad (75)$$

Fix $\epsilon > 0$, and choose $\theta \in (0, \pi)$ small enough to ensure that

$$\tau \in (-\theta, \theta) \implies \left| \frac{f(t-\tau) + f(t+\tau)}{2} - \check{f}(t) \right| < \epsilon, \quad (76)$$

and choose N large enough to ensure that

$$n > N \implies \sup_{\theta < \tau < 2\pi - \theta} K_n(\tau) < \epsilon. \quad (77)$$

Putting the estimates (76) and (77) into the expression (75) gives

$$|\sigma_n(f, t) - \check{f}(t)| < \epsilon + \epsilon \|f - \check{f}(t)\|_1,$$

which proves (a).

Part (b) follows at once from (a).

For (c), notice³ that f must be uniformly continuous on I . This means that (given $\epsilon > 0$) θ can be chosen so that (76) holds for all $t \in I$ and N depends only on θ and ϵ . This means that a uniform estimate of the form

$$|\sigma_n(f, t) - \check{f}(t)| < \epsilon + \epsilon \|f - \check{f}(t)\|_1,$$

can be found for all $t \in I$. □

6. Lebesgue's Theorem

The Fejér condition, that

$$\check{f}(t) = \lim_{h \rightarrow 0} \frac{f(t+h) + f(t-h)}{2} \quad (78)$$

exists is very strong, and is not preserved if the function f is modified on a null set. This means that property (78) is not really well-defined on L_1 . However, (78) implies another property: there is a number $\check{f}(t)$ for which

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_0^h \left| \frac{f(t+h) + f(t-h)}{2} - \check{f}(t) \right| d\tau = 0. \quad (79)$$

This is a more robust condition, better suited to integrable functions⁴.

THEOREM 6.9. *If f has property (79) at t , then $\sigma_n(f, t) \rightarrow \check{f}(t)$. In particular (by the footnote), for almost every value of t , $\sigma_n(f, t) \rightarrow \check{f}(t)$.*

COROLLARY 6.4. *If the Fourier series of $f \in L_1(\mathbb{T})$ converges on a set F of positive measure, then almost everywhere on F the Fourier series must converge to f . In particular, a Fourier series that converges to zero almost everywhere must have all its coefficients equal to zero.*

REMARK 6.1. The case of trigonometric series is different: a basic counter-example in the theory of trigonometric series is that there are non-zero trigonometric series that converge to zero almost everywhere. On the other hand, a trigonometric series that converges to zero everywhere must have all coefficients zero⁵.

PROOF OF 6.9. Recall the expression (75) in the proof of Theorem 6.8,

$$\sigma_n(f, t) - \check{f}(t) = \frac{1}{\pi} \left(\int_0^\theta + \int_\theta^\pi \right) K_n(\tau) \left(\frac{f(t-\tau) + f(t+\tau)}{2} - \check{f}(t) \right) d\tau. \quad (80)$$

Also, by (69),

$$K_n(\tau) = \frac{1}{n+1} \left\{ \frac{\sin \frac{n+1}{2}\tau}{\frac{1}{2}\tau} \right\}, \quad (81)$$

³A continuous function on a closed bounded interval is uniformly continuous.

⁴There are functions f with the property that Fejér's condition (78) does not hold anywhere, but (79) does hold for any $f \in L_1(\mathbb{T})$, for almost all t with $\check{f}(t) = f(t)$. This is described in volume 1 of *Trigonometric Series*, A. Zygmund, Cambridge University Press, Cambridge (1959).

⁵See Chapter 5 of *Ensembles parfaits et series trigonometriques*, J.-P. Kahane and R. Salem, Hermann (1963).

and $\sin \frac{\tau}{2} > \frac{\tau}{\pi}$ for $0 < \tau < \pi$, so

$$K_n(\tau) \leq \min \left\{ n+1, \frac{\pi^2}{(n+1)\tau^2} \right\}. \quad (82)$$

It follows that the second integral in (80) will converge to zero so long as $(n+1)\theta^2$ does. Pick $\theta = n^{-1/4}$; this guarantees that as $n \rightarrow \infty$ the second integral tends to zero.

Now consider the first integral. Write

$$\Psi(h) = \int_0^h \left| \frac{f(t+h) + f(t-h)}{2} - \check{f}(t) \right| d\tau.$$

Then

$$\left| \frac{1}{\pi} \int_0^\theta K_n(\tau) \left[\frac{f(t+\tau) + f(t-\tau)}{2} - \check{f}(t) \right] d\tau \right|$$

is bounded above by

$$\begin{aligned} \frac{1}{\pi} \left| \int_0^{1/n} \right| + \frac{1}{\pi} \left| \int_{1/n}^\theta \right| &\leq \frac{n+1}{\pi} \Psi\left(\frac{1}{n}\right) \\ &\quad + \frac{\pi}{n+1} \int_{1/n}^\theta \left| \frac{f(t+\tau) + f(t-\tau)}{2} - \check{f}(t) \right| \frac{d\tau}{\tau^2} \end{aligned}$$

(we have used the estimate for K_n from (82)). By the assumption (79), the first term $\frac{n+1}{\pi} \Psi\left(\frac{1}{n}\right)$ tends to zero. Apply integration by parts to the second term gives

$$\frac{\pi}{n+1} \int_{1/n}^\theta \left| \frac{f(t+\tau) + f(t-\tau)}{2} - \check{f}(t) \right| \frac{d\tau}{\tau^2} = \frac{\pi}{n+1} \left[\frac{\Psi(\tau)}{\tau^2} \right]_{1/n}^\theta + \frac{2\pi}{n+1} \int_{1/n}^\theta \frac{\Psi(\tau)}{\tau^3} d\tau. \quad (83)$$

For given $\epsilon > 0$ and $n > n(\epsilon)$ (79) gives

$$\Psi(\tau) < \epsilon\tau \text{ for } \tau \in (0, \theta = n^{-1/4}).$$

It follows that (83) is bounded above by

$$\frac{\pi\epsilon n}{n+1} + \frac{2\pi\epsilon}{n+1} \int_{1/n}^\theta \frac{d\tau}{\tau^2} < 3\pi\epsilon,$$

which completes the proof. \square

APPENDIX A

1. Zorn's lemma and Hamel bases

DEFINITION A.1. A *partially ordered set* or *poset* is a non-empty set S together with a relation \leq that satisfies the following conditions:

- (i) $x \leq x$ for all $x \in S$;
- (ii) if $x \leq y$ and $y \leq z$ then $x \leq z$ for all $x, y, z \in S$.

If in addition for any two elements x, y of S at least one of the relations $x \leq y$ or $y \leq x$ holds, then we say that S is a *totally ordered set*.

The set of subsets of a set X , with \leq meaning inclusion, defines a partially ordered set for example.

DEFINITION A.2. Let S be a partially ordered set, and T any subset of S . An element $x \in S$ is an *upper bound* of T if $y \leq x$ for all $y \in T$.

DEFINITION A.3. Let S be a partially ordered set. An element $S \in S$ is *maximal* if for any $y \in S$, $x \leq y \implies y \leq x$.

The next result, Zorn's lemma, is one of the formulations of the Axiom of Choice.

THEOREM A.1. *If S is a partially ordered set in which every totally ordered subset has an upper bound, then S has a maximal element.*

This result is used frequently to “construct” things – though whenever we use it all we really are able to do is assert that something must exist subject to assuming the Axiom of Choice. An example is the following result – as usual, trivial in finite dimensions.

To see that the following theorem is “constructing” something a little surprising, think of the following examples: \mathbb{R} is a linear space over \mathbb{Q} ; $L^2[0, 1]$ is a linear space over \mathbb{R} .

THEOREM A.2. *Let X be a linear space over any field. Then S contains a set \mathcal{A} of linearly independent elements such that the linear subspace spanned by \mathcal{A} coincides with X .*

Any such set \mathcal{A} is called a *Hamel basis* for X . It is quite a different kind of object to the usual spanning set or basis used, where X is the *closure* of the span of the basis. If the Hamel basis is $\mathcal{A} = \{x_\lambda\}_{\lambda \in \Lambda}$, then every element of X has a (unique) representation

$$x = \sum a_\lambda x_\lambda$$

in which *the sum is finite* and the a_λ are scalars.

PROOF. Let S be the set of subsets of X that comprise linearly independent elements, and write $S = \{\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots\}$. Define a partial ordering on S by $\mathcal{A} \leq \mathcal{B}$ if and only if $\mathcal{A} \subset \mathcal{B}$.

We first claim that if $\{\mathcal{A}_\alpha\}$ is a totally ordered subset of S , it has the upper bound $\mathcal{B} = \bigcup_\alpha \mathcal{A}_\alpha$. In order to prove this, we must show that any finite number of elements x_1, \dots, x_n of \mathcal{B} are linearly independent. Assume that $x_i \in \mathcal{A}_{\alpha_i}$ for $i = 1, \dots, n$. Since the set $\{\mathcal{A}_\alpha\}$ is totally ordered, one of the subsets \mathcal{A}_{α_j} contains all the others. It follows that $\{x_1, \dots, x_n\} \subset \mathcal{A}_{\alpha_j}$, so x_1, \dots, x_n are linearly independent.

We may therefore apply Theorem A.1 to conclude that S has a maximal element \mathcal{A} . If $y \in X$ is not a finite linear combination of elements of \mathcal{A} , then the set $\mathcal{B} = \mathcal{A} \cup \{y\}$ belongs to S (since it is linearly independent), and $\mathcal{A} \leq \mathcal{B}$, but it is not true that $\mathcal{B} \leq \mathcal{A}$, contradicting the maximality of \mathcal{A} .

It follows that every element of X is a finite linear combination of elements of \mathcal{A} . \square

2. Baire category theorem

Most of the facts assembled here are really about *metric spaces* – normed spaces are a special case of metric spaces.

A subset $S \subset X$ of a normed space is *nowhere dense* if for every point x in the closure of S , and for every $\epsilon > 0$ $B_\epsilon(x) \cap (X \setminus \bar{S})$ is non-empty.

The *diameter* of $S \subset X$ is defined by

$$\text{diam}(S) = \sup_{a,b \in S} \|a - b\|.$$

THEOREM A.3. Let $\{F_n\}$ be a decreasing sequence of non-empty closed sets (this means $F_n \supset F_{n+1}$ for all n) in a complete normed space X . If the sequence of diameters $\text{diam}(F_n)$ converges to zero, then there exists exactly one point in the intersection $\bigcap_{n=1}^{\infty} F_n$.

PROOF. If x and y are both in the intersection, then by the definition of the diameter, $\|x - y\| \leq \text{diam}(F_n) \rightarrow 0$ so $x = y$. It follows that there can be no more than one point in the intersection.

Now choose a point $x_n \in F_n$ for each n . Then $\|x_n - x_m\| \leq \text{diam}(F_n) \rightarrow 0$ as $n \geq m \rightarrow \infty$. Thus the sequence (x_n) is Cauchy, so has a limit x say by completeness. For any n , F_n is a closed set that contains all the x_m with $m \geq n$, so $x \in F_n$. It follows that $x \in \bigcap_{n=1}^{\infty} F_n$. \square

The next result is a version of the Baire¹ category theorem.

THEOREM A.4. A complete normed space cannot be written as a countable union of nowhere dense sets.

In the language of metric spaces, this means that a complete normed space is of *second category*.

¹Rene Baire (1874–1932) was one of the most influential French mathematicians of the early 20th century. His interest in the general ideas of continuity was reinforced by Volterra. In 1905, Baire became professor of analysis at the Faculty of Science in Dijon. While there, he wrote an important treatise on discontinuous functions. Baire's category theorem bears his name today, as do two other important mathematical concepts, Baire functions and Baire classes.

PROOF. Let X be a complete normed space, and suppose that $X = \bigcup_{j=1}^{\infty} X_j$ where each X_j is nowhere dense (that is, the sets \bar{X}_j all have empty interior). Fix a ball $B_1(x_0)$. Since \bar{X}_1 does not contain $B_1(x_0)$ there must be a point $x_1 \in B_1(x_0)$ with $x_1 \notin \bar{X}_1$. It follows that there is a ball $B_{r_1}(x_1)$ such that $\bar{B}_{r_1}(x_1) \subset B_1(x_0)$ and $\bar{B}_{r_1}(x_1) \cap \bar{X}_1 = \emptyset$. Assume without loss of generality that $r_1 < \frac{1}{2}$.

Similarly, there is a point x_2 and a radius r_2 such that $\bar{B}_{r_2}(x_2) \subset B_{r_1}(x_1)$, and $\bar{B}_{r_2}(x_2) \cap \bar{X}_2 = \emptyset$, and without loss of generality $r_2 < \frac{1}{3}$. Notice that $\bar{B}_{r_2}(x_2) \cap \bar{X}_1 = \emptyset$ automatically since $\bar{B}_{r_2}(x_2) \subset B_{r_1}(x_1)$.

Inductively, we construct a sequence of decreasing closed balls $\bar{B}_{r_n}(x_n)$ such that $\bar{B}_{r_n}(x_n) \cap \bar{X}_j = \emptyset$ for $1 \leq j \leq n$, and $r_n \rightarrow 0$ as $n \rightarrow \infty$.

Now by Theorem A.3, there must be a point x in the intersection of all the closed balls $\bar{B}_{r_n}(x_n)$, so $x \notin \bar{X}_j$ for all $j \geq 1$. This implies that $x \notin \bigcup_{j \geq 1} \bar{X}_j = X$, a contradiction. \square