Throughout the chapter, a variety of properties and insightful interpretations of the resulting algorithms were developed, and their performance characteristics were explored both analytically and empirically.

Finally, it is important to appreciate that many other algorithms, such as those for distinguishing among superimposed $1/f$ signals and for detecting $1/f$ signals with unknown parameters, can be similarly derived using the methods developed in this chapter. In addition, straightforward generalizations of many of the algorithms to two- and higher-dimensional data such as imagery can also be developed.

5

Deterministically Self-Similar Signals

5.1 INTRODUCTION

Signals $x(t)$ satisfying the deterministic scale-invariance property

$$x(t) = a^{-H} x(at) \tag{5.1}$$

for all $a > 0$, are generally referred to in mathematics as ***homogeneous*** functions, in particular of **degree** H. Homogeneous functions can be regular or nearly so, for example $x(t) = 1$ or $x(t) = u(t)$, or they can be generalized functions, such as $x(t) = \delta(t)$. In any case, as shown by Gel'fand [90], homogeneous functions can be parameterized with only a few constants. As such, they constitute a rather limited class of signal models for many engineering applications.

A comparatively richer class of signal models is obtained by considering waveforms that are required to satisfy (5.1) only for values of a that are integer powers of two. The homogeneous signals in this broader class then satisfy the **dyadic self-similarity** property

$$x(t) = 2^{-kH} x(2^k t) \tag{5.2}$$

for all integers k. It is this more general family of homogeneous signals of degree H whose properties and characterizations we study in this chapter, and our treatment follows that in Wornell and Oppenheim [91]. We will typically use the generic term "homogeneous signal" to refer to signals satisfying (5.2). However, when there is risk of confusion in our subsequent development we will specifically refer to signals satisfying (5.2) as ***bihomogeneous***.

Homogeneous signals constitute an interesting and potentially valuable class of signals for use in, for example, a variety of communications-based applications. As an illustration of potential, in Chapter 6 we explore their use in developing a diversity strategy for embedding information into a waveform "on all time scales." As a consequence of their intrinsic self-similarity, these waveforms have the property that an arbitrarily short duration time-segment is sufficient to recover the entire waveform, and hence the embedded information, given adequate bandwidth. Likewise an arbitrarily low-bandwidth approximation to the waveform is sufficient to recover the undistorted waveform, and again the embedded information, given adequate duration. Furthermore, we will see that these homogeneous waveforms have spectral characteristics very much like those of $1/f$ processes, and, in fact, have fractal properties as well.

Collectively, such properties make this modulation scheme an intriguing diversity paradigm for communication over highly unreliable channels of uncertain duration, bandwidth, and SNR, as well as in a variety of other contexts. We explore these and other issues, including implementation, in the next chapter. In the meantime, we turn our attention to developing a convenient and efficient mathematical framework for characterizing homogeneous signals that we will exploit.

Some important classes of homogeneous signals have spectral characteristics very much like those of $1/f$ processes, and, in fact, have fractal properties as well. Specifically, while all nontrivial homogeneous signals have infinite energy and many have infinite power, we will see that there are in fact classes of these signals with which one can associate a generalized $1/f$-like Fourier transform, and others with which one can associate a generalized $1/f$-like power spectrum. These are the homogeneous signals of interest in this chapter. We distinguish between these two classes of such signals in our subsequent treatment, denoting them *energy-dominated* and *power-dominated* homogeneous signals, respectively.

We begin our theoretical development by formalizing our notion of an energy-dominated homogeneous signal, and constructing vector space characterizations. In turn, these lead to some powerful constructions of orthonormal "self-similar" bases for homogeneous signals. In the process, it will become apparent that, as in the case of statistically self-similar $1/f$-type processes, orthonormal wavelet basis expansions constitute natural and efficient representations for these signals as well.

Before proceeding, we point out that our development relies heavily on a rather natural and efficient vector space perspective. In addition to facilitating the derivation of key results, this approach leads to powerful geometrical interpretations. Accessible treatments of the appropriate mathematical background can be found in, e.g., portions of Naylor and Sell [28] or Reed and Simon [29].

5.2 ENERGY-DOMINATED HOMOGENEOUS SIGNALS

Our definition of an energy-dominated homogeneous signal is reminiscent of the one we proposed for $1/f$ processes in Section 3.2. Specifically, we choose the following.

Definition 5.1 *A bihomogeneous signal $x(t)$ is said to be energy-dominated if when $x(t)$ is filtered by an ideal bandpass filter with frequency response*

$$B_0(\omega) = \begin{cases} 1 & \pi < |\omega| \le 2\pi \\ 0 & \textit{otherwise} \end{cases} \tag{5.3}$$

the resulting signal $\tilde{x}_0(t)$ has finite-energy, i.e.,

$$\int_{-\infty}^{\infty} \tilde{x}_0^2(t)\, dt < \infty.$$

Some preliminary remarks regarding this definition are worthwhile at this point. First, we note that the choice of passband edges at π and 2π in our definition is, in fact, somewhat arbitrary. In particular, substituting in the definition any passband that does not include $\omega = 0$ or $\omega = \infty$ but includes one entire frequency octave leads to precisely the same class of signals. Nevertheless, our particular choice is both sufficient and convenient.

It is also worth noting that the class of energy-dominated homogeneous signals includes both reasonably regular functions, such as the constant $x(t) = 1$, the ramp $x(t) = t$, the time-warped sinusoid $x(t) = \cos[2\pi \log_2 t]$, and the unit step function $x(t) = u(t)$, as well as singular functions, such as $x(t) = \delta(t)$ and its derivatives. However, although we are not always able to actually "plot" signals of this class, we are able to suitably characterize such functions in some useful ways. We begin by using $\mathbf{E}^H$ to denote the collection of all energy-dominated homogeneous signals of degree H. The following theorem allows us to interpret the notion of spectra for such signals. A straightforward but detailed proof is provided in Appendix D.1.

Theorem 5.2 *When an energy-dominated homogeneous signal $x(t)$ is filtered by an ideal bandpass filter with frequency response*

$$B(\omega) = \begin{cases} 1 & \omega_L < |\omega| \le \omega_U \\ 0 & \textit{otherwise} \end{cases} \tag{5.4}$$

for arbitrary $0 < \omega_L < \omega_U < \infty$, the resulting signal $y(t)$ has finite energy and a Fourier transform of the form

$$Y(\omega) = \begin{cases} X(\omega) & \omega_L < |\omega| \le \omega_U \\ 0 & \textit{otherwise} \end{cases} \tag{5.5}$$

where $X(\omega)$ is some function that is independent of ω_L and ω_U and has octave-spaced ripple; i.e., for all integers k,

$$|\omega|^{H+1} X(\omega) = |2^k\omega|^{H+1} X(2^k\omega). \tag{5.6}$$

Since in this theorem $X(\omega)$ does not depend on ω_L or ω_U, this function may be interpreted as the generalized Fourier transform of the infinite-energy signal $x(t)$. Furthermore, (5.6) implies that the generalized Fourier transform of signals in $\mathbf{E}^H$ obeys a $1/f$-like (power-law) relationship, viz.,

$$|X(\omega)| \sim \frac{1}{|\omega|^{H+1}}.$$

However, we continue to reserve the term "$1/f$ process" or "$1/f$ signal" for the *statistically* self-similar random processes defined in Chapter 3.

We also remark that because (5.5) excludes $\omega = 0$ and $\omega = \infty$, knowledge of $X(\omega)$ does not uniquely specify $x(t) \in \mathbf{E}^H$; i.e., the mapping

$$x(t) \longleftrightarrow X(\omega)$$

is not one to one. As an example, $x(t) = 1$ and $x(t) = 2$ are both in $\mathbf{E}^H$ for $H = 0$, yet both have $X(\omega) = 0$ for $\omega > 0$. In order to accommodate this behavior in our subsequent theoretical development, all signals having a common $X(\omega)$ are be combined into an equivalence class. For example, two homogeneous functions $f(t)$ and $g(t)$ are equivalent if they differ by a homogeneous function whose frequency content is concentrated at the origin, such as t^H in the case that H is an integer.

Because the dyadic self-similarity property (5.2) of bihomogeneous signals is very similar to the dyadic scaling relationship between basis functions in an orthonormal wavelet basis, wavelets provide a particularly nice representation for this family of signals. Specifically, with $x(t)$ denoting an energy-dominated homogeneous signal, the expansion in an orthonormal wavelet basis is

$$x(t) = \sum_m \sum_n x_n^m \psi_n^m(t) \tag{5.7a}$$

$$x_n^m = \int_{-\infty}^{\infty} x(t)\psi_n^m(t). \tag{5.7b}$$

Since $x(t)$ satisfies (5.2) and since $\psi_n^m(t)$ satisfies (2.6), it easily follows from (5.7b) that for homogeneous signals

$$x_n^m = \beta^{-m/2} x_n^0 \tag{5.8}$$

where

$$\beta = 2^{2H+1} = 2^{\gamma}. \tag{5.9}$$

Denoting x_n^0 by $q[n]$, (5.7a) then becomes

$$x(t) = \sum \sum_n \beta^{-m/2} q[n] \psi_n^m(t), \tag{5.10}$$

from which we see that $x(t)$ is completely specified in terms of $q[n]$. We term $q[n]$ a *generating sequence* for $x(t)$ since, as we will see, this representation

leads to techniques for synthesizing useful approximations to homogeneous signals in practice.

Let us now specifically choose the ideal bandpass wavelet basis, whose basis functions we denote by

$$\tilde{\psi}_n^m(t) = 2^{m/2}\,\tilde{\psi}(2^m t - n) \tag{5.11}$$

where $\tilde{\psi}(t)$ is the ideal bandpass wavelet whose Fourier transform is given by (2.7). If we sample the output $\tilde{x}_0(t)$ of the filter in Definition 5.1 at unit rate, we obtain the sequence $\tilde{q}[n] = \tilde{x}_n^0$, where $\tilde{x}_n^m$ denotes the coefficients of expansion of $x(t)$ in terms of the ideal bandpass wavelet basis. Since $\tilde{x}_0(t)$ has the orthonormal expansion

$$\tilde{x}_0(t) = \sum_n \tilde{q}[n]\tilde{\psi}_n^0(t) \tag{5.12}$$

we have

$$\int_{-\infty}^{\infty} \tilde{x}_0^2(t)\,dt = \sum_n \tilde{q}^2[n]. \tag{5.13}$$

Consequently, a homogeneous function is energy-dominated if and only if its generating sequence in terms of the ideal bandpass wavelet basis has finite energy, i.e.,

$$\sum_n \tilde{q}^2[n] < \infty.$$

A convenient inner product between two energy-dominated homogeneous signals $f(t)$ and $g(t)$ can be defined as

$$\langle f, g\rangle_{\tilde{\psi}} = \int_{-\infty}^{\infty} f_0(t)\, g_0(t)\, dt$$

where the signals $f_0(t)$ and $g_0(t)$ are the responses of the bandpass filter (5.3) to $f(t)$ and $g(t)$, respectively. Exploiting (5.12) we may more conveniently express this inner product in terms of $\tilde{a}[n]$ and $\tilde{b}[n]$, the respective generating sequences of $f(t)$ and $g(t)$ under the bandpass wavelet basis, as

$$\langle f, g\rangle_{\tilde{\psi}} = \sum_n \tilde{a}[n]\,\tilde{b}[n]. \tag{5.14}$$

With this inner product, $\mathbf{E}^H$ constitutes a Hilbert space and the induced norm on $\mathbf{E}^H$ is

$$\|x\|_{\tilde{\psi}}^2 = \int_{-\infty}^{\infty} \tilde{x}_0^2(t)\,dt = \sum_n \tilde{q}^2[n]. \tag{5.15}$$

One can readily construct "self-similar" bases for $\mathbf{E}^H$. Indeed, the ideal bandpass wavelet basis (5.11) immediately provides an orthonormal basis for $\mathbf{E}^H$. In particular, for any $x(t) \in \mathbf{E}^H$, we have the synthesis/analysis pair

$$x(t) = \sum_n \tilde{q}[n]\,\tilde{\theta}_n^H(t) \tag{5.16a}$$

$$\tilde{q}[n] = \langle x, \tilde{\theta}_n^H\rangle_{\tilde{\psi}} \tag{5.16b}$$

where

$$\tilde{\theta}_n^H(t) = \sum_m \beta^{-m/2}\tilde{\psi}_n^m(t). \tag{5.17}$$

One can readily verify that the basis functions (5.17) are self-similar, orthogonal, and have unit norm.

The fact that the ideal bandpass basis is unrealizable means that (5.16) is not a practical mechanism for synthesizing or analyzing homogeneous signals. However, more practical wavelet bases are equally suitable for defining an inner product for the Hilbert space $\mathbf{E}^H$. In fact, we now show that a broad class of wavelet bases can be used to construct such inner products, and that as a consequence some highly efficient algorithms arise for processing homogeneous signals.

We begin by noting that not every orthonormal wavelet basis can be used to define inner products for $\mathbf{E}^H$. In order to determine which orthonormal wavelet bases can be used for this purpose, we must determine for which wavelets $\psi(t)$

$$q[n] = \int_{-\infty}^{\infty} x(t)\,\psi_n^0(t)\,dt \in \ell^2(\mathbb{Z}) \Leftrightarrow x(t) = \sum_m \sum_n \beta^{-m/2} q[n]\psi_n^m(t) \in \mathbf{E}^H.$$

That is, we seek conditions on a wavelet basis such that the sequence

$$q[n] = \int_{-\infty}^{\infty} x(t)\,\psi_n^0(t)\,dt$$

has finite energy whenever the homogeneous signal $x(t)$ is energy-dominated, and simultaneously such that the homogeneous signal

$$x(t) = \sum_m \sum_n \beta^{-m/2} q[n]\psi_n^m(t)$$

is energy-dominated whenever the sequence $q[n]$ has finite energy. Our main result is presented in terms of the following theorem. A proof of this theorem is provided in Appendix D.2.

Theorem 5.3 *Consider an orthonormal wavelet basis such that $\psi(t)$ has R vanishing moments for some integer $R \geq 1$, i.e.,*

$$\Psi^{(r)}(0) = 0, \qquad r = 0, 1, \ldots, R-1 \tag{5.18}$$

and let

$$x(t) = \sum_m \sum_n \beta^{-m/2} q[n]\psi_n^m(t)$$

be a bihomogeneous signal whose degree H is such that $\gamma = \log_2 \beta = 2H+1$ satisfies $0 < \gamma < 2R-1$. Then $x(t)$ is energy-dominated if and only if $q[n]$ has finite energy.

This theorem implies that for our Hilbert space $\mathbf{E}^H$ we may choose from among a large number of inner products whose induced norms are all equivalent. In particular, for any wavelet $\psi(t)$ with sufficiently many vanishing moments, we may define the inner product between two functions $f(t)$ and $g(t)$ in $\mathbf{E}^H$ whose generating sequences are $a[n]$ and $b[n]$, respectively, as

$$\langle f, g\rangle_\psi = \sum_n a[n]b[n]. \tag{5.19}$$

Of course, this collection of inner products is almost surely not exhaustive. Even for wavelet-based inner products, Theorem 5.3 asserts only that the vanishing moment condition is sufficient to ensure that the inner product generates an equivalent norm. It seems unlikely that the vanishing moment condition is a necessary condition.

The wavelet-based norms for $\mathbf{E}^H$ constitute a highly convenient and practical collection from which to choose in applications involving the use of homogeneous signals. Indeed, each associated wavelet-based inner product leads immediately to an orthonormal self-similar basis for $\mathbf{E}^H$: if $x(t) \in \mathbf{E}^H$, then

$$x(t) = \sum_n q[n]\,\theta_n^H(t) \tag{5.20a}$$

$$q[n] = \langle x, \theta_n^H\rangle_\psi \tag{5.20b}$$

where, again, the basis functions

$$\theta_n^H(t) = \sum_m \beta^{-m/2}\,\psi_n^m(t) \tag{5.21}$$

are all self-similar, mutually orthogonal, and have unit norm.

As an example for the case $H = 0$, Fig. 5.1 depicts the self-similar basis functions $\theta_4^0(t)$, $\theta_5^0(t)$, $\theta_6^0(t)$, and $\theta_7^0(t)$ corresponding to the Daubechies 5th-order compactly supported wavelet basis. These functions were generated by evaluating the summation (5.21) over a large but finite range of scales m. We emphasize that $q[n]$ is only a unique characterization of $x(t)$ when we associate it with a particular choice of wavelet $\psi(t)$. In general, every different wavelet decomposition of $x(t)$ yields a different $q[n]$, though all have finite energy.

It is useful to note that for an arbitrary nonhomogeneous signal $x(t)$, the sequence

$$q[n] = \langle x, \theta_n^H\rangle_\psi$$

defines the projections of $x(t)$ onto $\mathbf{E}^H$, so that

$$\hat{x}(t) = \int_{-\infty}^{\infty} q[n]\,\theta_n^H(t)\,dt$$

represents the closest homogeneous signal to $x(t)$ with respect to the induced norm $\|\cdot\|_\psi$, i.e.,

$$\hat{x}(t) = \arg\min_{y(t)\in\mathbf{E}^H} \|y - x\|_\psi.$$

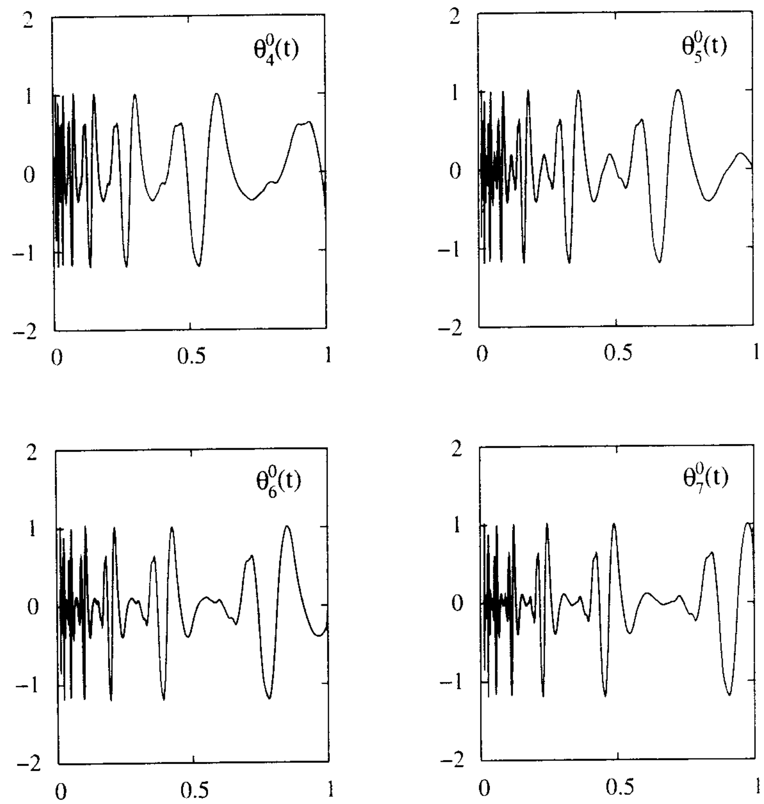

Figure 5.1. The self-similar basis functions $\theta_4^H(t)$, $\theta_5^H(t)$, $\theta_6^H(t)$, and $\theta_7^H(t)$ of an orthonormal basis for $\mathbf{E}^H$, $H = 0$.

In Chapter 6, it will be apparent how such projections arise rather naturally in treating problems of estimation with homogeneous signals.

Finally, we remark that wavelet-based characterizations also give rise to a convenient expression for the generalized Fourier transform of an energy-dominated homogeneous signal, $x(t)$. In particular, if we take the Fourier transform of (5.10) we get, via some routine algebra,

$$X(\omega) = \sum_m 2^{-(H+1)m} \Psi(2^{-m}\omega) Q(2^{-m}\omega) \tag{5.22}$$

where $Q(\omega)$ is the discrete-time Fourier transform of $q[n]$. This spectrum is to be interpreted in the sense of Theorem 5.2, i.e., $X(\omega)$ defines the spectral

content of the output of a bandpass filter at every frequency ω within the passband.

In summary, we have shown that a broad class of wavelet-based norms are equivalent for $\mathbf{E}^H$ in a mathematical sense, and that each of these norms is associated with a particular inner product. An interesting open question concerns whether *every* equivalent norm for $\mathbf{E}^H$ can be associated with a wavelet basis, in which case the basis functions associated with every orthonormal basis for $\mathbf{E}^H$ could be expressed in terms of some wavelet according to (5.21). In any case, regardless of whether the collection of inner products we construct is exhaustive or not, they at least constitute a highly convenient and practical collection from which to choose in any given application involving the use of homogeneous signals.

5.3 POWER-DOMINATED HOMOGENEOUS SIGNALS

Energy-dominated homogeneous signals have infinite energy. In fact, most have infinite power as well. However, there are other infinite-power homogeneous signals that are not energy-dominated. In this section, we consider a more general class of infinite-power homogeneous signals that find application as information-bearing waveforms in Chapter 6. The definition and properties closely parallel those for energy-dominated homogeneous signals.

Definition 5.4 *A bihomogeneous signal $x(t)$ is said to be power-dominated if when $x(t)$ is filtered by an ideal bandpass filter with frequency response (5.3) the resulting signal $\tilde{x}_0(t)$ has finite power, i.e.,*

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \tilde{x}_0^2(t)\, dt < \infty.$$

The notation $\mathbf{P}^H$ is used to designate the class of power-dominated homogeneous signals of degree H. Moreover, while our definition necessarily includes the energy-dominated signals, which have zero power, insofar as our discussion is concerned they constitute a degenerate case.

Analogous to Theorem 5.2 for the energy-dominated case, we can establish the following theorem describing the spectral properties of power-dominated homogeneous signals.

Theorem 5.5 *When a power-dominated homogeneous signal $x(t)$ is filtered by an ideal bandpass filter with frequency response (5.4), the resulting signal $y(t)$ has finite power and a power spectrum of the form*

$$S_y(\omega) = \lim_{T\to\infty} \frac{1}{2T} \left| \int_{-T}^{T} y(t)\, e^{-j\omega t}\, dt \right|^2 = \begin{cases} S_x(\omega) & \omega_L < |\omega| \le \omega_U \\ 0 & \textit{otherwise} \end{cases} \tag{5.23}$$

where $S_x(\omega)$ is some function that is independent of ω_L and ω_U and has octave-spaced ripple; i.e., for all integers k,

$$|\omega|^{2H+1} S_x(\omega) = |2^k \omega|^{2H+1} S_x(2^k \omega). \tag{5.24}$$

The details of the proof of this theorem are contained in Appendix D.3, although the approach is directly analogous to the proof of its counterpart, Theorem 5.2. Note that since $S_x(\omega)$ in the theorem does not depend on ω_L or ω_U, this function may be interpreted as the generalized power spectrum of $x(t)$. Furthermore, the relation (5.24) implies that signals in $\mathbf{P}^H$ have a generalized time-averaged power spectrum that is $1/f$-like, i.e.,

$$S_x(\omega) \sim \frac{1}{|\omega|^\gamma} \tag{5.25}$$

where, via (5.9), $\gamma = 2H + 1$.

Theorem 5.5 directly implies that a homogeneous signal $x(t)$ is power-dominated if and only if its generating sequence $\tilde{q}[n]$ in the ideal bandpass wavelet basis has finite power, i.e.,

$$\lim_{L\to\infty} \frac{1}{2L+1} \sum_{n=-L}^{L} \tilde{q}^2[n] < \infty.$$

Similarly we can readily deduce from the results of Section 5.2 that, in fact, for any orthonormal wavelet basis with $R > H + 1$ vanishing moments, the generating sequence for a homogeneous signal of degree H in that basis has finite power if and only if the signal is power-dominated. This implies that when we use (5.20a) with such wavelets to synthesize a homogeneous signal $x(t)$ using an arbitrary finite power sequence $q[n]$, we are assured that $x(t) \in \mathbf{P}^H$. Likewise, when we use (5.20b) to analyze any signal $x(t) \in \mathbf{P}^H$, we are assured that $q[n]$ has finite power.

Some general remarks are appropriate at this point in the discussion. Energy-dominated homogeneous signals of arbitrary degree H can be highly regular, at least away from $t = 0$. In contrast, power-dominated homogeneous signals typically have a fractal structure similar to the statistically self-similar $1/f$ processes of corresponding degree H, whose power spectra are also of the form (5.25) with $\gamma = 2H + 1$ [cf. (3.6)]. In turn, this suggests that, when defined, power-dominated homogeneous signals and $1/f$ processes of the same degree also have identical Hausdorff-Besicovitch dimensions [4]. Indeed, despite their obvious structural differences, power-dominated homogeneous signals and $1/f$ processes "look" remarkably similar in a qualitative sense. This is apparent in Fig. 5.2, where we depict the sample path of a $1/f$ process alongside a power-dominated homogeneous signal of the same degree whose generating sequence has been taken from a white random process. We stress, however, that in Fig. 5.2(a), the self-similarity of the $1/f$ process is statistical; i.e., a typical sample function does not satisfy (5.2) but its autocorrelation function does. In Fig. 5.2(b), the self-similarity of the homogeneous signal is deterministic. In fact, while the wavelet coefficients of homogeneous signals are identical from scale to scale to within an amplitude factor, i.e.,

$$x_n^m = \beta^{-m/2} q[n],$$

recall from Chapter 3 that the wavelet coefficients of $1/f$ processes have only the same second-order statistics from scale to scale to within an amplitude factor, i.e.,

$$E\left[x_n^m x_l^m\right] = \beta^{-m} \rho[n-l]$$

for some function $\rho[n]$ that is independent of m.

We can quantify the apparent similarity between the two types of signals through an observation about their spectra. In general, we remarked that for a given H, both exhibit power law spectral relationships with the same parameter γ. The following theorem further substantiates this for the case of randomly generated power-dominated homogeneous signals. The details of the proof are contained in Appendix D.4.

Theorem 5.6 *For any orthonormal wavelet basis in which $\psi(t)$ has Rth order regularity for some $R \geq 1$, the random process $x(t)$ synthesized according to*

$$x(t) = \sum_m \sum_n \beta^{-m/2} q[n] \psi_n^m(t) \tag{5.26}$$

using a correlation-ergodic (e.g., Gaussian), zero-mean, stationary white random sequence $q[n]$ of variance σ^2 has a generalized time-averaged power spectrum of the form

$$S_x(\omega) = \sigma^2 \sum_m 2^{-\gamma m} |\Psi(2^{-m}\omega)|^2. \tag{5.27}$$

Note that the time-averaged spectrum (5.27) is identical to the time-averaged spectrum (3.36) for the wavelet-based synthesis of $1/f$ processes described in Section 3.3.2. However, we must be careful not to misinterpret this result. It does not suggest that (5.26) is a reasonable approach for synthesizing $1/f$ processes. Indeed, it would constitute a very poor model for $1/f$-type behavior based on the analysis results of Section 3.3.2: when $1/f$ processes are decomposed into wavelet bases we get statistical rather than deterministic similarity from scale to scale. Instead, the theorem remarks that the *time-averaged* second order statistics of the two types of signals are the same. Consequently, one would anticipate that distinguishing $1/f$ processes from power-dominated homogeneous signals based on spectral analysis alone would be rather difficult. Nevertheless, the tremendous structural differences between the two means that they may be readily distinguished using other techniques such as, for example, wavelet-based analysis.

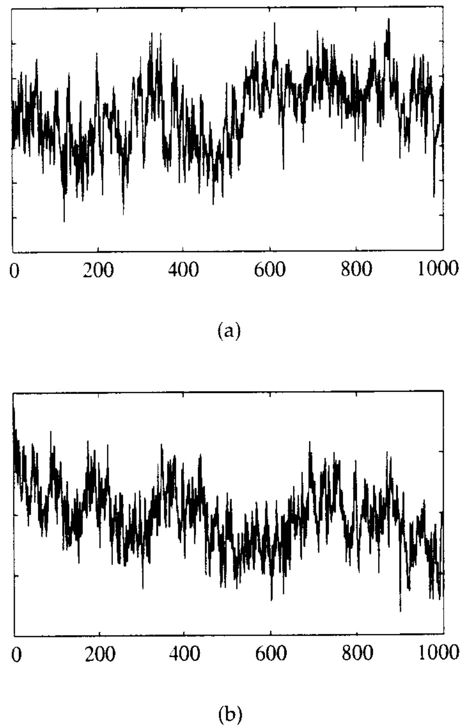

Figure 5.2. Comparison between the sample path of a $1/f$ process and a power-dominated homogeneous signal. Both correspond to $\gamma = 1$ (i.e., $H = 0$). (a) A sample function of a $1/f$ process. (b) A power-dominated homogeneous signal.

Note, too, that (5.27) corresponds to the superposition of the spectra associated with each scale or octave-band in the wavelet-based synthesis. In general, we would expect the spectrum of $x(t)$ to be the superposition of the spectra of the individual channels together with their cross-spectra. However, the time-averaged cross-spectra in this scenario are zero, which is a consequence of the fact that the white sequence $q[n]$ is modulated at different rates in each channel. Indeed, the time-averaged correlation is zero between $q[n]$ and $q[2^m n]$ for any $m \geq 1$ and $n \neq 0$; that is, white noise is uncorrelated with dilated and compressed versions of itself.

Finally, we note that because (5.27) and (3.36) are identical, we can use Theorem 3.4 to conclude that the spectra of a class of randomly generated

power-dominated homogeneous signals are bounded on any finite interval of the frequency axis that does not include $\omega = 0$. However, it is important to appreciate that not all power-dominated homogeneous signals have spectra that are bounded on $\pi \leq \omega \leq 2\pi$. An interesting subclass of power-dominated homogeneous signals with such unbounded spectra arises, in fact, in our application in Chapter 6. For these signals, $\tilde{x}(t)$ as defined in Definition 5.4 is *periodic*, so we refer to this class of power-dominated homogeneous signals as *periodicity-dominated.* It is straightforward to establish that these homogeneous signals have the property that when passed through an arbitrary bandpass filter of the form (5.4) the output is periodic as well. Furthermore, their power spectra consist of impulses whose *areas* decay according to a $1/|\omega|^\gamma$ relationship. An important class of periodicity-dominated homogeneous signals can be generated through a wavelet-based synthesis of the form (5.10) in which the generating sequence $q[n]$ is periodic.

5.4 DISCRETE-TIME ALGORITHMS FOR HOMOGENEOUS SIGNALS

Orthonormal wavelet representations provide some useful insights into homogeneous signals. For instance, because the sequence $q[n]$ is replicated at each scale in the representation (5.10) of a homogeneous signal $x(t)$, the detail signals

$$D_m x(t) = \beta^{-m/2} \sum_n q[n] \psi_n^m(t)$$

representing $q[n]$ modulated into a particular octave band are simply time-dilated versions of one another, to within an amplitude factor. The corresponding time-frequency portrait of a homogeneous signal is depicted in Fig. 5.3, from which the scaling properties are apparent. For purposes of illustration, the signal in this figure has degree $H = -1/2$ (i.e., $\beta = 1$), which corresponds to the case in which $q[n]$ is scaled by the same amplitude factor in each octave band. As always, the partitioning in such time-frequency portraits is idealized; in general, there is both spectral and temporal overlap between cells.

Wavelet representations also lead to some highly efficient algorithms for synthesizing, analyzing, and processing homogeneous signals just as they do for $1/f$ processes as discussed in Chapters 3 and 4. The signal processing structures we develop in this section are a consequence of applying the DWT algorithm to the highly structured form of the wavelet coefficients of homogeneous signals.

We have already encountered one discrete-time representation for a homogeneous signal $x(t)$, namely that in terms of a generating sequence $q[n]$ which corresponds to the coefficients of the expansion of $x(t)$ in an orthonormal basis $\{\theta_n^H(t)\}$ for $\mathbf{E}^H$. When the $\theta_n^H(t)$ are derived from a wavelet

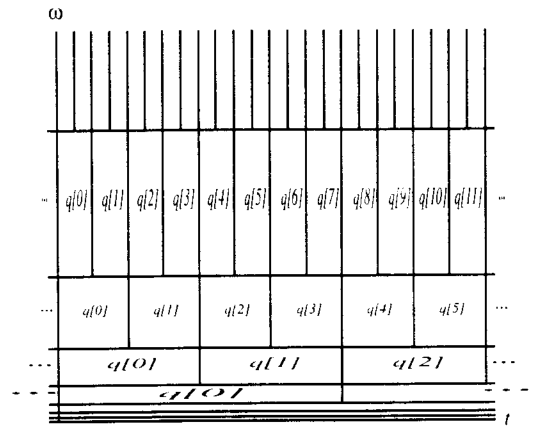

Figure 5.3. The time-frequency portrait of a homogeneous signal of degree $H = -1/2$.

basis according to (5.21), another useful discrete-time representation for $x(t)$ is available, which we now discuss.

Consider the coefficients a_n^m characterizing the resolution-limited approximation $A_m\{x(t)\}$ of a homogeneous signal $x(t)$ with respect to a particular wavelet-based multiresolution signal analysis. Since these coefficients are the projections of $x(t)$ onto dilations and translations of the scaling function $\phi(t)$ according to (2.13), it is straightforward to verify that they, too, are identical at all scales to within an amplitude factor, i.e.,

$$a_n^m = \beta^{-m/2} a_n^0. \tag{5.28}$$

Consequently, the sequence a_n^0 is an alternative discrete-time characterization of $x(t)$, since knowledge of it is sufficient to reconstruct $x(t)$ to arbitrary accuracy. For convenience, we refer to a_n^0 as the *characteristic sequence* and denote it as $p[n]$. As is true for the generating sequence, the characteristic sequence associated with $x(t)$ depends upon the particular multiresolution analysis used; distinct multiresolution signal analyses generally yield different characteristic sequences for any given homogeneous signal. In what follows, we restrict our attention to multiresolution analyses whose basic wavelet meets the vanishing moment conditions of Theorem 5.3.

The characteristic sequence $p[n]$ is associated with a resolution-limited approximation to the corresponding homogeneous signal $x(t)$. Specifically, $p[n]$ represents unit-rate samples of the output of the filter, driven by $x(t)$, whose frequency response is $\Phi^*(\omega)$, the complex conjugate of the Fourier transform of the scaling function. Because frequencies in a neighborhood of

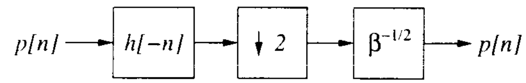

Figure 5.4. The discrete-time self-similarity identity for a characteristic sequence $p[n]$.

the spectral origin, where the spectrum of $x(t)$ diverges, are passed by such a filter, $p[n]$ often has infinite energy or, worse, infinite power, even when the generating sequence $q[n]$ has finite energy.

The characteristic sequence can, in fact, be viewed as a *discrete-time* homogeneous signal, and a theory can be developed following an approach directly analogous to that used in Sections 5.2 and 5.3 for the case of continuous-time homogeneous signals. The characteristic sequence satisfies the discrete-time self-similarity relation[1]

$$\beta^{1/2} p[n] = \sum_k h[k-2n] p[k], \tag{5.29}$$

which is readily obtained by substituting for a_n^m in the DWT analysis equation (2.21a) using (5.28). Indeed, as depicted in Fig. 5.4, (5.29) is a statement that when $p[n]$ is lowpass filtered with the conjugate filter whose unit-sample response is $h[-n]$ and then downsampled, we recover an amplitude-scaled version of $p[n]$. Although characteristic sequences are, in an appropriate sense, "generalized sequences," when highpass filtered with the corresponding conjugate highpass filter whose unit-sample response is $g[-n]$, the output is a finite energy or finite power sequence, depending on whether $p[n]$ corresponds to a homogeneous signal $x(t)$ that is energy-dominated or power-dominated, respectively. Consequently, we can analogously classify the sequence $p[n]$ as energy-dominated in the former case, and power-dominated in the latter case. In fact, when the output of such a highpass filter is downsampled at rate two, we recover the characteristic sequence $q[n]$ associated with the expansion of $x(t)$ in the corresponding wavelet basis, i.e.,

$$\beta^{1/2} q[n] = \sum_k g[k-2n] p[k]. \tag{5.30}$$

This can be readily verified by substituting for a_n^m and x_n^m in the DWT analysis equation (2.21b) using (5.28) and (5.8), and by recognizing that $a_0^m = p[n]$ and $x_0^m = q[n]$.

From a different perspective, (5.30) provides a convenient mechanism for obtaining the representation for a homogeneous signal $x(t)$ in terms of its

[1]Relations of this type may be considered discrete-time counterparts of the *dilation equations* considered by Strang [26].

generating sequence $q[n]$ from one in terms of its corresponding characteristic sequence $p[n]$, i.e.,

$$p[n] \longrightarrow q[n].$$

To obtain the reverse mapping

$$q[n] \longrightarrow p[n]$$

is less straightforward. For an arbitrary sequence $q[n]$, the associated characteristic sequence $p[n]$ is the solution to the linear equation

$$\beta^{-1/2}p[n] - \sum_k h[n-2k]p[k] = \sum_k g[n-2k]q[k], \qquad (5.31)$$

as can be verified by specializing the DWT synthesis equation (2.21c) to the case of homogeneous signals. There appears to be no direct method for solving this equation. However, the DWT synthesis algorithm suggests a convenient and efficient iterative algorithm for constructing $p[n]$ from $q[n]$. In particular, denoting the estimate of $p[n]$ on the ith iteration by $p^{[i]}[n]$, the algorithm is

$$p^{[0]}[n] = 0 \qquad (5.32a)$$

$$p^{[i+1]}[n] = \beta^{1/2}\sum_k \left\{h[n-2k]p^{[i]}[k] + g[n-2k]q[k]\right\}. \qquad (5.32b)$$

This recursive upsample-filter-merge algorithm, depicted in Fig. 5.5, can be interpreted as repeatedly modulating $q[n]$ with the appropriate gain into successively lower octave bands of the frequency interval $0 \leq |\omega| \leq \pi$. Note that the precomputable quantity

$$q_*[n] = \sum_k g[n-2k]q[k]$$

represents the sequence $q[n]$ modulated into essentially the upper half band of frequencies.

Any real application of homogeneous signals can ultimately exploit scaling properties over only a *finite* range of scales, so that it suffices in practice to modulate $q[n]$ into a finite range of contiguous octave bands. Consequently, only a finite number of iterations of the algorithm (5.32) are required. More generally, this also means that many of the theoretical issues associated with homogeneous signals concerning singularities and convergence do not present practical difficulties in the application of these signals, as will be apparent in our developments of Chapter 6.

As we conclude this chapter, it is worth mentioning that there may be useful connections to be explored between the self-similar signal theory described here and the work of Barnsley [92] on deterministically self-affine one-dimensional and multi-dimensional signals. Malassenet and Mersereau [93], for example, suggest that these so-called "iterated function systems" have efficient representations in terms of wavelet bases as well.

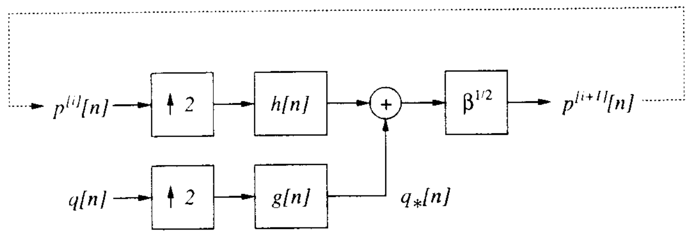

Figure 5.5. Iterative algorithm for the synthesis of the characteristic sequence $p[n]$ of a homogeneous signal $x(t)$ from its generating sequence $q[n]$. The notation $p^{[i]}[n]$ denotes the value of $p[n]$ at the ith iteration.

5.5 SUMMARY

In this chapter we focussed on fractal signals characterized by a deterministic scaling relation. We showed that these "homogeneous" signals, in contrast to the fractal random processes described in Chapter 3, have the property that the waveforms themselves remain invariant to within an amplitude factor under arbitrary scaling of the time axis.

We then introduced and developed a new and richer generalized family of homogeneous signals defined in terms of a *dyadic* scale-invariance property. When necessary to avoid confusion with traditional homogeneous signals, we specifically referred to this broader family as bihomogeneous signals. Motivated by our interest in using these signals as modulating waveforms in some communication applications in the next chapter, we proceeded to develop some of their important properties and representations.

We began by distinguishing between two classes: energy-dominated and power-dominated, and then developed their spectral properties. We then showed that the use of wavelet basis expansions leads to constructions of powerful orthonormal self-similar bases for homogeneous signals. From this perspective, we saw that wavelet representations play as natural and important a role in the representation of these signals as they did for the $1/f$ processes developed in Chapter 3.

In the latter portion of the chapter, we exploited the discrete wavelet transform algorithm to derive highly efficient discrete-time algorithms for both synthesizing and analyzing homogeneous signals using these representations. As we will see, these algorithms play an important role in the transmitters and receivers of the communication system we explore in the next chapter.