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Polarization Measurement

Soe-Mie F. Nee
U.S. Naval Air Warfare Center

60.1 Basic Concepts of Polarization

Polarization of *light* is a property of electromagnetic (EM) waves, which include heat, microwaves, radio waves, and x-rays. An EM wave has orthogonal electric and magnetic fields associated with it which vibrate in directions perpendicular to the direction of propagation. The *electric field* of a sinusoidal EM wave, in particular, can always be decomposed into two orthogonal components; each component has an *amplitude* and a *phase*. The amplitude is the maximum value of the field component, and the light *intensity* is proportional to the square of the amplitude. The phase, referred to a fixed position or time, tells what part of the cycle the electric field is vibrating in. G. G. Stokes pointed out in 1852 that these two orthogonal components do not interfere in amplitude but are additive according to vector algebra [1]. When the two orthogonal components are in phase, the EM wave is *linearly polarized.* When the two orthogonal components have the same amplitude and a relative phase of 90°, the EM wave is *circularly polarized.* In general, an EM wave has arbitrary amplitudes and phases for the two orthogonal fields and is elliptically polarized. The concept of polarization ellipse and the descriptions for the polarization of an EM wave in terms of Jones vectors and Stokes vectors are given in the subsection, "Polarization of an EM Wave," and also in References 1 through 9.

Light is composed of an ensemble of EM waves. A group of EM waves traveling in the same direction can have some linearly polarized waves, some circularly polarized waves, and some elliptically polarized waves. When they are combined, resulting light can be unpolarized, partially linearly polarized, or partially elliptically polarized. Unpolarized light occurs when there are no fixed directions of the electric field and also no fixed phase relations between the two orthogonal field components. In general, light is partially polarized and can be decomposed into unpolarized light and elliptically polarized light. These concepts are described in terms of Stokes vector in the subsection, "Polarization of Light," and also in References 1 through 6.

Polarized light can be produced by passing light through a polarizer. An ideal polarizer transmits only light whose electric field is parallel to the transmission axis of the polarizer and rejects light with the orthogonal field. Polarization of light can be observed by stacking two polarizers together and turning one with respect to the other. The transmitted light intensity through these two polarizers will vary, and at some particular positions it will vanish. In this case, light is linearly polarized after passing through the first polarizer. When the second polarizer is turned until its axis is perpendicular to the axis of the first polarizer, light cannot pass through the second polarizer. The transmitted intensity varies according to the square of the cosine of the angle between the two polarizers [5, 9–12]. Real polarizers are not perfect and transmit light with minimum intensity I_{min} when the polarizer axis is perpendicular to the polarization of purely linearly polarized incident light. This is caused by the small depolarization of a polarizer [12]. Depolarization is a mechanism that turns polarized light into unpolarized light and is the opposite effect of polarization. The maximum transmitted intensity I_{max} occurs when the polarizer axis is parallel to the incident polarization direction. The extinction ratio of a polarizer is defined as $I_{\text{min}}/I_{\text{max}}$. Other relations for polarizers can be found in References 9 through 12.

Besides the polarizer, another basic element in polarization measurements is the phase retarder or wave plate. A phase retarder changes the relative phase between the two orthogonal fields of an EM wave [4–11]. The change of relative phase between the two orthogonal components is called the *phase retardation* or *retardance*. The retardance of a quarter-wave retarder is 90°, and that of a half-wave retarder is 180°. Circularly polarized light can be generated by passing linearly polarized light through a quarter-wave plate whose axis is at 45° with respect to the incident linear polarization direction. A half-wave plate may change the polarization direction of linearly polarized light. In general, a phase retarder changes linearly polarized light into elliptically polarized light. Representations of the optical response of polarizers, retarders, and other materials in terms of the Müeller matrix and Jones matrix are given in the subsection "Polarization by the Response of a Medium" and also in References 4 through 8 and 12 through 24.

Polarization is generated by the anisotropic response of materials and/or anisotropic geometry of systems. The mechanisms for producing polarization include preferential absorption in a dichroic material, reflection and transmission at oblique incidence, double refraction in a birefringent material, diffraction by grating or wires, and scattering by particles [2–11, 13]. These properties can be utilized to make polarizers and phase retarders. For example, a wire-grid polarizer is made of parallel fine conducting wires. When light is incident on a wire-grid polarizer with the grid period smaller than the wavelength, the electric field parallel to the wires is shorted and absorbed so that only the electric field perpendicular to the wires passes through the polarizer. In a dichroic polarizer, anisotropic molecules are aligned in a preferential direction so that absorption is very different for the two orthogonal directions referred to the alignment direction. The nonpreferential field is absorbed by the molecules in the medium, while the preferential field passes through the medium [5, 8]. In a prism polarizer, the two orthogonal fields are separated by double refraction in a birefringent crystal, and the unwanted polarization is deflected away by the special geometry of a prism. A material is birefringent when it has different refractive indices for different field directions. When a light beam passes through a birefringent slab, a phase retardation is generated. Birefringent slabs can be used to make phase retarders or wave plates. Different kinds of polarizers and retarders are described in detail in References 5 and 9 through 11.

When special arrangements of polarizers and phase retarders are combined with a light source and a detector, polarized light can be generated and analyzed. Such an optical system is called a *polarimeter* or an *ellipsometer.* The subsection "Principles of Polarimetry" discusses the generation and analysis of polarized light and the operational principles for Polarizer-Sample-Analyzer (PSA) ellipsometry and

Polarizer-Compensator-Sample-Analyzer (PCSA) ellipsometry using the intensity approach associated with Stokes vectors and Müeller matrices [4, 17–23, 25, 26]. A phase retarder is also called a *compensator* because it was introduced into a polarimeter to compensate the phase change by a sample. The intensity approach was chosen because intensity, but not electric field, is measured in most experiments, and also because the electric field approach cannot treat depolarization. However, the electric field approach is convenient to use for highly polarized light when depolarization does not cause appreciable errors in the measurement. Discussion of ellipsometry using the electric field approach can be found in References 4, 15, 16, and 27 through 33].

Polarization effects are widely applied in modern optical technologies. The electro-optic modulator and shutter are based on tunable birefringence by applying a high voltage across a birefringent crystal to modulate the phase of transmitted light and hence to achieve intensity modulation [4, 7, 9, 34, 35]. Liquid crystal displays use similar principles [36]. Birefringence can also be modulated by the photo-elastic effect [37, 38]. The magneto-optical readout for laser disks utilizes the magneto-optical Kerr effect that generates phase retardation upon reflection from magnetic materials [34, 39]. Other applications of polarization are fiber optics, nonlinear optics, material characterization, medical optics, and many other fields. All of these applications utilize the anisotropic nature of materials or the anisotropic geometry of systems. This chapter is concerned with the application of polarization on material characterization. In this application, a polarimeter or ellipsometer is used to measure optical properties and surface properties of materials and thin films [40–48, see also Chapter 61, "Refractive Index"]. In the subsection "Polarization Instrumentation and Experiments," different components of polarimeters are discussed using an example of an automated reflection null ellipsometer, and two sample experiments are described to measure birefringence of a birefringent slab and the optical constants of a material.

60.2 Polarization of an Electromagnetic Wave

The electric field $E(z, t)$ of a monochromatic EM wave propagating along the z-direction with a frequency ω and an angular wave-number k can be decomposed into two orthogonal components E_x and E_y, and represented by

$$\begin{cases} \mathbf{E}(z,t) = \hat{x}E_x(z,t) + \hat{y}E_y(z,t) \\ E_x(z,t) = a_x e^{i(\omega t - kz + \delta_x)} \\ E_y(z,t) = a_y e^{i(\omega t - kz + \delta_y)} \end{cases} \tag{60.1}$$

where a_x and δ_x are the amplitude and phase, respectively, for E_x, and a_y and δ_y are for E_y [1–4]. k is related to wavelength λ by $k = 2\pi/\lambda$. In vacuum, $k = \omega/c$. Let $\delta = \delta_y - \delta_x$ be the relative phase between E_y and E_x. Then Equation 60.1 can be simplified to

$$\mathbf{E}(z,t) = (\hat{x}\, a_x + \hat{y}\, a_y\, e^{i\delta})e^{i(\omega t - kz + \delta_x)} \tag{60.2}$$

Polarization Ellipse

It is often convenient to express **E** in terms of a complex variable. The observed field is actually the real part of **E**. The projection of Re[**E**(z, t)] with $\delta_x = 0$ onto the xy-plane at $z = 0$ is given by

$$\mathbf{E}(0,t) = \hat{x}\, a_x \cos\omega t + \hat{y}\, a_y \cos(\omega t + \delta) \tag{60.3}$$

The loci of **E**(0,t) with $a_x = 3$, $a_y = 2$ and different values of δ are shown in Figure 60.1. For $\delta = 0$, the locus of **E** is a line with a slope of a_y/a_x. The EM wave is linearly polarized when E_x and E_y are in phase with each other. In other cases, the loci are ellipses, which are called *polarization ellipses,* and the EM wave is elliptically polarized. The instantaneous electric field can be visualized by drawing an arrow from the origin to a point on an ellipse. The electric field direction rotates in the clockwise direction for $0 < \delta < 180°$, while in the counter clockwise direction for $-180 < \delta < 0°$. When $\delta = \pm 90°$, the axes of the ellipse correspond to the x and y coordinate axes. If $a_x = a_y$, this ellipse then becomes a circle, and the EM wave is circularly polarized. The convention in ellipsometry defines the right-handed circularly polarized wave as the one whose field rotates in the clockwise direction with $\delta = 90°$ [4, 5]. A left-handed circularly polarized wave thus corresponds to $\delta = -90°$ for a counterclockwise rotating electric field.

In Figure 60.1, a polarization ellipse is specified by a set of three parameters: a_x, a_y, and δ. The ellipse can also be specified by the other set of three parameters: the major axis a, the minor axis b, and the orientation angle ϕ of the major axis measured from the x-axis. Figure 60.2 shows the geometry of an ellipse with these parameters. Parameters a and b can also be expressed in terms of the ellipticity e and ellipticity angle ε, defined by $e = b/a = \tan \varepsilon$. For linear polarization, $\delta = 0$, $b = 0 = e = \varepsilon$, and $\tan \phi = a_y/a_x$. For $\phi = 0$, the major and minor axes of the ellipse always correspond to the coordinate axes. For circular polarization, $a = b = a_x = a_y$ and $\delta = \pm 90°$. Right-handed circularly polarized light has a positive ellipticity with $e = 1$ and $\varepsilon = 45°$, and left-handed circularly polarized light has a negative ellipticity with $e = -1$ and $\varepsilon = -45°$. In general, ϕ represents the orientation of the ellipse, and ε indicates the shape of the ellipse and the direction of field rotation. References 2 through 7 give more details about this subject.

Jones Vector and Stokes Vector

The electric field expressed in vector form in Equation 60.1 can also be expressed as a column matrix. A polarization ellipse depends on a_x, a_y, δ_x, and δ_y, but not on k and ω. By neglecting the common factor of $e^{i(\omega t - kz)}$ in both E_x and E_y, the Jones vector is defined as [4, 7, 8]

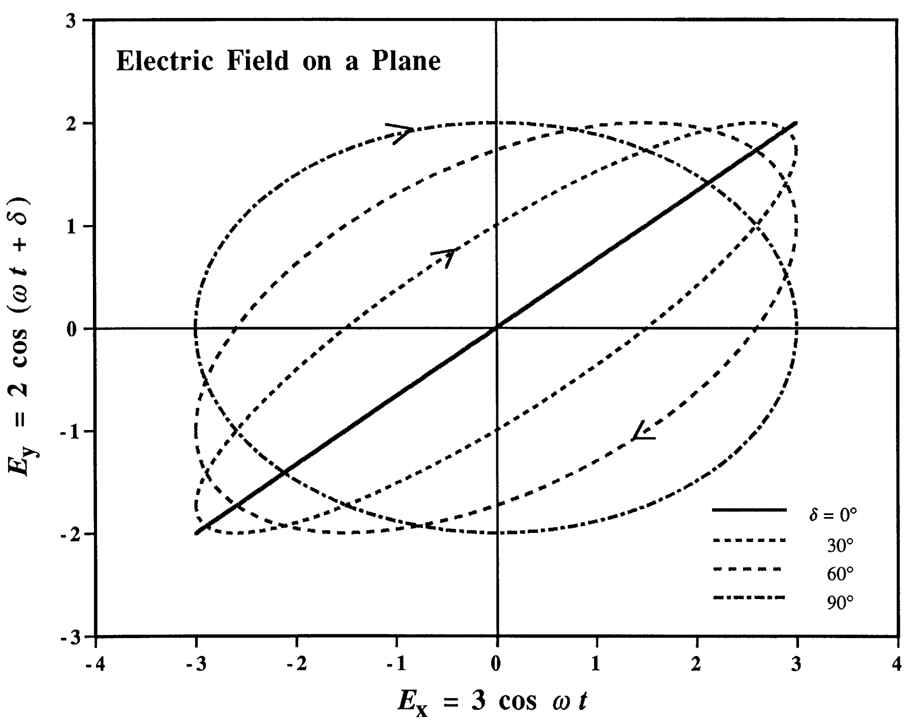

FIGURE 60.1 Projection of the electric field of an EM wave with amplitudes $a_x = 3$, $a_y = 2$ and different values of phase retardation δ onto the $z = 0$ plane. Most of these loci are ellipses and reduce to lines or circles in special cases. The electric field changes in the clockwise direction for δ between 0 and 180°.

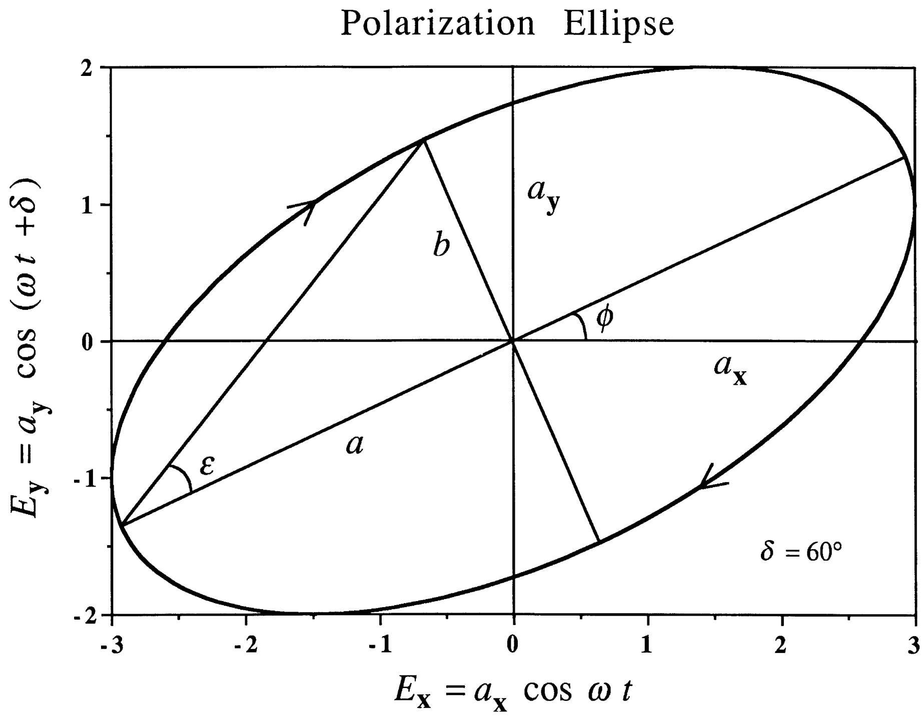

FIGURE 60.2 Characteristic parameters for a polarization ellipse. a_x and a_y are the field amplitudes in the x- and y-directions, and δ is the phase retardance; a and b are the major and minor axes of the ellipse, ϕ is the orientation of the major axis with respect to the x-axis, and ε is the ellipticity angle which is equal to $\tan^{-1}(b/a)$. A polarization ellipse can be characterized by (a_x, a_y, δ), (a, b, ϕ) or (I, ϕ, ε), where I is the intensity of the EM wave.

$$\mathbf{E} = \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} a_x e^{i\delta_x} \\ a_y e^{i\delta_y} \end{pmatrix} \tag{60.4}$$

Both elements of a Jones vector are complex numbers. Jones algebra is convenient for describing perfectly polarized light. Since a light sensor measures only intensity but not electric field in most cases, the Stokes vector is more convenient to use in metrology. The Stokes parameters are four intensity-based parameters used to describe the polarization state of light, represented by S_0, S_1, S_2, S_3, or by I, Q, U, V [1–6, 14]. The Stokes vector is the set of these Stokes parameters, defined as [4, 14]

$$\mathbf{S} = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} \equiv \begin{pmatrix} \langle E_x E_x^* \rangle + \langle E_y E_y^* \rangle \\ \langle E_x E_x^* \rangle - \langle E_y E_y^* \rangle \\ \langle E_x E_y^* \rangle + \langle E_y E_x^* \rangle \\ i\langle E_x E_y^* \rangle + i\langle E_y E_x^* \rangle \end{pmatrix} \tag{60.5}$$

For an EM wave, the average bracket in Equation 60.5 represents the time average. $\langle E_x E_x^* \rangle = I_x$ is the intensity of the component of light linearly polarized in the x-direction. Similarly, $\langle E_y E_y^* \rangle = I_y$. All of the Stokes parameters are real numbers and are measurable. For an ensemble of EM waves, the average brackets represent both time and ensemble averages.

Perfectly Polarized Light

Stokes vectors for different polarization states of light can be evaluated using Equation 60.5. Table 60.1 lists the Jones vectors and Stokes vectors for different states of perfectly polarized light. The Jones vector **E** and Stokes vector **S** expressed in terms of ellipticity angle ε and orientation angle ϕ are

$$\mathbf{E} = a\begin{pmatrix} \cos\varepsilon\cos\phi - i\sin\varepsilon\sin\phi \\ \cos\varepsilon\sin\phi + i\sin\varepsilon\cos\phi \end{pmatrix} \tag{60.6}$$

$$\mathbf{S} = I\begin{pmatrix} 1 \\ \cos 2\varepsilon\cos 2\phi \\ \cos 2\varepsilon\sin 2\phi \\ \sin 2\varepsilon \end{pmatrix} \tag{60.7}$$

In Equations 60.6 and 60.7, the amplitude a and intensity I are totally separated from the angles ϕ and ε which determine the polarization state. The degrees of linear and circular polarization are cos2ε and sin2ε, correspondingly. Stokes parameters for perfectly polarized light given in Equation 60.7 satisfy the identity

$$S_0^2 = S_1^2 + S_2^2 + S_3^2 \tag{60.8}$$

60.3 Polarization of Light

Light is composed of an ensemble of EM waves. A single EM wave has a certain electric field direction and phase. Unpolarized light can be visualized as an ensemble of EM waves with random field directions and phases. The field direction and phase for unpolarized light can not be defined then. The description of light in terms of Jones vector is therefore inadequate to describe the polarization of unpolarized light. For an ensemble of many EM waves, the electric field components in the Stokes vector given by Equation 60.5 is the sum of the corresponding components for all waves. In particular, for an ensemble of incoherent EM waves, the Stokes vectors for individual waves are additive:

$$\mathbf{S} = \sum_{i=1}^{N} \mathbf{S_i} \tag{60.9}$$

TABLE 60.1 Jones Vectors and Stokes Vectors for Different Polarization States for Perfectly Polarized Light

Polarization	Linear	Linear	Linear	Circular	Elliptical
Direction	0°	90°	±45°	right/left	
Phase	0°	0°	0°	±90°	δ
Jones vector	$\begin{pmatrix}1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\\pm1\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\\pm i\end{pmatrix}$	$\begin{pmatrix}a_x\\a_y e^{i\delta}\end{pmatrix}$
Stokes vector	$\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\\pm1\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\\\pm1\end{pmatrix}$	$\begin{pmatrix}a_x^2+a_y^2\\a_x^2-a_y^2\\2a_xa_y\cos\delta\\2a_xa_y\sin\delta\end{pmatrix}$

For an ensemble of EM waves with identical $\phi_i = \phi$ and $\varepsilon_i = \varepsilon$, the resultant polarization is still the same as the individual wave, as indicated by Equations 60.6 and 60.7, regardless of whether these waves are coherent.

Unpolarized and Partially Polarized Light

If an ensemble consists of randomly oriented linearly polarized waves, all $\varepsilon_i = 0$ and ϕ_i are random, then $S_1 = S_2 = S_3 = 0$, according to Equations 60.7 and 60.9. Light is thus unpolarized, and $\mathbf{S} = I(1, 0, 0, 0)$. If an ensemble consists of elliptically polarized waves with the same orientation $\phi_i = \phi$ or $\phi + \pi$ and perfectly random ellipticity angle ε_i, then the Stokes vector is $I(1, 0, 0, 0)$, and light is also unpolarized. Thus, the Stokes vector **S** in Equation 60.9 already implies the sense of the ensemble average of polarization. The average brackets in Eq. (5) can be considered as both the time average and ensemble average for incoherent waves.

For unpolarized light with $S_1 = S_2 = S_3 = 0$, Equation 60.8 does not hold. In general, light is partially polarized, i.e., part of it is perfectly polarized and the rest is unpolarized [1–6]. Stokes parameters for arbitrary polarizations satisfy

$$S_0^2 \geq S_1^2 + S_2^2 + S_3^2 \qquad (60.10)$$

The degree of polarization is given by

$$\mathcal{P} = \frac{\sqrt{S_1^2 + S_2^2 + S_3^2}}{S_0} \qquad (60.11)$$

Perfectly polarized light has $\mathcal{P} = 1$, and unpolarized light has $\mathcal{P} = 0$. Partially polarized light has $0 < \mathcal{P} < 1$. The intensity of the polarized part is $\mathcal{P}I$, and the intensity of the unpolarized part is $I(1 - \mathcal{P})$. By the superposition concept, the Stokes vector for partially polarized light can be obtained from Equation 60.7 as

$$\mathbf{S} = I \begin{pmatrix} 1 \\ \mathcal{P}\cos 2\varepsilon \cos 2\phi \\ \mathcal{P}\cos 2\varepsilon \sin 2\phi \\ \mathcal{P}\sin 2\varepsilon \end{pmatrix} \qquad (60.12)$$

The degree of linear polarization $\mathcal{P}_L$ and circular polarization $\mathcal{P}_C$ are

$$\begin{cases} \mathcal{P}_L = \mathcal{P}\cos 2\varepsilon = \dfrac{\sqrt{S_1^2 + S_2^2}}{S_0} \\ \mathcal{P}_C = \mathcal{P}\sin 2\varepsilon = \dfrac{S_3}{S_0} \end{cases} \qquad (60.13)$$

Polarization by the Response of a Medium

Jones Matrix

To measure polarization, light must interact with a medium to give a response. The response of a polarizer is to pass one polarization and reject the orthogonal one. The response of a phase retarder is to change the relative phase between the two polarizations. A medium can be any optical component, a test sample or any object under investigation. The response of a medium relates the state of output light to the state of incident light. The polarization state of light can be described by a complex vector EM field, a Jones vector, or a Stokes vector [4, 5, 7, 8]. Let an incident EM wave be specified by a complex field or Jones vector (E_x, E_y), and the output field be (E_x', E_y'), the general relations between these two fields are

$$\mathbf{E}' = \begin{pmatrix} E_x' \\ E_y' \end{pmatrix} = \begin{pmatrix} r_{xx} & r_{xy} \\ r_{yx} & r_{yy} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \mathbf{J}\,\mathbf{E} \qquad (60.14)$$

J in Equation 60.14 is a 2×2 matrix that relates the input Jones vector **E** to the output Jones vector **E**′ and is called the Jones matrix. The response of a medium is characterized by the elements of the Jones matrix, r_{xx}, r_{xy}, r_{yx}, and r_{yy}, which are all complex numbers.

Principal Coordinate System

Since the directions of an electric field are different in different rotated coordinate systems, $\{r_{ij}\}$ are not unique. For many symmetric media, there exists a coordinate system in which r_{xy} and r_{yx} are zero, and r_{xx} and r_{yy} are called the eigenvalues for $\{r_{ij}\}$. This is the principal coordinate system or principal frame whose *x*- and *y*-axes are the two principal axes. Finding the principal frame is an eigenvalue problem. If the incident polarization is along one of the principal-axis $\hat{x}$, then the output polarization is still along $\hat{x}$. In the principal frame, $\{r_{ij}\}$ are called the coefficients of response. For example, $\{r_{ij}\}$ may represent the reflection coefficients for a reflection response, or the scattering coefficients for a scattering response, etc. In the principal coordinate system, Equation 60.14is simplified to

$$\mathbf{E}' = \begin{pmatrix} E_x' \\ E_y' \end{pmatrix} = \begin{pmatrix} r_{xx} & 0 \\ 0 & r_{yy} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \mathbf{J}(0)\mathbf{E} \qquad (60.15)$$

J(0) is the diagonalized Jones matrix in the principal frame. For a polarizer, the principal axes are the transmission and extinction axes. The former is assigned to the *x*-axis. For a phase retarder, the principal axes are the fast- and slow-axes. The phase change for the EM wave with its field along the fast-axis is larger than the slow-axis. The fast-axis is usually assigned to the *x*-axis. The principal Jones matrices **p** for a perfect polarizer, and **c** for a perfect wave plate with a retardance τ are [4, 5, 7, 8]

$$\mathbf{p} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathbf{c} = \begin{pmatrix} e^{i\tau/2} & 0 \\ 0 & e^{-i\tau/2} \end{pmatrix} \qquad (60.16)$$

Ellipsometric Parameters

For reflection from a surface, the two principal axes are the *s*- and *p*-polarizations. The *s*-polarization field is along the *y*-axis which is chosen to be perpendicular to the plane of incidence. The *p*-polarization field is along the *x*-axis which is in the plane of incidence. The complex reflection coefficients for these two polarizations are designated as $r_p = r_{xx}$ and $r_s = r_{yy}$. The ellipsometric parameters ψ and Δ are defined by [4, 15, 16]

$$\frac{r_p}{r_s} = \frac{r_{xx}}{r_{yy}} = \tan\psi \exp(i\Delta) \qquad (60.17)$$

Δ is the phase change between reflected and incident light. If the electric field direction of incident light is given by ϕ_0, then the field direction ϕ of reflected light can be obtained from $\tan\phi = \tan\phi_0/\tan\psi$. At the Brewster angle, where $r_p = 0$ and $\psi = 0$, then $\phi = 90°$, and reflected light is vertically polarized. Equation 60.17 can also be applied to transmissive systems. Since a vacuum does not change the polarization of light, the ellipsometric parameters are $\psi = 45°$ and $\Delta = 0°$. A perfect polarizer with the polarization along the *x*-axis has $\psi = 90°$ and $\Delta = 0°$, a perfect quarter-wave plate has $\psi = 45°$ and $\Delta = 90°$, and a perfect half-wave plate has $\psi = 45°$ and $\Delta = 180°$.

Müeller Matrix

The Jones calculus is convenient for perfectly polarized light and a nondepolarizing response [8]. If unpolarized light is incident on a sample, the Jones vector can not describe the field direction and phase for unpolarized light. The Stokes vector and Müeller matrix are more convenient to use in treating polarization for general cases. A relation between the output Stokes vector **S**′ and the input Stokes vector **S** is

$$\mathbf{S}' = \mathbf{M}\,\mathbf{S} \tag{60.18}$$

The matrix **M** that relates the input and output Stokes vectors is called a Müeller matrix. **M** is a 4× 4 matrix of real numbers.

For the general transformation of electric field given by Equation 60.14, the components M_{ij} of **M** can be derived from Equations 60.5, 60.14, and 60.18. The expressions for M_{ij} have been obtained by van de Hulst [13] and are also given as Equation (2.243) of Reference 4. In a measurement, the EM waves of output light may come from many different area or volume elements of a medium, so that statistical averages must be considered in the evaluation of M_{ij}. The ensemble average of **M** can still be expressed by the same expressions, with the ensemble average bracket applying to all M_{ij}. To make the Müeller matrix meaningful, the new subscripts of Equation 60.14 are reassigned as 1: *xx*, 2: *yy*, 3: *xy*, 4: *yx*. Subscripts 1 and 2 correspond to the copolarized response, and subscripts 3 and 4 correspond to the cross-polarized response. The ensemble average of any two of the response coefficients is called a *correlation function* for these coefficients. Let us define the self-correlation functions to be $2F_j$ and the cross-correlation functions to be $G_{jm} + i\,g_{jm}$ as follows:

$$\begin{cases} 2F_j \equiv \langle r_j r_j^* \rangle \\ G_{jm} + i\,g_{jm} \equiv \langle r_j r_m^* \rangle \end{cases} \qquad \text{j,m} = 1,2,3,4;\ j \neq m \tag{60.19}$$

The cross-correlation functions have the properties of $G_{jm} = G_{mj} = \mathrm{Re}\,\langle r_j r_m^* \rangle$, and $g_{jm} = -g_{mj} = \mathrm{Im}\,\langle r_j r_m^* \rangle$. All F_j, G_{jm}, and g_{jm} are real numbers. **M** is then

$$\mathbf{M} = \begin{pmatrix} F_1+F_2+F_3+F_4 & F_1-F_2-F_3+F_4 & G_{13}+G_{24} & g_{13}-g_{24} \\ F_1-F_2+F_3-F_4 & F_1+F_2-F_3-F_4 & G_{13}-G_{24} & g_{13}+g_{24} \\ G_{14}+G_{23} & G_{14}-G_{23} & G_{12}+G_{34} & g_{12}-g_{34} \\ -g_{14}+g_{23} & -g_{14}-g_{23} & -g_{12}-g_{34} & G_{12}-G_{34} \end{pmatrix} \tag{60.20}$$

The upper left quadrant of **M** corresponds to the self-correlation terms. The lower right quadrant corresponds to the cross-correlations between the two co-polarized responses and between the two cross-polarized responses. The upper right and lower left quadrants correspond to the cross-correlations between the co-polarized and cross-polarized responses.

Principal Müeller Matrix

For the Jones matrix in the principal frame given by Equation 60.15, the cross-polarized responses are zero, so that the M_{jm} in the upper right and lower left quadrants of Equation 60.20 are zero. **M** can be expressed in terms of ψ and Δ using Equations 60.17 and 60.20 as [4, 17–19]

$$\mathbf{M} = R\begin{pmatrix} 1 & -\cos 2\psi & 0 & 0 \\ -\cos 2\psi & 1 & 0 & 0 \\ 0 & 0 & \sin 2\psi \cos\delta & \sin 2\psi \sin\Delta \\ 0 & 0 & -\sin 2\psi \sin\Delta & \sin 2\psi \cos\Delta \end{pmatrix} \qquad (60.21)$$

where $R = (r_{xx}\, r_{xx}^* + r_{yy}\, r_{yy}^*)/2$. For reflection, R is the average reflectance, and for transmission, R is the average transmittance. For a vacuum, $\psi = 45°$ and $\Delta = 0°$, **M** is a unit matrix. Using Equation 60.21 or Equations 60.16, 60.19, and 60.20 directly, matrix **P** for a perfect polarizer ($\psi = 90°$, $\Delta = 0°$) and matrix **C** for a perfect wave plate ($\psi = 45°$, $\Delta = \tau$) in the principal frame are obtained as [4, 5, 20]

$$\mathbf{P} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\tau & \sin\tau \\ 0 & 0 & -\sin\tau & \cos\tau \end{pmatrix} \qquad (60.22)$$

Depolarization

A very interesting example is the perfectly random response. Analogous to the conditions used for incoherent scattering [17–19], the random response coefficients δr_j have the properties that

$$\begin{cases} \langle \delta r_j \rangle = 0 & \\ \langle \delta r_j \delta r_m^* \rangle = \langle |\delta r_j|^2 \rangle \delta_{j,m} & \quad j,m = 1,2,3,4 \\ \langle |\delta r_j|^2 \rangle = \text{constant} & \end{cases} \qquad (60.23)$$

The first line states that all δr_j are each averaged to zero, so that they would not appear in the average of Equation 60.14. By substitution of r_j of Equation 60.19 by δr_j and using the conditions for δr_j given by Equation set 60.23, the correlation functions F_j and $G_{jm} + i\, g_{jm}$ can be evaluated. The second line of Equation set 60.23 states that all δr_j are uncorrelated with one another, so that all G_{ij} and g_{ij} are zero. The third line states that δr_j are isotropic, so that all F_j are the same. Eventually, all $M_{jm} = 0$ except $M_{00} = 4\, F_1$. The depolarization matrix **D** for a perfectly random response is then

$$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (60.24)$$

D is an ideal depolarizer as defined in Reference 20. The general Müeller matrix of Equation 60.20 satisfies the physical condition [14, 24]

$$\sum_{i,j=0}^{3} M_{ij}^2 \le 4 M_{00}^2 \qquad (60.25)$$

The sum of all the squares of the elements of **D** is M_{00}^2. **D** satisfies the inequality of criterion in Equation 60.25. The matrix of Equation 60.21 is nondepolarizing such that output light is still perfectly polarized if incident light is perfectly polarized. The equality in Equation 60.25 holds for **M** of Equation 60.21. This section discusses optical components and samples that are nondepolarizing. References 12 and 17 through 19 give more details about the Müeller matrices for samples that exhibit both polarization and depolarization properties.

Coordinate Transformation

In polarimetric measurements, polarizers and retarders are frequently rotated to desired positions. When a component is rotated, the incident field is not changed, but the representations of this field in the principal and laboratory coordinate systems are different. Transformations of the electric fields, Stokes vectors, Jones and Müeller matrices between these two coordinate systems or frames are basic exercises in polarimetry. Let the laboratory frame axes be x- and y-axes, and the principal frame axes be x'- and y'-axes, and the principal frame is rotated to an angle α with respect to the laboratory frame, as shown in Figure 60.3. The Jones vector (E_x', E_y') in the principal frame is related to (E_x, E_y) in the laboratory frame by

$$\mathbf{E}' = \begin{pmatrix} E_x' \\ E_y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \mathbf{r}(\alpha)\mathbf{E} \tag{60.26}$$

The rotation matrix $\mathbf{r}(\alpha)$ is the 2×2 matrix in Equation 60.26 for transformation of Jones vectors. The inverse transform is given by $\mathbf{E} = \mathbf{r}^{\mathbf{T}}(\alpha)\,\mathbf{E}'$, where the superscript **T** denotes the transpose of a matrix. In Figure 60.3, $\mathbf{E}'$ appears to be turned by an angle of $-\alpha$, since the coordinate system is rotated by an angle α. The Faraday rotation matrix that rotates **E** by an angle of α is equivalent to $\mathbf{r}(\alpha)$.

One can substitute $r_1 = r_2 = \cos\alpha$ and $r_3 = -r_4 = \sin\alpha$ into Equations 60.19 and 60.20 to construct the rotation matrix $\mathbf{R}(\alpha)$ for transformation of a Stokes vector **S** to a coordinate system oriented at an angle α.

$$\mathbf{R}(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\alpha & \sin 2\alpha & 0 \\ 0 & -\sin 2\alpha & \cos 2\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{60.27}$$

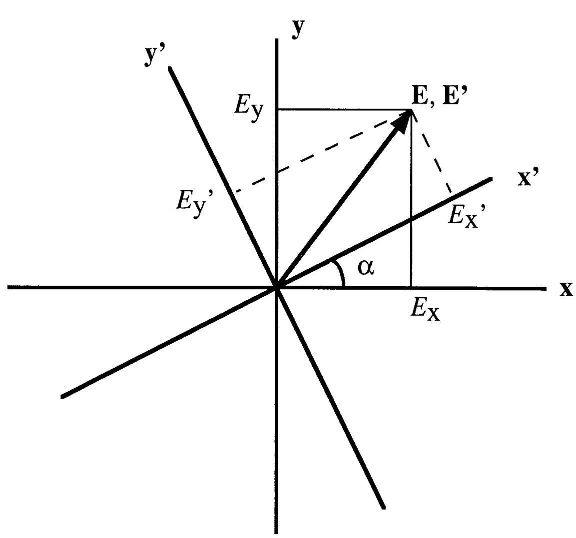

FIGURE 60.3 Coordinate transformation for the electric field components (E_x, E_y) in the laboratory system (x, y) and the components (E_x', E_y') in the principal coordinate system (x', y'). The principal frame is oriented at an angle a with respect to the laboratory frame.

The transformations between Stokes vector **S**′ in the principal frame and **S** in the laboratory frame are

$$\begin{cases} \mathbf{S}' = \mathbf{R}\,\mathbf{S} \\ \mathbf{S} = \mathbf{R}^{\mathbf{T}}\,\mathbf{S}' \end{cases} \tag{60.28}$$

The transformation of a Müeller matrix **M**(0) in the principal frame to **M**(α) in the laboratory frame can be obtained by a similarity transformation:

$$\mathbf{M}(\alpha) = \mathbf{R}^{\mathbf{T}}(\alpha)\mathbf{M}(0)\mathbf{R}(\alpha) \tag{60.29}$$

Equation 60.29 can also be used for the transformation of Jones matrix **J**(0) in the principal coordinate frame to *J*(α) in the laboratory frame, provided that **M** is replaced by **J** and **R**(α) by **r**(α) in Equation 60.29.

The Jones matrix for a polarimetric component orientated at an angle α is

$$\mathbf{J}(\alpha) = \begin{pmatrix} r_{xx}\cos^2\alpha + r_{yy}\sin^2\alpha & \sin\alpha\cos\alpha(r_{xx} - r_{yy}) \\ \sin\alpha\cos\alpha(r_{xx} - r_{yy}) & r_{xx}\sin^2\alpha + r_{yy}\cos^2\alpha \end{pmatrix} \tag{60.30}$$

The Müeller matrix **P**(*P*) for a perfect polarizer oriented at an angle *P* and **C**(*C*) for a perfect compensator with a retardance τ at an angle *C* are [4, 6, 21]

$$\mathbf{P}(P) = \frac{1}{2}\begin{pmatrix} 1 & \cos 2P & \sin 2P & 0 \\ \cos 2P & \cos^2 2P & \sin 2P\cos 2P & 0 \\ \sin 2P & \sin 2P\cos 2P & \sin^2 2P & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{60.31}$$

$$\mathbf{C}(C) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^2 2C + \sin^2 2C\cos\tau & \sin 2C\cos 2C(1-\cos\tau) & -\sin 2C\sin\tau \\ 0 & \sin 2C\cos 2C(1-\cos\tau) & \sin^2 2C + \cos^2 2C\cos\tau & \cos 2C\sin\tau \\ 0 & \sin 2C\sin\tau & -\cos 2C\sin\tau & \cos\tau \end{pmatrix} \tag{60.32}$$

Equations 60.16, 60.22, and 60.30 through 60.32 can be used to calculate the Jones matrices and Müeller matrices for polarizers and wave plates at arbitrary orientations. Table 60.2 lists some of these matrices for the most frequently used devices.

For a light beam passing through successive components oriented at different angles, Müeller matrices or Jones matrices in the laboratory frame must be used for successive multiplications. According to Equation 60.18, the matrix $\mathbf{M_1}(\alpha_1)$ for the component that light first passes through should be placed at the extreme right, and the matrix $\mathbf{M_n}(\alpha_n)$ for the last component at the extreme left. The combined matrix **M** is

$$\mathbf{M} = \mathbf{M_n}(\alpha_n)\ldots\mathbf{M}_1(\alpha_1) \tag{60.33}$$

TABLE 60.2 The Jones Matrices and Müeller Matrices for Perfect Polarizer and Wave Plates at Different Orientation Angles

Device	Angle	Jones matrix	Müeller matrix
Polarizer	0°	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
Polarizer	90°	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
Polarizer	±45°	$\frac{1}{2}\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 1 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & 0 \\ \pm 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
λ/4-plate	0°	$\begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$
λ/4-plate	±45°	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \pm i \\ \pm i & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm(-1) \\ 0 & 0 & 1 & 0 \\ 0 & \pm 1 & 0 & 0 \end{pmatrix}$
λ/2-plate	0°	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$
λ/2-plate	±45°	$\begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

60.4 Principles of Polarimetry

Polarimetry is a method for measuring the polarization of light and the polarization response of materials. An optical system used for such purposes is called a polarimeter or an ellipsometer. To measure the polarization response of a sample, polarized light is generated and incident on the sample. By examining the polarization states of both incident and reflected or transmitted light, the characteristics of a sample can be determined. A schematic diagram of a polarimeter used to measure the polarization response of a sample is shown in Figure 60.4. The light source and polarizer are used to generate polarized light, and the analyzer and detector are used to analyze the polarization of light [4, 20]. An analyzer is a polarizer used to analyze polarized light.

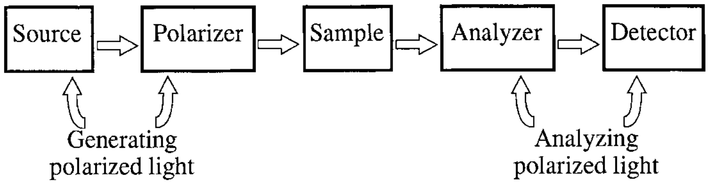

FIGURE 60.4 Schematic diagram of a polarimeter to measure polarization response. The light source and polarizer are used to generate polarized light, and the analyzer and detector are used to analyze the state of polarization of the light.

Analysis of Polarized Light

Measurement of polarization of light is essential in polarimetry, since polarized light to be examined is not limited to that generated in a laboratory. The instrument to measure the four Stokes parameters is called a photo-polarimeter or a Stokesmeter. To measure linear polarization, pass the light beam through a linear analyzer oriented at angle $A = 0°$, 90°, and ±45°, and measure the corresponding intensities I_x, I_y, I_+ and I_-. To measure circular polarization, first pass the light beam through a quarter-wave retarder with $C = 0°$, then through an analyzer oriented at $A = \pm 45°$, and measure the intensities I_R and I_L. The pair of quarter-wave retarder and analyzer constitutes a circular analyzer. A detector measuring intensity corresponds to an operation given by a row vector $\mathbf{I} = (1, 0, 0, 0)$. The combined operation of a detector following an analyzer is $\mathbf{IA} = 0.5\,(1, \cos 2A, \sin 2A, 0)$. The operations for the linear and circular analyzers on a Stokes vector S and the intensities obtained are given in Table 60.3. The Stokes parameters can be obtained from the difference and sum of the intensities for these pair operations, and are given by

$$\mathbf{S} = \begin{pmatrix} S_o \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} I_x + I_y \\ I_x - I_y \\ I_+ - I_- \\ I_R - I_L \end{pmatrix} \qquad (60.34)$$

Equation 60.34 is a general expression that is good for any polarization states and is also the operational principle for most Stokesmeters. The four-detector Stokesmeter designed by Azzam is an exception that contains no moving components and can measure the four Stokes parameters in real time [26].

Generation of Polarized Light

Characterization of polarization response of a sample requires incident polarized light whose polarization state is controllable. A convenient source is a laser, which may be constructed to emit polarized light directly without the help of extra devices. A half-wave plate may be used to rotate the laser polarization

TABLE 60.3 Intensities I for a Light Beam with Stokes Parameters S_0, S_1, S_2, and S_3 Analyzed by Linear and Circular Analyzers. A circular analyzer consists of a quarter-wave plate oriented at $C = 0°$, followed by an analyzer oriented at $A = \pm 45°$.

Analyzer	Linear	Linear	Circular
C (°)	NA	NA	0
A (°)	0, 90	45, –45	45, –45
Operation	0.5(1, ±1, 0, 0)	0.5(1, 0, ±1, 0)	0.5(1, 0, 0, ±1)
Intensity	I_x, I_y	I_+, I_-	I_R, I_L
I =	$(S_0 \pm S_1)/2$	$(S_0 \pm S_2)/2$	$(S_0 \pm S_3)/2$

to a desired direction by placing the fast-axis bisecting the new and old directions, as shown in Figure 60.5a. Light generated from a lamp and a monochromator is usually partially polarized [12]. To generate linearly polarized light at an angle P, a linear polarizer oriented at an angle P is placed behind the monochromator as shown in Figure 60.5b. To generate circularly polarized light, first generate linearly polarized light, and then put a quarter-wave plate behind with the fast-axis oriented at an angle of 45° or –45° with respect to the linear polarization as shown in Figure 60.5c. Such a combination of polarizer and quarter-wave plate is called a circular polarizer. When a phase retarder has an arbitrary retardance or is placed at an arbitrary angle relative to the polarizer, elliptically polarized light is then generated. Given an incident Stokes vector of (S_0, S_1, S_2, S_3), the Stokes vector $\mathbf{S}'$ for polarized light generated by the polarizers mentioned above can be obtained from Equations 60.18 and 60.31 through 60.33. The obtained $\mathbf{S}'$ are listed in Table 60.4. Note that $\mathbf{S}'$ is not directly proportional to S_0 unless incident light is unpolarized. Care must be taken in generating polarized light in an ellipsometer because incident light is rarely completely unpolarized.

Polarizer–Sample–Analyzer Ellipsometry

An ellipsometer is an instrument to measure the ellipsometric parameters ψ and Δ of a sample. It can be used for both reflection and transmission. An ellipsometer is usually referred to as the reflection system, and a polarimeter as the transmissive system [4]. Different ellipsometers are designed to measure different responses for different kinds of samples. It is important to know about the sample when designing an experiment. The simplest ellipsometer is a polarizer–sample–analyzer (PSA) ellipsometer. A more general one is a polarizer–compensator–sample–analyzer (PCSA) ellipsometer. Figure 60.6 shows

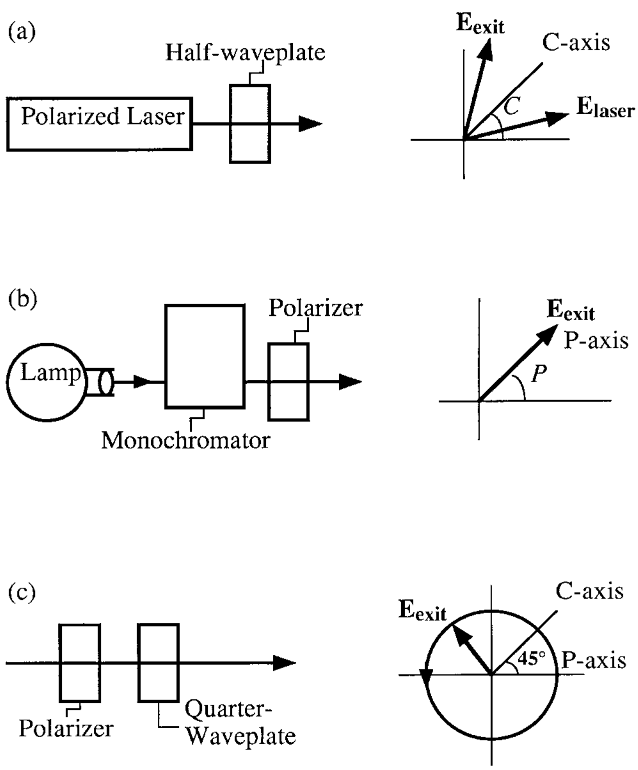

FIGURE 60.5 Generation of light linearly polarized at a desired direction using (a) a laser source and a half-wave plate and (b) a lamp, monochromator, and a polarizer, plus (c) generation of circularly polarized light using a polarizer and a quarter-wave plate.

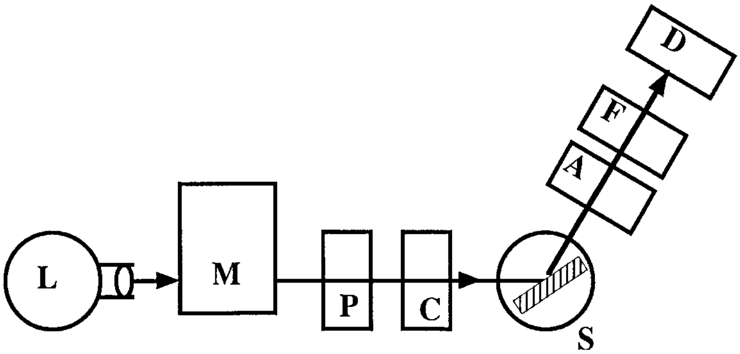

FIGURE 60.6 Schematic diagram of a PCSA ellipsometer.

TABLE 60.4 Stokes Vectors **S′** for Linearly and Circularly Polarized Light Generated by Specific Combinations of a Polarizer Oriented at an Angle *P* and a Quarter-Wave Retarder at an Angle *C*. The incident Stokes vector is $S = (S_0, S_1, S_2, S_3)$.

Polarizer	Linear	Linear	Circular
P(°)	0, 90	45, –45	–45, 45
C(°)	NA	NA	0
Operation	**P**(P)S	**P**(P)S	**Q**(C) **P**(P) **S**
Stokes Vector	$\mathbf{S}_x'$, $\mathbf{S}_y'$	$\mathbf{S}_+'$, $\mathbf{S}_-'$	$\mathbf{S}_R'$, $\mathbf{S}_L'$
S′	$\frac{S_0 \pm S_1}{2}\begin{pmatrix}1\\ \pm 1\\ 0\\ 0\end{pmatrix}$	$\frac{S_0 \pm S_2}{2}\begin{pmatrix}1\\ 0\\ \pm 1\\ 0\end{pmatrix}$	$\frac{S_0 \pm S_3}{2}\begin{pmatrix}1\\ 0\\ 0\\ \pm 1\end{pmatrix}$

a schematic diagram of a PCSA reflection ellipsometer. A PSA system can be visualized by removing the compensator *C* in Figure 60.6.

Let the oriented angles of the polarizer and analyzer be *P* and *A*, respectively, as measured from the plane of incidence. For unpolarized incident light and a sample matrix **M** given by Equation 60.21, the measured intensity *I* for a PSA ellipsometer is [22]

$$\begin{aligned} I(P, A) &= \mathbf{I}\,\mathbf{A}(A)\,\mathbf{M}(R, \psi, \Delta)\mathbf{P}(P)\mathbf{S} \\ &= T_p T_a R I_0 [1 - \cos 2\psi(\cos 2P + \cos 2A)] + \cos 2P \cos 2A + \sin 2\psi \cos\Delta \sin 2P \sin 2A \end{aligned} \quad (60.35)$$

I_0 is the intensity of incident light, T_p and T_a are the transmittance of the polarizer and analyzer, respectively. If incident light is partially polarized with a Stokes vector (S_0, S_1, S_2, S_3), then the right-hand side of Equation 60.35 should be multiplied by the factor $(S_0 + S_1 \cos 2P + S_2 \sin 2P)/S_0$. In such a case, the dependence on *P* is more complicated. A good practice is to keep *P* fixed and vary only *A*. Many different ways can be devised to extract ψ and Δ from Equation 60.35, such as the Stokes polarimeter, null polarimeter, and rotating-analyzer ellipsometer.

For a Stokes polarimeter, *P* is set at 45° or -45°, and *A* at 0°, 90°, and ±45°. The ellipsometric parameters ψ and Δ can be solved from the four equations evaluated at these *P* and *A* positions via the relations

$$\begin{cases} \cos 2\psi = \dfrac{I_y - I_x}{I_y + I_x} \\ \sin 2\psi \cos\Delta = \dfrac{I_+ - I_-}{I_+ + I_-} \end{cases} \tag{60.36}$$

For a null polarimeter, set $P = \pm 45°$, and vary A to find the null positions. This method is excellent for reflection from transparent materials with Δ equal to 0 or π. The value of ψ is related to the null position $A_\pm$ as [22]

$$\tan 2A_\pm = \mp \tan 2\psi \cos\Delta \qquad \text{for } P = \pm 45° \tag{60.37}$$

Average of $A_\pm$at $P = \pm 45°$ can eliminate errors from the misalignment of analyzer and polarizer.

In a rotating-analyzer ellipsometer (RAE), the analyzer is rotated at an angular frequency ω_r. Set P = 45° or –45°, and $A = \omega_r t$. The measured intensity is

$$I = T_p R T_a I_0 [1 - \cos 2\psi \cos 2\omega_r t + \sin 2\psi \cos\Delta \sin 2\omega_r t] \tag{60.38}$$

The Fourier coefficients, being equal to $-\cos 2\psi$ and $\sin 2\psi \cos\Delta$, can be recovered from the demodulated signals or from a fast Fourier transform (FFT) technique [28, 29]. However, the measured Δ cannot be distinguished from $-\Delta$. This system can be fully automated for realtime operation. With a white light source and monochromator, an RAE can serve as a spectroscopic ellipsometer [15, 16, 28, 29].

Polarizer–Compensator–Sample–Analyzer Ellipsometry

In a PCSA or a PSCA ellipsometer, a compensator of retardance τ is inserted in front of or following the sample. Figure 60.6 shows a schematic diagram of a PCSA ellipsometer. For unpolarized incident light in a PCSA system, the measured intensity for general conditions of P, C, and A is given by [21]

$$\begin{aligned} I(P, C, A) &= \mathbf{I}\,\mathbf{A}(A)\,\mathbf{M}(R, \psi, \Delta)\,\mathbf{C}(C)\,\mathbf{P}(P)\,\mathbf{S} \\ &= T_p T_c R T_a I_0 [Y_0 + Y_1 \cos 2A + Y_2 \sin 2A] \end{aligned} \tag{60.39}$$

where

$$\begin{cases} Y_0 = \{1 - \cos 2\psi [\cos 2C \cos 2(P - C) - \cos\tau \sin 2C \sin 2(P - C)]\} \\ Y_1 = \{-\cos 2\psi + [\cos 2C \cos 2(P - C) - \cos\tau \sin 2(P - C)]\} \\ Y_2 = \sin 2\psi \{\cos\Delta [\sin 2C \cos 2(P - C) + \cos\tau \cos 2C \sin 2(P - C)]\} - \sin\tau \sin\Delta \sin 2(P - C) \end{cases} \tag{60.40}$$

If incident light is partially polarized, then I_0 in Equation 60.39 should be replaced by ($S_0 + S_1 \cos 2P + S_2 \sin 2P$). For a PSCA system, interchange P and A in Equations 60.39 and 60.40. These formulas can be used to design different kinds of PCSA ellipsometers by choosing different conditions and different types of modulation. For certain special conditions, Equations 60.40 can be greatly simplified.

Null Ellipsometry

Consider a PCSA null ellipsometer (NE) in which the compensator is a perfect quarter-wave retarder ($\tau = 90°$) which is set at $C_\pm = \pm 45°$. The measured intensity is

$$I(P, C_{\pm}, A) = T_p T_c R T_a I_0[1 - \cos 2\psi \cos 2A \pm \sin 2\psi \sin 2A \cos \Delta \pm (2P - 90°)] \qquad (60.41)$$

An NE is an instrument to find the null positions in order to determine ψ and Δ. The four zones that will null the intensity in Equation 60.41 and the null positions are listed in Table 60.5. The null positions of *A* give ψ directly, and the null positions of *P* give Δ directly. Although ψ and Δ can be determined from measurements in only one zone, systematic errors caused by imperfect components, misalignment, and partially polarized incident light can be nonnegligible. By taking the average of four zones, many of these linear systematic errors can be cancelled [4, 21, 30, 31]. To look for the null positions manually is a slow process; automation of the nulling process can speed up the measurements. Different methods can be used to automate the NE: (1) both polarizer and analyzer are controlled by servo-motors with the feedback from the detector to find the null intensity, (2) the intensity is digitized and fed into a computer which is used to find the null positions, (3) Faraday rotators are used to rotate and modulate the polarization directions of light incident on and reflected from the sample to get the slopes of intensity versus angle until the slopes are zero at the null positions [4, 27]. The advantage of NE is its simplicity in obtaining ψ and Δ. Also, its accuracy is unbeatable by other kinds of ellipsometry.

TABLE 60.5 The Null Positions of Polarizer Angle *P* and Analyzer Angle *A* for the Four Different Zones at *C* = ±45° in a Null Ellipsometer.

Zone	C(°)	P	A
1	–45	–45° + Δ/2	ψ
2	45	–45° – Δ/2	ψ
3	–45	45° + Δ/2	–ψ
4	45	45° – Δ/2	–ψ
Average	$\psi = (A_1 + A_2 - A_3 - A_4)/4$,	$\Delta = (P_1 - P_2 + P_3 - P_4)/2$	

Phase-Modulated Ellipsometry

A phase-modulated ellipsometer (PME) uses a phase retarder whose retardance is modulative. For a PME, a good choice of *P*, *C*, and *A* in Equations 60.39 and 60.40 is *P* = 0°, *C* = 45°, and *A* = 45°. The intensity is then

$$I(\tau) = T_p T_c R T_a I_0[1 - \cos 2\psi \cos\tau + \sin 2\psi \sin\Delta \sin\tau] \qquad (60.42)$$

The retardance τ is modulated, and the modulated intensity is detected. If τ is modulated according to $\tau = \tau_o \cos\Omega_t$, then *I*(τ) can be expanded in a Fourier series with the Fourier coefficients, depending on cos2ψ, sin2ψ cosΔ and the Bessel functions of τ_o. These coefficients can be recovered from the demodulated signals, and the values of ψ and Δ can then be solved. τ can be modulated by electro-optic modulation using the Pockels effect, or piezoelectric modulation using the photoelastic effect [32, 33]. These modulations are fast, and demodulation using a lock-in amplifier is convenient. The advantage of a PME is that it contains no moving components and offers real-time measurement.

60.5 Polarization Instrumentation and Experiments

Figure 60.7 shows a schematic diagram of a PCSA reflection null ellipsometer. The instrument is composed of five systems: the source system, polarimetric system, sample system, detection system, and computer system for automatic control, data acquisition, and processing.

The simplest source is a polarized laser. The monochromatic source system in Figure 60.7 is usually used in visible, ultraviolet and infrared spectrometers. Light from a lamp source L is focused by a lens system or a spherical mirror system onto the entrance slit of a grating monochromator M. Light leaving

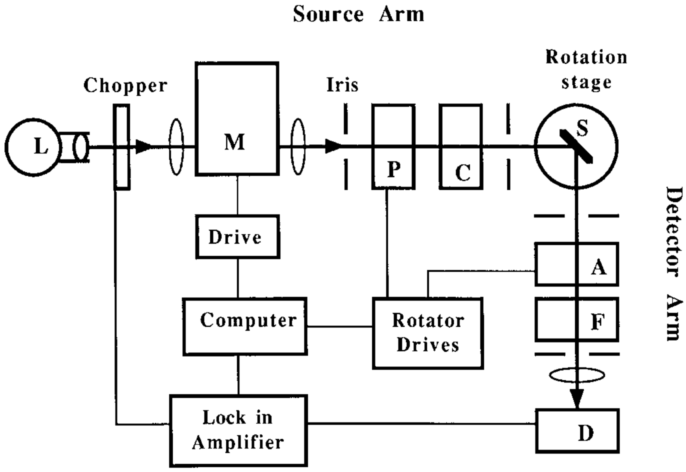

FIGURE 60.7 Schematic diagram of a PCSA reflection null ellipsometer, which is composed of five systems: the source system, polarimetric system, sample system, detection system, and computer system. The symbols are L = light source, M = monochromator, F = wavelength filter, and D detector. The symbols P, C, S, and A have their customary meanings.

the exit slit of M is collimated by another lens system and a set of iris apertures. The long-pass filter F in the detector arm is used with a grating monochromator to remove undesired short-wavelength radiation. Choices of monochromators include a grating monochromator, prism monochromator or Fourier transform spectrometer. A spectrometer is a necessary component for a spectroscopic ellipsometer. Synchrotron radiation is also a continuum source and is used to replace the lamp in the vacuum ultraviolet region [25]. The synchrotron radiation beam is intense and polarized. Grazing incidence reflection optics are usually used to avoid absorption in the components.

The polarimetric system shown in Figure 60.7 is a PCSA ellipsometer. For a PSA ellipsometer, remove the compensator C. For a PCSCA system, add another compensator in the detector arm [20]. It is better to mount the polarizers and phase retarders on automatic rotators so that their orientations can be easily aligned and read. For simple experiments, manual rotators can also do the job. References 10 and 11 give detailed descriptions and references for different kinds of polarizers and phase retarders. Many of the well known optical companies sell polarizers and wave plates (retarders) for use in the visible, ultraviolet, and near infrared spectral regions. The Buyers Guide of Laser Focus World [49] and the Photonics Buyers' Guide [50] list companies that manufacture and sell polarizers, phase retarders and polarimeters. Tables 60.6 and 60.7 list some companies that make these products. Commonly used polarizers in the visible are Glan prisms and dichroic sheets or plates. The best polarizer in the visible is the calcite Glan-Thompson prism, which has a very small extinction ratio and a large acceptance angle. It is more difficult to find a good broadband polarizer in the mid-infrared spectral region (λ: 3 to 5 μm). Wire-grid polarizers are good in the long-wave infrared region (λ > 8 μm). For a broadband phase retarder, a Babinet-Soleil compensator is convenient, since it can be set to any retardance value using a micrometer adjustment. Inexpensive wave plates are good for laser sources. Phase modulation can be achieved by modulation of the birefringence of electro-optic or photo-elastic retarders.

The sample system depends on the type of experiment to be performed. Components on the detector side of Figure 60.7 are normally mounted on a rail that is rotatable about the axis of the rotation stage. The sample holder S on the rotation stage should have enough degrees of freedom for easy alignment. In a reflection ellipsometer, the x-axes for P, C, and A should be well aligned to lie in the plane of incidence. A transmission polarimeter is much simpler to align.

The detection system consists of a detector and a noise suppression system. Diode detectors for use in the visible and near-infrared spectral regions are inexpensive. Photomultipliers for the visible and

TABLE 60.6 Companies That Make Polarizers and Phase Retarders

Company	Tel/Fax	Address
Cleveland Crystals	(216) 486-6100 (216) 486-6103	19306 Redwood Ave., Cleveland, OH 44110
Corning, Inc.	(607) 974-7966 (607) 974-7210	POLARCOR Team, Advance Materials, HP-CB, Corning, NY 14831
CVI Laser	(505) 296-9541 (505) 298-9908	P.O. Box 11308, Albuquerque, NM 87192
Hinds Instruments	(503) 690-2000 (503-690-3000	3175 NW Aloclek Drive, Hillsboro, OR 97124
Karl Lambrecht	(312) 472-5442 (312) 472-2724	4204 N. Lincoln Ave., Chicago, IL 60618
Meadowlark	(303) 833-4333 (303) 833-4335	P.O. Box 1000, 5964 Iris Parkway, Frederick, CO 80530
Molectron, Inc.	(503) 620-9069 (503) 620-8964	7470 SW Bridgeport Rd., Portland, OR 97224
New Focus, Inc.	(408) 980-8088 (408) 980-8883	2630 Walsh Ave., Santa Clara, CA 95051-0905
Rocky Mountain Instrument	(303) 651-2211 (303) 651-2648	1501 S. Sunset St., Longmont, Colorado 80501
Special Optics	(201) 785-4015 (201) 785-0166	P.O. Box 163, Little Falls, NJ 07424
Tower Opt. Corp.	(201) 305-9626 (201) 305-1175	130 Ryerson Ave., Wayne, NJ 07470
II-VI, Inc.	(412) 352-1504 (412) 352-4980	375 Saxonburg Blvd., Saxonburg, PA 16056

TABLE 60.7 Companies That Make Ellipsometers

Company	Tel/Fax	Address
Gaertner Scientific	847-673-5006 847-673-5009	8228 McCormick Blvd., Skokie, IL 60076
Instrument SA, Inc.	908-494-8660 908-494-8796	6 Olsen Ave., Edison, NJ 08820
J. A. Woolam Co., Inc.	402-477-7501 402-477-8214	650 J. Street, Suite 39, Lincoln, NE 68508
Leonard Research	937-426-1222 937-426-3642	2792 Indian Riffle Rd., Beavercreek, OH 45440 P.O. Box 607, Beavercreek, OH 45434-0607
Rudolph Research	201-691-1300 201-691-5480	1 Rudolph Rd., P.O. Box 1000, Flanders, NJ 07836
SOPRA	(1) 47 81 09 49 (1) 42 42 29 34	26, rue Pierre Joigneaux, 92270 Bois-Colombes, FRANCE
Tencor Instrument	415-969-6767 415-969-6731	2400 Charleston Rd., Mountain View, CA 94043-9958

near-infrared regions have high sensitivity, and cooled semiconductor detectors give good performance in the mid-infrared to far-infrared regions. In Figure 60.7, the noise suppression system includes a chopper and a lock-in amplifier. The chopper modulates incident light intensity, and the intensity detected by detector D is demodulated by a lock-in amplifier. This combination eliminates most broadband noise and greatly improves the signal-to-noise ratio. RAE and PME have their own modulations, and do not need a chopper.

A computer system provides the automatic functions to control the polarizers, retarder, and monochromator, and to acquire and process data. A computer system is essential for making accurate and rapid measurements. In Figure 60.7, the computer records the polarizer angle *P*, the analyzer angle *A*, the intensity *I* from the detector, and the wavelength λ of the monochromator. The data of $I(P)$ and $I(A)$ can be used to find the null positions of *P* and *A*. Then the computer controls the drivers to move *P* or *A* to the null positions. In a spectroscopic RAE, data of $I(t)$ are recorded as the analyzer is rotating. The computer uses the FFT program to find the Fourier coefficients, solves for ψ and Δ, records the results, and then drives the monochromator to a new wavelength and repeats the process. A spectroscopic PME uses similar computer process as RAE, besides the different modulation and demodulation.

Measurement of Birefringence

Retardance can be measured using a transmission PSA ellipsometer. A wave plate is a good sample for this experiment. The retardance δ for a birefringent slab of thickness d and principal refractive indices ne and no is

$$\delta = \frac{2\pi d(n_e - n_o)}{\lambda} \tag{60.43}$$

in the absence of multiple reflections. For a wave plate, the value of ψ is close to 45°, and the Δ value is equal to δ. Use a lamp source with a monochromator to scan the wavelength. Put a polarizer at 45° and an analyzer at –45° with respect to the fast-axis of the wave plate [40]. When λ is scanned, the transmitted intensity *I* through the PSA ellipsometer will vary, in proportion to $(1 + \cos\delta)$ according to Equation 60.35. *I* is a maximum when δ is 0°, and is a minimum when δ = 180°. From the wavelengths at which a maximum or a minimum intensity occurs, the birefringence $n_e - n_o$ can be determined.

Measurement of Optical Constants

A reflection ellipsometer can be used to measure optical constants *n* and *k* of materials. Light is incident obliquely on a sample at an angle of θ. The refractive index n and the extinction coefficient k can be calculated from the measured ψ and Δ values using the following formula [2, 4, 16, 22]

$$(n + \mathrm{i}k)^2 = \sin^2\theta\left\{1 + \tan^2\theta\left[\frac{1 - \tan\psi\exp(\mathrm{i}\Delta)}{1 + \tan\psi\exp(\mathrm{i}\Delta)}\right]^2\right\} \tag{60.44}$$

If an automated system is not available, try an NE to obtain ψ and Δ manually. Automated systems such as an RAE or a PME can take data much faster. For transparent materials, Δ is either 0 or π, the simple null polarimeter with a PSA system offers satisfactory results [22]. The method is effective near the Brewster angle region. At the Brewster angle θ_B, $\psi = 0$, then $n = \tan\theta_B$ according to Equation 60.44. For metals whose ψ is large when $\Delta = -90°$, the principal angle ellipsometry (PAE) can be used [2]. At the principal angle θ_P, $\cos\Delta = 0$, Equation 60.44 can be simplified to

$$(n + \mathrm{i}k)^2 = \sin^2\theta_P[1 + \tan^2\theta_P(\cos 4\psi + \mathrm{i}\sin 4\psi)] \tag{60.45}$$

In PAE, the principal angle θ_P is searched and then ψ is measured. The Stokes polarimeter with a PSA system is suitable to search θ_P and measure ψ for PAE.

Determination of optical constants using reflection ellipsometry is subject to errors caused by surface roughness, natural oxides and surface contamination [15, 16, 41–48]. These effects can be corrected by assuming that there is an effective layer on the surface and then using a least-square regression method to fit the ellipsometric data to the appropriate model [16, 43–48]. Ellipsometry is also used to measure refractive index and thickness of a thin film on a bulk substrate whose optical constants are known [4, 15, 16, 27]. Other applications of polarimetry and ellipsometry can be found in recent proceedings about polarization [51–53].

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