7

Linear Self-Similar Systems

7.1 INTRODUCTION

In preceding chapters we have explored several useful classes of statistically and deterministically self-similar signals that arise in engineering applications. This chapter represents a preliminary investigation into the relationships between self-similar signals and an underlying self-similar system theory. In particular, we explore not only how we may interpret some of our methods for synthesizing self-similar signals in the context of driven self-similar systems, but also the role that the wavelet transform plays in characterizing such systems. In the end, this leads to some interesting and potentially important insights and perspectives into the results of the book, and in the process suggests some promising future directions for work in this area.

The self-similar systems we ultimately discuss in this chapter have the property that they are linear and jointly time- and scale-invariant. In the first half of the chapter we define this class of systems, develop several properties, and show how both the Laplace and Mellin transforms can be used in their analysis. In the latter half of the chapter we develop wavelet-based characterizations of this class of systems to illustrate that the wavelet transform is in some sense best matched to these systems—that such characterizations are as natural and as useful for these systems as Fourier-based characterizations are for linear time-invariant systems.

Overall, our treatment is rather informal in style, reflecting a conscious effort to emphasize the conceptual themes over mathematical rigor and gener-

ality. To facilitate this, our development focuses, for example, on input-output descriptions of systems. Our development begins with a brief summary of some results in the theory of linear time-invariant systems. For more extensive treatments, see, e.g., Oppenheim and Willsky [1], Siebert [2], or Kailath [100].

Linear systems are typically defined as follows. Suppose $y(t)$, $y_1(t)$, and $y_2(t)$ are the responses of a system $\mathcal{S}\{\cdot\}$ to arbitrary inputs $x(t)$, $x_1(t)$, and $x_2(t)$, respectively. Then the system is linear when it satisfies, for any a and b, the superposition principle

$$\mathcal{S}\{ax_1(t) + bx_2(t)\} = ay_1(t) + by_2(t). \tag{7.1}$$

Linear systems are often conveniently described in terms of the integral

$$y(t) = \mathcal{S}\{x(t)\} = \int_{-\infty}^{\infty} x(\tau)\,\kappa(t,\tau)\,d\tau,$$

where $\kappa(t,\tau)$ is the kernel of the linear system and represents the response of the system at time t to a unit impulse at time τ, i.e.,

$$\kappa(t,\tau) \triangleq \mathcal{S}\{\delta(t-\tau)\}.$$

7.2 LINEAR TIME-INVARIANT SYSTEMS

An important class of linear systems are those that are also time-invariant. A system is time-invariant when it satisfies, for any constant τ,

$$\mathcal{S}\{x(t-\tau)\} = y(t-\tau). \tag{7.2}$$

Collectively the properties (7.1) and (7.2) characterize a linear time-invariant (LTI) system.

A linear system is time-invariant if and only if its kernel $\kappa(t,\tau)$ satisfies, for any b,

$$\kappa(t,\tau) = \kappa(t-b,\tau-b). \tag{7.3}$$

For this class of systems, the kernel has the form

$$\kappa(t,\tau) = v(t-\tau)$$

where $v(t)$ is the familiar impulse response of the system. Furthermore, the corresponding input-output relation is, of course, described in terms of the usual convolution integral,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,v(t-\tau)\,d\tau \triangleq x(t) * v(t).$$

The eigenfunctions of LTI systems are complex exponentials of the form e^{st}, from which we get that the Laplace transform

$$X(s) = \int_{-\infty}^{\infty} x(t)\,e^{-st}\,dt$$

possesses the so-called convolution property; i.e., for signals $x(t)$ and $y(t)$ with Laplace transforms $X(s)$ and $Y(s)$, respectively, we have

$$x(t) * y(t) \longleftrightarrow X(s)\, Y(s).$$

7.3 LINEAR SCALE-INVARIANT SYSTEMS

In contrast to linear time-invariant systems, linear *scale*-invariant system theory has been comparatively less explored, though it has received occasional attention in the systems [101] and pattern recognition [102] literature, and in the broader mathematics literature in connection with the Mellin transform [103] [104] [105].

To explore these systems, suppose that $y(t)$ is the response of a system $\mathcal{S}\{\cdot\}$ to an arbitrary input $x(t)$. Then a system $\mathcal{S}\{\cdot\}$ is said to be scale-invariant whenever, for any constant $\tau > 0$,

$$\mathcal{S}\{x(t/\tau)\} = y(t/\tau). \tag{7.4}$$

A system satisfying both (7.1) and (7.4) is referred to as a *linear scale-invariant* (LSI) system.

It is straightforward to show that a necessary and sufficient condition for the kernel $\kappa(t,\tau)$ of a linear system to correspond to a scale-invariant system is that it satisfies

$$\kappa(t,\tau) = a\kappa(at, a\tau) \tag{7.5}$$

for any $a > 0$.

An LSI system is generally characterized in terms of the *lagged-impulse response* pair

$$\xi_+(t) = \mathcal{S}\{\delta(t-1)\} \tag{7.6a}$$
$$\xi_-(t) = \mathcal{S}\{\delta(t+1)\}. \tag{7.6b}$$

Indeed, when an input $x(t)$ can be decomposed, except at $t = 0$, as

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} x(\tau)\,\delta\left(\frac{t}{\tau}-1\right)\frac{d\tau}{\tau} \\ &= \int_{0}^{\infty} x(\tau)\,\delta\left(\frac{t}{\tau}-1\right)\frac{d\tau}{\tau} - \int_{0}^{\infty} x(-\tau)\,\delta\left(\frac{t}{\tau}+1\right)\frac{d\tau}{\tau}, \end{aligned} \tag{7.7}$$

we can exploit the superposition principle (7.1) together with (7.4) to obtain the following input-output relation

$$y(t) = \int_0^\infty x(\tau)\,\xi_+\left(\frac{t}{\tau}\right)\frac{d\tau}{\tau} - \int_0^\infty x(-\tau)\,\xi_-\left(\frac{t}{\tau}\right)\frac{d\tau}{\tau}. \tag{7.8}$$

For simplicity of exposition, we restrict our subsequent discussion to the case of causal inputs

$$x(t) = 0, \qquad t \le 0$$

and LSI systems whose outputs are causal

$$y(t) = \mathcal{S}\{x(t)\} = 0, \qquad t \le 0.$$

From the development, it will be apparent how to accommodate the more general scenario of (7.8).

For the causal case, only one of the lagged impulse responses (7.6) is required to characterize the system, and, in particular, the input-output relation (7.8) simplifies to

$$y(t) = \int_0^\infty x(\tau)\,\xi\left(\frac{t}{\tau}\right)\frac{d\tau}{\tau} \triangleq x(t) \star \xi(t) \tag{7.9}$$

where we let $\xi(t) = \xi_+(t)$ to simplify our notation, and where we use the symbol $\star$ to distinguish this convolutional relationship from the usual convolution $*$ associated with LTI systems. Note that for these LSI systems the kernel is

$$\kappa(t,\tau) = \frac{1}{\tau}\,\xi\left(\frac{t}{\tau}\right).$$

This new convolution operation possesses many of the properties of the usual convolution operation. For example, it is straightforward to show that it is commutative for well-behaved operands, i.e.,

$$x(t) \star \xi(t) = \xi(t) \star x(t) = \int_0^\infty x\left(\frac{t}{\tau}\right)\xi(\tau)\,\frac{d\tau}{\tau}. \tag{7.10}$$

As a consequence, the cascade of two LSI systems with lagged-impulse responses $\xi_1(t)$ and $\xi_2(t)$, respectively, is typically equivalent to a single system with lagged-impulse response $\xi_1(t) \star \xi_2(t)$. Furthermore, such systems may be cascaded in either order without changing the overall system.

Likewise, it is straightforward to show that the new convolution operation is distributive for well-behaved operands, i.e.,

$$x(t) \star \{\xi_1(t) + \xi_2(t)\} = x(t) \star \xi_1(t) + x(t) \star \xi_2(t). \tag{7.11}$$

Hence, the parallel connection of two LSI systems with lagged-impulse responses $\xi_1(t)$ and $\xi_2(t)$, respectively, is equivalent to a single system with lagged-impulse response $\xi_1(t) + \xi_2(t)$.

The eigenfunctions of linear scale-invariant systems are homogeneous functions of degree s; specifically they are the complex power functions defined by

$$x(t) = t^s. \tag{7.12}$$

where s is a complex number. Indeed, from (7.9) and (7.10) the response of an LSI system to (7.12) is readily obtained as

$$y(t) = \check{\Xi}(s)\, t^s$$

with the associated complex eigenvalue given by

$$\check{\Xi}(s) = \int_0^\infty \xi(\tau)\, \tau^{-s-1}\, d\tau \tag{7.13}$$

whenever this integral converges. Eq. (7.13) is referred to as the Mellin transform[1] of the signal $\xi(t)$ [105] [106] [107].

The eigenfunction property of the complex power functions implies that the Mellin transform constitutes an important tool in the analysis of LSI systems. Indeed, it is particularly convenient to compute the response of an LSI system to any input that is the superposition of eigenfunctions. Fortunately, a broad class of signals $x(t)$ can be expressed as a superposition of eigenfunctions of LSI systems according to

$$x(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \check{X}(s)\, t^s\, ds \tag{7.14a}$$

for $t > 0$, where

$$\check{X}(s) = \int_0^\infty x(\tau)\, \tau^{-s-1}\, d\tau \tag{7.14b}$$

and c is in the region of convergence of $\check{X}(s)$. Eqs. (7.14) collectively constitute the Mellin representation of a signal $x(t)$; the Mellin inverse formula (7.14a) is the synthesis relation, while the Mellin transform (7.14b) is the analysis formula. Interestingly, we may interpret the Mellin transformation as a representation of $x(t)$ by its "fractional" moments.

The Mellin inverse formula implies that a broad class of linear scale-invariant systems are completely characterized by the Mellin transforms $\check{\Xi}(s)$ of their respective lagged-impulse responses. Consequently, we can refer to this quantity as the *system function* associated with the LSI system. As a consequence of the eigenfunction property of the complex power functions, the input-output relation for a linear scale-invariant system with system function $\check{\Xi}(s)$ can be expressed in the Mellin domain as

$$\check{Y}(s) = \check{\Xi}(s)\check{X}(s) \tag{7.15}$$

whenever both terms on the right-hand side have a common region of convergence. Hence, via the Mellin transform, we can map our convolution operation (7.9) into a convenient multiplicative operation (7.15).

The Mellin transform, its inversion formula, properties, and numerous transform pairs are well documented in the literature [105] [106] [107]. One

[1] Actually, we have chosen a slight but inconsequential variant of the Mellin transform—the usual Mellin transform has s replaced by $-s$ in our definition.

basic Mellin transform pair is given by

$$t^{-s_0}\, u(t-1) \longleftrightarrow \frac{1}{s+s_0}, \quad Re(s) > -s_0 \tag{7.16}$$

for arbitrary s_0.

From this pair we are able to show that the Mellin transform plays an important role in the solution of a class of scale-differential equations that give rise to linear scale-invariant systems. We begin by quantifying the notion of a "derivative operator in scale." A reasonable definition of the derivative in scale of a signal $x(t)$ is given by

$$\nabla_s x(t) \triangleq \lim_{\varepsilon \to 1} \frac{x(\varepsilon t) - x(t)}{\ln \varepsilon}.$$

One can readily interpret this definition in the context of traditional derivatives as

$$\nabla_s x(t) = \frac{d}{d \ln t} x(t) = t \frac{d}{dt} x(t).$$

Differentiation in scale corresponds to a multiplication by s in the Mellin domain, which suggests that the Mellin transform can be used to efficiently solve what can be described as a class of "dynamical systems in scale."[2] Consider the following Nth-order linear constant-coefficient scale-differential equation

$$\sum_{k=0}^{N} a_k \nabla_s^k y(t) = \sum_{k=0}^{M} b_k \nabla_s^k x(t),$$

where we denote the kth derivative in scale, obtained by iterative application of the derivative operator, by ∇_s^k. Then, via the convolution property of the Mellin transform, we obtain

$$\check{Y}(s) = \check{\Xi}(s)\check{X}(s)$$

where $\check{\Xi}(s)$ is rational, i.e.,

$$\check{\Xi}(s) = \frac{\prod_{k=0}^{M} b_k s^k}{\prod_{k=0}^{N} a_k s^k},$$

in the corresponding region of convergence. The usual partial fraction expansion approach, together with Mellin pairs of the form (7.16), can be used to derive $y(t)$ from its Mellin transform.

[2] In fact, this development raises some interesting questions regarding connections to the more general literature that is evolving on multiscale systems [84] [108]. Exploring such relationships, however, is beyond the scope of this chapter.

It is interesting to note that in the 1950s, such an approach was developed for the synthesis and analysis of time-varying networks governed by scale-differential and Euler-Cauchy equations [101], although the relationship to linear scale-invariant system theory was not recognized. Nevertheless, the convolution relationship (7.9) does appear in this work.

Before we turn our attention to a more broadly defined class of LSI systems, we remark that there is, in fact, a natural homomorphism between linear scale-invariant and linear time-invariant (LTI) systems. This relationship allows us to derive virtually all the results described in this section, in addition to many others, by mapping corresponding properties from the theory of LTI systems. Specifically, by replacing time t with exponential time e^t, we find, for example, that LSI systems become LTI systems, complex power functions become complex exponentials, the Mellin transform becomes the bilateral Laplace transform, and linear constant-coefficient scale-differential equations become familiar linear constant-coefficient differential equations.

We next consider a somewhat broader notion of LSI system that will be useful in the sequel.

7.3.1 Generalized Linear Scale-Invariant Systems

Suppose $y(t)$ is the response of a system $\mathcal{S}\{\cdot\}$ to an arbitrary input $x(t)$. Then we say the system $\mathcal{S}\{\cdot\}$ is *scale-invariant with parameter* λ whenever, for any constant $\tau > 0$,

$$\mathcal{S}\{x(t/\tau)\} = \tau^{\lambda} y(t/\tau). \tag{7.17}$$

We denote systems that satisfy the superposition principle (7.1) and the generalized scale-invariance relation (7.17) as LSI(λ) systems. Obviously, strict-sense LSI systems correspond to the special case $\lambda = 0$. It can be easily established that a necessary and sufficient condition for a linear system to be scale-invariant with parameter λ is that the kernel satisfy, for any $a > 0$,

$$\kappa(t, \tau) = a^{-(\lambda-1)} \kappa(at, a\tau). \tag{7.18}$$

Such generalized linear scale-invariant systems are also completely characterized in terms of their lagged-impulse response pair (7.6). And, again when we are able to decompose our input according to (7.7) and restrict our attention to the case of causal signals, we can exploit (7.1) and (7.17) to get the following input-output relation

$$y(t) = \int_0^\infty x(\tau)\, \xi\left(\frac{t}{\tau}\right) \frac{d\tau}{\tau^{1-\lambda}}. \tag{7.19}$$

Rewriting (7.19) as

$$y(t) = \int_0^\infty \left\{x(\tau)\tau^{\lambda}\right\} \xi\left(\frac{t}{\tau}\right) \frac{d\tau}{\tau}$$

we observe that in principle any LSI(λ) system can be implemented as the cascade of a system that multiplies the input by $|t|^{\lambda}$, followed by a strict-sense LSI system with lagged-impulse response $\xi(t)$. However, in many cases, this may not be a particularly convenient implementation, either conceptually or practically.

7.4 LINEAR TIME- AND SCALE-INVARIANT SYSTEMS

We say that a system is *linear time- and scale-invariant with parameter* λ, denoted LTSI(λ), whenever it jointly satisfies the properties of superposition (7.1), time-invariance (7.2), and generalized scale-invariance (7.17). In this case, the time-invariance constraint (7.3) requires the kernel to be of the form

$$\kappa(t, \tau) = v(t - \tau)$$

for some impulse response $v(\cdot)$; and the scale-invariance constraint (7.18) imposes, in turn, that this impulse response be a generalized homogeneous function of degree $\lambda - 1$, i.e.,

$$v(t) = a^{-(\lambda-1)} v(at)$$

for all t and all $a > 0$. Following Gel'fand et al. [90], we can parameterize the entire class of impulse responses for such systems. In particular, provided $\lambda \neq 0, -1, -2, \ldots$, we get that $v(t)$ takes the form

$$v(t) = C_1|t|^{\lambda-1}\, u(t) + C_2|t|^{\lambda-1}\, u(-t). \tag{7.20a}$$

For the special case $\lambda = -n$ for $n = 0, 1, 2, \ldots$,

$$v(t) = C_3|t|^{-(n-1)}\, u(t) + C_4|t|^{-(n-1)}\, u(-t) + C_5\delta^{(n)}(t) \tag{7.20b}$$

where $\delta^{(n)}(t)$ denotes the nth derivative of the unit impulse and $u(t)$ the unit step function. In both cases, the $C_1, \ldots C_5$ are arbitrary constants.

There are many familiar LTSI(λ) systems. For example, the identity system, for which

$$v(t) = \delta(t),$$

corresponds to $\lambda = 0$, $C_3 = C_4 = 0$ and $C_5 = 1$. In fact, as is apparent from the parameterizations (7.20), the identity system is the *only* stable LTSI(λ) system. A second example is the integrator. This system has a regular impulse response

$$v(t) = u(t)$$

and corresponds to $\lambda = 1$, $C_1 = 1$, and $C_2 = 0$. As a final example, consider a differentiator, which has for an impulse response the unit doublet

$$v(t) = \delta'(t).$$

This choice corresponds to $\lambda = -1$, $C_3 = C_4 = 0$, and $C_5 = 1$.

Linear time- and scale-invariant systems are natural candidates for modeling and processing self-similar signals as we begin to show in the next section.

7.4.1 Self-Similar Signals and LTSI(λ) Systems

In this section, we explore some relationships between self-similar signals and systems. In particular, we show how LTSI(λ) systems preserve the time-invariance and scale-invariance of their inputs, and point out how these properties have been exploited in some of the models for self-similar signals described earlier in the book.

Our result in the deterministic case is as follows. Let $v(t)$ be the impulse response of an LTSI(λ) system, so that $v(t)$ is homogeneous of degree $\lambda - 1$, i.e., for any $a > 0$

$$v(t) = a^{-(\lambda-1)}\, v(at),$$

and consider driving the system with a scale-invariant input signal $x(t)$ that is homogeneous of degree H. Then it is straightforward to establish that the output $y(t)$ of the system

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, v(t-\tau)\, d\tau$$

when well defined, is scale-invariant as well. In fact, it is homogeneous of degree $H + \lambda$, so that, for any $a > 0$

$$y(t) = a^{-(H+\lambda)}\, y(at). \tag{7.21}$$

Two obvious special cases are immediately apparent. The first corresponds to the case in which the system is the identity system ($\lambda = 0$). Here the output and input are identical, and (7.21) yields the appropriate result. The second corresponds to the case in which the input is an impulse ($H = -1$). Here, the output is $v(t)$, and, again, (7.21) yields the correct result. This, of course, suggests that at least one synthesis for a class of homogeneous signals is in terms of an LTSI(λ) system driven by an impulse.

Note that we can derive analogous results for deterministically *time-invariant* inputs. However, in this case the results are somewhat degenerate. In particular, except in trivial cases, for a time-invariant (i.e., constant) input, the output of such a system is only well defined if the system is an identity system since any other LTSI(λ) system is unstable. Nevertheless, in this unique case, the output is, obviously, time-invariant as well.

Consider, next, the case of an input that is either wide- or strict-sense statistically scale-invariant as defined in Chapter 3. In this case, it is also straightforward to show that the output, when well defined, is also statistically scale-invariant and satisfies

$$y(t) \stackrel{\mathcal{P}}{=} a^{-(H+\lambda)}\, y(at)$$

with equality in the corresponding statistical sense.

For wide- or strict-sense stationary (i.e., statistically time-invariant) inputs, the outputs, when well defined, are also stationary. This is, of course, a well-known result from LTI system theory. Note, however, that, again from stability considerations, the only nontrivial system for which the output is well defined is the identity system. This implies, for instance, that, in general, when driven with stationary white noise, the outputs of such systems are not well defined.

Many of these issues surfaced in Chapter 3, where we considered the modeling of $1/f$ processes through a synthesis filter formulation. Specifically, the system with impulse response (3.9) we first proposed as a synthesis filter for $1/f$ processes is precisely an example of an LTSI(λ) system with $\lambda = H + 1/2$. A similar filter with $\lambda = H - 1/2$ appears in the conceptual synthesis for fractional Brownian motion illustrated in Fig. 3.2. Furthermore, the fractional integrator used in the Barnes-Allan synthesis for $1/f$ processes, which we described in Section 3.2, has properties similar to those of an LTSI(λ) system. More generally, there would appear to be a number of potentially important connections between the operators of fractional calculus [56] and linear jointly time- and scale-invariant system theory. Finally, the ARMA filter used in the lumped RC-line synthesis of $1/f$-like behavior that we discussed in Section 3.3.1 can be viewed as an approximation to an LTSI(λ) system. Specifically, this filter is linear and time-invariant, but satisfies the scale-invariance relation (7.17) only for dilation factors τ of the form $\tau = \Delta^m$.

The impulse-response constitutes one important means for characterizing LTSI(λ) systems. However, from an implementational perspective, it is not always the most convenient. In the next section, we develop a canonical representation for a class of LTSI(λ) systems in terms of wavelet bases. As we will see, this characterization not only provides additional insight into such systems, but ultimately leads to some important techniques for realizing and approximating them. In fact, we will see that it is possible to interpret many of the wavelet-based representations for self-similar signals we derived in Chapters 3 and 5 in the context of these results.

7.5 WAVELET-BASED LTSI(λ) SYSTEMS

Consider a system that computes the continuous-parameter wavelet transform of the input $x(t)$ via (2.1), multiplies the resulting field X_ν^μ by some regular field K_ν^μ in time-scale space, then inverts the result according to the synthesis formula (2.4), so that

$$y(t) = \mathcal{W}^{-1}\left\{K_\nu^\mu \mathcal{W}\left\{x(t)\right\}\right\}. \tag{7.22}$$

It is straightforward to establish that such a linear system has a kernel

$$\kappa(t,\tau) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi_\nu^\mu(t)\, K_\nu^\mu\, \psi_\nu^\mu(\tau)\, \mu^{-2} d\mu d\nu. \tag{7.23}$$

The structure of this kernel imposes certain constraints on the linear system; for example, such systems are symmetric, i.e.,

$$\kappa(t,\tau) = \kappa(\tau,t). \tag{7.24}$$

However, the structure of the kernel is sufficiently general that one can implement LTI, LSI, or LTSI systems using this framework.

For instance, using the readily derived identity

$$v_\nu^\mu(t-b) = v_{\nu+b}^\mu(t)$$

valid for all b, the system is time-invariant [i.e., satisfies (7.2)] whenever the multiplier field satisfies

$$K_\nu^\mu = K_{\nu-b}^\mu$$

for all b. In other words, (7.22) implements an LTI system whenever the field K_ν^μ is independent of ν. In this case, K_ν^μ can be expressed as

$$K_\nu^\mu = k(\mu) \tag{7.25}$$

for some regular function of scale $k(\cdot)$.

Likewise, using the identity

$$v_\nu^\mu(at) = |a|^{-1/2} v_{\nu/a}^{\mu/a}(t) \tag{7.26}$$

valid for all $a \neq 0$, the system is scale-invariant with parameter λ [i.e., satisfies (7.17)] whenever the multiplier field satisfies

$$K_\nu^\mu = a^{-\lambda} K_{a\nu}^{a\mu} \tag{7.27}$$

for all $a > 0$.

For the system to be jointly time- and scale-invariant with parameter λ, (7.25) and (7.27) require that

$$k(\mu) = a^{-\lambda} k(a\mu),$$

i.e., that $k(\cdot)$ be homogeneous of degree λ. The imposition of regularity on $k(\cdot)$ precludes it from containing impulses or derivatives of impulses. Again using Gel'fand's parameterization of the homogeneous functions, we conclude that the system (7.22) is LTSI(λ) whenever the multiplier field has the form

$$K_\nu^\mu = k(\mu) = C_1|\mu|^\lambda\, u(\mu) + C_2|\mu|^\lambda\, u(-\mu) \tag{7.28}$$

for some constants C_1 and C_2. Note that even if these constants are chosen so that $k(\cdot)$ is asymmetric, the impulse response $v(t)$ of the resulting system is even, i.e.,

$$v(t) = v(-t). \tag{7.29}$$

This is a consequence of the symmetry constraint (7.24). In fact, since we can rewrite (7.23) using (7.26) as

$$\kappa(t,\tau) = \frac{1}{C_\psi}\int_{-\infty}^{\infty} d\nu \int_0^{\infty} v_\nu^\mu(t)\,[k(\mu)+k(-\mu)]\, v_\nu^\mu(\tau)\,\mu^{-2} d\mu.$$

we see that the kernel of the system is really only a function of the *even* part of the function $k(\cdot)$. Hence, without loss of generality we may set $C_2 = 0$ in (7.28) and choose

$$K_\nu^\mu = k(\mu) = C\mu^\lambda\, u(\mu) \tag{7.30}$$

where C is an arbitrary constant.

Finally, combining (7.29) with (7.20), we can conclude that whenever (7.22) implements an LTSI(λ) system, i.e., whenever $k(\cdot)$ is chosen according to (7.30), the impulse response corresponding to the system of (7.22) must take the form

$$v(t) = C_r|t|^{\lambda-1}$$

for $\lambda \neq 0, -2, -4, \ldots$, or the form

$$v(t) = C_r|t|^{-(n-1)} + C_s\delta^{(n)}(t)$$

for $\lambda = -n$ for $n = 0, 2, 4, \ldots$. In both cases, C_r and C_s are parameters determined by the constant C in (7.30). Note in particular that, at least for the case $\lambda = 0$, we must have

$$\begin{aligned} C_s &= C \\ C_r &= 0. \end{aligned}$$

This follows from the fact that, since $K_\nu^\mu \equiv C$, the overall system (7.22) is just a scaled identity system. Fig. 7.1 summarizes the resulting wavelet-based realization of a linear jointly time- and scale-invariant system with parameter λ. Note that this is analogous to implementing an LTI system by computing the Fourier transform of the input, multiplying by some frequency response, and applying the inverse Fourier transform to the result. As is the case for Fourier-based implementations of LTI systems, not all LTSI(λ) systems may be realized using the wavelet-based implementation of Fig. 7.1. For example, the symmetry constraint (7.29) precludes us from being able to implement either the differentiator or integrator system examples discussed in Section 7.4 since these systems have impulse responses that are not even.

As a final remark, it is important to emphasize that the actual choice of wavelet basis plays no significant role in the representation of LTSI(λ) systems discussed in this section. However, while the choice of basis does not enter into the theoretical development, it is reasonable to expect it to be a factor in any practical implementation. In the next section, we consider a strategy for approximating LTSI(λ) systems that exploits orthonormal wavelet bases. As we will see, these quasi-LTSI(λ) systems are particularly convenient to implement and can be made computationally efficient.

7.5.1 Dyadic Approximations to LTSI(λ) Systems

A practical approximation to a linear time-scale invariant system can be constructed via orthonormal wavelet bases of the type described in Section 2.3.

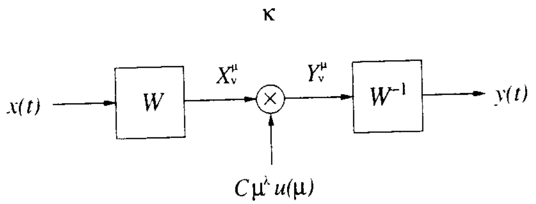

Figure 7.1. Wavelet-based implementation of an LTSI(λ) system.

Because signal reconstructions in terms of such bases require only a countable collection of samples of the wavelet coefficient field, the system turns out to be fundamentally more practical from an implementational perspective. In addition, using an implementation based on the DWT, the system can be made computationally highly efficient as well. In fact, in some sense, using the discrete wavelet transform to implement an LTSI(λ) system is analogous to implementing an LTI system using the discrete Fourier transform (DFT).

Consider a system which computes the orthonormal wavelet decomposition of the input $x(t)$ according to the analysis formula (2.5b), i.e.,

$$x_n^m = \int_{-\infty}^{\infty} x(t)\,\psi_n^m(t)\,dt,$$

then scales the resulting collection of wavelet coefficients by a factor k_n^m,

$$y_n^m = k_n^m x_n^m$$

then resynthesizes a signal from these modified coefficients to generate an output according to the synthesis formula (2.5a), i.e.,

$$y(t) = \sum_m \sum_n y_n^m \psi_n^m(t).$$

It is a straightforward exercise to show that the overall system, described via

$$y(t) = W_d^{-1}\left\{k_n^m W_d\left\{x(t)\right\}\right\}, \tag{7.31}$$

corresponds to a symmetric linear system with kernel

$$\tilde{\kappa}(t,\tau) = \sum_m \sum_n \psi_n^m(t) k_n^m \psi_n^m(\tau).$$

A close inspection reveals that, as a consequence of the nature of the discretization inherent in the system, one cannot choose the multiplier coefficients k_n^m such that the resulting system is time-invariant. Likewise, one cannot choose the coefficients so that the overall system is scale-invariant for

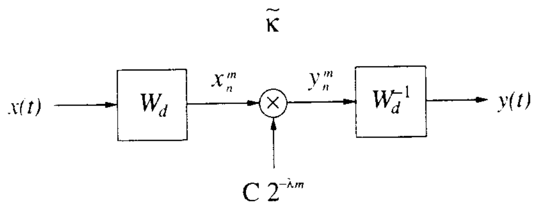

Figure 7.2. A dyadic approximation of an LTSI(λ) system as implemented via an orthonormal wavelet basis.

any degree λ. However, we can show that if the k_n^m are chosen in a manner consistent with the discussion of the previous section, viz.,

$$k_n^m = C\,\mu^\lambda\Big|_{\mu=2^{-m}} = C2^{-\lambda m}, \tag{7.32}$$

then the system defined via (7.31) obeys some associated notions of time- and scale-invariance.

We begin by noting that this system, which is depicted in Fig. 7.2, has a kernel satisfying, for any m,

$$\tilde{\kappa}(t,\tau) = 2^{-(\lambda-1)m}\tilde{\kappa}(2^m t, 2^m \tau), \tag{7.33}$$

where we have used the identity

$$\psi_n^m(2^i t) = 2^{-i/2}\psi_n^{m+i}(t)$$

valid for any integer i. However, since (7.33) can be restated in terms of the generalized scale invariance condition (7.18) as

$$\tilde{\kappa}(t,\tau) = a^{-(\lambda-1)}\tilde{\kappa}(at, a\tau), \qquad a = 2^m$$

we see that the system obeys a weaker, *dyadic* scale invariance condition. In particular, the system satisfies (7.17) only for dilation factors τ of the form

$$\tau = 2^m$$

for integers m.

Likewise, the system obeys a somewhat weaker time-invariance property. Consider a class of input signals $x(t)$ to the system that have no detail at scales coarser than 2^M for some integer M, so that

$$x_n^m = 0, \qquad m < M.$$

In this case, the multiplier coefficients k_n^m for $m < M$ for the system are irrelevant and we may arbitrarily assume them to be zero. For this class of inputs, the effective kernel is

$$\tilde{\kappa}_{\text{eff}}(t,\tau) = \sum_{m \geq M} \sum_n \psi_n^m(t) C 2^{-\lambda m} \psi_n^m(\tau).$$

Using the identity

$$\psi_n^m(t - 2^l - M) = \psi_{n+l2^{m-M}}^m(t)$$

valid for $m \geq M$ and l an integer, we see that this kernel satisfies

$$\tilde{\kappa}(t, \tau) = \tilde{\kappa}(t - l2^{-M}, \tau - l2^{-M}) \tag{7.34}$$

for all integers l. Since (7.34) can be re-expressed as

$$\tilde{\kappa}(t, \tau) = \tilde{\kappa}(t - b, \tau - b), \qquad b = l2^{-M}$$

we see that for this class of input signals the system is *periodically time-varying*, i.e., satisfies (7.2) for any shift factor τ of the form

$$\tau = l2^{-M}, \qquad l = \ldots, -1, 0, 1, 2, \ldots.$$

Note that in contrast to the wavelet-based systems discussed in the previous section, in this case the actual choice of wavelet affects the characteristics of the overall system. Indeed, with respect to scaling behavior, the choice of wavelet affects how the system behaves under nondyadic scale changes at the input. Furthermore, the choice of wavelet affects the class of inputs for which our time-invariance relation is applicable, as well as the behavior of the system under input translations that are not multiples of 2^{-M}.

7.6 SUMMARY

In this chapter, we undertook a preliminary investigation into the system theoretic foundations of the concepts developed in this book. This was aimed toward developing some unifying perspectives on the results we have obtained. After defining scale-invariant systems, we explored the relationships between such systems, self-similar signals, and the wavelet transform. Our results provide additional evidence that wavelet-based synthesis, analysis, and processing of self-similar signals are rather natural. Indeed, the $1/f$ synthesis and whitening filters described in Section 4.2—which play an important role in detection and estimation problems involving $1/f$ processes—are specific examples of linear systems that are effectively jointly time- and scale-invariant. Interpreting the transmitter and receiver structures for fractal modulation discussed in Chapter 6 in terms of such systems also has the potential to lead to some potentially useful additional insights into both homogeneous signals and fractal modulation.

More generally, a system theory perspective provides some novel insights into the relationships between Laplace, Fourier, Mellin and wavelet transformations, both as signal analysis tools and as representations for characterizing linear systems. In particular, our results suggest that while Laplace transforms are naturally suited to the analysis of linear time-invariant systems, and while Mellin transforms are naturally suited to the analysis of scale-invariant systems, it is the wavelet transform that plays the corresponding

role for linear systems that are jointly time- and scale-invariant. Moreover, we showed that wavelet representations lead to some very efficient and practical computational structures for characterizing and implementing such systems. Ultimately, the ideas developed in this chapter may well lead to a basis for a unified development of fractal signal and system theory. As such, this represents one of several interesting and potentially rich open directions for further research.