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The Hankel Transform

17.1 The Hankel Transform

17.1.1 Definition of the ν^{th} Order Hankel Transform

$$F_\nu(s) \equiv Ha_\nu\{f(r)\} = \int_0^\infty rf(r)J_\nu(sr)\,dr, \qquad r = \sqrt{x^2+y^2}$$

$$f(r) \equiv Ha_\nu^{-1}\{F_\nu(s)\} = \int_0^\infty sF_\nu(s)J_\nu(sr)\,ds$$

17.1.2 The Zero-Order Hankel Transform

$$F(s) \equiv Ha_0\{f(r)\} = \int_0^\infty rf(r)J_0(sr)\,dr$$

$$f(r) \equiv Ha_0^{-1}\{F(s)\} = \int_0^\infty sF(s)J_0(sr)\,ds$$

17.1.3 Relation to Fourier Transform with Function of Circular Symmetry

$$\mathcal{F}\{f(\sqrt{x^2+y^2})\} = F(u,v)$$

$$F(u,v) = 2\pi F(s) = 2\pi F(\sqrt{u^2+v^2})$$

with $\mathcal{F}\{f(x,y)\} = \iint_{-\infty}^{\infty} f(x,y)\exp[-j(xu+yv)]\,dx\,dy.$

Example

$\mathcal{F}\{\exp[-a(x^2+y^2)]\} = \frac{\pi}{a}\exp[-(u^2+v^2)/4a]$ and, therefore,

$$F(s)=\frac{1}{2\pi}F(u,v)=\frac{1}{2a}\exp[-s^2/4a],\quad s^2=(u^2+v^2),\quad a>0$$

17.2 Properties of Hankel Transform

17.2.1 Derivatives

$$F_v(s)=Ha_v\{f(r)\}$$

$$G_v(s)=Ha_v\{f'(r)\}=s\left[\frac{v+1}{2v}F_{v-1}(s)-\frac{v-1}{2v}F_{v+1}(s)\right]$$

$$Ha_v\left\{\frac{d^2f(r)}{dr^2}+\frac{1}{r}\frac{df(r)}{dr}-\left(\frac{v}{r}\right)^2 f(r)\right\}=-s^2Ha_v\{f(r)\}$$

Note: $\frac{d}{dr}[rJ_v(sr)]=\frac{sr}{2v}[(\nu+1)J_{v-1}(sr)-(\nu-1)J_{v+1}(sr)]$

Example

$$F\left\{\frac{1}{r}\frac{d}{dr}\left[r\frac{df(r)}{dr}\right]\right\}=-(u^2+v^2)F(u,v).$$

But from 17.1.3

$$F\left\{\frac{1}{r}\frac{d}{dr}\left[r\frac{df(r)}{dr}\right]=\frac{d^2f(r)}{dr^2}+\frac{1}{r}\frac{df(r)}{dr}\right\}=-2\pi s^2F(s)$$

and hence

$$Ha_0\left\{\frac{d^2f(r)}{dr^2}+\frac{1}{r}\frac{df(r)}{dr}\right\}=-s^2F(s)=-s^2Ha_0\{f(r)\}$$

17.2.2 Similarity

$$Ha_v\{f(ar)\}=\frac{1}{a^2}F_v\left(\frac{s}{a}\right)$$

Example (see 17.1.3)

$$F\left\{f(a\sqrt{x^2+y^2})\right\}=F\left\{f(\sqrt{(ax)^2+(ay)^2})\right\}=\iint f(\sqrt{(ax)^2+(ay)^2})\exp[-jux-jvy]\,dx\,dy$$

$$=\frac{1}{a^2}\iint f(\sqrt{t^2+\tau^2})\exp[-j\tfrac{u}{a}t-j\tfrac{v}{a}\tau]\,dt\,d\tau=\frac{1}{a^2}2\pi F\left(\frac{s}{a}\right).$$

Hence

$$Ha_0\{f(ar)\}=\frac{1}{a^2}F\left(\frac{s}{a}\right)$$

17.2.3 Division by r

1. $\mathcal{H}a_v\{r^{-1}f(r)\} = \frac{s}{2v}\left[F_{v-1}(s) + F_{v+1}(s)\right]$

2. $\mathcal{H}a_v\{r^{v-1}\frac{d}{dr}[r^{1-v}f(r)]\} = sF_{v-1}(s)$

3. $\mathcal{H}a_v\left\{r^{-v-1}\frac{d}{dr}\left[r^{v+1}f(r)\right]\right\} = sF_{v+1}(s)$

17.2.4 Parseval's Theorem

$$F_v(s) = \mathcal{H}a_v\{f(r)\}, \quad G_v(s) = \mathcal{H}a_v\{g(r)\}$$

1. $\int_0^\infty F_v(s)G_v(s)s\,ds = \int_0^\infty rg(r)\,f(r)\,dr$

2. $\int_0^\infty F_v(s)G_v^*(s)s\,ds = \int_0^\infty rf(r)g^*(r)\,dr$ for complex signals

3. $\int_0^\infty r|f(r)|^2\,dr = \int_0^\infty s|F(s)|^2\,ds$

17.2.5 Convolution Identity

$$\mathcal{F}_{(2)}\left\{\iint_{-\infty}^{\infty} f_1(\sqrt{x_1^2+y_1^2})\,f_2(\sqrt{(x-x_1)^2+(y-y_1)^2})\,dx_1\,dy_1\right\} = 4\pi^2 F_1(s)F_2(s)$$

Hence $\mathcal{H}a_0\{f_1(r)**f_2(r)\} = \frac{1}{2\pi}\mathcal{F}_{(2)}\{f_1(r)**f_2(r)\} = 2\pi F_1(s)F_2(s)$

Also $\mathcal{H}a_0\{2\pi f_1(r)\,f_2(r)\} = F_1(s)**F_2(s)$

17.2.6 Moment

$$m_n = \int_0^\infty r^n f(r)\,dr$$

But

$$J_0(sr) = 1 - \left(\frac{sr}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{sr}{2}\right)^4 - \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{sr}{2}\right)^{2n}$$

hence

$$F(s) = \mathcal{H}a_0\{f(r)\} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{s}{2}\right)^{2n}\int_0^\infty r^{2n+1}f(r)\,dr = \sum_{n=0}^{\infty}\frac{(-1)^n m_{2n+1}}{(n!)^2 2^{2n}}s^{2n}$$

17.3 Examples of Hankel Transform

17.3.1 Example

From

$$\int_0^a rJ_0(sr)\,dr = \int_0^a \frac{1}{s}\frac{d}{dr}[rJ_1(sr)] = [aJ_1(as)]/s$$

implies that $Ha_0\{p_a(r)\} = \frac{a}{s}J_1(as)$, where $p_a(r) = 1$ for $r < a$ and zero otherwise.

17.3.2 Example

From $\int_0^a J_0(sr)dr = \frac{1}{s}$, $s > 0$ we obtain $Ha_0\{1/r\} = 1/s$.

17.3.3 Example

From $\int_0^a r\delta(r-a)\,J_0(sr)\,dr = aJ_0(as)$ we obtain $Ha_0\{\delta(r-a)\} = aJ_0(as)$, $s > 0$ and because of symmetry $Ha_0\{aJ_0(ar)\} = \delta(s-a)$, $a > 0$

17.3.4 Example

If $f_1(r) = f_2(r) = [J_1(ar)]/r$ then from 17.2.5 $Ha_0\{2\pi J_1^2(ar)/r^2\} = \frac{1}{a^2}p_a(s)**p_a(s)$ where

$$p_a(s)**p_a(s) = \left(2\cos^{-1}\frac{s}{2a} - \frac{s}{a}\sqrt{1-\frac{s^2}{4a^2}}\right)a^2.$$

Hence

$$Ha_0\{2\pi J_1^2(ar)/r^2\} = \left(2\cos^{-1}\frac{s}{2a} - \frac{s}{a}\sqrt{1-\frac{s^2}{4a^2}}\right)p_{2a}(s),$$

where $p_{2a}(s) = 1$ for $|s| \le 2a$ and 0 otherwise.

17.3.5 Example

From the relationship

$$\int_0^a rJ_0(br)\,J_0(sr)\,dr = a[bJ_1(ab)\,J_0(as) - sJ_0(ab)\,J_1(as)]/(b^2 - s^2)$$

we find

$$Ha_0\{J_0(br)\,p_a(r)\} = [abJ_1(ab)\,J_0(as) - asJ_0(ab)\,J_1(as)]/(b^2 - s^2)\,.$$

17.3.6 Example

From $\delta(s-a)**\delta(s-a) = 4a^2/(s\sqrt{4a^2 - s^2})$ for $s < 2a$ and equals zero for $s > 2a$, 17.2.5 and 17.3.3 we obtain $Ha_0\{J_0(ar)\,J_0(ar)\} = 2p_{2a}(s)/(\pi s\sqrt{4a^2 - s^2})$.

17.3.7 Example

From $p_a(s) ** \delta(s-a) = 2a\cos^{-1}(s/2a)$ for $s \le 2a$ and cycles zero for $s > 2a$, from $\mathcal{H}a_0\{J_0(ar)\} = \delta(s-a)/a$; from $\mathcal{H}a_0\{J_1(ar)/r\} = p_a(s)/a$ and 17.2.5 we obtain $\mathcal{H}a_0\{J_0(ar)J_1(ar)/r\} = p_{2a}(s)\ \cos^{-1}(s/2a)/(a\pi)$.

17.3.8 Example

$\mathcal{H}a_v\{r^v u(a-r)\} = \int_0^a r^{v+1} J_v(sr)\,dr = \frac{1}{s^{v+2}}\int_0^{as} x^{v+1} J_v(x)\,dx,\ a>0$ where $u(a-r)=1$ for $r \le a$ and 0 for $r > a$ is the unit step function. But $\int x^v J_{v-1}(x)\,dx = x^v J_v(x) + C$ (see 25.3.2) and, hence,

$$\mathcal{H}a_v\{r^v h(a-r)\} = \frac{(as)^{v+1}}{s^{v+2}} J_{v+1}(as) = a^{v+1} J_{v+1}(as),\ \ a>0,\ \ v > -1/2.$$

17.3.9 Example

$\mathcal{H}a_0\{e^{-ar}\} = L\{rJ_0(sr)\} = -\frac{d}{da}[(s^2+a^2)^{-1/2}] = a/[s^2+a^2]^{3/2},\ \ a>0$

17.4 Relation to Fourier Transform

17.4.1 Relationship Between Fourier and Hankel Transform

If $F(s)$ is the Hankel transform of $f(r)$, then

1. $2\pi F(\sqrt{u^2+v^2}) = F(u,v) = \mathcal{F}\{f\sqrt{x^2+y^2})$
2. $\Phi(\omega) = \int_{-\infty}^{\infty} e^{-j\omega x}\varphi(x)\,dx$

   $\varphi(x) = \int_{-\infty}^{\infty} f(\sqrt{x^2+y^2})\,dy$

   $\mathcal{F}\{\varphi(x)\} = 2\pi F(s)\big|_{s=\omega}$

17.4.2 Example

If $f(r) = p_a(r)$, then

$$\varphi(x) = \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy = 2\sqrt{a^2-x^2}$$

for $|x| < a$, and $\varphi(x) = 0$ for $|x| > a$. But $\mathcal{H}a_0\{p_a(r)\} = aJ_1(as)/s$ (see 17.3.1) and, hence,

$$\mathcal{F}\{2\sqrt{a^2-x^2}\ p_a(x)\} = 2\pi J_1(a\omega)/\omega\ .$$

17.4.3 Example

If $f(r) = p_a(r)/\sqrt{a^2-x^2}$, then

$$\varphi(x) = \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{dy}{\sqrt{a^2-(x^2+y^2)}} = \int_{-\pi/2}^{\pi/2} d\theta = \pi$$

for $|x| < a$ and equals zero for $|x| > a$.
Hence

$$\Phi(\omega) = \int_{-a}^{a} \pi e^{-j\omega x} d = \frac{2\pi \sin a\omega}{\omega} = 2\pi F(s)\Big|_{s=\omega}$$

which implies that $Ha_0\{p_a(r)/\sqrt{a^2-r^2}\} = \sin as/s$.

17.5 Hankel Transforms of Order Zero

Table 17.1 lists the Hankel transforms of some particular functions for the important special case $\nu = 0$. Table 17.2 lists Hankel transforms of general order ν. In these tables, $u(x)$ is the unit step function, I_ν and K_ν are modified Bessel functions, $\boldsymbol{L}_0$ and $\boldsymbol{H}_0$ are Struve functions, and Ker and Kei are Kelvin functions as defined in Abramowitz and Stegun.

TABLE 17.1 Hankel Transform of Order Zero

$f(r)$	$F_0(s) = Ha_0\{f(r);s\}$
Algebraic function	
$1/r$	$1/s$
$1/r^\mu, \quad \frac{1}{2} < \mu < 2$	$2^{1-\mu} \dfrac{\Gamma\left(1-\frac{\mu}{2}\right)}{\Gamma\left(\frac{\mu}{2}\right)} \dfrac{1}{s^{2-\mu}}$
$\dfrac{1}{(a^2+r^2)^{1/2}}, \quad \mathrm{Re}(a) > 0$	$\dfrac{e^{-as}}{s}$
$\begin{cases} \dfrac{1}{(a^2-r^2)^{1/2}}, & 0 < r < a \\ 0, & a < r < \infty \end{cases}$	$\dfrac{\sin(as)}{s}$
$\dfrac{1}{(r^2+a^2)^{3/2}}, \quad \mathrm{Re}(a) > 0$	$a^{-1}e^{-as}$
$\dfrac{1}{r(r+a)}$	$\dfrac{\pi}{2}[\mathbf{H}_0(as) - Y_0(as)]$
$\dfrac{1}{r^2+a^2}$	$K_0(as)$
$\dfrac{1}{r(r^2+a^2)}$	$\dfrac{\pi}{2a}[I_0(as) - \mathbf{L}_0(as)]$
$\dfrac{1}{1+r^4}$	$-\mathrm{Kei}(s)$
$\dfrac{1}{a^4+r^4} \quad \lvert\arg a\rvert < \pi/4$	$-a^{-2}\mathrm{Kei}(as)$

TABLE 17.1 Hankel Transform of Order Zero (continued)

$f(r)$	$F_0(s) = Ha_0\{f(r);s\}$
$\dfrac{r^2}{1+r^4}$	$\mathrm{Ker}(s)$
$\dfrac{1}{\sqrt{r^4+a^4}}$	$K_0(as/\sqrt{2})\,J_0(as/\sqrt{2})$
Exponential function	
$e^{-ar}, \quad \mathrm{Re}(a) > 0$	$\dfrac{s}{(s^2+a^2)^{3/2}}$
$\dfrac{e^{-ar}}{r}$	$\dfrac{1}{(s^2+a^2)^{1/2}}$
$(1-e^{-ar})r^{-2}, \quad \mathrm{Re}(a) > 0$	$\sinh^{-1}(a/s)$
$r^{n-\frac{1}{2}}\,e^{-ar}, \quad \mathrm{Re}(a) > 0$	$\dfrac{n!}{(a^2+s^2)^{\frac{1}{2}n+\frac{1}{2}}} P_n\left[\dfrac{a}{(a^2+s^2)^{1/2}}\right]$
$e^{-a^2r^2}$	$\dfrac{e^{-s^2/4a^2}}{2a^2}$
Trigonometric function	
$\dfrac{\sin ar}{r}, \quad a > 0$	$\begin{cases} \dfrac{1}{(a^2-s^2)^{1/2}}, & 0 < s < a \\ 0 & a < s < \infty \end{cases}$
$\dfrac{\sin ar}{r^2}, \quad a > 0$	$\begin{cases} \frac{1}{2}\pi, & 0 < s < a \\ \sin^{-1}(a/s), & a < s < \infty \end{cases}$
$\dfrac{\sin ar}{b^2+r^2}, \quad a > 0 \quad \mathrm{Re}(b) > 0$	$\dfrac{\pi}{2} e^{-ab}\, I_0(sb), \quad 0 < s < a$
$\dfrac{\cos ar}{r}, \quad a > 0$	$\begin{cases} 0, & 0 < s < a \\ (s^2-a^2)^{-1/2}, & a < s < \infty \end{cases}$
$\dfrac{1-\cos ar}{r^2}, \quad a > 0$	$\begin{cases} \cosh^{-1}(a/s), & 0 < s < a \\ 0, & a < s < \infty \end{cases}$
$\dfrac{\cos ar}{b^2+r^2}, \quad a > 0 \quad \mathrm{Re}(b) > 0$	$\cosh(ab)K_0(bs), \quad a < s < \infty$
$\cos(a^2r^2/2), \quad a > 0$	$a^{-2}\sin(a^{-2}s^2/2)$
Other functions	
$\dfrac{1-J_0(ar)}{r^2}, \quad a > 0$	$\begin{cases} \ln\dfrac{a}{s}, & s < a \\ 0, & s > a \end{cases}$
$\dfrac{J_1(ar)}{r}, \quad a > 0$	$\begin{cases} a^{-1}, & 0 < s < a \\ 0, & a < s < \infty \end{cases}$

TABLE 17.2 Hankel Transform of Order ν^{th}

$f(r)$	$F_\nu(s) = Ha_\nu\{f(r);s\}$
Algebraic functions	
$1/r, \quad \text{Re}(\nu) > -1$	$1/s$
$1/r^\mu, \quad \frac{1}{2} < \mu < \nu + 2$	$\dfrac{2^{1-\mu}\Gamma\left(\frac{\nu+2-\mu}{2}\right)}{s^{2-\mu}\Gamma\left(\frac{\nu+\mu}{2}\right)}$
$\begin{cases} r^\nu, & 0<r<1, \ \text{Re}(\nu)>-1 \\ 0, & 1<r<\infty \end{cases}$	$\dfrac{J_{\nu+1}(s)}{s}$
$\dfrac{r^\nu}{r^2+a^2}, \quad \text{Re}(a)>0, \ -1<\text{Re}(\nu)<\frac{3}{2}$	$a^\nu K_\nu(as)$
$r^\nu(a^2-r^2)^\mu\, u(a-r), \quad \text{Re}(\nu)>-1, \quad 0<r<a, \quad \text{Re}(\mu)>-1$	$2^\mu\Gamma(\mu+1)s^{-\mu-1}a^{\nu+\mu+1}J_{\nu+\mu+1}(as)$
$\dfrac{r^\nu}{(r^2+a^2)^{\nu+\frac{1}{2}}}, \quad \text{Re}(a)>0, \ \text{Re}(\nu)>-\frac{1}{2}$	$\dfrac{\sqrt{\pi}\, s^{\nu-1}}{2^\nu e^{as}\Gamma(\nu+\frac{1}{2})}$
$\dfrac{r^\nu}{(r^2+a^2)^{\mu+1}}, \quad \text{Re}(a)>0, \quad -1<\text{Re}(\nu)<2\,\text{Re}(\mu)+\frac{3}{2}$	$\dfrac{a^{\nu-\mu}s^\mu K_{\nu-\mu}(as)}{2^\mu\Gamma(\mu+1)}$
$\dfrac{r^\nu}{(a^2-r^2)^{\nu+\frac{1}{2}}}u(a-r), \quad 0<r<a, \quad \lvert\text{Re}(\nu)\rvert<\frac{1}{2}$	$\pi^{-1/2}2^{-\nu}\Gamma(\frac{1}{2}-\nu)s^{\nu-1}\sin(as)$
$\dfrac{r^\nu}{(r^4+4a^4)^{\nu+\frac{1}{2}}}, \quad \lvert\arg(a)\rvert<\pi/4, \quad \lvert\text{Re}(\nu)\rvert>-\frac{1}{2}$	$\dfrac{s^\nu\sqrt{\pi}\, J_\nu(as)K_\nu(as)}{a^{2\nu}2^{3\nu}e^{as}\Gamma(\nu+\frac{1}{2})}$
$\dfrac{r^{\nu+2}}{(r^4+4a^4)^{\nu+\frac{1}{2}}}, \quad \lvert\arg(a)\rvert<\pi/4, \quad \lvert\text{Re}(\nu)\rvert>\frac{1}{6}$	$\dfrac{\sqrt{\pi}\, s^\nu J_{\nu-1}(as)K_{\nu-1}(as)}{2^{3\nu-1}a^{2\nu-1}\Gamma(\nu+\frac{1}{2})}$
Exponential function	
$\dfrac{e^{-ar}}{r}, \quad \text{Re}(a)>0, \ \text{Re}(\nu)>-1$	$s^{-\nu}(s^2+a^2)^{-\frac{1}{2}}[(a^2+s^2)^{\frac{1}{2}}-a]^\nu$
$\dfrac{e^{-ar}}{r^2}, \quad \text{Re}(a)>0, \ \text{Re}(\nu)>0$	$\nu^{-1}s^{-\nu}[(a^2+s^2)^{\frac{1}{2}}-a]^\nu$
$r^\nu e^{-ar}, \quad \text{Re}(a)>0, \ \text{Re}(\nu)>-1$	$\dfrac{1}{\sqrt{\pi}}2^{\nu+1}\Gamma(\nu+\frac{3}{2})as^\nu\dfrac{1}{(a^2+s^2)^{\nu+\frac{3}{2}}}$
$r^{\nu-1}e^{-ar}, \quad \text{Re}(a)>0, \ \text{Re}(\nu)>-\frac{1}{2}$	$\dfrac{1}{\sqrt{\pi}}2^\nu\Gamma(\nu+\frac{1}{2})s^\nu\dfrac{1}{(a^2+s^2)^{\nu+\frac{1}{2}}}$
$r^\nu e^{-ar^2}, \quad \text{Re}(a)>0, \ \text{Re}(\nu)>-1$	$\dfrac{s^\nu}{(2a)^{\nu+1}}\exp\left(-\dfrac{s^2}{4a}\right)$
Trigonometric functions	
$\dfrac{\sin ar}{r}, \quad a>0, \quad \text{Re}(\nu)>-2$	$\cos(\pi\nu/2)s^\nu(a^2-s^2)^{-\frac{1}{2}}[a+(a^2-s^2)^{\frac{1}{2}}]^{-\nu}, \quad 0<s<a$ $(s^2-a^2)^{-\frac{1}{2}}\sin[\nu\sin^{-1}(a/s)], \quad a<s<\infty$

TABLE 17.2 Hankel Transform of Order ν^{th} (continued)

$f(r)$	$F_\nu(s) = Ha_\nu\{f(r);s\}$
$\frac{\sin ar}{r^2}, \quad a>0, \quad \mathrm{Re}(\nu)>-1$	$\nu^{-1}\sin(\nu\pi/2)s^\nu[a+(a^2-s^2)^{\frac{1}{2}}]^{-\nu}, \quad 0<s<a$ $\nu^{-1}\sin[\nu\sin^{-1}(a/s)], \quad a<s<\infty$
$r^\nu \sin ar, \quad a>0, \quad -\frac{3}{2}<\mathrm{Re}(\nu)<-\frac{1}{2}$	$\frac{-2^{\nu+1}\sin\nu\pi\,\Gamma(\nu+\frac{3}{2})as^\nu}{\sqrt{\pi}\,(a^2-s^2)^{-\nu-\frac{3}{2}}}, \quad 0<s<a$ $\frac{-2^{\nu+1}\Gamma(\nu+\frac{3}{2})as^\nu}{\sqrt{\pi}\,(s^2-a^2)^{-\nu-\frac{3}{2}}}, \quad a<s<\infty$
$r^{\nu-1}\sin ar, \quad a>0, \quad -1<\mathrm{Re}(\nu)<\frac{1}{2}$	$\frac{\sqrt{\pi}\;2^\nu\;s^\nu}{\Gamma(\frac{1}{2}-\nu)\;(a^2-s^2)^{-\nu-\frac{1}{2}}}, \quad 0<s<a$ $0, \quad a<s<\infty$
$r^\nu \cos ar, \quad a>0, \quad -1<\mathrm{Re}(\nu)<-\frac{1}{2}$	$\frac{2^{1+\nu}\sqrt{\pi}\;a\,s^\nu}{\Gamma(-\frac{1}{2}-\nu)\;(a^2-s^2)^{-\nu-\frac{3}{2}}}, \quad 0<s<a$ $0, \quad a<s<\infty$
Other functions	
$\frac{J_{\nu-1}(ar)}{r}, \quad a>0, \quad \mathrm{Re}(\nu)>-1$	$0, \quad 0<s<a$ $a^{\nu-1}s^{-\nu}, \quad a<s<\infty$
$\frac{J_\nu(ar)}{r^2}, \quad a>0, \quad \mathrm{Re}(\nu)>0$	$\frac{1}{2\nu}\frac{s^\nu}{a^\nu}, \quad 0<s<a$ $\frac{1}{2\nu}\frac{a^\nu}{s^\nu}, \quad a\le s<\infty$
$\frac{J_{\nu+1}(ar)}{r}, \quad a>0, \quad \mathrm{Re}(\nu)>-\frac{3}{2}$	$a^{-\nu-1}s^\nu, \quad 0<s<a$ $0, \quad a<s<\infty$

References

Bracewell, R. N., *The Fourier Transform and Its Applications*, McGraw-Hill Book Co., New York, 1978.

Davies, B., Integral *Transforms and Their Applications*, 2nd ed., Springer-Verlag, New York, 1984.

Erdelyi, A., W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Tables of Integral Transforms*, McGraw-Hill Book Co., New York, 1954.

Papoulis, A., *Systems and Transforms with Applications in Optics*, McGraw-Hill Book Co., New York, 1968.

Piessens, R., The Hankel Transforms, in *The Transforms and Applications Handbook*, ed. Alexander D. Poularikas, CRC Press, Boca Raton, FL, 1996.

Sneddon, Ian N., *The Use of Integral Transforms*, McGraw-Hill Book Co., New York, 1972.

Wolf, K., B., *Integral Transforms in Science Engineering*, Plenum Press, New York, 1979.