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8
Two-Dimensional Z-Transform

8.1 The Z-Transform

8.1.1 Definition

$$X(z_1,z_2)=\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}x(n_1,n_2)z_1^{-n_1}z_2^{-n_2}$$

8.1.2 Relationship to Discrete-Time Fourier Transform

$$X(z_1,z_2)\Big|_{z_1=e^{j\omega_1},z_2=e^{j\omega_2}}=\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}x(n_1,n_2)e^{-j\omega_1 n_1}e^{-j\omega_2 n_2}=X(\omega_1,\omega_2)$$

evaluated at $z_1=e^{j\omega_1}$ and $z_2=e^{j\omega_2}$.

8.1.3 Region of Convergence (ROC)

Points (z_1,z_2) for which

$$\sum_{n_1}\sum_{n_2}|x(n_1,n_2)||z_1|^{-n_1}|z_2|^{-n_2}<\infty$$

are located in the ROC. This implies that

$$|X(z_1,z_2)|<\infty$$

If (z_{01},z_{02}) point lies in the ROC, then all points (z_1,z_2) that satisfy

$$|z_1| \geq |z_{01}|, \qquad |z_2| \geq |z_{02}|$$

also lie in the ROC.

For the first quadrant sequences, the boundary of the ROC must have nonpositive slope.

8.1.4 Sequences with Finite Support

$$X(z_1,z_2) = \sum_{n_1=N_1}^{M_1} \sum_{n_2=N_2}^{M_2} x(n_1,n_2) z_1^{-n_1} z_2^{-n_2}$$

The Z-transform converges for all finite values of z_1 and z_2, except possibly for $z_1 = 0$ and $z_2 = 0$.

8.1.5 Sequences with support on a wedge

If a sequence has support shown in Figure 8.1a, then its ROC is shown in Figure 8.1b (see Dudgeon and Mersereau, 1984). If the point (z_{01},z_{02}) belongs to the ROC, then

$$\ell n|z_1| \geq \ell n|z_{01}| \quad \text{and} \quad \ell n|z_2| \geq L\ell n|z_1| + \{\ell n|z_{02}| - L\ell n|z_{01}|\}$$

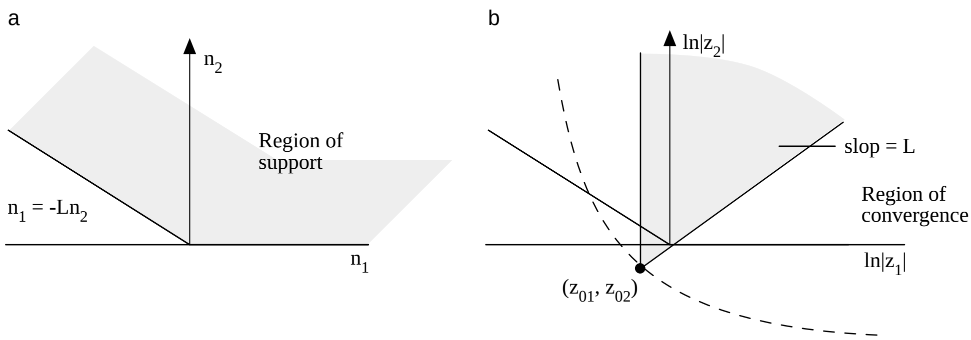

FIGURE 8.1

8.1.6 Sequences with Support on a Half-Plane

The boundary of the region of convergence is constrained to be a single-valued function of $|z_1|$ (or $\ell n|z_1|$) (see Dudgeon and Mersereau, 1984).

8.1.7 Sequences with Support Everywhere

- $x(n_1,n_2) = e^{-n_1^2-n_2^2}$ converges for all values of (z_1,z_2)
- $x(n_1,n_2) = 2^{|n_1|}\,2^{|n_2|}$ will not converge for any value of (z_1,z_2)
- Often the z-transform of a sequence with support everywhere will converge in a region of finite area.
- A sequence with support everywhere can be split into four quadrant sequences, e. g.,

$$x(n_1,n_2) = x_1(n_1,n_2) + x_2(n_1,n_2) + x_3(n_1,n_2) + x_4(n_1,n_2)$$

where

$$x_1(n_1,n_2) = \begin{cases} x(n_1,n_2) & \text{for } \; n_1 > 0, \; n_2 > 0 \\ \tfrac{1}{2}x(n_1,n_2) & \text{for } \; n_1 = 0, \; n_2 > 0 \;\; \text{or} \;\; n_1 > 0, \; n_2 = 0 \\ \tfrac{1}{4}x(n_1,n_2) & \text{for } \; n_1 = n_2 = 0 \\ 0 & \text{for } \; n_1 < 0 \;\; \text{or} \;\; n_2 < 0 \end{cases}$$

Similarly are defined $x_2(n_1,n_2)$, $x_3(n_1,n_2)$, and $x_4(n_1,n_2)$, . The region of convergence of $x(n_1,n_2)$ is the *intersection* of the region of convergence of the four z-transforms of the four quadrants.

8.1.8 ROCs of Different Supports

Figure 8.2 shows the support of the function and its ROC (Lim, 1990).

8.2 Properties of the Z-Transform

8.2.1 Properties of the Z-Transform

TABLE 8.1 Properties of the 2-D Z-Transform

$$x(n_1,n_2) \longleftrightarrow X(z_1,z_2), \quad ROC: R_x$$

$$y(n_1,n_2) \longleftrightarrow Y(z_1,z_2), \quad ROC: R_y$$

1. Linearity

$$ax(n_1,n_2)+by(n_1,n_2) \longleftrightarrow aX(z_1,z_2)+bY(z_1,z_2), \quad ROC: \text{ at least } R_x \cap R_y$$

2. Convolution

$$x(n_1,n_2) * y(n_1,n_2) = \sum_{k_1}\sum_{k_2} x(n_1-k_1,n_2-k_2)\,y(k_1,k_2) \longleftrightarrow X(z_1,z_2)Y(z_1,z_2), \quad ROC: \; R_x \cap R_y$$

3. Separable Signals

$$x(n_1,n_2) = x_1(n_1)\,x_2(n_2) \longleftrightarrow X(z_1,z_2) = X_1(z_1)\,X_2(z_2), \quad ROC: |z_1| \in ROC\, X_1(z_1) \text{ and } |z_2| \in ROC\, X_2(z_2)$$

4. Shift Property

$$x(n_1 \pm m_1, \; n_2 \pm m_2) \longleftrightarrow X(z_1,z_2) = z_1^{\pm m_1} z_2^{\pm m_2} \, X(z_1,z_2),$$

$$ROC: R_x \text{ with possible exceptions } |z_1| = 0,\infty \text{ and } |z_2| = 0,\infty$$

5. Differentiation Property

$$-n_1 x(n_1,n_2) \longleftrightarrow z_1 \frac{\partial \, X(z_1,z_2)}{\partial z_1}, \quad ROC: R_x$$

$$-n_2 x(n_1,n_2) \longleftrightarrow z_2 \frac{\partial \, X(z_1,z_2)}{\partial z_2}, \quad ROC: R_x$$

$$n_1 n_2 x(n_1,n_2) \longleftrightarrow z_1 \, z_2 \frac{\partial^2 \, X(z_1,z_2)}{\partial z_1 \partial z_2}, \quad ROC: R_x$$

6. Modulation Property

$$w(n_1,n_2) = a^{n_1} b^{n_2} \, x(n_1,n_2) \longleftrightarrow X(a^{-1}z_1, b^{-1}z_2), \quad ROC: W(z_1,z_2) \text{ has the same as } X(z_1,z_2),$$

but scaled by $|a|$ in z_1 variable and by $|b|$ in the z_2 variable.

7. Conjugate Properties

$$x(n_1,n_2) = \text{complex function} \longleftrightarrow X(z_1,z_2)$$

TABLE 8.1 Properties of the 2-D Z-Transform (continued)

	$x^*(n_1,n_2) \longleftrightarrow X^*(z_1^*,z_2^*)$
	$\mathrm{Re}\{x(n_1,n_2)\} \longleftrightarrow \frac{1}{2}[X(z_1,z_2)+X^*(z_1^*,z_2^*)]$
	$\mathrm{Im}\{x(n_1,n_2)\} \longleftrightarrow \frac{1}{2j}[X(z_1,z_2)-X^*(z_1^*,z_2^*)]$
	ROC: same as $X(z_1,z_2)$
8.	Reflection Properties
	$x(n_1,n_2) \longleftrightarrow X(z_1,z_2)$
	$x(-n_1,n_2) \longleftrightarrow X(z_1^{-1},z_2)$
	$x(n_1,-n_2) \longleftrightarrow X(z_1,z_2^{-1})$
	$x(-n_1,-n_2) \longleftrightarrow X(z_1^{-1},z_2^{-1})$ $\quad ROC: \lvert z_1^{-1}\rvert, \lvert z_2^{-1}\rvert$ in R_x
9.	Multiplication Property
	$x(n_1,n_2)y(n_1,n_2) \longleftrightarrow \left(\frac{1}{2\pi j}\right)^2 \oint_{C_2}\oint_{C_1} X\left(\frac{z_i}{\upsilon_1},\frac{z_2}{\upsilon_2}\right)Y(z_1,z_2)\frac{d\upsilon_1}{\upsilon_1}\frac{d\upsilon_2}{\upsilon_2}$
10.	Parseval's Theorem
	$\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} x(n_1,n_2)y^*(n_1,n_2) \longleftrightarrow \left(\frac{1}{2\pi j}\right)^2 \oint_{C_2}\oint_{C_1} X(z_1,z_2)Y^*\left(\frac{1}{z_1^*},\frac{1}{z_2^*}\right)\frac{dz_1}{z_1}\frac{dz_2}{z_2}$
	Contours must: closed, counter-clockwise, encircle the origin, lie totally within *ROC*.
11.	Initial Value Theorems
	$x(n_1,n_2)=0 \qquad n_1<0,\; n_2<0$
	$\lim_{z_1\to\infty} X(z_1,z_2) = \sum_{n_2} x(0,n_2)z_2^{-n_2}$
	$\lim_{z_2\to\infty} X(z_1,z_2) = \sum_{n_1} x(n_1,0)z_1^{-n_1}$
	$\lim_{\substack{z_1\to\infty \\ z_2\to\infty}} X(z_1,z_2) = x(0,0)$
12.	Linear Mapping of Variables
	$x(n_1,n_2)=y(m_1,m_2) \qquad n_1 = I\,m_1 + J\,m_2, \quad n_2 = Km_1 + Lm_2$
	I, J, K, L are integers
	$IL-KJ \neq 0$
	$X(z_1,z_2) = Y(z_1^I z_2^K, z_1^J z_2^L) \qquad ROC: (\lvert z_1^I z_2^K\rvert, \lvert z_1^J z_2^L\rvert)$ in R_x

8.3. Inverse Z-Transform

8.3.1 Inverse Z-Transform

$$x(n_1,n_2) = \left(\frac{1}{2\pi j}\right)^2 \oint_{C_2}\oint_{C_1} X(z_1,z_2)z_1^{n_1-1}z_2^{n_2-1}\,dz_1\,dz_2$$

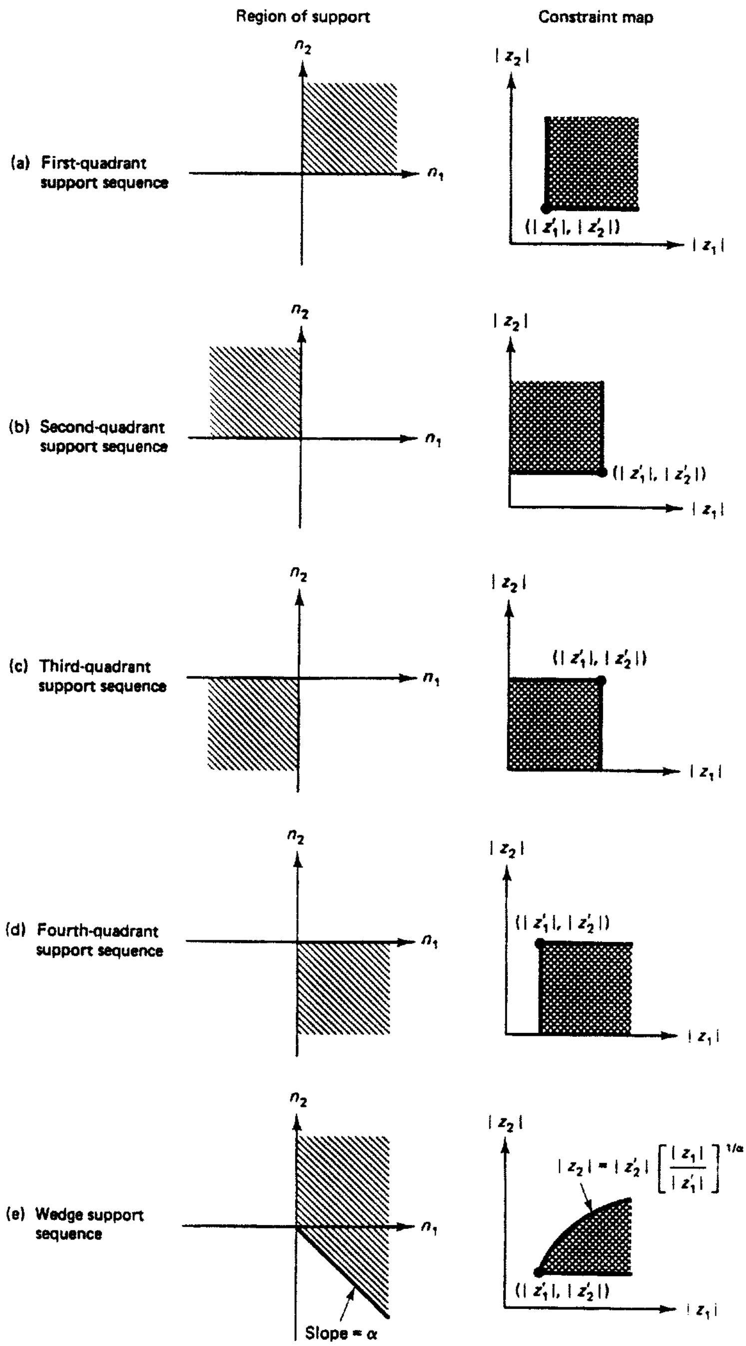

FIGURE 8.2

C_1, C_2 both in the ROC

C_1 counter-clockwise encircling the origin in the z_1 plane with z_2 fixed

C_2 counter-clockwise encircling the origin in the z_2 plane with z_1 fixed

8.4 System Function

8.4.1 System Function

$$\sum_{k_1}\sum_{k_2} a(k_1,k_2)\,y(n_1-k_1,n_2-k_2) = \sum_{k_1}\sum_{k_2} b(k_1,k_2)\,x(n_1-k_1,n_2-k_2)$$

$$a(k_1,k_2),\;\; b(k_1,k_2) \equiv \text{finite} - \text{extent sequences}$$

$$Y(z_1,z_2)\sum_{k_1}\sum_{k_2} a(k_1,k_2)\,z_1^{-k_1}\,z_2^{-k_2} = X(z_1,z_2)\sum_{k_1}\sum_{k_2} b(k_1,k_2)\,z_1^{-k_1}\,z_2^{-k_2}$$

or

$$H(z_1,z_2) = \frac{Y(z_1,z_2)}{X(z_1,z_2)} = \frac{\sum_{k_1}\sum_{k_2} b(k_1,k_2)\,z_1^{-k_1}\,z_2^{-k_2}}{\sum_{k_1}\sum_{k_2} a(k_1,k_2)\,z_1^{-k_1}\,z_2^{-k_2}} = \frac{B(z_1,z_2)}{A(z_1,z_2)}$$

8.5 Stability Theorems

8.5.1 Theorem 8.5.1.1 (Shanks, 1972)

Let $H(z_1,z_2) = 1/A(z_1,z_2)$ be a first quadrant recursive filter. This filter is stable if, and only if, $A(z_1,z_2) \neq 0$ for every point (z_1,z_2) such that $|z_1| \geq 1$ or $|z_2| \geq 1$.

8.5.2 Theorem 8.5.1.2 (Shanks, 1972)

Let $H(z_1,z_2) = 1/A(z_1,z_2)$ be a first quadrant recursive filter. Then $H(z_1,z_2)$ is stable if, and only if, the following conditions are true:

a) $A(z_1,z_2) \neq 0,\;\; |z_1| \geq 1,\;\; |z_2| = 1$
b) $A(z_1,z_2) \neq 0,\;\; |z_1| = 1,\;\; |z_2| \geq 1$

8.5.3 Theorem 8.5.1.3 (Huang, 1972)

Let $H(z_1,z_2) = 1/A(z_1,z_2)$ be a first-quadrant recursive filter. The filter is stable if, and only if, $A(z_1,z_2)$ satisfies the following two conditions:

a) $A(z_1,z_2) \neq 0,\;\; |z_1| \geq 1,\;\; |z_2| = 1$
b) $A(a,z_2) \neq 0,\;\; |z_2| \geq 1$ for any a such that $|a| \geq 1$

8.5.4 Theorem 8.5.1.4 (DeCarlo, 1977; Strintzis, 1977)

Let $H(z_1,z_2) = 1/A(z_1,z_2)$ be a first quadrant recursive filter. The filter is stable if, and only if, $A(z_1,z_2)$ satisfies the following three conditions:

a) $A(z_1,z_2) \neq 0,\;\; |z_1| = 1,\;\; |z_2| = 1$
b) $A(a,z_2) \neq 0,$ for $|z_2| \geq 1$ for any a such that $|a| = 1$
c) $A(z_1,b) \neq 0,$ for $|z_1| \geq 1$ for any b such that $|b| = 1$

References

DeCarlo, R., J. Murray, and R. Saeks, Multivariate Nyquist theory, *Int. J. Control*, 25, 657-75, 1977.

Dudgeon, Dan E. and Russell M. Mersereau, *Multidimensional Digital Signal Processing*, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1984.

Huang, Thomas S., Stability of two-dimensional recursive filters, *IEEE Trans. Audio and Electroacoustics*, AU-20, 158-63, 1972.

Lim, Jae S., *Two-Dimensional Signal and Image Processing*, Prentice-Hall Inc., Englewood Cliffs, NJ, 1990.

Shanks, John L., Sven Treitel, and James H. Justice, Stability and synthesis of two-dimensional recursive filters, *IEEE Trans. Audio and Electroacoustics*, AU-20, 115-28, 1972.

Strintzis, Michael G., Test of stability of multidimensional filters, *IEEE Trans. Circuits and Systems*, CAS-24, 432-37, 1977.

Appendix 1

Examples

1.1 Two-Dimensional Z-transforms

Example 8.1

The Z-transform of $x(n_1,n_2) = a^{n_1} b^{n_2} u(n_1,n_2) = a^{n_1} b^{n_2} u(n_1)u(n_2)$ is

$$X(z_1,z_2) = \sum_{n_1=-\infty}^{\infty} a^{n_1} u(n_1) z_1^{-n_1} \sum_{n_2=-\infty}^{\infty} b^{n_2} u(n_2) z_2^{-n_2} = \sum_{n_1=0}^{\infty} a^{n_1} z_1^{-n_1} \sum_{n_2=0}^{\infty} b^{n_2} z_2^{-n_2}$$

$$= \frac{1}{1-az_1^{-1}} \frac{1}{1-bz_2^{-1}}$$

with region of convergence (ROC) $|az_1^{-1}| < 1$ and $|bz_2^{-1}| < 1$, or $|z_1| > |a|$ and $|z_2| > |b|$ (see Figure 8.3).

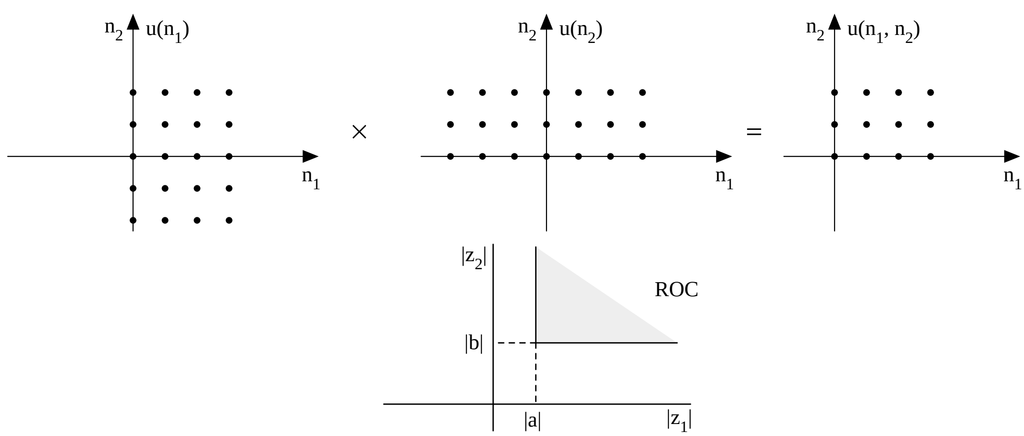

FIGURE 8.3

Example 8.2

The Z-transform of $x(n_1,n_2) = a^{n_1}\,\delta(n_1 - n_2)u(n_1,n_2)$ is

$$X(z_1,z_2) = \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} a^{n_1}\,u(n_1)u(n_2)\delta(n_1-n_2)z_1^{-n_1}\,z_2^{-n_2} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} a^{n_1}\,\delta(n_1-n_2)z_1^{-n_1}\,z_2^{-n_2}$$

$$= \sum_{n_2=0}^{\infty} a^{n_2}\,z_1^{-n_2}\,z_2^{-n_2} = \frac{1}{1-az_1^{-1}\,z_2^{-1}}$$

From last summation the ROC is $\left|az_1^{-1}\,z_2^{-1}\right| < 1$ or $|a| < |z_1|\,|z_2|$ equivalently

$$\ell n|a| < \ell n|z_1| + \ell n|z_2|$$

The function and its ROC are shown in Figure 8.4.

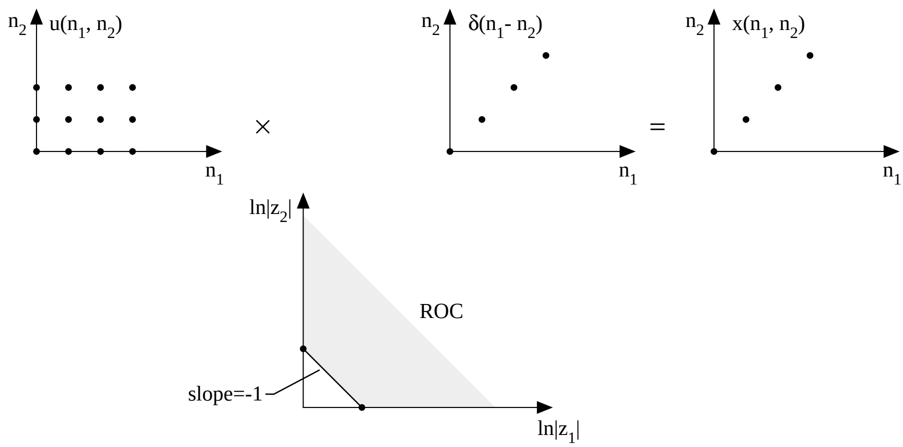

FIGURE 8.4

Example 8.3

The Z-transform of $x(n_1,n_2) = a^{n_1}\,u(n_1,n_2)u(n_1 - n_2)$ is

$$X(z_1,z_2) = \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} a^{n_1}\,u(n_1,n_2)u(n_1-n_2)z_1^{-n_1}\,z_2^{-n_2}$$

$$= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} a^{n_1}\,u(n_1-n_2)z_1^{-n_1}\,z_2^{-n_2} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} a^{n_1}\,z_1^{-n_1}\,z_2^{-n_2}$$

$$= \sum_{n_1=0}^{\infty} a^{n_1}\,z_1^{-n_1} = \frac{1-\left(z_2^{-1}\right)^{n_1+1}}{1-z_2^{-1}} = \frac{1}{\left(1-az_1^{-1}\right)\left(1-az_1^{-1}z_2^{-1}\right)}$$

$$\text{ROC}: \ |z_1| > |a|\ ,\quad |z_1\,z_2| > |a|$$

The function and its ROC are shown in Figure 8.5.

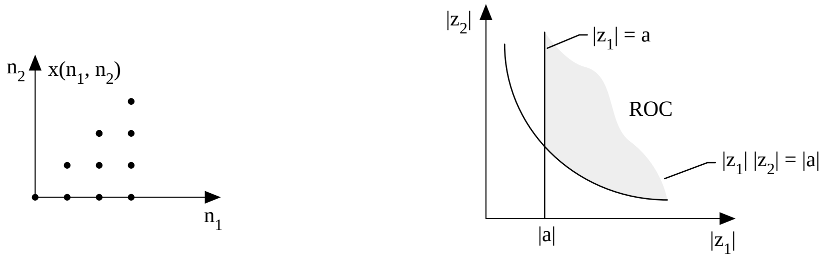

FIGURE 8.5

Example 8.4 (inverse integration)

If $X(z_1,z_2)$ has a region of convergence in the 2-D unit surface,

$$x(n_1,n_2)=\left(\frac{1}{2\pi j}\right)^2\oint_{C_2}\oint_{C_1}\frac{z_1^{n_1}z_2^{n_2}}{1-\frac{1}{2}z_1^{-1}-\frac{1}{4}z_2^{-2}}\,dz_1\,dz_2$$

$$=\left(\frac{1}{2\pi j}\right)^2\oint_{C_2}\oint_{C_1}\frac{(z_2-\frac{1}{4})^{-1}z_1^{n_1}z_2^{n_2}}{z_1-[\frac{1}{2}z_2/(z_2-\frac{1}{4})]}\,dz_1\,dz_2$$

$$=\frac{2\pi j}{(2\pi j)^2}\oint_{C_2}(z_2-\tfrac{1}{4})^{-1}\frac{(\frac{1}{2})^{n_1}z_2^{n_1}}{(z_2-\frac{1}{4})^{n_1}}z_2^{n_2}\,dz_2=(\tfrac{1}{2})^{n_1}(\tfrac{1}{4})^{n_2}\frac{(n_1+n_2)!}{n_1!\,n_2!},\quad n_1,n_2\ge 0$$