Poularikas A. D. “Windows”
*The Handbook of Formulas and Tables for Signal Processing.*
Ed. Alexander D. Poularikas
Boca Raton: CRC Press LLC, 1999
Windows

7.1 Introductory Material

7.2 Figures of Merit

7.3 Window (Filter) Descriptions

Rectangle (Dirichlet) Window • Triangle (Feyer, Bartlet) Window • \( \cos^2(t) \) Windows • Hann • Hamming • Short Cosine series • Blackman • Harris-Nuttall • Sampled Kaiser-Bessel • Parabolic (Riesz, Bochner, Parzen) • Riemann • de la Valle-Poussin (Jackson, Parzen) • Cosine taper (Tukey) • Bohman • Poisson • Hann-Poisson • Cauchy (Abel, Poisson) • Gaussian (Weierstrass) • Dolph-Chebyshev • Kaiser-Bessel Barcilon-Themes • Highest Sidelobe Level versus Worst-Case Processing Loss

References

7.1 Introductory Material

7.1.1 Introduction

\[ N = \text{number of samples} \quad T = \text{interval between samples} \]

\[ NT = \text{total time duration of the signal} \quad \frac{1}{NT} = \text{minimum spectral resolution (} s^{-1} \text{)} \]

\[ DFT = \text{discrete Fourier transform} \]

Leakage = Spectral leakage takes place when the signal has frequencies other than those of the basis set. These other frequencies will exhibit non-zero properties on the entire basis set known as leakage.

7.2 Figures of Merit (see Table 7.1)

7.2.1 Equivalent Noise Bandwidth

\[ ENBW = \frac{\sum w^2(nT)}{\sum w(nT)^2} = \text{equivalent noise bandwidth} \]

the width of an equivalent ideal rectangular spectral response that will pass the same noise power as the window (filter) under test.

\[ w(nT) = \text{window samples} \]
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<th>Coherent gain (bins)</th>
<th>Equivalent noise BW (bins)</th>
<th>3.0-dB BW (bins)</th>
<th>Scallop loss (dB)</th>
<th>Worst-case process loss (dB)</th>
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7.2.2 Coherent Gain

\[ CG = \text{coherent gain} = \frac{1}{N} \sum_{n=0}^{N-1} w(nT) = \text{zero frequency gain (dc gain) of the window} \]

7.2.3 Processing Gain

\[ PG = \frac{1}{ENBW} = \frac{\text{output signal-to-noise ratio}}{\text{input signal-to-noise ratio}} \]

7.2.4 Scalloping Loss

\[ \text{scalloping loss} = \frac{\sum_{n} w(nT) \exp \left( -j \frac{n \pi}{N} \right)}{\sum_{n} w(nT)} = \frac{\text{coherent gain for a tone located half a bin from DFT sample point}}{\text{coherent gain for a tone located at a DFT sample point}} = \text{maximum reduction in PG due to signal frequency} \]

7.2.5 Mainlobe Spectral Response

mainlobe spectral response = spectral interval between the peak gain and the –3.0 dB and –6.0 dB response level

7.2.6 Overlap Correlation

Correlation coefficients represent the degree of correlation of filter output points that are separated by 25% and 50% of the filter length. These terms are useful in quantifying the estimation uncertainty (or variance reduction) related to incoherent averaging of filter (window) data.

7.3 Window (Filter) Descriptions

7.3.1 Introduction

- \( T = 1 \)
- \( -\pi \leq \omega \leq \pi \) or \( 0 \leq \omega \leq 2\pi \)
- \( \text{DFT bin} = 2\pi/N \)
- Windows are even (about the origin) sequences with an odd number of points.
- The right-most point of the window will be discarded.
- \( N \) will be taken to be even, and the total points will be odd, and hence
  \[ N = 2 \times (\text{total points}) = \text{even} \]

7.3.2 Rectangle (Dirichlet) Window

\[ w(n) = 1.0 \quad n = -\frac{N}{2}, \cdots, -1, 0, 1, \cdots, \frac{N}{2} \]
To make it realizable shift the sequence by \( N/2 \) to the right. Hence we obtain

\[ w(n) = 1 \quad n = 0,1,\cdots, N - 1 \]

\[ W(\omega) = \sum_{n=0}^{N/2} e^{-j\omega n} = e^{-j\frac{N-1}{2}n} \frac{\sin \left( \frac{N \omega}{2} \right)}{\sin \frac{\omega}{2}} \]

Figure 7.1 shows the rectangular window and its amplitude spectrum \( |W(\omega)| \).

**FIGURE 7.1** a) Rectangular window. b) Amplitude spectrum of rectangular window.

### 7.3.3 Triangle (Fejer, Bartlet) Window

\[ w(n) = 1.0 - \frac{|n|}{N/2} \quad n = -\frac{N}{2},\cdots,-1,0,1,\cdots,\frac{N}{2} \]

For DFT the window is

\[ w(n) = \begin{cases} \frac{n}{N/2} & n = 0,1,\cdots,\frac{N}{2} \\ \frac{N-n}{N/2} & n = \frac{N}{2}+1,\cdots,N-1 \end{cases} \]

and its DFT is

\[ W(\omega) = e^{-j\frac{N-1}{2}n} \left[ \frac{\sin \left( \frac{N \omega}{4} \right)}{\sin \frac{\omega}{2}} \right]^2 \]

since the symmetric function \( w(n) \) is shifted by \( \frac{N}{2} - 1 \) positions to produce the DFT sequence. Figure 7.2 shows the triangular window and its DFT.
7.3.4 \( \cos^{\alpha} (t) \) Windows

\[
w(n) = \cos^{\alpha}\left(\frac{n}{N}\pi\right) \quad n = -\frac{N}{2}, \ldots, 0, 1, \ldots, \frac{N}{2}
\]

\[
w(n) = \sin^{\alpha}\left(\frac{n}{N}\pi\right) \quad n = 0, 1, \ldots, N - 1
\]

Common values of \( \alpha : 1 \leq \alpha \leq 4 \)

Figures 7.3 through 7.5 show the \( \cos^{\alpha}\left(\frac{n\pi}{N}\right) \), \( \cos^{\beta}\left(\frac{n\pi}{N}\right) \), and \( \cos^{\gamma}\left(\frac{n\pi}{N}\right) \) windows and their Fourier transform.

FIGURE 7.2 a) Triangular window. b) Amplitude spectrum of triangular window.

FIGURE 7.3 a) Cosine window with \( \alpha = 1 \). b) Amplitude spectrum of cosine window with \( \alpha = 1 \).

FIGURE 7.4 a) Cosine window with \( \alpha = 3 \). b) Amplitude spectrum of cosine window with \( \alpha = 3 \).
7.3.5 Hann Window

\[ w(n) = \cos^2 \left( \frac{n}{N} \pi \right) = \frac{1}{2} \left[ 1 + \cos \left( \frac{2n}{N} \pi \right) \right] \quad n = -\frac{N}{2}, \ldots, -1, 0, 1, \ldots, \frac{N}{2} \]

\[ w(n) = \sin^2 \left( \frac{n}{N} \pi \right) = \frac{1}{2} \left[ 1 - \cos \left( \frac{2n}{N} \pi \right) \right] \quad n = 0, 1, \ldots, N - 1 \]

DFT of the window is

\[ W(\omega) = \frac{1}{2} D(\omega) + \frac{1}{4} \left[ D \left( \omega - \frac{2\pi}{N} \right) + D \left( \omega + \frac{2\pi}{N} \right) \right] \]

\[ D(\omega) = e^{i\omega} \frac{\sin \left( \frac{N}{2} \omega \right)}{\sin \frac{\omega}{2}} = \text{Dirichlet Kernel} \quad -\pi \leq \omega \leq \pi \]

See Figure 7.6 for the Hann window.

7.3.6 Hamming Window

\[ w(n) = \alpha + (1 - \alpha) \cos \frac{2\pi n}{N} \quad n = -\frac{N}{2}, \ldots, -1, 0, 1, \ldots, \frac{N}{2} \]
\[ W(\omega) = a D(\omega) + \frac{1}{2} (1 - a) \left[ D(\omega - \frac{2\pi}{N}) + D(\omega + \frac{2\pi}{N}) \right] \quad -\pi \leq \omega \leq \pi \]

\[ D = \text{Dirichlet Kernel (see 7.3.5)} \]

\[ w(n) = 0.54 + 0.46 \cos \frac{2\pi}{N} n \quad n = -\frac{N}{2}, \ldots, -1, 0, 1, \ldots, \frac{N}{2} \]

\[ w(n) = 0.54 - 0.46 \cos \frac{2\pi}{N} n \quad n = 0, 1, \ldots, N - 1 \]

Figure 7.7 depicts the Hamming window and its amplitude spectrum.

\[ W(\omega) = a(0) D(\omega) + \sum_{k=0}^{K/2} a(k) \cos \left( \frac{2\pi}{N} kn \right) \quad n = -\frac{N}{2}, \ldots, -1, 0, 1, \ldots, \frac{N}{2} \]

\[ \sum_{k=0}^{K/2} a(k) = 1 \quad \text{constraint} \]

\[ W(\omega) = a(0) D(\omega) + \sum_{k=0}^{K/2} a(k) \left[ D(\omega - k \frac{2\pi}{N}) + D(\omega + k \frac{2\pi}{N}) \right] \quad -\pi \leq \omega \leq \pi \]

7.3.8 Blackman Window

\[ a(0) = 0.42659071 \equiv 0.42, \quad a(1) = 0.49656062 \equiv 0.50, \quad a(2) = 0.07684867 \equiv 0.08 \]

\[ w(n) = 0.42 + 0.5 \cos \left( \frac{2\pi}{N} n \right) + 0.08 \cos \left( \frac{2\pi}{N} 2n \right) \quad n = -\frac{N}{2}, \ldots, -1, 0, 1, \ldots, \frac{N}{2} \]

\[ w(n) = 0.42 + 0.5 \cos \left( \frac{2\pi}{N} (n - 25) \right) + 0.08 \cos \left( \frac{2\pi}{N} 2(n - 25) \right) \quad n = 0, 1, \ldots, N - 1 \]

Figure 7.8 shows the characteristics of the Blackman window.
7.3.9 Harris-Nutall Window

Table 7.2 gives the coefficients for short cosine series windows.

<table>
<thead>
<tr>
<th></th>
<th>3-Term (-61 dB)</th>
<th>3-Term (-67 dB)</th>
<th>4-Term (-74 dB)</th>
<th>4-Term (-94 dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a(0)</td>
<td>0.44959</td>
<td>0.42323</td>
<td>0.40217</td>
<td>0.35875</td>
</tr>
<tr>
<td>a(1)</td>
<td>0.49364</td>
<td>0.49755</td>
<td>0.49703</td>
<td>0.48829</td>
</tr>
<tr>
<td>a(2)</td>
<td>0.05677</td>
<td>0.07922</td>
<td>0.09392</td>
<td>0.14128</td>
</tr>
<tr>
<td>a(3)</td>
<td>0</td>
<td>0</td>
<td>0.00183</td>
<td>0.01168</td>
</tr>
</tbody>
</table>

Figures 7.9 and 7.10 show the Harris-Nutall window characteristics.
7.3.10 Sampled Kaiser-Bessel Window

Kaiser-Bessel spectrum window \( W(\omega) = \frac{\sinh(\pi \sqrt{\alpha^2 - (\omega N/2)^2})}{\pi \sqrt{\alpha^2 - (\omega N/2)^2}} \quad 0 \leq \alpha \leq 4 \)

\[ H_i(m) = \frac{\sinh(\pi \sqrt{\alpha^2 - m^2})}{\pi \sqrt{\alpha^2 - m^2}} \quad \omega = m(2\pi / N) \]

\[ c = H_i(0) + 2H_i(1) + 2H_i(2) + [2H_i(3)] \]

\[ a(0) = \frac{H_i(0)}{c}, \quad a(m) = \frac{2H_i(0)}{c}, \quad m = 1, 2, 3 \]

\[ a(0) = 0.40243, \quad a(1) = 0.49804, \quad a(2) = 0.09831, \quad a(3) = 0.00122 \]

7.3.11 Parabolic (Riesz, Bochner, Parzen) Window

\[ w(n) = 1.0 - \left( \frac{n}{N/2} \right)^2 \quad 0 \leq |n| \leq \frac{N}{2} \]

\[ w(n) = 1.0 - \left( \frac{n - N}{N/2} \right)^2 \quad n = 0, 1, 2, \ldots, N - 1 \]

Figure 7.11 shows the parabolic window and its spectrum characteristics.

**FIGURE 7.11** a) Parabolic window. b) Amplitude spectrum of Parabolic window.

7.3.12 Riemann Window

\[ w(n) = \frac{\sin \left( \frac{2\pi n}{N} \right)}{\frac{2\pi n}{N}} \quad 0 \leq |n| \leq \frac{N}{2} \]
Figure 7.12 shows the window’s characteristics.

\[
w(n) = \begin{cases} 
\frac{\sin \left( \frac{2\pi (n - N/2)}{N} \right)}{2\pi (n - N/2)} & n = 0, 1, 2, \ldots, N-1 \\
\frac{2\pi (n - N/2)}{N} & 0 \leq |n| \leq \frac{N}{2} \\
2 \left[ 1 - \frac{|n|}{N/2} \right]^3 & \frac{N}{2} \leq |n| \leq N 
\end{cases}
\]

7.3.13 de la Vallé-Poussin (Jackson, Parzen) Window

\[
w(n) = \begin{cases} 
1 - 6 \left[ \frac{n}{N/2} \right]^2 \left[ 1 - \frac{|n|}{N/2} \right] & 0 \leq |n| \leq \frac{N}{2} \\
2 \left[ 1 - \frac{|n|}{N/2} \right]^3 & \frac{N}{2} \leq |n| \leq N 
\end{cases}
\]

Figure 7.13 shows the window and its frequency response.

7.3.14 Cosine Taper (Tukey) Window

The Tukey window is equal to one over \((1 - \alpha/2)N\) points, with a cosine taper from one to zero for the remaining points \((\alpha/2)N\).
\[
    w(n) = \begin{cases} 
    0.5 \left( 1 + \cos \left( \frac{\pi}{(1 - \alpha)(N/2)} \right) \right) & 0 \leq |n| \leq \alpha \frac{N}{2} \\
    1 - \frac{|n|}{N/2} \cos \left( \pi \frac{|n|}{N/2} \right) + \frac{1}{\pi} \sin \left( \pi \frac{|n|}{N/2} \right) & \alpha \frac{N}{2} < |n| \leq \frac{N}{2}
    \end{cases}
\]

Figures 7.14 and 7.15 show the window and its frequency responses for \( \alpha = 8/25 \) and \( \alpha = 12/25 \).

7.3.15 Bohman Window

\[
    w(n) = \left( 1 - \frac{|n|}{N/2} \right) \cos \left( \pi \frac{|n|}{N/2} \right) + \frac{1}{\pi} \sin \left( \pi \frac{|n|}{N/2} \right) \quad 0 \leq |n| \leq \frac{N}{2}
\]

Figure 7.16 shows the window and its spectrum.

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7.3.16 Poisson Window

\[ w(n) = \exp\left(-\alpha \frac{|n|}{N/2}\right), \quad 0 \leq |n| \leq \frac{N}{2} \]

Figures 7.17 through 7.19 show the window and its spectrum with \( \alpha = 1.5 \), \( \alpha = 3 \), and \( \alpha = 4 \).

**FIGURE 7.17** a) Poisson window with \( \alpha = 1.5 \). b) Amplitude spectrum of Poisson window with \( \alpha = 1.5 \).

**FIGURE 7.18** a) Poisson window with \( \alpha = 3.0 \). b) Amplitude spectrum of Poisson window with \( \alpha = 3.0 \).

**FIGURE 7.19** a) Poisson window with \( \alpha = 4.0 \). b) Amplitude spectrum of Poisson window with \( \alpha = 4.0 \).

7.3.17 Hann-Poisson Window

\[ w(n) = 0.5 \left[ 1 + \cos\left(\pi \frac{n}{N/2}\right)\right] \exp\left(-\alpha \frac{|n|}{N/2}\right), \quad 0 \leq |n| \leq \frac{N}{2} \]
Figures 7.20 and 7.21 show the window and its spectrum with \( \alpha = 0.5 \) and \( \alpha = 1.0 \), respectively.

7.3.18 Cauchy (Abel, Poisson) Window

\[
  w(n) = \frac{1}{1 + \left( \frac{n}{N/2} \right)^\alpha}, \quad 0 \leq |n| \leq \frac{N}{2}
\]

Figures 7.22 through 7.24 show the window and its spectrum with \( \alpha = 3.0 \), \( \alpha = 4.0 \), and \( \alpha = 6.0 \), respectively.

Figures 7.20 a) Hann-Poisson window with \( \alpha = 0.5 \). b) Amplitude spectrum of Hann-Poisson window with \( \alpha = 0.5 \).

Figures 7.21 a) Hann-Poisson window with \( \alpha = 1.0 \). b) Amplitude spectrum of Hann-Poisson window with \( \alpha = 1.0 \).

Figures 7.22 a) Cauchy window with \( \alpha = 3.0 \). b) Amplitude spectrum of Cauchy window with \( \alpha = 3.0 \).
7.3.19 Gaussian (Weierstrass) Window

\[ w(n) = \exp\left\{ -\frac{1}{2} \left( \frac{\alpha}{N/2} \right)^2 \right\}. \quad 0 \leq |n| \leq \frac{N}{2} \]

Figures 7.25 and 7.26 show the window and its spectrum for \( \alpha = 2.5 \) and \( \alpha = 3.5 \).

7.3.20 Dolph-Chebyshev Window

\[ W(k) = \frac{\cosh(N\cosh^{-1}(\beta \cosh(\pi k / N)))}{\cosh(N\cosh^{-1}(\beta))} \]
where
\[
\cosh^{-1}(X) = \ln\left(X + \sqrt{X^2 - 1.0}\right), \quad |X| > 1.0
\]

and
\[
W(k) = \frac{\cos(N\cos^{-1}(\beta\cos(\pi k / N)))}{\cosh(N\cosh^{-1}(\beta))}
\]

where
\[
\cos^{-1}(X) = \frac{\pi}{2} - \tan^{-1}\left(\frac{X}{\sqrt{X^2 - 1.0}}\right), \quad |X| \leq 1.0
\]

where \(\beta\) satisfies
\[
\beta = \cosh\left(\frac{1}{N} \cosh^{-1} 10^n\right)
\]

and
\[
w(n) = \sum_{k=0}^{N-1} W(k) \exp\left(\frac{2\pi}{N} nk\right)
\]

7.3.21 Kaiser-Bessel Window

\[
w(n) = \frac{\pi \alpha}{I_0[\pi \alpha]} \left[1.0 - \left(\frac{n}{N/2}\right)^2\right], \quad 0 \leq |n| \leq \frac{N}{2}
\]

where
\[
I_\alpha(x) = \sum_{k=0}^{\infty} \left[\frac{x^k}{k!}\right]^2 = \text{zero-order modified Bessel function}
\]

Figures 7.27 and 7.28 show the window and its spectrum for \(\alpha = 2\) and \(\alpha = 3\).
7.3.22 Barcilon-Themes Window

\[
W(k) = \frac{A \cos[y(k)] + B[y(k)/C] \sin[y(k)]}{(C + AB)[(y(k)/C)^2 + 1.0]}
\]

where

\[
A = \sinh C = \sqrt{10^{2\alpha}} - 1.0
\]

\[
B = \cosh C = 10^\alpha
\]

\[
C = \cosh^{-1}(10^\alpha)
\]

\[
\beta = \cosh(C/N)
\]

\[
y(k) = N\cos^{-1}\left[\beta \cos\left(\frac{\pi k}{N}\right)\right]
\]

7.3.23 Highest Sidelobe Level versus Worst-Case Processing Loss

Figure 7.29 shows the highest sidelobe level versus worst-case processing loss. Shaped DFT filters in the lower left tend to perform well.
FIGURE 7.29 Highest sidelobe level versus worst-case processing loss. Shaped DFT filters in the lower left tend to perform well.

References