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4

Discrete-Time Fourier Transform, One- and Two-Dimensional

4.1 One-Dimensional Discrete-Time Fourier Transform

4.1.1 Definitions

$$\mathcal{F}\{x(n)\} \doteq X(\omega) \doteq X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \qquad -\pi \le \omega \le \pi$$

$X(\omega)$ = periodic with period 2π

$$x(n) = \mathcal{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{-j\omega n}\, d\omega$$

The Fourier transform function appears as a function of $e^{j\omega}$ and, hence, in sequences we will use both representations for convenience.

4.1.2 Properties

TABLE 4.1 Properties of One-Dimensional Discrete-Time Fourier Transform

Properties	Sequence	Transform $[X(\omega) \equiv X(e^{j\omega})]$
Discrete-time Fourier transform	$x(n)$	$X(\omega)$
Linearity	$ax(n)+by(n)$	$aX(\omega)+bY(\omega)$
Time reversal	$x(-n)$	$X(-\omega)$
Complex conjugation	$x^*(n)$	$X^*(-\omega)$
Reversal and complex conjugate	$x^*(-n)$	$X^*(\omega)$

TABLE 4.1 Properties of One-Dimensional Discrete-Time Fourier Transform (continued)

Properties	Sequence	Transform $[X(\omega) \equiv X(e^{j\omega})]$
Time shifting	$x(n \pm m)$	$e^{\pm j\omega m} X(\omega)$
Frequency shift	$e^{\pm j\omega_o n} x(n)$	$X(\omega \mp \omega_o)$
Modulation	$\cos\omega_o n\ x(n)$	$\frac{1}{2}[X(\omega+\omega_o)+X(\omega-\omega_o)]$
	$\sin\omega_o n\ x(n)$	$\frac{1}{2j}[X(\omega+\omega_o)-X(\omega-\omega_o)]$
Convolution	$x(n) * h(n)$	$X(\omega)Y(\omega)$
Multiplication	$x(n)h(n)$	$\frac{1}{2\pi} X(\omega) * Y(\omega)$
Delta function	$\delta(n-n_o) = \begin{cases} 1, & n = n_o \\ 0, & \text{otherwise} \end{cases}$	$e^{-jn_o\omega}$
Frequency domain delta function	$e^{j\omega_o n}$	$\frac{1}{2\pi}\delta(\omega-\omega_o)$
Cosine function	$\cos\omega_o n$	$\pi\delta(\omega+\omega_o)+\pi\delta(\omega-\omega_o)$
Sine function	$\sin\omega_o n$	$\frac{\pi}{j}(\delta\omega+\omega_o)-\frac{1}{2}\delta(\omega-\omega_o)$
N sample step sequence	$u_N(n) = \begin{cases} 1, & n = 0,1,2,\ldots \\ 0, & \text{otherwise} \end{cases}$	$e^{-j\omega(N-1)/2}\dfrac{\sin(\omega N/2)}{\sin(\omega/2)}$
Symmetric pulse	$p_N(n) = \begin{cases} 1, & \lvert n\rvert \le N \\ 0, & \lvert n\rvert \ge N \end{cases}$	$\dfrac{\sin[\omega(N+1/2)]}{\sin(\omega/2)}$
Triangle sequence	$\wedge_N(n) = \begin{cases} N-\lvert n\rvert, & \lvert n\rvert \le N \\ 0, & \lvert n\rvert > N \end{cases}$	$\dfrac{\sin^2(\omega N/2)}{\sin^2(\omega/2)}$
Real sequence	$x(n)$	$X(\omega) = X^*(-\omega)$
Decomposition of real $x(n)$ in even $x_e(n)$ and $x_o(n)$ parts.	$x(n) = x_e(n)+x_o(n)$ $x_e = \frac{1}{2}[x(n)+x(-n)]$ $x_o = \frac{1}{2}[x(n)-x(-n)]$	$X(\omega)$ $\mathrm{Re}\{X(\omega)\}$ $j\,\mathrm{Im}\{X(\omega)\}$
Decomposition of a complex sequence $x(n)$ into a conjugate symmetric part $x_e(n)$ and conjugate antisymmetric part $x_o(n)$	$x(n) = x_e(n)+x_o(n)$ $x_e = \frac{1}{2}[x(n)+x^*(-n)]$ $x_o = \frac{1}{2}[x(n)-x^*(-n)]$	$X(\omega)$ $\mathrm{Re}\{X(\omega)\}$ $j\,\mathrm{Im}\{X(\omega)\}$
Decomposition of a complex transform $X(\omega)$	$x(n)$ $\mathrm{Re}\{x(n)\}$ $j\,\mathrm{Im}\{x(n)\}$	$X(\omega) = X_e(\omega)+X_o(\omega)$ $X_e(\omega) = \frac{1}{2}[X(\omega)+X^*(-\omega)]$ $X_o(\omega) = \frac{1}{2}[X(\omega)-X^*(-\omega)]$
Increasing sampling frequency by m; i.e., transforming a data sequence $x_1(n)$ padded with zeros $x_1(n)$ by a factor of M	$x(n) = \begin{cases} x_1(n), & \text{if } n/M = m \\ 0, & \text{otherwise} \end{cases}$	$X_1(M\omega)$
Reducing sampling frequency by M; i.e., decimating a sequence $x_1(n)$ by a factor of M	$x(n) = x_1(Mn)$ $n = 0, \pm 1, \pm 2, \ldots$	$\dfrac{1}{M}\sum_{\ell=0}^{M-1} X_1\left(\omega - \dfrac{2\pi\ell}{M}\right)$
Parseval's theorem	$\sum_{n=-\infty}^{\infty} x(n)h^*(n)$ $\sum_{n=-\infty}^{\infty} \lvert x(n)\rvert^2$	$= \dfrac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)H^*(\omega)\,d\omega$ $= \dfrac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(\omega)\rvert^2\,d\omega$
Correlation	$x(n) \star h(n)$	$X(\omega)H(\omega)$

4.1.3 Finite Sequence

$$F_N(\omega) = \sum_{n=0}^{N-1} f(n) e^{-j\omega n} = e^{-j\omega(N-1)/2} \frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}}$$

4.1.4 Approximation to Continuous-Time Fourier Transform

$$F(\omega_c) = \int_{-\infty}^{\infty} f(t) e^{-j\omega_c t}\, dt \qquad \omega_c = \text{frequency for continuous Fourier transform}$$

$$F(\omega_c) \cong \sum_{n=-\infty}^{\infty} T\, f(nt) e^{-j\omega_c nT} \qquad T = \text{sampling time such that } F(\omega_c) \cong 0 \text{ for all } |\omega_c| > \pi / T$$

4.2 Two-Dimensional Discrete-Time Fourier Transform

4.2.1 Definition

$$X(\omega_1, \omega_2) \doteq \sum_{m,n=-\infty}^{\infty}\sum x(m,n) e^{-j(m\omega_1 + n\omega_2)} \qquad -\pi \le \omega_1, \omega_2 \le \pi$$

$$x(m,n) = \frac{1}{(2\pi)^2} \iint_{-\pi}^{\pi} X(\omega_1, \omega_2) e^{j(m\omega_1 + n\omega_2)}$$

4.2.2 Properties of Two-Dimensional Discrete-Time Fourier Transform

TABLE 4.2 Properties of Two-Dimensional Discrete-Time Fourier Transform

Properties	Sequence	Transform
	$x(m,n), y(m,n), h(m,n), \ldots$	$X(\omega_1,\omega_2), Y(\omega_1,\omega_2), H(\omega_1,\omega_2), \ldots$
Linearity	$a_1 x_1(m,n) + a_2 x_2(m,n)$	$a_1 X_1(\omega_1,\omega_2) + a_2 X_2(\omega_1,\omega_2)$
Conjugation	$x^*(m,n)$	$X^*(-\omega_1,-\omega_2)$
Separability	$x_1(m) x_2(m)$	$X_1(\omega_1) X_2(\omega_2)$
Shifting	$x(m \pm m_0, n \pm n_0)$	$\exp[\pm j(m_0\omega_1 + n_0\omega_2)] X(\omega_1,\omega_2)$
Modulation	$\exp[\pm j(\omega_{01} m + \omega_{02} n)] x(m,n)$	$X(\omega_1 \mp \omega_{01}, \omega_2 \mp \omega_{02})$
Convolution	$y(m,n) = h(m,n) ** x(m,n)$	$Y(\omega_1,\omega_2) = H(\omega_1,\omega_2) X(\omega_1,\omega_2)$
Multiplication	$h(m,n) x(m,n)$	$\left(\frac{1}{4\pi^2}\right) H(\omega_1,\omega_2) ** X(\omega_1,\omega_2)$
Spatial correlation	$c(m,n) = h(m,n) \star\star\, x(m,n)$	$C(\omega_1,\omega_2) = H(-\omega_1,-\omega_2) X(\omega_1,\omega_2)$
Inner product	$I = \sum_{m,n=-\infty}^{\infty}\sum x(m,n) y^*(m,n)$	$I = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} X(\omega_1,\omega_2) Y^*(\omega_1,\omega_2)\, d\omega_1\, d\omega_2$
Energy conservation	$\mathsf{E} = \sum_{m,n=-\infty}^{\infty}\sum \lvert x(m,n)\rvert^2$	$\mathsf{E} = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \lvert X(\omega_1,\omega_2)\rvert^2\, d\omega_1\, d\omega_2$
	$\sum_{m,n=-\infty}^{\infty}\sum \exp[j(m\omega_{01} + n\omega_{02})]$	$4\pi^2 \delta(\omega_1 - \omega_{01}, \omega_2 - \omega_{02})$

TABLE 4.2 Properties of Two-Dimensional Discrete-Time Fourier Transform (continued)

Properties	Sequence	Transform
	$\delta(m,n)$	$\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\exp[-j(\omega_1 m+\omega_2 n)\,d\omega_1\,d\omega_2$
Differentiation	$-jmx(m,n)$	$\frac{\partial X(\omega_1,\omega_2)}{\partial\omega_1}$
	$-jnx(m,n)$	$\frac{\partial X(\omega_1,\omega_2)}{\partial\omega_2}$
	$mnx(m,n)$	$\frac{\partial^2 X(\omega_1,\omega_2)}{\partial\omega_1\partial\omega_2}$
Transportation	$x(m,n)$	$X(\omega_2,\omega_1)$
Reflection	$x(-m,n)$	$X(-\omega_1,\omega_2)$
	$x(m,-n)$	$X(\omega_1,-\omega_2)$
	$x(-m,-n)$	$X(-\omega_1,-\omega_2)$
Real and Imaginary	$\mathrm{Re}[x(m,n)]$	$\frac{1}{2}[X(\omega_1,\omega_2)+X^*(-\omega_1,-\omega_2)]$
Parts	$j\,\mathrm{Im}[x(m,n)]$	$\frac{1}{2}[X(\omega_1,\omega_2)-X^*(-\omega_1,-\omega_2)]$
	$\frac{1}{2}[x(m,n)+x^*(-m,-n)]$	$\mathrm{Re}[X(\omega_1,\omega_2)]$
	$\frac{1}{2}[x(m,n)-x^*(-m,-n)]$	$j\mathrm{Im}[X(\omega_1,\omega_2)]$
Real-valued sequence		$X(\omega_1,\omega_2) = X^*(-\omega_1,-\omega_2)$ $\mathrm{Re}[X(\omega_1,\omega_2)] = \mathrm{Re}[X(-\omega_1,-\omega_2)]$ $\mathrm{Im}[X(\omega_1,\omega_2)] = -\mathrm{Im}[X(-\omega_1,-\omega_2)]$
		$\mathrm{Re}[X(\omega_1,\omega_2)]$, $\lvert X(\omega_1,\omega_2)\rvert$: even (symmetric with respect to the origin)
		$\mathrm{Im}[X(\omega_1,\omega_2)]$; $\tan^{-1}\frac{\mathrm{Im}[X(\omega_1,\omega_2)]}{\mathrm{Re}[X(\omega_1,\omega_2)]}$: odd (antisymmetric with respect to the origin)
	$x(m,n)$: real and even	$X(\omega_1,\omega_2)$: real and even
	$x(m,n)$: real and odd	$X(\omega_1,\omega_2)$: pure imaginary and odd

References

Dudgeon, Dan E., and Russell M. Mersereau, *Multidimensional Digital Signal Processing*, Prentice Hall, Englewood Cliffs, NJ, 1984.

Jain, Aril K., *Fundamentals of Digital Image Processing*, Prentice Hall, Englewood Cliffs, NJ, 1989.

Lim, Jae S., *Two-Dimensional Signal and Image Processing*, Prentice Hall, Englewood Cliffs, NJ, 1990.

Appendix 1

1.1 Two-Dimensional Discrete-Time Fourier Transform

Example

The transform of $x(m,n) = \frac{1}{5}\delta(m-1)\delta(n) + \frac{1}{5}\delta(m+1)\delta(n) + \frac{1}{5}\delta(m)\delta(n-1) + \frac{1}{5}\delta(m)\delta(n+1) + \frac{2}{5}\delta(m)\delta(n)$ which is shown in Figure 4.1 is $X(\omega_1,\omega_2) = \sum\sum_{m,n=-\infty}^{\infty} x(m,n)e^{-jm\omega_1-jn\omega_2} = \frac{1}{5}e^{-j\omega_1} + \frac{1}{5}e^{j\omega_1} + \frac{1}{5}e^{-j\omega_2} + \frac{1}{5}e^{j\omega_2} + \frac{1}{2} = \frac{2}{5}\cos\omega_1 + \frac{2}{5}\cos\omega_2 + \frac{2}{5}$ and is shown in Figure 4.2.

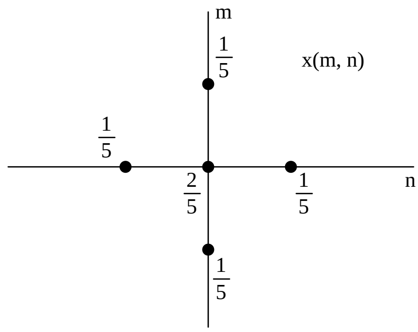

FIGURE 4.1

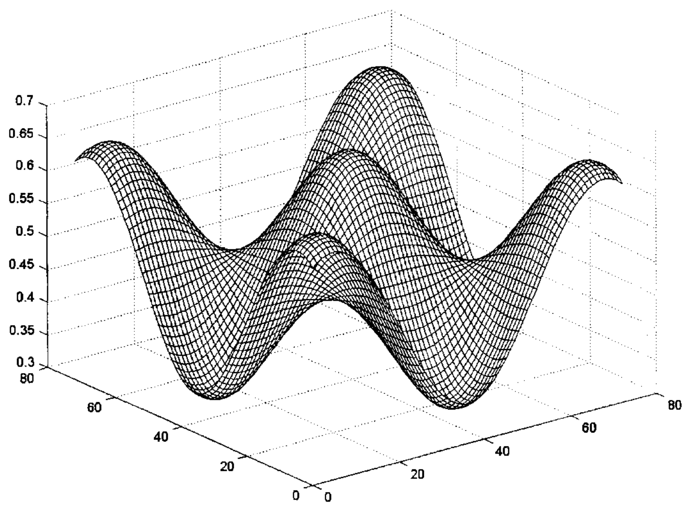

FIGURE 4.2

1.2 Two-Dimensional Discrete-Time Fourier Transform

Example

(Ideal Lowpass Filter). To find the inverse of $H(\omega_1,\omega_2)$ shown in Figure 4.3 we observe that it is equal to the multiplication of two pulse filters whose one direction extends from minus infinity to infinity. Hence (a and b are less than π)

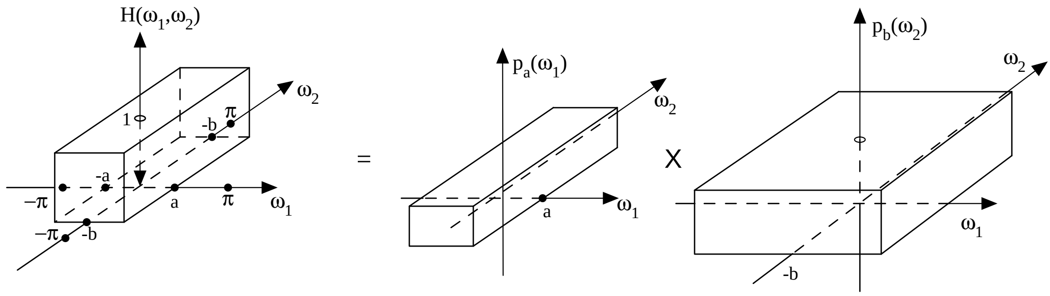

FIGURE 4.3

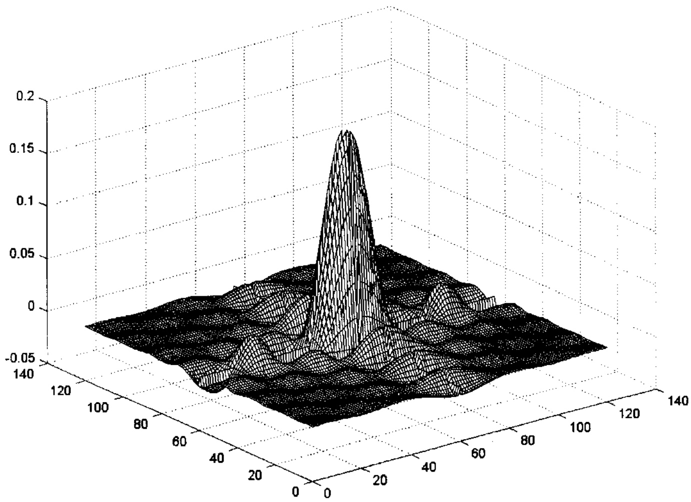

FIGURE 4.4

$$h(m,n) = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} H(\omega_1,\omega_2)e^{j\omega_1 m}e^{j\omega_2 n}\,d\omega_1\,d\omega_2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} p_a(\omega_1)e^{j\omega_1 m}\,d\omega_1\,\frac{1}{2\pi}\int_{-\pi}^{\pi} p_b(\omega_2)e^{j\omega_2 n}\,d\omega_2$$

$$= \frac{1}{2\pi}\int_{-a}^{a} e^{j\omega_1 m}\,d\omega_1\,\frac{1}{2\pi}\int_{-b}^{b} e^{j\omega_2 n}\,d\omega_2 = \frac{\sin am}{\pi m}\,\frac{\sin bn}{\pi n}$$

which is plotted in Figure 4.4.

Example

Let the ideal filter be of the form $H(\omega_1,\omega_2) = 1$ for $\omega_1^2 + \omega_2^2 \le \omega_c^2$ and zero otherwise (see Figure 4.5). Hence

$$h(m,n) = \frac{1}{(2\pi)^2}\iint_{\substack{(\omega_1,\omega_2)\in \\ [\omega_1^1+\omega_2^2\le\omega_c^2]}} 1e^{j\omega_1 m}e^{j\omega_2 n}\,d\omega_1\,d\omega_2$$

$$= \frac{1}{(2\pi)^2}\int_{r=0}^{\omega_c} r\,dr \int_{\theta=a}^{a+2\pi} e^{jr(m\cos\theta+n\sin\theta)}\,d\theta \qquad a = \text{real constant}$$

where we set $r\cos\theta = \omega_1$, $r\sin\theta = \omega_2$. Next change to: $m = q\cos\phi$ and $n = q\sin\phi$. Hence

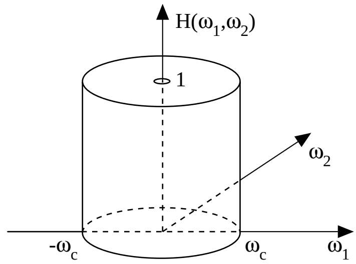

FIGURE 4.5

$$h(m,n) = \frac{1}{(2\pi)^2} \int_{r=0}^{\omega_c} r \int_{\theta=a}^{a+2\pi} e^{jrn\cos(\theta-\phi)}\, d\theta = \frac{1}{(2\pi)^2} \int_{r=0}^{\omega_c} r\, 2\pi\, J_o(r\sqrt{m^2+n^2})\, dr = \frac{\omega_c}{2\pi\sqrt{m^2+n^2}} J_1(\omega_c \sqrt{m^2+n^2})$$