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# 2

# Fourier Transforms

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## 2.1 Introduction and Basic Definitions

The Fourier transform is certainly one of the best known of the integral transforms and vies with the Laplace transform as being the most generally useful. Since its introduction by Fourier in the early 1800s, it has found use in innumerable applications and has, itself, led to the development of other transforms. Today the Fourier transform is a fundamental tool in engineering science. Its importance has been enhanced by the development in the twentieth century of generalizations extending the set of functions that can be Fourier transformed and by the development of efficient algorithms for computing the discrete version of the Fourier transform.

There are two parts to this article on the Fourier transform. The first (Sections 2.1 through 2.4) contains the fundamental theory necessary for the intelligent use of the Fourier transform in practical problems arising in engineering. The second part (Sections 2.5 through 2.8) is devoted to applications in which the Fourier transform plays a significant role. This part contains both fairly detailed descriptions of specific applications and fairly broad overviews of classes of applications.

This particular section deals with the basic definition of the Fourier transform and some of the integrals used to compute Fourier transforms. Two definitions for the transform are given. First, the classical definition is given in Subsection 2.1.1. This is the integral formula for directly computing transforms generally found in elementary texts. From this formula many of the basic formulas and identities involving the Fourier transform can be derived. Inherent in the classical definition, however, are integrability conditions that cannot be satisfied by many functions routinely arising in applications. For this reason, more general definitions of the Fourier transform are briefly discussed in Subsections 2.1.3 and 2.1.4. These general definitions will also help clarify the role of generalized functions in Fourier analysis.

The computation of Fourier transforms often involves the evaluation of integrals, many of which cannot be evaluated by the elementary methods of calculus. For this reason, this section also contains a brief discussion illustrating the use of the residue theorem in computing certain integrals as well as a brief discussion of how to deal with certain integrals containing singularities in the integrand.

### 2.1.1 Basic Definition, Notation, and Terminology

If $\phi(s)$ is an absolutely integrable function on $(-\infty, \infty)$ (i.e., $\int_{-\infty}^{\infty} |\phi(s)| \, ds < \infty$), then the (direct) Fourier transform of $\phi(s)$, $\mathcal{F}[\phi]$, and the Fourier inverse transform of $\phi(s)$, $\mathcal{F}^{-1}[\phi]$, are the functions given by

$$\mathcal{F}[\phi]\Big|_x = \int_{-\infty}^{\infty} \phi(s) e^{-jxs} ds \tag{2.1.1.1}$$

and

$$\mathcal{F}^{-1}[\phi]\Big|_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(s) e^{jxs} ds . \tag{2.1.1.2}$$

**Example 2.1.1.1**

If $\phi(s) = e^{-s}\, u(s)$, then

$$\mathcal{F}[\phi]\Big|_x = \int_{-\infty}^{\infty} e^{-s} u(s) e^{-jxs} ds = \int_0^{\infty} e^{-(1+jx)s} ds = \frac{1}{1+jx}$$

and

$$\mathcal{F}^{-1}[\phi]\Big|_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-s} u(s) e^{jxs} ds = \frac{1}{2\pi}\int_0^{\infty} e^{-(1-jx)s} ds = \frac{1}{2\pi - j2\pi x}.$$

**Example 2.1.1.2**

For $\alpha > 0$, the transform of the corresponding pulse function,

$$p_\alpha(s) = \begin{cases} 1, & \text{if } \quad |s| < \alpha \\ 0, & \text{if } \quad \alpha < |s| \end{cases},$$

is

$$\mathcal{F}[p_\alpha]\Big|_x = \int_{-\alpha}^{\alpha} e^{-jxs} ds = \frac{e^{j\alpha x} - e^{-j\alpha x}}{jx} = \frac{2}{x}\sin(\alpha x).$$

---

A function, $\psi$, is said to be "classically transformable" if either

1. $\psi$ is absolutely integrable on the real line, or
2. $\psi$ is the Fourier transform (or Fourier inverse transform) of an absolutely integrable function, or
3. $\psi$ is a linear combination of an absolutely integrable function and a Fourier transform (or Fourier inverse transform) of an absolutely integrable function.

If $\phi$ is classically transformable but not absolutely integrable, then it can be shown that formulas (2.1.1.1) and (2.1.1.2) can still be used to define $\mathcal{F}[\phi]$ and $\mathcal{F}^{-1}[\phi]$ provided the limits are taken symmetrically; that is,

$$\mathcal{F}[\phi]\Big|_x = \lim_{a\to\infty}\int_{-a}^{a} \phi(s) e^{-jxs} ds$$

and

$$\mathcal{F}^{-1}[\phi]\Big|_x = \frac{1}{2\pi}\lim_{a\to\infty}\int_{-a}^{a} \phi(s) e^{jxs} ds.$$

In most applications involving Fourier transforms, the functions of time, $t$, or position, $x$, are denoted using lower case letters — for example: $f$ and $g$. The Fourier transforms of these functions are denoted using the corresponding upper case letters — for example: $F = \mathcal{F}[f]$ and $G = \mathcal{F}[g]$. The transformed functions can be viewed as functions of angular frequency, $\omega$. Along these same lines it is standard practice to view a signal as a pair of functions, $f(t)$ and $F(\omega)$, with $f(t)$ being the "time domain representation of the signal" and $F(\omega)$ being the "frequency domain representation of the signal."

## 2.1.2 Alternate Definitions

Pairs of formulas other than formulas (2.1.1.1) and (2.1.1.2) are often used to define $\mathcal{F}[\phi]$ and $\mathcal{F}^{-1}[\phi]$. Some of the other formula pairs commonly used are:

$$\mathcal{F}[\phi]\Big|_x = \int_{-\infty}^{\infty} \phi(s) e^{-j2\pi xs}\, ds, \quad \mathcal{F}^{-1}[\phi]\Big|_x = \int_{-\infty}^{\infty} \phi(s) e^{j2\pi xs}\, ds \tag{2.1.2.1}$$

and

$$\mathcal{F}[\phi]\Big|_x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(s) e^{-jxs}\, ds, \quad \mathcal{F}^{-1}[\phi]\Big|_x = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(s) e^{jxs}\, ds. \tag{2.1.2.2}$$

Equivalent analysis can be performed using the theory arising from any of these pairs; however, the resulting formulas and equations will depend on which pair is used. For this reason care must be taken to ensure that, in any particular application, all the Fourier analysis formulas and equations used are derived from the same defining pair of formulas.

### Example 2.1.2.1

Let $\phi(t) = e^{-t}\, u(t)$ and let $\psi_1$, $\psi_2$, and $\psi_3$ be the Fourier transforms of $\phi$ as defined, respectively, by formulas (2.1.1.1), (2.1.2.1), and (2.1.2.2). Then,

$$\psi_1(\omega) = \int_{-\infty}^{\infty} e^{-t} u(t) e^{-jt\omega}\, dt = \frac{1}{1+j\omega},$$

$$\psi_2(\omega) = \int_{-\infty}^{\infty} e^{-t} u(t) e^{-j2\pi t\omega}\, dt = \frac{1}{1+j2\pi\omega},$$

and

$$\psi_3(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t} u(t) e^{-jt\omega}\, dt = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{1+j\omega}.$$

## 2.1.3 The Generalized Transforms

Many functions and generalized functions* arising in applications are not sufficiently integrable to apply the definitions given in subsection 2.1.1 directly. For such functions it is necessary to employ a generalized definition of the Fourier transform constructed using the set of "rapidly decreasing test functions" and a version of Parseval's equation (see subsection 2.2.14).

A function, $\phi$, is a "rapidly decreasing test function" if

1. every derivative of $\phi$ exists and is a continuous function on $(-\infty, \infty)$ and
2. for every pair of nonnegative integers, $n$ and $p$,

$$\left|\phi^{(n)}(s)\right| = O(|s|^{-p}) \quad \text{as } |s| \to \infty.$$

The set of all rapidly decreasing test functions is denoted by $\mathcal{S}$ and includes the Gaussian functions as well as all test functions that vanish outside of some finite interval (such as those discussed in the first

*For a detailed discussion of generalized functions, see the first chapter in this handbook.

chapter of this handbook. If $\phi$ is a rapidly decreasing test function then it is easily verified that $\phi$ is classically transformable and that both $\mathcal{F}[\phi]$ and $\mathcal{F}^{-1}[\phi]$ are also rapidly decreasing test functions. It can also be shown that $\mathcal{F}^{-1}[\mathcal{F}[\phi]] = \phi$. Moreover, if $f$ and $G$ are classically transformable, then

$$\int_{-\infty}^{\infty} \mathcal{F}[f]\Big|_x \phi(x)dx = \int_{-\infty}^{\infty} f(y)\mathcal{F}[\phi]\Big|_y dy \tag{2.1.3.1}$$

and

$$\int_{-\infty}^{\infty} \mathcal{F}^{-1}[G]\Big|_x \phi(x)dx = \int_{-\infty}^{\infty} G(y)\mathcal{F}^{-1}[\phi]\Big|_y dy. \tag{2.1.3.2}$$

If $f$ is a function or a generalized function for which the right-hand side of equation (2.1.3.1) is well defined for every rapidly decreasing test function, $\phi$, then the generalized Fourier transform of $f$, $\mathcal{F}[f]$, is that generalized function satisfying (2.1.3.1) for every $\phi$ in $\mathcal{S}$. Likewise, if $G$ is a function or generalized function for which the right-hand side of (2.1.3.2) is well defined for every rapidly decreasing test function, $\phi$, then the generalized inverse Fourier transform of $G$, $\mathcal{F}^{-1}[G]$, is that generalized function satisfying equation (2.1.3.2) for every $\phi$ in $\mathcal{S}$.

### Example 2.1.3.1

Let $\alpha$ be any real number. Then, for every rapidly decreasing test function $\phi$,

$$\begin{aligned}\int_{-\infty}^{\infty} \mathcal{F}\left[e^{j\alpha y}\right]\Big|_x \phi(x)dx &= \int_{-\infty}^{\infty} e^{j\alpha y}\mathcal{F}[\phi]\Big|_y dy \\ &= 2\pi\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}[\phi]\Big|_y e^{j\alpha y}dy\right] \\ &= 2\pi\mathcal{F}^{-1}\left[\mathcal{F}[\phi]\right]\Big|_\alpha \\ &= 2\pi\phi(\alpha) \\ &= \int_{-\infty}^{\infty} 2\pi\delta(x-\alpha)\phi(x)dx\end{aligned}$$

where $\delta(x)$ is the delta function. This shows that, for every $\phi$ in $\mathcal{S}$,

$$\int_{-\infty}^{\infty} 2\pi\delta(x-\alpha)\phi(x)dx = \int_{-\infty}^{\infty} e^{j\alpha y}\mathcal{F}[\phi]\Big|_y dy$$

and thus,

$$\mathcal{F}[e^{j\alpha y}]\big|_x = 2\pi\delta(x-\alpha).$$

---

Any (generalized) function whose Fourier transform can be computed via the above generalized definition is called "transformable." The set of all such functions is sometimes called the set of "tempered generalized functions" or the set of "tempered distributions." This set includes any piecewise continuous function, $f$, which is also polynomially bounded, that is, which satisfies

$$|f(s)| = O(|s|^p) \quad \text{as } |s| \to \infty$$

for some $p < \infty$. Finally, it should also be noted that if $f$ is classically transformable, then it is transformable, and the generalized definition of $\mathcal{F}[f]$ yields exactly the same function as the classical definition.

### 2.1.4 Further Generalization of the Generalized Transforms

Unfortunately, even with the theory discussed in the previous subsection, it is not possible to define or discuss the Fourier transform of the real exponential, $e^t$. It may be of interest to note, however, that a further generalization that does permit all exponentially bounded functions to be considered "Fourier transformable" is currently being developed using a recently discovered alternate set of test functions. This alternate set, denoted by $\mathcal{G}$, is the subset of the rapidly decreasing test functions that satisfy the following two additional properties:

1. Each test function is an analytic test function on the entire complex plane.
2. Each test function, $\phi(x + jy)$, satisfies

$$\phi(x + jy) = O(e^{-\alpha|x|}) \quad \text{as} \quad x \to \pm\infty$$

for every real value of $y$ and $\alpha$.

The second additional property of these test functions ensures that all exponentially bounded functions are covered by this theory. The very same computations given in example 2.1.3.1 can be used to show that, for any **complex** value, $\alpha + j\beta$,

$$\mathcal{F}[e^{j(\alpha+j\beta)t}]|_\omega = 2\pi\delta_{\alpha+j\beta}(\omega),$$

where $\delta_{\alpha+j\beta}(t)$ is "the delta function at $\alpha + j\beta$." This delta function, $\delta_{\alpha+j\beta}(t)$, is the generalized function satisfying

$$\int_{-\infty}^{\infty} \delta_{\alpha+j\beta}(t)\phi(t)\,dt = \phi(\alpha + j\beta)$$

for every test function $\phi(t)$, in $\mathcal{G}$. In particular, letting $\alpha + j\beta = -j$,

$$\mathcal{F}[e^t]\big|_\omega = 2\pi\delta_{-j}(\omega)$$

and

$$\mathcal{F}[\delta_j(t)]\big|_\omega = e^\omega.$$

In addition to allowing delta functions to be defined at complex points, the analyticity of the test functions allows a generalization of translation. Let $\alpha + j\beta$ be any complex number and $f(t)$ any (exponentially bounded) (generalized) function. The "generalized translation of $f(t)$ by $\alpha + j\beta$," denoted by $T_{\alpha+j\beta}f(t)$, is that generalized function satisfying

$$\int_{-\infty}^{\infty} T_{\alpha+j\beta}f(t)\phi(t)\,dt = \int_{-\infty}^{\infty} f(t)\phi\big(t + (\alpha + j\beta)\big)\,dt \qquad (2.1.4.1)$$

for every test function, $\phi(t)$, in $\mathcal{G}$. So long as $\beta = 0$ or $f(t)$ is, itself, an analytic function on the entire complex plane, then the generalized translation is exactly the same as the classical translation.

$$T_{\alpha+j\beta}f(t) = f(t - (\alpha + j\beta)).$$

It may be observed, however, that equation (2.1.4.1) defines the generalized function $T_{\alpha+j\beta}f$ even when $f(z)$ is not defined for nonreal values of $z$.

## 2.1.5 Use of the Residue Theorem

Often a Fourier transform or inverse transform can be described as an integral of a function that either is analytic on the entire complex plane, or else has a few isolated poles in the complex plane. Such integrals can often be evaluated through intelligent use of the reside theorem from complex analysis (see Appendix I). Two examples illustrating such use of the residue theorem will be given in this subsection. The first example illustrates its use when the function is analytic throughout the complex plane, while the second example illustrates its use when the function has poles off the real axis. The use of the residue theorem to compute transforms when the function has poles on the real axis will be discussed in the next subsection.

### Example 2.1.5.1 Transform of an Analytic Function

Consider computing the Fourier transform of $g(t) = e^{-t^2}$,

$$G(\omega) = \mathcal{F}\left[g(t)\right]\Big|_{\omega} = \int_{-\infty}^{\infty} e^{-t^2} e^{-j\omega t}\,dt .$$

Because

$$t^2 + j\omega t = \left(t + j\frac{\omega}{2}\right)^2 + \frac{\omega^2}{4},$$

it follows that

$$\begin{aligned} G(\omega) &= e^{-\frac{1}{4}\omega^2} \int_{-\infty}^{\infty} \exp\left[-\left(t + j\frac{\omega}{2}\right)^2\right] dt \\ &= e^{-\frac{1}{4}\omega^2} \int_{-\infty + j\frac{\omega}{2}}^{\infty + j\frac{\omega}{2}} e^{-z^2}\,dz . \end{aligned} \tag{2.1.5.1}$$

Consider, now, the integral of $e^{-z^2}$ over the contour $C_\gamma$ where, for each $\gamma > 0$, $C_\gamma = C_{1,\gamma} + C_{2,\gamma} + C_{3,\gamma} + C_{4,\gamma}$ is the contour in Figure 2.1. Because $e^{-z^2}$ is analytic everywhere on the complex plane, the residue theorem states that

$$\begin{aligned} 0 &= \int_{C_\gamma} e^{-z^2}\,dz \\ &= \int_{C_{1,\gamma}} e^{-z^2}\,dz + \int_{C_{2,\gamma}} e^{-z^2}\,dz + \int_{C_{3,\gamma}} e^{-z^2}\,dz + \int_{C_{4,\gamma}} e^{-z^2}\,dz . \end{aligned}$$

Thus,

$$-\int_{C_{3,\gamma}} e^{-z^2}\,dz = \int_{C_{1,\gamma}} e^{-z^2}\,dz + \int_{C_{2,\gamma}} e^{-z^2}\,dz + \int_{C_{4,\gamma}} e^{-z^2}\,dz . \tag{2.1.5.2}$$

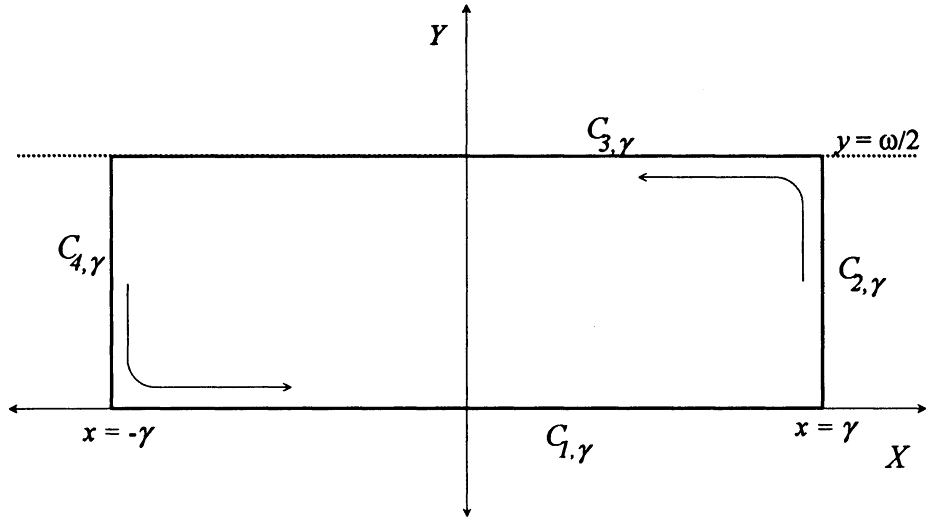


**FIGURE 2.1** Contour for computing $\mathscr{F}[e^{-t^2}]$.

Now,

$$\lim_{\gamma\to\infty}\int_{C_{2,\gamma}} e^{-z^2}dz = \lim_{\gamma\to\infty}\int_{y=0}^{\omega/2} e^{-(\gamma+jy)^2}dy$$

$$= \lim_{\gamma\to\infty} e^{-\gamma^2}\int_{y=0}^{\omega/2} e^{y^2-j2\gamma y}dy$$

$$= 0\,.$$

Likewise,

$$\lim_{\gamma\to\infty}\int_{C_{4,\gamma}} e^{-z^2}dz = 0\,,$$

while

$$\lim_{\gamma\to\infty}\int_{C_{3,\gamma}} e^{-z^2}dz = \lim_{\gamma\to\infty}\int_{\gamma+j\frac{\omega}{2}}^{-\gamma+j\frac{\omega}{2}} e^{-z^2}dz = -\int_{-\infty+j\frac{\omega}{2}}^{\infty+j\frac{\omega}{2}} e^{-z^2}dz$$

and

$$\lim_{\gamma\to\infty}\int_{C_{1,\gamma}} e^{-z^2}dz = \lim_{\gamma\to\infty}\int_{x=-\gamma}^{\gamma} e^{-x^2}dz = \int_{-\infty}^{\infty} e^{-x^2}dx\,.$$

The last integral is well known and equals $\sqrt{\pi}$ . Combining equations (2.1.5.1) and (2.1.5.2) with the above limits yields

$$
\begin{aligned}
G(\omega) &= e^{-\frac{1}{4}\omega^2} \int_{-\infty + j\frac{\omega}{2}}^{\infty + j\frac{\omega}{2}} e^{-z^2}\,dz \\
&= e^{-\frac{1}{4}\omega^2} \lim_{\gamma\to\infty} \left[ -\int_{C_{3,\gamma}} e^{-z^2}\,dz \right] \\
&= e^{-\frac{1}{4}\omega^2} \lim_{\gamma\to\infty} \left[ \int_{C_{1,\gamma}} e^{-z^2}\,dz + \int_{C_{2,\gamma}} e^{-z^2}\,dz + \int_{C_{4,\gamma}} e^{-z^2}\,dz \right] \\
&= e^{-\frac{1}{4}\omega^2} \sqrt{\pi}\,.
\end{aligned}
$$

So,

$$
\mathscr{F}\left[e^{-t^2}\right]\Big|_{\omega} = G(\omega) = \sqrt{\pi}\, e^{-\frac{1}{4}\omega^2}\,.
$$

### Example 2.1.5.2 Transform of a Function with a Pole Off the Real Axis

Consider computing the Fourier inverse transform of $F(\omega) = \left(1+\omega^2\right)^{-1}$,

$$
f(t) = \mathscr{F}^{-1}\left[F(\omega)\right]\Big|_{t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{jt\omega}}{1+\omega^2}\,d\omega\,. \tag{2.1.5.3}
$$

For $t = 0$,

$$
f(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{1+\omega^2}\,d\omega = \frac{1}{2\pi}\arctan\omega\Big|_{-\infty}^{\infty} = \frac{1}{2}\,. \tag{2.1.5.4}
$$

To evaluate $f(t)$ when $t \neq 0$, first observe that the integrand in formula (2.1.5.3), viewed as a function of the complex variable,

$$
\Phi(z) = \frac{e^{jtz}}{1+z^2}\,,
$$

has simple poles at $z = \pm j$. The residue at $z = j$ is

$$
\operatorname{Res}_j[\Phi] = \lim_{z\to j}(z-j)\Phi(z) = \lim_{z\to j}(z-j)\left[\frac{e^{jtz}}{(z-j)(z+j)}\right] = \frac{1}{2j}e^{-t}\,,
$$

while the residue at $z = -j$ is

$$
\operatorname{Res}_{-j}[\Phi] = \lim_{z\to -j}(z+j)\Phi(z) = -\frac{1}{2j}e^{t}\,.
$$

For each $\gamma > 1$, let $C_\gamma$, $C_{+,\gamma}$, and $C_{-,\gamma}$ be the curves sketched in Figure 2.2. By the residue theorem:

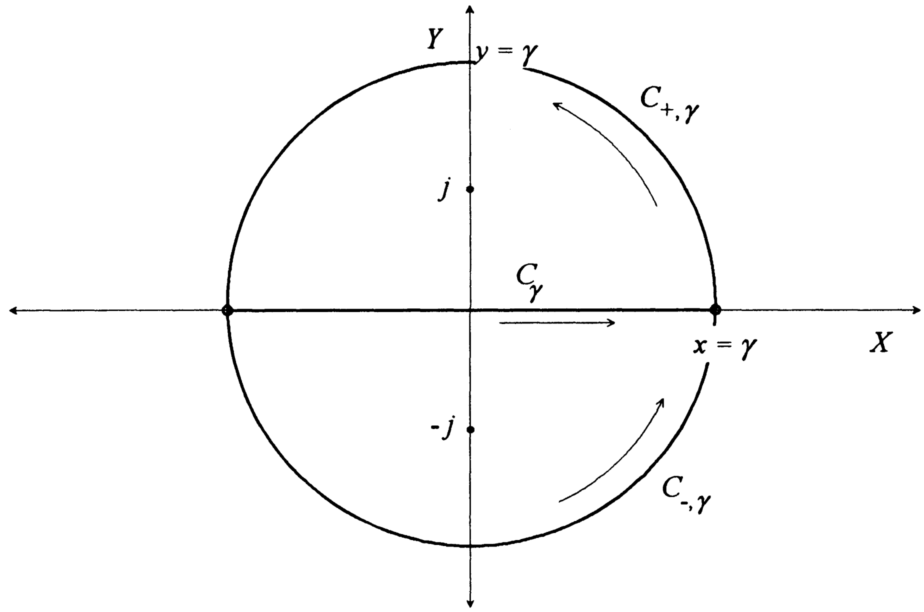


**FIGURE 2.2** Contours for computing $\mathcal{F}^{-1}[(1+\omega^2)^{-1}]$.

$$\int_{C_\gamma} \frac{e^{jtz}}{1+z^2}\,dz + \int_{C_{+,\gamma}} \frac{e^{jtz}}{1+z^2}\,dz = 2\pi j\,\mathrm{Res}_j[\Phi] = \pi e^{-t}$$

and

$$-\int_{C_\gamma} \frac{e^{jtz}}{1+z^2}\,dz + \int_{C_{-,\gamma}} \frac{e^{jtz}}{1+z^2}\,dz = 2\pi j\,\mathrm{Res}_{-j}[\Phi] = \pi e^{t}.$$

Combining these calculations with equation (2.1.5.3) yields

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{jt\omega}}{1+\omega^2}\,d\omega \\ &= \frac{1}{2\pi}\lim_{\gamma\to\infty}\int_{C_\gamma} \frac{e^{jtz}}{1+z^2}\,dz \\ &= \frac{1}{2\pi}\left[\pi e^{-t} \lim_{\gamma\to\infty}\int_{C_{+,\gamma}} \frac{e^{jtz}}{1+z^2}\,dz\right] \end{aligned} \tag{2.1.5.5}$$

and

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{jt\omega}}{1+\omega^2}\,d\omega \\ &= \frac{1}{2\pi}\lim_{\gamma\to\infty}\int_{C_\gamma} \frac{e^{jtz}}{1+z^2}\,dz \\ &= \frac{1}{2\pi}\left[\pi e^{t} + \lim_{\gamma\to\infty}\int_{C_{-,\gamma}} \frac{e^{jtz}}{1+z^2}\,dz\right]. \end{aligned} \tag{2.1.5.6}$$

Now,

$$\left|\int_{C_{+,\gamma}} \frac{e^{jtz}}{1+z^2}\,dz\right| = \left|\int_0^{\pi} \frac{e^{jt\gamma(\cos\theta + j\sin\theta)}}{1+\gamma^2 e^{j2\theta}}\gamma\, e^{j\theta}\,d\theta\right|$$

$$< \int_0^{\pi} \left|\frac{e^{jt\gamma(\cos\theta + j\sin\theta)}}{1+\gamma^2 e^{j2\theta}}\gamma\, e^{j\theta}\right| d\theta$$

$$< \int_0^{\pi} \frac{e^{-t\gamma\sin\theta}}{\gamma^2 - 1}\gamma\,d\theta\,.$$

So long as $t > 0$ and $0 \le \theta \le \pi$,

$$0 \le e^{-t\gamma\sin\theta} \le 1.$$

Thus, for $t > 0$,

$$\lim_{\gamma\to\infty}\left|\int_{C_{+,\gamma}} \frac{e^{jtz}}{1+z^2}\,dz\right| \le \lim_{\gamma\to\infty}\int_0^{\pi} \frac{e^{-t\gamma\sin\theta}}{\gamma^2 - 1}\gamma\,d\theta$$

$$\le \lim_{\gamma\to\infty}\int_0^{\pi} \frac{\gamma}{\gamma^2 - 1}\,d\theta$$

$$\le \lim_{\gamma\to\infty} \frac{2\pi\gamma}{\gamma^2 - 1}$$

$$= 0\,.$$

Combining this last result with equation (2.1.5.5) gives

$$f(t) = \frac{1}{2\pi}\left[\pi e^{-t} - \lim_{\gamma\to\infty}\int_{C_{+,\gamma}} \frac{e^{jtz}}{1+z^2}\,dz\right] = \frac{1}{2}e^{-t} \tag{2.1.5.7}$$

whenever $t > 0$.

In a similar fashion, it is easy to show that if $t < 0$,

$$\lim_{\gamma\to\infty}\left|\int_{C_{-,\gamma}} \frac{e^{jtz}}{1+z^2}\,dz\right| \le \lim_{\gamma\to\infty}\int_{\pi}^{2\pi} \frac{e^{-t\gamma\sin\theta}}{\gamma^2 - 1}\gamma\,d\theta$$

$$\le \lim_{\gamma\to\infty}\int_{\pi}^{2\pi} \frac{\gamma}{\gamma^2 - 1}\,d\theta$$

$$= 0\,,$$

which, combined with equation (2.1.5.6), yields

$$f(t) = \frac{1}{2\pi}\left[\pi e^{t} + \lim_{\gamma\to\infty}\int_{C_{-,\gamma}} \frac{e^{jtz}}{1+z^2}\,dz\right] = \frac{1}{2}e^{t} \tag{2.1.5.8}$$

whenever $t < 0$.

Finally, it should be noted that formulas (2.1.5.4), (2.1.5.7), and (2.1.5.8) can be written more concisely as

$$f(t) = \frac{1}{2} e^{-|t|}.$$

## 2.1.6 Cauchy Principal Values

The Cauchy principal value (CPV) at $x = x_0$ of an integral, $\int_{-\infty}^{\infty} \phi(x)\, dx$, is

$$\mathrm{CPV} \int_{-\infty}^{\infty} \phi(x) dx = \lim_{\varepsilon \to 0^+} \left[ \int_{-\infty}^{x_0 - \varepsilon} \phi(x) dx + \int_{x_0 + \varepsilon}^{\infty} \phi(x) dx \right]$$

provided the limit exists. So long as $\phi$ is an integrable function, it should be clear that

$$\mathrm{CPV} \int_{-\infty}^{\infty} \phi(x) dx = \int_{-\infty}^{\infty} \phi(x) dx.$$

It is when $\phi$ is not an integrable function that the Cauchy principal value is useful. In particular, the Fourier transform and Fourier inverse transform of any function with a singularity of the form $(x - x_0)^{-1}$ can be evaluated as the Cauchy principal values at $x = x_0$ of the integrals in formulas (2.1.1.1) and (2.1.1.2).

**Example 2.1.6.1**

Consider evaluating the inverse transform of $F(\omega) = \omega^{-1}$. Because of the $\omega^{-1}$ singularity, $f = \mathcal{F}^{-1}[F]$ is given by

$$f(t) = \frac{1}{2\pi} \mathrm{CPV} \int_{-\infty}^{\infty} \frac{1}{\omega} e^{j\omega t} d\omega$$

or, equivalently, by

$$f(t) = \frac{1}{2\pi} \lim_{\substack{\varepsilon \to 0^+ \\ R \to +\infty}} \left[ \int_{-R}^{-\varepsilon} \frac{1}{z} e^{jtz} dz + \int_{\varepsilon}^{R} \frac{1}{z} e^{jtz} dz \right]. \tag{2.1.6.1}$$

Because $\omega^{-1}$ is an odd function, $f(0)$ is easily evaluated,

$$f(0) = \frac{1}{2\pi} \lim_{\substack{\varepsilon \to 0^+ \\ R \to +\infty}} \left[ \int_{-R}^{-\varepsilon} \frac{1}{\omega} d\omega + \int_{\varepsilon}^{R} \frac{1}{\omega} d\omega \right] = 0. \tag{2.1.6.2}$$

To evaluate $f(t)$ when $t > 0$, first observe that the only pole of the integrand in formula (2.1.6.1),

$$\Phi(z) = \frac{1}{z} e^{jtz},$$

is at $z = 0$. For each $0 < \varepsilon < R$, let $C_\varepsilon$ and $C_R$ be the semicircles indicated in Figure 2.3. By the residue theorem,

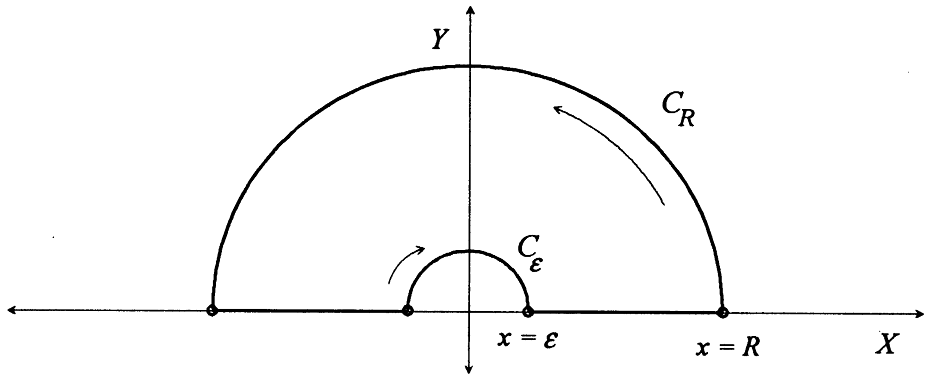


**FIGURE 2.3** Contour for computing $\mathscr{F}^{-1}[\omega^{-1}]$.

$$\int_{-R}^{-\varepsilon}\frac{1}{z}e^{jtz}dz+\int_{\varepsilon}^{R}\frac{1}{z}e^{jtz}dz+\int_{C_\varepsilon}\frac{1}{z}e^{jtz}dz+\int_{C_R}\frac{1}{z}e^{jtz}dz=0.$$

This, combined with equation (2.1.6.1), yields

$$f(t)=-\frac{1}{2\pi}\left[\lim_{\varepsilon\to 0^+}\int_{C_\varepsilon}\frac{1}{z}e^{jtz}dz+\lim_{R\to\infty}\int_{C_R}\frac{1}{z}e^{jtz}dz\right], \qquad (2.1.6.3)$$

provided the limits exist. Now,

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\int_{C_\varepsilon}\frac{1}{z}e^{jtz}dz &= \lim_{\varepsilon\to 0^+}\int_{\pi}^{0}\frac{1}{\varepsilon e^{j\theta}}e^{jt\varepsilon(\cos\theta+j\sin\theta)}j\varepsilon e^{j\theta}d\theta \\ &= j\lim_{\varepsilon\to 0^+}\int_{\pi}^{0}e^{-\varepsilon t(\sin\theta+j\cos\theta)}d\theta \\ &= j\int_{\pi}^{0}e^{0}d\theta \\ &= -j\pi .\end{aligned} \qquad (2.1.6.4)$$

Similarly,

$$\int_{C_R}\frac{1}{z}e^{jtz}dz=j\int_{0}^{\pi}e^{-Rt(\sin\theta+j\cos\theta)}d\theta.$$

Here, because $t > 0$, the integrand is uniformly bounded and vanishes as $R \to \infty$. Thus,

$$\lim_{R\to\infty}\int_{C_R}\frac{1}{z}e^{jtz}dz=0. \qquad (2.1.6.5)$$

With equations (2.1.6.4) and (2.1.6.5), equation (2.1.6.3) becomes

$$f(t)=-\frac{1}{2\pi}\left[\lim_{\varepsilon\to 0^+}\int_{C_\varepsilon}\frac{1}{z}e^{jtz}dz+\lim_{R\to\infty}\int_{C_R}\frac{1}{z}e^{jtz}dz\right]=\frac{j}{2}. \qquad (2.1.6.6)$$

By replacing $C_\varepsilon$ and $C_R$ with corresponding semicircles in the lower half-plane, the approach used to evaluate $f(t)$ when $0 < t$, can be used to evaluate $f(t)$ when $t < 0$. The computations are virtually identical, except for a reversal of the orientation of the contour of integration, and yield

$$f(t) = -\frac{j}{2}, \tag{2.1.6.7}$$

when $t < 0$.

Finally, it should be noted that formulas (2.1.6.2), (2.1.6.6), and (2.1.6.7) can be written more concisely as

$$\mathcal{F}^{-1}\left[\frac{1}{\omega}\right]\bigg|_t = f(t) = \frac{j}{2}\operatorname{sgn}(t).$$

## 2.2 General Identities and Relations

Some of the more general identities commonly used in computing and manipulating Fourier transforms and inverse transforms are described here. Brief (nonrigorous) derivations of some are presented, usually employing the classical transforms (formulas [2.1.1.1] and [2.1.1.2]). Unless otherwise stated, however, each identity may be assumed to hold for the generalized transforms as well.

### 2.2.1 Invertibility

The Fourier transform and the Fourier inverse transform, $\mathcal{F}$ and $\mathcal{F}^{-1}$, are operational inverses, that is,

$$\psi = \mathcal{F}[\phi] \Leftrightarrow \mathcal{F}^{-1}[\psi] = \phi .$$

Equivalently,

$$\mathcal{F}^{-1}[\mathcal{F}[f]] = f \quad \text{and} \quad \mathcal{F}[\mathcal{F}^{-1}[F]] = F.$$

**Example 2.2.1.1**

Because $\mathcal{F}[e^{-t}\,u(t)]\big|_\omega = (1 + j\omega)^{-1}$ (see Example 2.1.1.1),

$$\mathcal{F}^{-1}\left[\frac{1}{1+j\omega}\right]\bigg|_t = e^{-t}u(t).$$

### 2.2.2 Near-Equivalence (Symmetry of the Transforms)

Computationally, the classical formulas for $\mathcal{F}[\phi(s)]\big|_x$ and $\mathcal{F}^{-1}[\phi(s)]\big|_x$ (formulas [2.1.1.1] and [2.1.1.2] are virtually the same, differing only by the sign in the exponential and the factor of $(2\pi)^{-1}$ in [2.1.1.2]). Observing that

$$\int_{-\infty}^{\infty}\phi(s)e^{-jxs}ds = 2\pi\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(s)e^{j(-x)s}ds\right] = 2\pi\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(s)e^{jx(-s)}ds\right]$$

leads to the "near equivalence" identity,

$$\mathcal{F}[\phi(s)]\big|_x = 2\pi\ \mathcal{F}^{-1}[\phi(s)]\big|_{-x} = 2\pi\mathcal{F}^{-1}[\phi(-s)]\big|_x. \qquad (2.2.2.1)$$

Likewise,

$$\mathcal{F}^{-1}\left[\phi(s)\right]\Big|_x = \frac{1}{2\pi}\mathcal{F}\left[\phi(s)\right]\Big|_{-x} = \frac{1}{2\pi}\mathcal{F}\left[\phi(-s)\right]\Big|_x . \qquad (2.2.2.2)$$

**Example 2.2.2.1**

Using near-equivalence and results of example 2.1.1.1,

$$\mathcal{F}\left[e^{s}u(-s)\right]\Big|_x = 2\pi\mathcal{F}^{-1}\left[e^{-s}u(s)\right]\Big|_x = 2\pi\left[\frac{1}{2\pi - j2\pi x}\right] = \frac{1}{1-jx} .$$

## 2.2.3 Conjugation of Transforms

Using $z^*$ to denote the complex conjugate of any complex quantity, $z$, it can be observed that

$$\left(\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt\right)^* = \int_{-\infty}^{\infty} f^*(t)e^{j\omega t}dt .$$

Thus,

$$\mathcal{F}[\,f]^* = 2\pi\mathcal{F}^{-1}[f^*]. \qquad (2.2.3.1)$$

Likewise,

$$\mathcal{F}^{-1}\left[f\right]^* = \frac{1}{2\pi}\mathcal{F}\left[f^*\right]. \qquad (2.2.3.2)$$

## 2.2.4 Linearity

If $\alpha$ and $\beta$ are any two scalar constants, then it follows from the linearity of the integral that

$$\mathcal{F}[\alpha f + \beta g] = \alpha\mathcal{F}[f] + \beta\mathcal{F}[g]$$

and

$$\mathcal{F}^{-1}[\alpha F + \beta G] = \alpha\mathcal{F}^{-1}[F] + \beta\mathcal{F}^{-1}[G].$$

**Example 2.2.4.1**

Using linearity and the transforms computed in Examples 2.1.1.1 and 2.2.2.1,

$$\mathcal{F}\left[e^{-|t|}\right]\Big|_\omega = \mathcal{F}\left[e^{-t}u(t) + e^{t}u(-t)\right]\Big|_\omega = \frac{1}{1+j\omega} + \frac{1}{1-j\omega} = \frac{2}{1+\omega^2}$$

and

$$\mathcal{F}\left[\operatorname{sgn}(t)e^{-|t|}\right]\Big|_\omega = \mathcal{F}\left[e^{-t}u(t) - e^{t}u(-t)\right]\Big|_\omega = \frac{1}{1+j\omega} - \frac{1}{1-j\omega} = \frac{-2\omega j}{1+\omega^2} .$$

### 2.2.5 Scaling

If $\alpha$ is any nonzero real number, then, using the substitution $\tau = \alpha t$,

$$\int_{-\infty}^{\infty} f(\alpha t) e^{-jt\omega} dt = \frac{1}{|\alpha|} \int_{-\infty}^{\infty} f(\tau) e^{-j\frac{\tau\omega}{\alpha}} d\tau \,.$$

Letting $F(\omega) = \mathscr{F}[f(t)]\big|_{\omega}$, this can be rewritten as

$$\mathscr{F}\left[f(\alpha t)\right]\Big|_{\omega} = \frac{1}{|\alpha|} F\left(\frac{\omega}{\alpha}\right). \tag{2.2.5.1}$$

Likewise,

$$\mathscr{F}^{-1}\left[F(\alpha\omega)\right]\Big|_{t} = \frac{1}{|\alpha|} f\left(\frac{t}{\alpha}\right). \tag{2.2.5.2}$$

**Example 2.2.5.1**

Using identity (2.2.5.1) and the results from example 2.2.4.1:

$$\mathscr{F}\left[e^{-|\alpha t|}\right]\Big|_{\omega} = \frac{1}{|\alpha|} \cdot \frac{2}{1+\left(\dfrac{\omega}{\alpha}\right)^2} = \frac{2|\alpha|}{\alpha^2+\omega^2} \,.$$

### 2.2.6 Translation and Multiplication by Exponentials

If $F(\omega) = \mathscr{F}[f(t)]\big|_{\omega}$ and $\alpha$ is any real number, then

$$\mathscr{F}[f(t-\alpha)]\big|_{\omega} = e^{-j\alpha\omega} F(\omega), \tag{2.2.6.1}$$

$$\mathscr{F}[e^{j\alpha t} f(t)]\big|_{\omega} = F(\omega-\alpha), \tag{2.2.6.2}$$

$$\mathscr{F}^{-1}[F(\omega-\alpha)]\big|_{t} = e^{j\alpha t} f(t), \tag{2.2.6.3}$$

and

$$\mathscr{F}^{-1}[e^{j\alpha\omega} F(\omega)]\big|_{t} = f(t+\alpha). \tag{2.2.6.4}$$

These formulas are easily derived from the classical definitions. Identity (2.2.6.2), for example, comes directly from the observation that

$$\int_{-\infty}^{\infty} e^{j\alpha t} f(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} f(t) e^{-j(\omega-\alpha)t} dt \,.$$

In general, identities (2.2.6.1) through (2.2.6.4) are not valid when $\alpha$ is not a real number. An exception to this occurs when $f$ is an analytic function on the entire complex plane. Then identities (2.2.6.1) and (2.2.6.4) do hold for all complex values of $\alpha$. Likewise, identities (2.2.6.2) and (2.2.6.3) may be used whenever $\alpha$ is complex provided $F$ is an analytic function on the entire complex plane.

**Example 2.2.6.1**

Let $g(t) = e^{-t^2}$. It can be shown that $g(t)$ is analytic on the entire complex plane and that its Fourier transform is

$$G(\omega) = \sqrt{\pi}\exp\left[-\frac{1}{4}\omega^2\right]$$

(see example 2.1.5.1 or example 2.2.11.2). If $\beta$ is any real value, then

$$\begin{aligned}\mathcal{F}\left[e^{-t^2+2\beta t}\right]\Big|_{\omega} &= \mathcal{F}\left[e^{j(-j2\beta)t}e^{-t^2}\right]\Big|_{\omega} \\ &= \sqrt{\pi}\exp\left[-\frac{1}{4}\left(\omega-(-j2\beta)\right)^2\right] \\ &= \sqrt{\pi}e^{\beta^2}\exp\left[-\frac{1}{4}\omega^2+j\beta\omega\right].\end{aligned}$$

## 2.2.7 Complex Translation and Multiplication by Real Exponentials

Using the "generalized" notion of translation discussed in subsection 2.1.4, it can be shown that for any complex value, $\alpha + j\beta$,

$$\mathcal{F}[T_{\alpha+j\beta}f(t)]\big|_{\omega} = e^{-j(\alpha+j\beta)\omega}F(\omega),$$

$$\mathcal{F}[e^{j(\alpha+j\beta)t}f(t)]\big|_{\omega} = T_{\alpha+j\beta}F(\omega),$$

$$\mathcal{F}^{-1}[T_{\alpha+j\beta}F(\omega)]\big|_{t} = e^{j(\alpha+j\beta)t}f(t),$$

and

$$\mathcal{F}^{-1}[e^{j(\alpha+j\beta)\omega}F(\omega)\big|_{t} = T_{-(\alpha+j\beta)}f(t).$$

Letting $\alpha = 0$ and $\beta = -\gamma$, these identities become

$$\mathcal{F}[T_{-j\gamma}f(t)]\big|_{\omega} = e^{-\gamma\omega}F(\omega),$$

$$\mathcal{F}[e^{\gamma t}f(t)]\big|_{\omega} = T_{-j\gamma}F(\omega),$$

$$\mathcal{F}^{-1}[T_{-j\gamma}F(\omega)]\big|_{t} = e^{\gamma t}f(t),$$

and

$$\mathcal{F}^{-1}[e^{\gamma\omega}F(\omega)]\big|_{t} = T_{j\gamma}f(t).$$

Caution must be exercised in the use of these formulas. It is true that $T_{\alpha+j\beta}f(t) = f(t-(\alpha+j\beta))$ whenever $\beta = 0$ or $f(z)$ is analytic on the entire complex plane. However, if $f(z)$ is not analytic and $\beta \neq 0$, then it is quite possible that $T_{\alpha+j\beta}f(t) \neq f(t-(\alpha+j\beta))$, even if $f(t-(\alpha+j\beta))$ is well defined. In these cases $T_{\alpha+j\beta}f(t)$ should be treated formally.

**Example 2.2.7.1**

By the above

$$\mathcal{F}\left[e^{t}u(t)\right]\Big|_{\omega} = \mathcal{F}\left[e^{2t}e^{-t}u(t)\right]\Big|_{\omega} = T_{-2j}\left[\frac{1}{1+j\omega}\right].$$

Note, however, that

$$\mathcal{F}\left[-e^{t}u(-t)\right]\Big|_{\omega} = \frac{-1}{1-j\omega} = \frac{1}{1+j\left(\omega-(-2j)\right)} .$$

Because $e^t\, u(t)$ and $-e^t\, u(-t)$ certainly are not equal, it follows that their transforms are not equal,

$$T_{-2j}\left[\frac{1}{1+j\omega}\right] \neq \frac{1}{1+j\left(\omega-(-2j)\right)}.$$

## 2.2.8 Modulation

The "modulation formulas,"

$$\mathcal{F}\left[\cos(\omega_0 t) f(t)\right]\Big|_{\omega} = \frac{1}{2}\left[F(\omega-\omega_0) + F(\omega+\omega_0)\right] \tag{2.2.8.1}$$

and

$$\mathcal{F}\left[\sin(\omega_0 t) f(t)\right]\Big|_{\omega} = \frac{1}{2j}\left[F(\omega-\omega_0) - F(\omega+\omega_0)\right] \tag{2.2.8.2}$$

are easily derived from identity (2.2.6.2) using the well-known formulas

$$\cos(\omega_0 t) = \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$$

and

$$\sin(\omega_0 t) = \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right].$$

**Example 2.2.81**

For $\alpha > 0$, the function

$$f(t) = \begin{cases} \cos\left(\dfrac{\pi}{2\alpha}t\right), & \text{if } -\alpha \le t \le \alpha \\ 0, & \text{otherwise} \end{cases}$$

can be written as

$$f(t) = \cos\left(\frac{\pi}{2\alpha}t\right)p_\alpha(t).$$

Thus, using identity (2.2.8.1) and the results of example 2.1.1.2,

$$\begin{aligned} F(\omega) &= \mathcal{F}\left[\cos\left(\frac{\pi}{2\alpha}t\right)p_\alpha(t)\right]\Big|_\omega \\ &= \frac{1}{2}\left[\frac{2}{\omega - \dfrac{\pi}{2\alpha}}\sin\left(\alpha\left[\omega - \frac{\pi}{2\alpha}\right]\right) + \frac{2}{\omega + \dfrac{\pi}{2\alpha}}\sin\left(\alpha\left[\omega + \frac{\pi}{2\alpha}\right]\right)\right] \\ &= \frac{4\alpha\pi}{\pi^2 - 4\alpha^2\omega^2}\cos(\alpha\omega). \end{aligned}$$

## 2.2.9 Products and Convolution

If $F = \mathcal{F}[f]$ and $G = \mathcal{F}[g]$, then the corresponding transforms of the products, $fg$ and $FG$, can be computed using the identities

$$\mathcal{F}[fg] = \frac{1}{2\pi}F * G \tag{2.2.9.1}$$

and

$$\mathcal{F}^{-1}[FG] = f * g, \tag{2.2.9.2}$$

provided the convolutions, $F * G$ and $f * g$, exist. Conversely, as long as the convolutions exist,

$$\mathcal{F}[f * g] = FG \tag{2.2.9.3}$$

and

$$\mathcal{F}^{-1}[F * G] = 2\pi\, fg. \tag{2.2.9.4}$$

Identity (2.2.9.1) can be derived as follows:

$$\begin{aligned} \int_{-\infty}^{\infty} f(t)g(t)e^{-j\omega t}\,dt &= \int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}F(s)e^{jst}\,ds\right)g(t)e^{-j\omega t}\,dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}F(s)\int_{-\infty}^{\infty}g(t)e^{-j(\omega - s)t}\,dt\,ds \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}F(s)G(\omega - s)\,ds. \end{aligned}$$

The other identities can be derived in a similar fashion.

**Example 2.2.9.1**

From direct computation, if $\beta > 0$, then

$$\mathcal{F}^{-1}\left[e^{-\beta\omega}u(\omega)\right]\Big|_t = \frac{1}{2\pi}\int_0^{\infty} e^{(jt-\beta)\omega}\,d\omega = \frac{1}{2\pi}\cdot\frac{1}{\beta - jt}.$$

And so,

$$\mathcal{F}\left[\frac{1}{\beta - jt}\right]\Bigg|_\omega = 2\pi e^{-\beta\omega}u(\omega).$$

Applying identity (2.2.9.1),

$$\begin{aligned}\mathcal{F}\left[\frac{1}{10-7jt-t^2}\right]\Bigg|_\omega &= \mathcal{F}\left[\frac{1}{2-jt}\cdot\frac{1}{5-jt}\right]\Bigg|_\omega \\ &= \frac{1}{2\pi}\left[2\pi e^{-2\omega}u(\omega)\right]*\left[2\pi e^{-5\omega}u(\omega)\right] \\ &= 2\pi\int_{-\infty}^{\infty} e^{-2s}u(s)e^{-5(\omega-s)}u(\omega-s)\,ds \\ &= \begin{cases} 0, & \text{if } \omega < 0 \\ \dfrac{2\pi}{3}\left[e^{-2\omega} - e^{-5\omega}\right], & \text{if } 0 < \omega \end{cases}.\end{aligned}$$

**Example 2.2.9.2**

By straightforward computations it is easily verified that for $\alpha > 0$,

$$\mathcal{F}\left[p_{\alpha/2}(t)\right]\Big|_\omega = \frac{2}{\omega}\sin\left(\frac{\alpha}{2}\omega\right)$$

and

$$p_{\alpha/2}(t) * p_{\alpha/2}(t) = \alpha\Lambda_\alpha(t),$$

where $p_{\alpha/2}(t)$ is the pulse function,

$$p_{\alpha/2}(t) = \begin{cases} 1, & \text{if } |t| < \dfrac{\alpha}{2} \\ 0, & \text{if } \dfrac{\alpha}{2} < |t| \end{cases},$$

and $\Lambda_\alpha(t)$ is the triangle function,

$$\Lambda_\alpha(t) = \begin{cases} 1 - \dfrac{|t|}{\alpha}, & \text{if } |t| < \alpha \\ 0, & \text{if } \alpha < |t| \end{cases}.$$

Using identity (2.2.9.3)

$$\mathcal{F}\left[\Lambda_\alpha(t)\right]\Big|_\omega = \frac{1}{\alpha}\mathcal{F}\left[p_{\alpha/2}(t) * p_{\alpha/2}(t)\right]\Big|_\omega$$
$$= \frac{1}{\alpha}\left(\frac{2}{\omega}\sin\left(\frac{\alpha}{2}\omega\right)\right)\left(\frac{2}{\omega}\sin\left(\frac{\alpha}{2}\omega\right)\right)$$
$$= \frac{4}{\alpha\omega^2}\sin^2\left(\frac{\alpha}{2}\omega\right).$$

## 2.2.10 Correlation

The cross-correlation of two functions, $f(t)$ and $g(t)$, is another function, denoted by $f(t) \star g(t)$, given by

$$f(t) \star g(t) = \int_{-\infty}^{\infty} f^*(s)\, g(t+s)\, ds, \tag{2.2.10.1}$$

where $f^*(s)$ denotes the complex conjugate of $f(s)$. The notation $\rho_{fg}(t)$ is often used instead of $f(t) \star g(t)$. The Wiener–Khintchine theorem states that, provided the correlations exist,

$$\mathcal{F}[f(t) \star g(t)]\big|_\omega = F^*(\omega)\, G(\omega) \tag{2.2.10.2}$$

and

$$\mathcal{F}\left[f^*(t)\, g(t)\right]\Big|_\omega = \frac{1}{2\pi} F(\omega) \star G(\omega), \tag{2.2.10.3}$$

where $F = \mathcal{F}[f]$ and $G = \mathcal{F}[g]$. Derivations of these formulas are similar to the analogous identities involving convolution.

For a given function, $f(t)$, the corresponding autocorrelation function is simply the cross-correlation of $f(t)$ with itself,

$$f(t) \star f(t) = \int_{-\infty}^{\infty} f^*(s)\, f(t+s)\, ds. \tag{2.2.10.4}$$

Often autocorrelation is denoted by $\rho_f(t)$ instead of $f(t) \star f(t)$. For autocorrelation, formulas (2.2.10.2) and (2.2.10.3) simplify to

$$\mathcal{F}[f(t) \star f(t)]\big|_\omega = |F(\omega)|^2 \tag{2.2.10.5}$$

and

$$\mathcal{F}\left[\left|f(t)\right|^2\right]\Big|_\omega = \frac{1}{2\pi} F(\omega) \star F(\omega). \tag{2.2.10.6}$$

## 2.2.11 Differentiation and Multiplication by Polynomials

If $f(t)$ is differentiable for all $t$ and vanishes as $t \to \pm\infty$, then the Fourier transform of the derivative of the function can be related to the transform of the undifferentiated function through the use of integration by parts,

$$\int_{-\infty}^{\infty} f'(t) e^{-j\omega t} dt = f(t) e^{-j\omega t} \Big|_{-\infty}^{\infty} + j\omega \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$= j\omega \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt .$$

In more concise form this can be written

$$\mathscr{F}[f'(t)]\big|_{\omega} = j\omega F(\omega), \tag{2.2.11.1}$$

where $F = \mathscr{F}[f]$. By near equivalence, if $G(\omega)$ is differentiable for all $\omega$ and vanishes as $\omega \to \pm\infty$, then

$$\mathscr{F}^{-1}[G'(\omega)]\big|_{t} = -jtg(t), \tag{2.2.11.2}$$

where $g = \mathscr{F}^{-1}[G]$. Similar derivations yield

$$\mathscr{F}[tf(t)]\big|_{\omega} = jF'(\omega) \tag{2.2.11.3}$$

and

$$\mathscr{F}^{-1}[\omega \ G(\omega)]\big|_{t} = -jg'(t), \tag{2.2.11.4}$$

provided $t\,f(t)$ and $\omega \ G(\omega)$ are suitably integrable.

**Example 2.2.11.1**

Using identity (2.2.11.1),

$$\mathscr{F}\left[\frac{j}{(1-jt)^2}\right]\Bigg|_{\omega} = \mathscr{F}\left[\frac{d}{dt}\left(\frac{1}{1-jt}\right)\right]\Bigg|_{\omega}$$

$$= j\omega \mathscr{F}\left[\frac{1}{1-jt}\right]\Bigg|_{\omega}$$

$$= j\omega 2\pi e^{-\omega} u(\omega) .$$

**Example 2.2.11.2**

Let $\alpha > 0$ and $g(t) = e^{-\alpha t^2}$. It is easily verified that

$$\frac{dg}{dt} = -2\alpha t g(t). \tag{2.2.11.5}$$

Taking the Fourier transform of each side and using identities (2.2.11.1) and (2.2.11.3) yields

$$j\omega G(\omega) = -2\alpha j \frac{dG}{d\omega}.$$

The solution to this first-order differential equation is easily computed. It is

$$G(\omega) = A\exp\left[-\frac{1}{4\alpha}\omega^2\right].$$

The value of the constant of integration, $A$, can be determined* by noting that

$$A = G(0) = \int_{-\infty}^{\infty} e^{-\alpha t^2}\,dt.$$

The value of this last integral is well known to be $\left(\sqrt{\pi/\alpha}\right)^{1/2}$. Thus,

$$G(\omega) = \sqrt{\frac{\pi}{\alpha}}\exp\left[-\frac{1}{4\alpha}\omega^2\right].$$

---

It should be noted that if $f'$ and $F'$ are assumed to be the classical derivatives of $f$ and $F$, that is

$$f'(t) = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

and

$$F'(\omega) = \lim_{\Delta\omega \to 0} \frac{F(\omega+\Delta\omega) - F(\omega)}{\Delta\omega},$$

then application of the above identities is limited by requirements that the functions involved be suitably smooth and that they vanish at infinity. These limitations can be eliminated, however, by interpreting $f'$ and $F'$ in a more generalized sense. In this more generalized interpretation, $f'$ and $F'$ are defined to be the (generalized) functions satisfying the "generalized" integration by parts formulas,

$$\int_{-\infty}^{\infty} f'(t)\phi(t)\,dt = -\int_{-\infty}^{\infty} f(t)\phi'(t)\,dt$$

and

$$\int_{-\infty}^{\infty} F'(\omega)\phi(\omega)\,d\omega = -\int_{-\infty}^{\infty} F(\omega)\phi'(\omega)\,d\omega,$$

*A method for determining $A$ using Bessel's equality is described in Subsection 2.2.15.

for every test function, $\phi$ ( with $\phi'$ denoting the classical derivative of $\phi$). As long as the function being differentiated is piecewise smooth and continuous, then there is no difference between the classical and the generalized derivative. If, however, the function, $f(x)$, has jump discontinuities at $x = x_1, x_2, \ldots, x_N$, then

$$f'_{\text{generalized}} = f'_{\text{classical}} + \sum_k J_k \delta_{x_k},$$

where $J_k$ denotes the "jump" in $f$ at $x = x_k$,

$$J_k = \lim_{\Delta x \to 0^+} f(x_k + \Delta x) - f(x_k - \Delta x).$$

It is not difficult to show that the product rule, $(fg)' = f'g + fg'$, holds for the generalized derivative as well as the classical derivative.

**Example 2.2.11.3**

Consider the step function, $u(t)$. The classical derivative of $u$ is clearly 0, because the graph of $u$ consists of two horizontal half-lines (with slope zero). Using the generalized integration by parts formula, however,

$$\begin{aligned}\int_{-\infty}^{\infty} u'(t)\phi(t)\,dt &= -\int_{-\infty}^{\infty} u(t)\phi'(t)\,dt \\ &= -\int_{o}^{\infty} \phi'(t)\,dt \\ &= \phi(0) \\ &= \int_{-\infty}^{\infty} \delta(t)\phi(t)\,dt,\end{aligned}$$

showing that $\delta(t)$ is the generalized derivative of $u(t)$.

**Example 2.2.11.4**

Using the generalized derivative and identity (2.2.11.3),

$$\begin{aligned}\mathcal{F}\left[\frac{t}{1-jt}\right]\Bigg|_{\omega} &= j\frac{d}{d\omega}\left(\mathcal{F}\left[\frac{1}{1-jt}\right]\Bigg|_{\omega}\right) \\ &= j\frac{d}{d\omega}\left(2\pi e^{-\omega}u(\omega)\right) \\ &= 2\pi j\left[\frac{de^{-\omega}}{d\omega}u(\omega) + e^{-\omega}u'(\omega)\right] \\ &= 2\pi j\left[-e^{-\omega}u(\omega) + \delta(\omega)\right].\end{aligned}$$

---

The extension of formulas (2.2.11.1) through (2.2.11.4) to the corresponding identities involving higher-order derivatives is straightforward. If $n$ is any positive integer, then

$$\mathcal{F}[f^{(n)}(t)]\big|_{\omega} = (j\omega)^n F(\omega), \tag{2.2.11.6}$$

$$\mathcal{F}^{-1}[F^{(n)}(\omega)]\big|_t = (-jt)^n\, f(t), \tag{2.2.11.7}$$

$$\mathcal{F}[t^n f(t)]\big|_\omega = j^n\, F^{(n)}(\omega), \tag{2.2.11.8}$$

and

$$\mathcal{F}^{-1}[\omega^n F(\omega)]\big|_t = (-j)^n\, f^{(n)}(t). \tag{2.2.11.9}$$

Again, these identities hold for all transformable functions as long as the derivatives are interpreted in the generalized sense.

### 2.2.12 Moments

For any suitably integrable function, $f(t)$, and nonnegative integer, $n$, the "$n$th moment of $f$" is the quantity

$$m_n(f) = \int_{-\infty}^{\infty} t^n f(t)\, dt.$$

Because

$$\int_{-\infty}^{\infty} t^n f(t)\, dt = \mathcal{F}\left[t^n f(t)\right]\Big|_0 ,$$

it is clear from identity (2.2.11.8) that

$$m_n(f) = j^n F^{(n)}(0).$$

### 2.2.13 Integration

If $F(\omega)$ and $G(\omega)$ are the Fourier transforms of $f(t)$ and $g(t)$, and $g(t) = t^{-1} f(t)$, then $tg(t) = f(t)$ and, by identity (2.2.11.3), $jG'(\omega) = F(\omega)$. Integrating this gives

$$G(\omega) - G(\alpha) = -j\int_{\alpha}^{\omega} F(s)\, ds ,$$

where $\alpha$ can be any real number. This can be written

$$\mathcal{F}\left[\frac{f(t)}{t}\right]\Bigg|_\omega = -j\int_{\alpha}^{\omega} F(s)\, ds + c_\alpha \tag{2.2.13.1}$$

where $c_\alpha = G(\alpha)$. For certain general types of functions and choices of $\alpha$, the value of $c_\alpha$ is easily determined. For example, if $f(t)$ is also absolutely integrable, then

$$\mathcal{F}\left[\frac{f(t)}{t}\right]\Bigg|_\omega = -j\int_{-\infty}^{\omega} F(s)\, ds , \tag{2.2.13.2}$$

while if $f(t)$ is an even function

$$\mathcal{F}\left[\frac{f(t)}{t}\right]\bigg|_{\omega} = -j\int_0^{\omega} F(s)\,ds\,, \tag{2.2.13.3}$$

provided the integrals are well defined.

It can also be shown that as long as the limit of $\omega^{-1} F(\omega)$ exists as $\omega \to 0$, then for each real value of $\alpha$ there is a constant, $c_\alpha$, such that

$$\mathcal{F}\left[\int_{\alpha}^{t} f(s)\,ds\right]\bigg|_{\omega} = -j\frac{F(\omega)}{\omega} + c_\alpha \delta(\omega)\,. \tag{2.2.13.4}$$

If $f(t)$ is an even function, then

$$\mathcal{F}\left[\int_{0}^{t} f(s)\,ds\right]\bigg|_{\omega} = -j\frac{F(\omega)}{\omega}\,, \tag{2.2.13.5}$$

while if $f(t)$ and $\int_{-\alpha}^{t} f(s)\ ds$ are absolutely integrable, then

$$\mathcal{F}\left[\int_{-\infty}^{t} f(s)\,ds\right]\bigg|_{\omega} = -j\frac{F(\omega)}{\omega}\,. \tag{2.2.13.6}$$

**Example 2.2.13.1**

Let $\alpha$ and $\beta$ be positive,

$$f(t) = e^{-\alpha|t|} - e^{-\beta|t|},$$

and

$$g(t) = \frac{f(t)}{t} = \frac{e^{-\alpha|t|} - e^{-\beta|t|}}{t}.$$

Both functions are easily verified to be transformable with

$$F(\omega) = \mathcal{F}\left[e^{-\alpha|t|} - e^{-\beta|t|}\right]\bigg|_{\omega} = \frac{2\alpha}{\alpha^2 + \omega^2} - \frac{2\beta}{\beta^2 + \omega^2}\,.$$

Because $f(t)$ is even, formula (2.2.13.3) applies, and

$$
\begin{aligned}
G(\omega) &= \mathcal{F}\left[\frac{e^{-\alpha|t|} - e^{-\beta|t|}}{t}\right]\Bigg|_{\omega} \\
&= -j\int_0^{\omega} F(s)\,ds \\
&= -j\int_0^{\omega}\left(\frac{2\alpha}{\alpha^2+s^2} - \frac{2\beta}{\beta^2+s^2}\right)ds \\
&= -2j\left(\arctan\left(\frac{\omega}{\alpha}\right) - \arctan\left(\frac{\omega}{\beta}\right)\right).
\end{aligned}
\tag{2.2.13.7}
$$

**Example 2.2.13.2**

Applying the same analysis done in the previous example but using

$$f(t) = 1 - e^{-\beta|t|}$$

leads, formally, to

$$
\begin{aligned}
\mathcal{F}\left[\frac{1-e^{-\beta|t|}}{t}\right]\Bigg|_{\omega} &= -j\int_0^{\omega}\left(2\pi\delta(s) - \frac{2\beta}{\beta^2+s^2}\right)ds \\
&= -2\pi j\int_0^{\omega}\delta(s)\,ds + 2j\arctan\left(\frac{\omega}{\beta}\right).
\end{aligned}
$$

Unfortunately, this is of little value because

$$\int_{\alpha}^{\omega}\delta(s)\,ds$$

is not well defined if $\alpha = 0$. However, because

$$\lim_{\alpha\to 0^+} e^{-\alpha|t|} = 1,$$

and

$$
\begin{aligned}
\lim_{\alpha\to 0^+}\arctan\left(\frac{\omega}{\alpha}\right) &= \begin{cases} \dfrac{\pi}{2}, & \text{if } 0 < \omega \\ -\dfrac{\pi}{2}, & \text{if } \omega < 0 \end{cases} \\
&= \frac{\pi}{2}\operatorname{sgn}(\omega),
\end{aligned}
$$

it can be argued, using equation (2.2.13.7), that

$$\begin{aligned}\mathcal{F}\left[\frac{1-e^{-\beta|t|}}{t}\right]\Bigg|_{\omega} &= \lim_{\alpha\to 0^+}\mathcal{F}\left[\frac{e^{-\alpha|t|}-e^{-\beta|t|}}{t}\right]\Bigg|_{\omega} \\ &= \lim_{\alpha\to 0^+} -2j\left(\arctan\left(\frac{\omega}{\alpha}\right)-\arctan\left(\frac{\omega}{\beta}\right)\right) \qquad (2.2.13.8) \\ &= -j\pi\,\mathrm{sgn}(\omega)+2j\arctan\left(\frac{\omega}{\beta}\right).\end{aligned}$$

## 2.2.14 Parseval's Equality

Parseval's equality is

$$\int_{-\infty}^{\infty} f(t)g^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)G^*(\omega)d\omega \qquad (2.2.14.1)$$

and is valid whenever the integrals make sense. Closely related to Parseval's equality are the two "fundamental identities,"

$$\int_{-\infty}^{\infty} f(x)\mathcal{F}[h]\big|_x dx = \int_{-\infty}^{\infty}\mathcal{F}[f]\big|_y h(y)dy \qquad (2.2.14.2)$$

and

$$\int_{-\infty}^{\infty} f(y)\mathcal{F}^{-1}[H]\big|_y dy = \int_{-\infty}^{\infty}\mathcal{F}^{-1}[F]\big|_x H(x)dx. \qquad (2.2.14.3)$$

Derivations of these identities are straightforward. Identity (2.2.14.2), for example, follows immediately from

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\left(\int_{-\infty}^{\infty} h(y)e^{-jxy}dy\right)dx &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)h(y)e^{-jxy}dy\,dx \\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x)e^{-jxy}dx\right)h(y)dy.\end{aligned}$$

Parseval's equality can then, in turn, be derived from identity (2.2.14.2) and the observation that

$$g^*(t) = \left(\mathcal{F}^{-1}[G]\big|_t\right)^* = \frac{1}{2\pi}\int_{-\infty}^{\infty} G^*(\omega)e^{-j\omega t}d\omega = \frac{1}{2\pi}\mathcal{F}[G^*]\big|_t.$$

## 2.2.15 Bessel's Equality

Bessel's equality,

$$\int_{-\infty}^{\infty} \left| f(t) \right|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| F(\omega) \right|^2 d\omega, \qquad (2.2.15.1)$$

is obtained directly from Parseval's equality by letting $g = f$.

**Example 2.2.15.1**

Let $\alpha > 0$ and $f(t) = p_\alpha(t)$, where $p_\alpha(t)$ is the pulse function. It is easily verified that

$$F(\omega) = \mathcal{F}\left[p_\alpha(t)\right]\Big|_\omega = \int_{-\alpha}^{\alpha} e^{-j\omega t}\, dt = \frac{2}{\omega} \sin(\alpha\omega).$$

So, using Bessel's equality,

$$\int_{-\infty}^{\infty} \left| \frac{\sin(\alpha\omega)}{\alpha\omega} \right|^2 d\omega = 2\pi \int_{-\infty}^{\infty} \left| \frac{1}{2\alpha} p_\alpha(t) \right|^2 dt$$

$$= \frac{2\pi}{4\alpha^2} \int_{-\alpha}^{\alpha} dt$$

$$= \frac{\pi}{\alpha}.$$

**Example 2.2.15.2**

Let $\alpha > 0$. In Example 2.2.11.2 it was shown that the Fourier transform of $g(t) = e^{-\alpha t^2}$ is $G(\omega) = A \exp\left[-\frac{1}{4\alpha}\omega^2\right]$. The positive constant $A$ can be determined by noting that, by Bessel's equality,

$$\int_{-\infty}^{\infty} \left| e^{-\alpha t^2} \right|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| A \exp\left[-\frac{1}{4\alpha}\omega^2\right] \right|^2 d\omega.$$

Letting $\omega = 2\alpha\tau$ this becomes, after a little simplification,

$$\int_{-\infty}^{\infty} e^{-2\alpha t^2} dt = \frac{\alpha}{\pi} A^2 \int_{-\infty}^{\infty} e^{-2\alpha\tau^2} d\tau.$$

Dividing out the integrals and solving for $A$ yields

$$A = \sqrt{\frac{\pi}{\alpha}},$$

where the positive square root is taken because

$$A = G(0) = \int_{-\infty}^{\infty} e^{-\alpha t^2} dt > 0.$$

## 2.2.16 The Bandwidth Theorem

If $f(t)$ is a function whose value may be considered as "negligible" outside of some interval, $(t_1, t_2)$, then the length of that interval, $\Delta t = t_2 - t_1$, is the effective duration of $f(t)$. Likewise, if $F(\omega)$ is the Fourier transform of $f(t)$, and $F(\omega)$ can be considered as "negligible" outside of some interval, $(\omega_1, \omega_2)$, then $\Delta\omega = \omega_2 - \omega_1$ is the effective bandwidth of $f(t)$.

The essence of the bandwidth theorem is that there is a universal positive constant, $\gamma$, such that the effective duration, $\Delta t$, and effective bandwidth, $\Delta\omega$, of any function (with finite $\Delta t$ or finite $\Delta\omega$) satisfies

$$\Delta t \Delta\omega \geq \gamma .$$

Thus, it is not possible to find a function whose effective bandwidth and effective duration are both arbitrarily small.

There are, in fact, several versions of the bandwidth theorem, each applicable to a particular class of functions. The two most important versions involve absolutely integrable functions and finite energy functions. They are described in greater detail in Subsections 2.3.3 and 2.3.5, respectively. Also in these subsections are appropriate precise definitions of effective duration and effective bandwidth.

Because it is the basis of the Heisenberg uncertainty principle of quantum mechanics, the bandwidth theorem is often, itself, referred to as the uncertainty principle of Fourier analysis.

# 2.3 Transforms of Specific Classes of Functions

In many applications one encounters specific classes of functions in which either the functions or their transforms satisfy certain particular properties. Several such classes of functions are discussed below.

## 2.3.1 Real/Imaginary Valued Even/Odd Functions

Let $F(\omega)$ be the Fourier transform of $f(t)$. Then, assuming $f(t)$ is integrable,

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt \\ &= \int_{-\infty}^{\infty} f(t)\left[\cos(\omega t) - j\sin(\omega t)\right] dt \qquad (2.3.1.1) \\ &= \int_{-\infty}^{\infty} f(t)\cos(\omega t)\, dt - j\int_{-\infty}^{\infty} f(t)\sin(\omega t)\, dt. \end{aligned}$$

If $f(t)$ is an even function, then

$$\int_{-\infty}^{\infty} f(t)\sin(\omega t)\, dt = 0$$

and equation (2.3.1.1) becomes

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\cos(\omega t)\, dt = 2\int_{0}^{\infty} f(t)\cos(\omega t)\, dt ,$$

which is clearly an even function of $\omega$ and is real valued whenever $f$ is real valued. Likewise, if $f(t)$ is an odd function, then

$$\int_{-\infty}^{\infty} f(t)\cos(\omega t)dt = 0,$$

and equation (3.1.1) reduces to

$$F(\omega) = -j\int_{-\infty}^{\infty} f(t)\sin(\omega t)dt = -2j\int_{0}^{\infty} f(t)\sin(\omega t)dt,$$

which is clearly an odd function of $\omega$ and is imaginary valued as long as $f$ is real valued.

These and related relations are summarized in Table 2.1.

**TABLE 2.1** ($F = \mathscr{F}[f]$)

| | | |
|---|---|---|
| $f(t)$ is even | ⇔ | $F(\omega)$ is even |
| $f(t)$ is real and even | ⇔ | $F(\omega)$ is real and even |
| $f(t)$ is imaginary and even | ⇔ | $F(\omega)$ is imaginary and even |
| $f(t)$ is odd | ⇔ | $F(\omega)$ is odd |
| $f(t)$ is real and odd | ⇔ | $F(\omega)$ is imaginary and odd |
| $f(t)$ is imaginary and odd | ⇔ | $F(\omega)$ is real and odd |

On occasion it is convenient to decompose a function, $f(t)$, into its even and odd components, $f_e(t)$ and $f_o(t)$,

$$f(t) = f_e(t) + f_o(t),$$

where

$$f_e(t) = \frac{1}{2}\left[f(t) + f(-t)\right] \quad \text{and} \quad f_0(t) = \frac{1}{2}\left[f(t) - f(-t)\right].$$

If $f(t)$ is a real-valued function with Fourier transform

$$F(\omega) = R(\omega) + jI(\omega),$$

where $R(\omega)$ and $I(\omega)$ denote, respectively, the real and imaginary parts of $F(\omega)$, then, by the above discussion it follows that

$$F_e(\omega) = R(\omega) = \mathscr{F}[f_e(t)]\big|_{\omega}\,, \tag{2.3.1.2}$$

$$F_o(\omega) = jI(\omega) = \mathscr{F}[f_o(t)]\big|_{\omega}\,, \tag{2.3.1.3}$$

$$f_e(t) = \mathscr{F}^{-1}\left[F_e(\omega)\right]\Big|_t = \frac{1}{\pi}\int_0^{\infty} R(\omega)\cos(\omega t)d\omega\,, \tag{2.3.1.4}$$

and

$$f_o(t) = \mathscr{F}^{-1}\left[F_o(\omega)\right]\Big|_t = -\frac{1}{\pi}\int_0^{\infty} I(\omega)\sin(\omega t)d\omega\,. \tag{2.3.1.5}$$

Rewriting $F(\omega)$ in terms of its amplitude, $A(\omega) = |F(\omega)|$, and phase, $\phi(\omega)$,

$$F(\omega) = A(\omega)e^{j\phi(\omega)},$$

it is easily seen that

$$\begin{aligned} R(\omega)\cos(\omega t) - I(\omega)\sin(\omega t) &= A(\omega)[\cos\phi(\omega)\cos(\omega t) - \sin\phi(\omega)\sin(\omega t)] \\ &= A(\omega)\cos(\omega t + \phi(\omega)). \end{aligned}$$

Thus, by equations (2.3.1.4) and (2.3.1.5), if $f(t)$ is real, then

$$f(t) = f_e(t) + f_o(t) = \frac{1}{\pi}\int_0^{\infty} A(\omega)\cos(\omega t + \phi(\omega))\,d\omega. \tag{2.3.16}$$

## 2.3.2 Absolutely Integrable Functions

If $f(t)$ is absolutely integrable (i.e., $\int_{-\infty}^{\infty} |f(t)|\,dt < \infty$) then the integral defining $F(\omega)$,

$$F(\omega) = \mathcal{F}[f(t)]\Big|_{\omega} = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

is well defined and well behaved. As a consequence, $F(\omega)$ is well defined for every $\omega$ and is a reasonably well behaved function on $(-\infty, \infty)$. One immediate observation is that for such functions,

$$F(0) = \int_{-\infty}^{\infty} f(t)\,dt.$$

It is also worth noting that for any $\omega$,

$$|F(\omega)| = \left|\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt\right| \le \int_{-\infty}^{\infty} \left|f(t)e^{-j\omega t}\right|dt = \int_{-\infty}^{\infty} |f(t)|\,dt.$$

The following can also be shown:

1. $F(\omega)$ is a continuous function of $\omega$ and for each $-\infty < \omega_0 < \infty$,

$$\lim_{\omega\to\omega_0} F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega_0 t}\,dt$$

2. (The Riemann-Lebesgue lemma)

$$\lim_{\omega\to\pm\infty} F(\omega) = 0.$$

As shown in the next example, care must be exercised not to assume these facts when $f(t)$ is not absolutely integrable.

**Example 2.3.2.1**

Consider the transform, $F(\omega)$ of $f(t) = \text{sinc}(t) = t^{-1}\sin(t)$. The function $f(t)$ is not absolutely integrable. Because

$$\mathcal{F}^{-1}\left[\pi\, p_1(t)\right]\Big|_{\omega} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\pi\, p_1(t)e^{j\omega t}dt = \text{sinc}(\omega)$$

it follows that

$$F(\omega) = \mathcal{F}[\text{sinc}(t)]\big|_{\omega} = \pi\, p_1(\omega).$$

Clearly

$$\lim_{\omega\to 1^+} F(\omega) = 0 \quad \text{and} \quad \lim_{\omega\to 1^-} F(\omega) = \pi,$$

while, using the residue theorem, it is easily shown that

$$F(1) = \mathcal{F}\left[\text{sinc}(t)\right]\Big|_{\omega=1} = \int_{-\infty}^{\infty}\text{sinc}(t)e^{jt}dt = \frac{\pi}{2}.$$

Thus, $F(\omega)$ is not continuous.

---

Analogous results hold when taking inverse transforms of absolutely integrable functions. If $F(\omega)$ is absolutely integrable and $f = \mathcal{F}^{-1}[F]$, then

$$f(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)d\omega,$$

and, for all real $t$,

$$\left|f(t)\right| \le \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|F(\omega)\right|d\omega.$$

Furthermore,

1′. $f(t)$ is a continuous function of $t$ and for each $-\infty < t_0 < \infty$,

$$\lim_{t\to t_0} f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{j\omega t_0}d\omega.$$

2′. (The Riemann-Lebesgue lemma)

$$\lim_{t\to\pm\infty} f(t) = 0.$$

### 2.3.3 The Bandwidth Theorem for Absolutely Integrable Functions

Assume that both $f(t)$ and its Fourier transform, $F(\omega)$, are absolutely integrable. Let $\bar{t}$ and $\bar{\omega}$ be any two fixed values for $t$ and $\omega$ such that $f(\bar{t}) \neq 0$ and $F(\bar{\omega}) \neq 0$. The corresponding effective duration, $\Delta t$, and the corresponding effective bandwidth, $\Delta\omega$, are the satisfying

$$\int_{-\infty}^{\infty} \left|f(t)\right| dt = \left|f(\bar{t})\right| \Delta t$$

and

$$\int_{-\infty}^{\infty} \left|F(\omega)\right| d\omega = \left|F(\bar{\omega})\right| \Delta\omega.$$

The bandwidth theorem for absolutely integrable functions states that

$$\Delta t \, \Delta\omega \geq 2\pi.$$

Moreover, using $\bar{t} = \bar{\omega} = 0$,

$$\Delta t \, \Delta\omega = 2\pi$$

whenever $f(t)$ and $F(\omega)$ are both real nonnegative functions (or real nonpositive functions) and neither $f(0)$ nor $F(0)$ vanishes.

The choice of the values for $\bar{t}$ and $\bar{\omega}$ depends on the use to be made of the bandwidth theorem. One standard choice for $\bar{t}$ and $\bar{\omega}$ is as the centroids of $|f(t)|$ and $|F(\omega)|$,

$$\bar{t} = \frac{\int_{-\infty}^{\infty} t\left|f(t)\right| dt}{\int_{-\infty}^{\infty} \left|f(t)\right| dt} \quad \text{and} \quad \bar{\omega} = \frac{\int_{-\infty}^{\infty} \omega\left|F(\omega)\right| d\omega}{\int_{-\infty}^{\infty} \left|F(\omega)\right| d\omega}.$$

Alternatively, to minimize the values used for the effective duration and effective bandwidth, $\bar{t}$ and $\bar{\omega}$ can be chosen to maximize the values of $|f(\bar{t})|$ and $|F(\bar{\omega})|$. Clearly, choosing $\bar{t} = 0$ and $\bar{\omega} = 0$ is especially appropriate if both $f(t)$ and $F(\omega)$ are real valued, even functions with maximums at the origin.

The above version of the bandwidth theorem is very easily derived. Because $f(t)$ and $F(\omega)$ are both absolutely integrable,

$$\left|F(\bar{\omega})\right| \leq \int_{-\infty}^{\infty} \left|f(t)\right| dt = \left|f(\bar{t})\right| \Delta t$$

and

$$\left|f(\bar{t})\right| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|F(\omega)\right| d\omega = \frac{1}{2\pi} \left|F(\bar{\omega})\right| \Delta\omega.$$

Thus,

$$\Delta t \, \Delta\omega \geq \frac{\left|F(\bar{\omega})\right|}{\left|f(\bar{t})\right|} \times \frac{2\pi\left|f(\bar{t})\right|}{\left|F(\bar{\omega})\right|} = 2\pi .$$

Clearly, if both $f(t)$ and $F(\omega)$ are real and nonnegative and neither $f(0)$ or $F(0)$ vanish, then the above inequalities can be replaced with

$$F(0) = \int_{-\infty}^{\infty} f(t)\,dt = f(0)\Delta t,$$

$$f(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\,d\omega = F(0)\Delta\omega,$$

and

$$\Delta t\ \Delta\omega = \frac{F(0)}{f(0)} \times \frac{2\pi f(0)}{F(0)} = 2\pi\ .$$

**Example 2.3.3.1**

Let $\alpha > 0$ and $f(t) = e^{-\alpha|t|}$. The transform of $f(t)$ is

$$F(\omega) = \frac{2\alpha}{\alpha^2 + \omega^2}\ .$$

Observe that both $f(t)$ and $F(\omega)$ are even functions with maximums at the origin. It is therefore appropriate to use $\bar{t} = 0$ and $\bar{\omega} = 0$ to compute the effective duration and effective bandwidth,

$$\Delta t = \frac{1}{|f(0)|}\int_{-\infty}^{\infty} |f(t)|\,dt = \int_{-\infty}^{\infty} e^{-\alpha|t|}\,dt = 2\int_{0}^{\infty} e^{-\alpha t}\,dt = \frac{2}{\alpha}$$

and

$$\Delta\omega = \frac{1}{|F(0)|}\int_{-\infty}^{\infty} |F(\omega)|\,d\omega = \frac{\alpha}{2}\int_{-\infty}^{\infty} \frac{2\alpha}{\alpha^2 + \omega^2}\,d\omega = \alpha\pi.$$

The products of these measures of effective bandwidth and duration are

$$\Delta t\ \Delta\omega = \left(\frac{2}{\alpha}\right)(\alpha\pi) = 2\pi\ .$$

as predicted by the bandwidth theorem.

### 2.3.4 Square Integrable ("Finite Energy") Functions

A function, $f(t)$, is square integrable if

$$\int_{-\infty}^{\infty} |f(t)|^2\,dt < \infty.$$

For many applicatons, it is natural to define the energy, $E$, in a function (or signal), $f(t)$, by

$$E = E[f] = \int_{-\infty}^{\infty} |f(t)|^2 \, dt.$$

For this reason, square integrable functions are also called finite energy functions. By Bessel's equality,

$$E[f] = \int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega, \tag{2.3.4.1}$$

where $F(\omega)$ is the Fourier transform of $f(t)$. This shows that a function is square integrable if and only if its transform is also square integrable. It also indicates why $|F(\omega)|^2$ is often referred to as either the "energy spectrum" or the "energy spectral density" of $f(t)$.

## 2.3.5 The Bandwidth Theorem for Finite Energy Functions

Assume that $f(t)$ and its Fourier transform, $F(\omega)$, are finite energy functions, and let the effective duration, $\Delta t$, and the effective bandwidth, $\Delta\omega$, be given by the "standard deviations,"

$$(\Delta t)^2 = \frac{\int_{-\infty}^{\infty} (t - \bar{t})^2 |f(t)|^2 \, dt}{\int_{-\infty}^{\infty} |f(t)|^2 \, dt}$$

and

$$(\Delta\omega)^2 = \frac{\int_{-\infty}^{\infty} (\omega - \bar{\omega})^2 |F(\omega)|^2 \, d\omega}{\int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega},$$

where $\bar{t}$ and $\bar{\omega}$ are the mean values of $t$ and $\omega$,

$$\bar{t} = \frac{\int_{-\infty}^{\infty} t |f(t)|^2 \, dt}{\int_{-\infty}^{\infty} |f(t)|^2 \, dt} \quad \text{and} \quad \bar{\omega} = \frac{\int_{-\infty}^{\infty} \omega |F(\omega)|^2 \, d\omega}{\int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega}.$$

Using the energy of $f(t)$,

$$E = \int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega,$$

the effective duration and effective bandwidth can be written more concisely as

$$\Delta t = \sqrt{\frac{1}{E} \int_{-\infty}^{\infty} (t - \bar{t})^2 |f(t)|^2 \, dt}$$

and

$$\Delta\omega = \sqrt{\frac{1}{2\pi E}\int_{-\infty}^{\infty}(\omega-\overline{\omega})^2 |F(\omega)|^2\, d\omega}\,.$$

The bandwidth theorem for finite energy functions states that, if the above quantities are well defined (and finite) and

$$\lim_{t\to\pm\infty} t|f(t)|^2 = 0,$$

then

$$\Delta t\,\Delta\omega \geq \frac{1}{2}.$$

Moreover, when $\bar{t} = 0$ and $\overline{\omega} = 0$, then

$$\Delta t\,\Delta\omega = \frac{1}{2}$$

if and only if $f(t)$ is a Gaussian,

$$f(t) = Ae^{-\alpha t^2},$$

for some $\alpha > 0$.

The reader should be aware that the effective duration and effective bandwidth defined in this subsection are not the same as the effective duration and effective bandwidth previously defined in Subsection 2.3.3. Nor do these definitions necessarily agree with the definitions given for the analogous quantities defined later in the subsections on reconstructing sampled functions.

### Example 2.3.5.1

Let $\alpha > 0$ and $f(t) = e^{-\alpha|t|}$. The transform of $f(t)$ is

$$F(\omega) = \frac{2\alpha}{\alpha^2+\omega^2}.$$

Because $tf(t)$ and $\omega F(\omega)$ are both odd functions, it is clear that $\bar{t} = 0$ and $\overline{\omega} = 0$. The energy is

$$E = \int_{-\infty}^{\infty}\left|e^{-\alpha|t|}\right|^2 dt = 2\int_0^{\infty} e^{-2\alpha t}\,dt = \frac{1}{\alpha}.$$

Using integration by parts, the corresponding effective duration and effective bandwidth are easily computed,

$$\begin{aligned}\Delta t &= \sqrt{\frac{1}{E}\int_{-\infty}^{\infty}(t-\bar{t})^2|f(t)|^2\,dt}\\ &= \sqrt{2\alpha\int_0^{\infty} t^2 e^{-2\alpha t}\,dt}\\ &= \frac{\sqrt{2}}{2\alpha}\end{aligned}$$

and

$$\begin{aligned}\Delta\omega &= \sqrt{\frac{1}{2\pi E}\int_{-\infty}^{\infty}(\omega-\overline{\omega})^2|F(\omega)|^2\,d\omega}\\ &= \sqrt{\frac{\alpha}{2\pi}\int_{-\infty}^{\infty}\omega^2\left(\frac{2\alpha}{\alpha^2+\omega^2}\right)^2 d\omega}\\ &= \sqrt{\frac{\alpha^3}{\pi}\int_{-\infty}^{\infty}\omega\frac{2\omega}{(\alpha^2+\omega^2)^2}\,d\omega}\\ &= \alpha.\end{aligned}$$

(By comparison, treating $f(t)$ and $F(\omega)$ as absolutely integrable functions [Example 2.3.3.1] led to an effective duration of $2\alpha^{-1}$ and an effective bandwidth of $\alpha\pi$.)

The products of these measures of bandwidth and duration computed here are

$$\Delta t\,\Delta\omega = \frac{\sqrt{2}}{2\alpha}\alpha = \frac{\sqrt{2}}{2} > \frac{1}{2},$$

as predicted by the bandwidth theorem for finite energy functions.

### 2.3.6 Functions with Finite Duration

A function, $f(t)$, has finite duration (with duration $2T$) if there is a $0 < T < \infty$ such that

$$f(t) = 0 \quad \text{whenever} \quad T < |t|.$$

The transform, $F(\omega)$, of such a function is given by a proper integral over a finite interval,

$$F(\omega) = \int_{-T}^{T} f(t)e^{-j\omega t}\,dt. \tag{2.3.6.1}$$

Any piecewise continuous function with finite duration is automatically absolutely integrable and automatically has finite energy, and, so, the discussions in the previous subsections apply to such functions. In addition, if $f(t)$ is a piecewise continuous function of finite duration (with duration $2T$), then, for every nonnegative integer, $n$, $t^n f(t)$ is also a piecewise continuous finite duration function with duration $2T$, and using identity (2.2.11.8),

$$F^{(n)}(\omega) = \mathcal{F}\left[(-jt)^n f(t)\right]\Big|_{\omega} = \int_{-T}^{T}(-jt)^n f(t)e^{-j\omega t}\,dt.$$

From the discussion in Subsection 2.3.2, it is apparent that the transform of a piecewise continuous function with finite duration must be classically differentiable up to any order, and that every derivative is continuous.

It should be noted that the integral defining $F(\omega)$ in formula (2.3.6.1) is, in fact, well defined for every complex $\omega = x + jy$. It is not difficult to show that the real and imaginary parts of $F(x + jy)$ satisfy the

Cauchy-Riemann equations of complex analysis (see Appendix 1). Thus, $F(\omega)$ is an analytic function on both the real line and the complex plane. As a consequence, it follows that the transform of a finite duration function cannot vanish (or be any constant value) over any nontrivial subinterval of the real line. In particular, no function of finite duration can also be band limited (see Subsection 2.3.7).

Another important feature of finite duration functions is that their transforms can be reconstructed using a discrete sampling of the transforms. This is discussed more fully in Section 2.5.

### 2.3.7 Band-Limited Functions

Let $f(t)$ be a functon with Fourier transform $F(\omega)$. The function, $f(t)$, is said to be band limited if there is a $0 < \Omega < \infty$, such that

$$F(\omega) = 0 \quad \text{whenever } \Omega < |\omega|.$$

The quantity $2\Omega$ is called the bandwidth of $f(t)$.

By the near equivalence of the Fourier and inverse Fourier transforms, it should be clear that $f(t)$ satisfies properties analogous to those satisfied by the transforms of finite duration functions. In particular

$$f(t) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} F(\omega)e^{j\omega t}\,d\omega\,, \tag{2.3.7.1}$$

and, for any nonnegative integer, $n$, $f^{(n)}(t)$ is a well-defined continuous function given by

$$f^{(n)}(t) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} (j\omega)^n F(\omega)e^{j\omega t}\,d\omega\,.$$

Letting $t = x + jy$ in equaton (2.3.7.1), it is easily verified that $f(x + jy)$ is a well-defined analytic function on both the real line and on the entire complex plane. From this it follows that if $f(t)$ is band limited, then $f(t)$ cannot vanish (or be any constant value) over any nontrivial subinterval of the real line. Thus, no band-limited function can also be of finite duration. This fact must be considered in many practical applicatons where it would be desirable (but, as just noted, impossible) to assume that the functions of interest are both band-limited and of finite duration.

Another most important feature of band-limited functions is that they can be reconstructed using a discrete sampling of their values. This is discussed more thoroughly in Section 2.5.

### 2.3.8 Finite Power Functions

For a given function, $f(t)$, the average autocorrelation function, $\bar{\rho}_f(t)$, is defined by

$$\bar{\rho}_f(t) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f^*(s)f(t+s)\,ds\,, \tag{2.3.8.1}$$

or, equivalently, by

$$\bar{\rho}_f(t) = \lim_{T\to\infty}\frac{1}{2T} f_T(t) \star f_T(t) \tag{2.3.8.2}$$

where the $\star$ denotes correlation (see Subsection 2.2.10), and $f_T(t)$ is the truncation of $f(t)$ at $t = \pm T$,

$$f_T(t) = f(t)p_T(t) = \begin{cases} f(t), & \text{if } -T \le t \le T \\ 0, & \text{otherwise} \end{cases}. \tag{2.3.8.3}$$

If $\bar{\rho}_f(t)$ is a well-defined function (or generalized function), then $f(t)$ is called a finite power function.

The power spectrum or power spectral density, $P(\omega)$, of a finite power function, $f(t)$ is defined to be the Fourier transform of its average autocorrelation,

$$P(\omega) = \mathcal{F}\left[\bar{\rho}_f(t)\right]\Big|_{\omega} = \int_{-\infty}^{\infty} \bar{\rho}_f(t) e^{-j\omega t}\, dt\,. \tag{2.3.8.4}$$

Using formula (2.3.8.2) for $\bar{\rho}_f(t)$ and recalling the Wiener-Khintchine theorem (Subsection 2.2.10).

$$\mathcal{F}\left[\bar{\rho}_f(t)\right]\Big|_{\omega} = \lim_{T\to\infty} \frac{1}{2T} \mathcal{F}\left[f_T(t) \star f_T(t)\right]\Big|_{\omega} = \lim_{T\to\infty} \frac{1}{2T} \left|F_T(\omega)\right|^2$$

where $F_T(\omega)$ is the Fourier transform of $f_T(t)$,

$$F_T(\omega) = \int_{-\infty}^{\infty} f(t) p_T(t) e^{-j\omega t}\, dt = \int_{-T}^{T} f(t) e^{-j\omega t}\, dt.$$

Thus, an alternate formula for the power spectrum is

$$P(\omega) = \lim_{T\to\infty} \frac{1}{2T} \left| \int_{-T}^{T} f(t) e^{-j\omega t}\, dt \right|^2. \tag{2.3.8.5}$$

The average power in $f(t)$ is defined to be

$$\bar{\rho}_f(0) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left|f(s)\right|^2 ds. \tag{2.3.8.6}$$

Because $P(\omega) = \mathcal{F}[\bar{\rho}_f(t)]\big|_{\omega}$, this is equivalent to

$$\bar{\rho}_f(0) = \mathcal{F}^{-1}\left[P(\omega)\right]\Big|_0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} P(\omega)\, d\omega\,.$$

A number of properties of the average autocorrelation should be noted. They are

1. $\bar{\rho}_f(t)$ is invariant under a shift in $f(t)$, that is, if $g(t) = f(t - t_0)$, then $\bar{\rho}_g(t) = \bar{\rho}_f(t)$.
2. $\bar{\rho}_f(t)$ and $|\bar{\rho}_f(t)|$ each has a maximum value at $t = 0$.
3. $(\bar{\rho}_f(t))^* = \bar{\rho}_f(-t)$. Thus, as is often the case, if $f(t)$ is a real-valued function, then $\bar{\rho}_f(t)$ is an even real-valued function.

As a consequence of the second property above, any function, $f(t)$, satisfying

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left|f(s)\right|^2 ds < \infty$$

is a finite power function.

The three properties listed above are easily derived. For the first,

$$\begin{aligned}
\bar{\rho}_g(t) &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f^*(s-t_0) f(s-t_0+t)\,ds \\
&= \lim_{T\to\infty} \frac{1}{2T}\int_{-T-t_0}^{T-t_0} f^*(\sigma) f(\sigma+t)\,ds \\
&= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f^*(\sigma) f(\sigma+t)\,d\sigma \\
&\quad + \lim_{T\to\infty} \frac{1}{2T}\int_{T}^{T-t_0} f^*(\sigma) f(\sigma+t)\,d\sigma \\
&\quad + \lim_{T\to\infty} \frac{1}{2T}\int_{-T-t_0}^{-T} f^*(\sigma) f(\sigma+t)\,d\sigma .
\end{aligned}$$

The first limit in the last line above equals $\bar{\rho}_f(t)$ while the other limits, involving integrals over intervals of fixed bounded length, must vanish.

From an application of the Schwarz inequality,

$$\left|\int_{-T}^{T} f^*(s) f(s+t)\,ds\right|^2 \le \int_{-T}^{T} \left|f^*(s)\right|^2 ds \int_{-T}^{T} \left|f(s+t)\right|^2 ds ,$$

it follows, after taking the limit, that

$$\left|\bar{\rho}_f(t)\right|^2 \le \left|\bar{\rho}_f(0)\right|^2 .$$

Hence, at $t = 0$, $|\bar{\rho}_f(t)|$ has a maximum (as does $\bar{\rho}_f(t)$, because $\bar{\rho}_f(0) = |\bar{\rho}_f(0)|$).

Finally, using the substitution $\sigma = s + t$,

$$\begin{aligned}
\left(\bar{\rho}_f(t)\right)^* &= \left(\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f^*(s) f(t+s)\,ds\right)^* \\
&= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(s) f^*(t+s)\,ds \\
&= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(\sigma-t) f^*(\sigma)\,d\sigma \\
&= \bar{\rho}_f(-t).
\end{aligned}$$

If $f(t)$ is a finite energy function, then, trivially, it is also a finite power function (with zero average power). Nontrivial examples of finite power functions include periodic functions, nearly periodic functions, constants, and step functions. Finite energy functions also play a significant role in signal-processing problems dealing with noise.

**Example 2.3.8.1**

Consider the step function,

$$u(t) = \begin{cases} 0, & \text{if} \quad t < 0 \\ 1, & \text{if} \quad 0 < t \end{cases}.$$

For $0 \le t$,

$$\begin{aligned} \bar{\rho}_u(t) &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} u(s)u(s+t)\,ds \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{0}^{T} ds \\ &= \frac{1}{2}. \end{aligned}$$

Because the step function is a real function, its average autocorrelation function must be an even function. Thus, for all $t$,

$$\bar{\rho}_u(t) = \frac{1}{2},$$

showing that the step function is a finite power function. Its average power $\bar{\rho}_u(0)$, is equal to 1/2, and its power spectrum is

$$P(\omega) = \mathscr{F}\left[\frac{1}{2}\right]\Big|_{\omega} = \pi\delta(\omega).$$

**Example 2.3.8.2**

Consider now the function

$$f(t) = \begin{cases} 0, & \text{if } t \le 0 \\ \sin t, & \text{if } 0 \le t \end{cases}.$$

For $0 \le t$,

$$\begin{aligned} \bar{\rho}_f(t) &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(s)f(s+t)\,ds \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{0}^{T} \sin(s)\sin(s+t)\,ds \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{0}^{T} \sin(s)[\sin(s)\cos(t) + \cos(s)\sin(t)]\,ds \\ &= \lim_{T\to\infty} \frac{1}{2T}\left[\cos(t)\int_{0}^{T} \sin^2(s)\,ds + \sin(t)\int_{0}^{T} \sin(s)\cos(s)\,ds\right] \\ &= \lim_{T\to\infty} \frac{1}{2T}\left[\cos(t)\left(\frac{T}{2} - \frac{\sin(2T)}{4}\right) - \sin(t)\frac{\sin^2(T)}{2}\right] \\ &= \frac{1}{4}\cos(t). \end{aligned}$$

Because $\bar{\rho}_f(t)$ is even,

$$\bar{\rho}_f(t) = \frac{1}{4}\cos(t)$$

for all $t$. The average power is

$$\bar{\rho}_f(0) = \frac{1}{4},$$

and the power spectrum is

$$P(\omega) = \mathscr{F}\left[\frac{1}{4}\cos(t)\right]\Big|_\omega = \frac{\pi}{4}\left[\delta(\omega-1)+\delta(\omega+1)\right].$$

### 2.3.9 Periodic Functions

Let $0 < p < \infty$. A function, $f(t)$, is periodic (with period $p$) if

$$f(t + p) = f(t)$$

for every real value of $t$. The Fourier series, $FS[f]$, for such a function is given by

$$FS[f]\Big|_t = \sum_{n=-\infty}^{\infty} c_n e^{jn\Delta\omega t}, \tag{2.3.9.1}$$

where

$$\Delta\omega = \frac{2\pi}{p}$$

and, for each $n$,

$$c_n = \frac{1}{p}\int_{\text{period}} f(t)e^{-jn\Delta\omega t}\,dt. \tag{2.3.9.2}$$

(Because of the periodicity of the integrand, the integral in formula (2.3.9.2) can be evaluated over any interval of length $p$.)

As long as $f(t)$ is at least piecewise smooth, its Fourier series will converge, and at every value of $t$ at which $f(t)$ is continuous,

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\Delta\omega t}.$$

At points where $f(t)$ has a "jump" discontinuity, the Fourier series converges to the midpoint of the jump. In any immediate neighborhood of a jump discontinuity any finite partial sum of the Fourier series,

$$\sum_{n=-N}^{N} c_n e^{jn\Delta\omega t},$$

will oscillate wildly and will, at points, significantly over- and undershoot the actual value of $f(t)$ ("Ringing" or Gibbs phenomena).

Because periodic functions are not at all integrable over the entire real line, the standard integral formula, formula (2.1.1.1), cannot be used to find the Fourier transform of $f(t)$. Using the generalized theory, however, it can be shown that as generalized functions,

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\Delta\omega t} \tag{2.3.9.3}$$

and that the Fourier transform of $f(t)$ is given by

$$\begin{aligned} F(\omega) &= \mathscr{F}\left[\sum_{n=-\infty}^{\infty} c_n e^{jn\Delta\omega t}\right]\Bigg|_{\omega} \\ &= \sum_{n=-\infty}^{\infty} c_n \mathscr{F}\left[e^{jn\Delta\omega t}\right]\Big|_{\omega} \\ &= \sum_{n=-\infty}^{\infty} c_n 2\pi\delta(\omega - n\Delta\omega). \end{aligned}$$

It should be noted that $F(\omega)$ is a regular array of delta functions with spacing inversely proportional to the period of $f(t)$ (see Subsection 2.3.10).

If $f(t)$ is periodic (with period $p$), then $f(t)$ is a finite power function (but is not, unless $f(t)$ is the zero function, a finite energy function). The average autocorrelation, $\bar{\rho}_f(t)$, will also be periodic and have period $p$. Formula (2.3.8.1) reduces to

$$\bar{\rho}_f(t) = \frac{1}{p}\int_{\text{period}} f^*(s) f(s+t)\,ds. \tag{2.3.9.4}$$

Because $\bar{\rho}_f(t)$ is periodic, it can also be expanded as a Fourier series,

$$\bar{\rho}_f(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\Delta\omega t}, \tag{2.3.9.5}$$

and the power spectrum is the regular array of delta functions,

$$P(\omega) = \sum_{n=-\infty}^{\infty} a_n 2\pi\delta(\omega - n\Delta\omega).$$

A useful relation between the Fourier coefficients of $\bar{\rho}_f(t)$,

$$a_n = \frac{1}{p}\int_{\text{period}} \bar{\rho}_f(t)e^{-jn\Delta\omega t}dt\,, \tag{2.3.9.6}$$

and the Fourier coefficients of $f(t)$,

$$c_n = \frac{1}{p}\int_{\text{period}} f(t)e^{-jn\Delta\omega t}dt\,, \tag{2.3.9.7}$$

is easily derived. Inserting formula (2.3.9.4) for $\bar{\rho}_f(t)$ into formula (2.3.9.6), rearranging, and using the substitution $\tau = s + t$,

$$\begin{aligned}
a_n &= \frac{1}{p}\int_{\text{period}}\left[\frac{1}{p}\int_{\text{period}} f^*(s)f(s+t)ds\right]e^{-jn\Delta\omega t}dt \\
&= \frac{1}{p}\int_{\text{period}} \frac{1}{p} f^*(s)\left[\int_{\text{period}} f(s+t)e^{-jn\Delta\omega t}dt\right]ds \\
&= \frac{1}{p}\int_{\text{period}} \frac{1}{p} f^*(s)\left[\int_{\text{period}} f(\tau)e^{-jn\Delta\omega(\tau-s)}d\tau\right]ds \\
&= \left[\frac{1}{p}\int_{\text{period}} f^*(s)e^{jn\Delta\omega s}ds\right]\left[\frac{1}{p}\int_{\text{period}} f(\tau)e^{-jn\Delta\omega\tau}d\tau\right] \\
&= c_n^* c_n\,.
\end{aligned}$$

Thus, $a_n = |c_n|^2$.

In summary, if $f(t)$ is periodic with period $p$, then so is its average autocorrelation function, $\bar{\rho}_f(t)$. Moreover (as generalized functions)

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\Delta\omega t}\,, \tag{2.3.9.8}$$

$$F(\omega) = 2\pi\sum_{n=-\infty}^{\infty} c_n\delta(\omega - n\Delta\omega), \tag{2.3.9.9}$$

$$\bar{\rho}_f(t) = \sum_{n=-\infty}^{\infty} |c_n|^2 e^{jn\Delta\omega t}\,, \tag{2.3.9.10}$$

and

$$P(\omega) = 2\pi\sum_{n=-\infty}^{\infty} |c_n|^2\delta(\omega - n\Delta\omega), \tag{2.3.9.11}$$

where $F(\omega)$ is the Fourier transform of $f(t)$, $P(\omega)$ is the power spectrum of $f(t)$,

$$\Delta\omega = \frac{2\pi}{p}, \tag{2.3.9.12}$$

and, for each $n$,

$$c_n = \frac{1}{p}\int_{\text{period}} f(t)e^{-jn\Delta\omega t}\,dt. \tag{2.3.9.13}$$

Analogous formulas are valid if $G(\omega)$ is a periodic function with period $P$. In particular, its inverse transform is

$$g(t) = \sum_{k=-\infty}^{\infty} C_k\delta(t - k\Delta t), \tag{2.3.9.14}$$

where

$$\Delta t = \frac{2\pi}{P}$$

and, for each $k$,

$$C_k = \frac{1}{P}\int_{\text{period}} G(\omega)e^{jk\Delta t\omega}\,d\omega\,.$$

Again, because of periodicity, the integral can be evaluated over any interval of length $P$.

**Example 2.3.9.1 Fourier Series and Transform of a Periodic Function**

Consider the "saw" function,

$$\text{saw}(t) = \begin{cases} t, & \text{if } -1 \le t < 1 \\ \text{saw}(t+2), & \text{for all } t \end{cases}$$

The graph of this saw function is sketched in Figure 2.4. Here, because the period is $p = 2$, formula (2.3.9.12) becomes

$$\Delta\omega = \frac{2\pi}{p} = \pi,$$

and formula (2.3.9.13) becomes

$$c_n = \frac{1}{2}\int_{-1}^{1} te^{-jn\pi t}\,dt = \begin{cases} 0, & \text{if } n = 0 \\ (-1)^n \dfrac{j}{n\pi}, & \text{if } n = \pm 1, \pm 2, \pm 3, \dots \end{cases}.$$

Using equations (2.3.9.8) and (2.3.9.9),

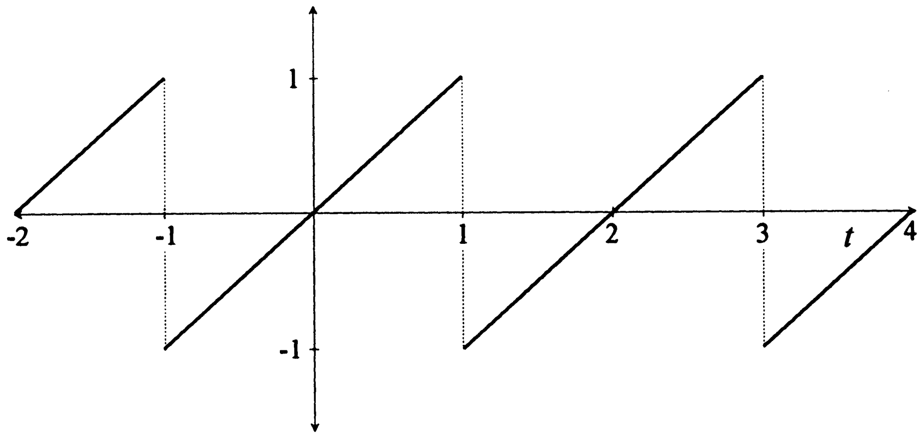


**FIGURE 2.4** The saw function.

$$\operatorname{saw}(t)=\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}(-1)^n\frac{j}{n\pi}e^{jn\pi t}$$

and

$$\mathcal{F}[\operatorname{saw}(t)]\Big|_{\omega}=j\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}(-1)^n\frac{2}{n}\delta(\omega-n\pi).$$

The graph of the $N$th partial sum approximation to saw($t$),

$$\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}(-1)^n\frac{j}{n\pi}e^{jn\pi t},$$

is sketched in Figure 2.5 (with $N = 20$), and the graph of the imaginary part of $\mathcal{F}[\operatorname{saw}(t)]\big|_{\omega}$ is sketched in Figure 2.6. The Gibbs phenomenon is evident in Figure 2.5. Formulas (2.3.9.10) and (2.3.9.11) for the autocorrelation function, $\bar{\rho}_{\text{saw}}(t)$, and the power spectrum, $P(\omega)$, yield

$$\bar{\rho}_{\text{saw}}(t)=\frac{1}{\pi^2}\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\frac{1}{n^2}e^{jn\pi t}$$

and

$$P(\omega)=\frac{2}{\pi}\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\frac{1}{n^2}\delta(\omega-n\pi).$$

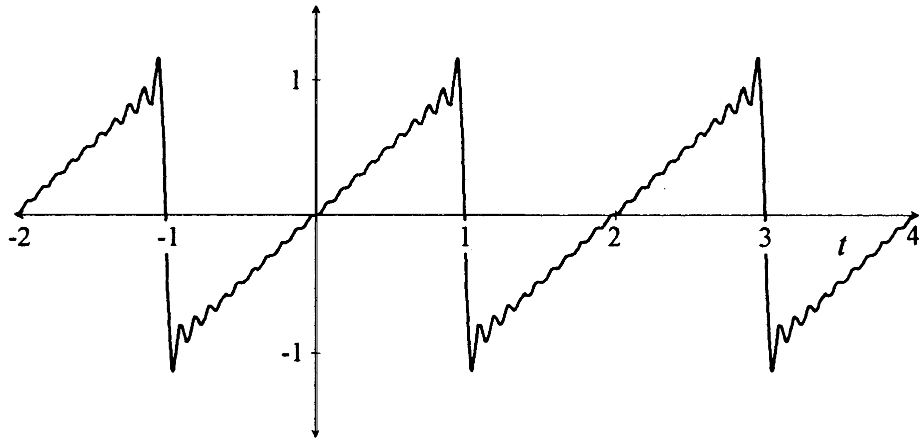


**FIGURE 2.5** Partial sum of the saw function's Fourier series.

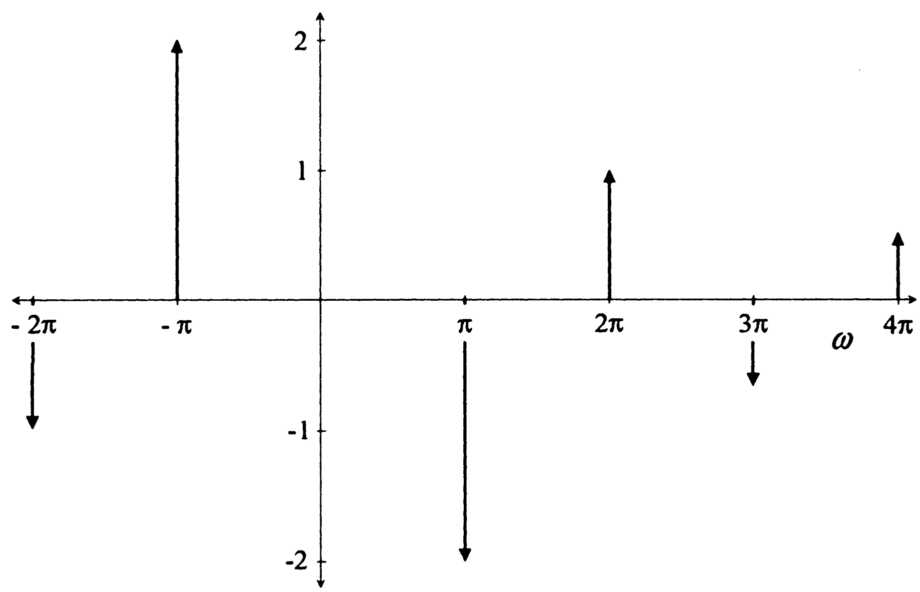


**FIGURE 2.6** Fourier transform of the saw function (imaginary part).

## 2.3.10 Regular Arrays of Delta Functions

Let $\Delta x > 0$. A function $\phi(x)$ is called a regular array of delta functions (with spacing $\Delta x$) if

$$\phi(x) = \sum_{n=-\infty}^{\infty} \phi_n \delta(x - n\Delta x),$$

where the $\phi_n$'s denote fixed values. Such arrays arise in sampling and as transforms of periodic functions. They are also useful in describing discrete probability distributions (see Examples 2.3.10.2 and 2.3.10.3 below).

### Example 2.3.10.1

The transform of the saw function from Example 2.3.9.1,

$$\mathscr{F}\left[\text{saw}(t)\right]\Big|_{\omega} = j\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}(-1)^n \frac{2}{n}\delta(\omega - n\pi),$$

is a regular array of delta functions with spacing $\Delta\omega = \pi$.

---

Let $f(t)$ be a function with Fourier transform $F(\omega)$. A straightforward extension and restatement of the results in the previous subsection is that $f(t)$ is periodic if and only if $F(\omega)$ is a regular array of delta functions. The period, $p$, of $f(t)$, and the spacing, $\Delta\omega$, of $F(\omega)$ are related by

$$p\Delta\omega = 2\pi.$$

Moreover,

$$f(t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} F_n e^{jn\Delta\omega t}$$

and

$$F(\omega) = \sum_{n=-\infty}^{\infty} F_n \delta(\omega - n\Delta\omega),$$

where, for each $n$,

$$F_n = \frac{2\pi}{p}\int_{\text{period}} f(t)e^{-jn\Delta\omega t}\,dt. \tag{2.3.10.1}$$

Conversely, if $g(t)$ is a function with Fourier transform $G(\omega)$, then $g(t)$ is a regular array of delta functions if and only if $G(\omega)$ is periodic. The spacing of $g(t)$, $\Delta t$, and the period of $G(\omega)$, $P$, are related by

$$P\Delta t = 2\pi.$$

Moreover,

$$g(t) = \sum_{k=-\infty}^{\infty} g_k \delta(t - k\Delta t)$$

and

$$G(\omega) = \sum_{k=-\infty}^{\infty} g_{-k} e^{jk\Delta t\omega},$$

where, for each $k$,

$$g_k = \frac{1}{P}\int_{\text{period}} G(\omega)e^{jk\Delta t\omega}\,d\omega.$$

### Example 2.3.10.2

For any $\lambda > 0$, the corresponding Poisson probability distribution is given by

$$\phi_\lambda(t) = e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \delta(t-n).$$

Its Fourier transform, $\psi_\lambda(\omega)$, is given by

$$\psi_\lambda(\omega) = e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-jn\omega} .$$

Recalling the Taylor series for the exponential,

$$\begin{aligned}\psi_\lambda(\omega) &= e^{-\lambda} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\lambda e^{-j\omega}\right)^n \\ &= e^{-\lambda} e^{\lambda e^{-j\omega}} \\ &= e^{-\lambda\left(1-\cos\omega + j\sin\omega\right)} ,\end{aligned}$$

which is clearly a periodic function with period $P = 2\pi$. It can also be seen that the amplitude, $A(\omega)$, and the phase, $\Theta(\omega)$, of $\psi_\lambda(\omega)$ are given by

$$A(\omega) = e^{-\lambda(1-\cos\omega)} \quad \text{and} \quad \Theta(\omega) = -\lambda \sin \omega.$$

### Example 2.3.10.3

For any nonnegative integer, $n$, and $0 \le p \le 1$, the corresponding binomial probability distribution is given by

$$b_{n,p}(t) = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \delta(t-k)$$

where $q = 1 - p$. The Fourier transform of $b_{n,p}$ is given by

$$B_{n,p}(\omega) = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} e^{-jk\omega} = \sum_{k=0}^{n} \binom{n}{k} \left(pe^{-j\omega}\right)^k q^{n-k} .$$

By the binomial theorem, this can be rewritten as

$$B_{n,p}(\omega) = (pe^{-j\omega} + q)^n,$$

which is clearly periodic with period $P = 2\pi$.

---

A regular array of delta functions,

$$g(t) = \sum_{k=-\infty}^{\infty} g_k \delta(t - k\Delta t),$$

cannot be a finite energy function (unless all the $g_k$'s vanish), but, if the $g_k$'s are bounded, can be treated as a finite power function with average autocorrelation function, $\bar{\rho}_g(t)$, and power spectrum, $P(\omega)$, given by

$$\bar{\rho}_g(t) = \sum_{k=-\infty}^{\infty} A_k \delta(t - k\Delta t)$$

and

$$P(\omega) = \sum_{k=-\infty}^{\infty} A_k e^{-jk\Delta t\omega},$$

where

$$A_k = \lim_{M\to\infty} \frac{1}{2M\Delta t} \sum_{m=-M}^{M} g_m^* g_{m+k} .$$

It should be noted, however, that if

$$\sum_{m=-\infty}^{\infty} |g_m|^2 < \infty ,$$

then the $A_k$'s will all be zero.

### 2.3.11 Periodic Arrays of Delta Functions

Regular periodic arrays of delta functions are of considerable importance because the formulas for the discrete Fourier transforms can be based directly on formulas derived in computing transforms of regular arrays that are also periodic. For an array with spacing $\Delta x$,

$$\phi(x) = \sum_{k=-\infty}^{\infty} \phi_k \delta(x - k\Delta x),$$

to also be periodic with period $p$,

$$\phi(x + p) = \phi(x),$$

it is necessary that there be a positive integer, $N$, called the index period, such that

$$\phi_{k+N} = \phi_k \quad \text{for all } k.$$

The index period, spacing, and period of $\phi(x)$ are related by

$$\text{period of } \phi(x) = (\text{index period of } \phi(x)) \times (\text{spacing of } \phi(x)).$$

**Example 2.3.11.1**

The regular periodic array,

$$f(t) = \sum_{k=-\infty}^{\infty} f_k \delta(t - k\Delta t),$$

with spacing $\Delta t = 1/2$, index period $N = 4$, and $(f_0, f_1, f_2, f_3) = (1, 2, 3, 3)$, is sketched in Figure 2.7. Note that $f_4 = f_0, f_5 = f_1, \ldots$, and that the period of $f(t)$ is $4\Delta t = 2$.

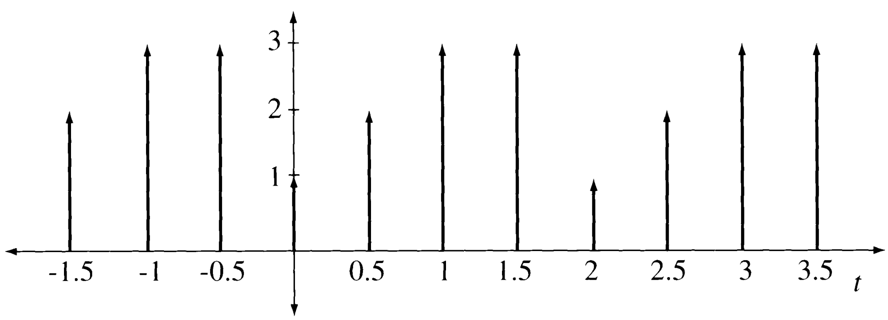


**FIGURE 2.7** A regular periodic array of delta functions.

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Let

$$f(t) = \sum_{k=-\infty}^{\infty} f_k \delta(t - k\Delta t)$$

be a regular periodic array with spacing $\Delta t$, index period $N$, and period $p = N\Delta t$. From the discussion in the subsection on regular arrays, it is evident that the Fourier transform of $f(t)$ is also a regular periodic array of delta functions.

$$F(\omega) = \sum_{n=-\infty}^{\infty} F_n \delta(\omega - n\Delta\omega). \tag{2.3.11.1}$$

Also, $f(t)$ can be expressed as a corresponding Fourier series,

$$f(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} F_n e^{jn\Delta\omega t}. \tag{2.3.11.2}$$

The spacing, $\Delta\omega$, and period, $P$, of $F(\omega)$ are related to the spacing, $\Delta t$, and period, $p$, of $f(t)$ by

$$\Delta\omega = \frac{2\pi}{p} \quad \text{and} \quad P = \frac{2\pi}{\Delta t}.$$

The index period, $M$, of $F(\omega)$ is given by

$$M = \frac{P}{\Delta\omega} = \frac{(2\pi/\Delta t)}{(2\pi/p)} = \frac{p}{\Delta t} = N.$$

Using equation (2.3.10.1),

$$F_n = \frac{2\pi}{p}\int_{t=-\frac{\Delta t}{2}}^{p-\frac{\Delta t}{2}} \left(\sum_{k=-\infty}^{\infty} f_k \delta(t-k\Delta t)\right) e^{-jn\Delta\omega t} dt. \qquad (2.3.11.3)$$

But, as is easily verified,

$$\int_{t=-\frac{\Delta t}{2}}^{p-\frac{\Delta t}{2}} \delta(t-k\Delta t) e^{-jn\Delta\omega t} dt = \begin{cases} e^{-jnk\Delta\omega\Delta t}, & \text{if } 0 \le k \le N-1 \\ 0, & \text{otherwise} \end{cases},$$

and

$$\Delta\omega\,\Delta t = \frac{2\pi\Delta t}{p} = \frac{2\pi}{N}.$$

Thus, equation (2.3.11.3) reduces to

$$F_n = \frac{2\pi}{N\Delta t}\sum_{k=0}^{N-1} f_k e^{-j\frac{2\pi}{N}nk}. \qquad (2.3.11.4)$$

A similar set of calculations yields the inverse relation,

$$f_k = \frac{1}{N\Delta\omega}\sum_{n=0}^{N-1} F_n e^{j\frac{2\pi}{N}kn}. \qquad (2.3.11.5)$$

Formulas for the autocorrelation function, $\bar{\rho}_f(t)$, and the power spectrum, $P(\omega)$, follow immediately from the above and the discussion in Subsections 2.3.9 and 2.3.10. They are

$$\bar{\rho}_f(t) = \sum_{k=-\infty}^{\infty} A_k \delta(t-k\Delta t), \qquad (2.3.11.6)$$

where

$$A_k = \frac{1}{N\Delta t}\sum_{m=0}^{N-1} f_m^* f_{m+k}, \qquad (2.3.11.7)$$

and

$$P(\omega) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} |F_n|^2 \delta(\omega - n\Delta\omega). \qquad (2.3.11.8)$$

**Example 2.3.11.2 The Comb Function**

For each $\Delta x > 0$, the corresponding comb function is

$$\text{comb}_{\Delta x}(x) = \sum_{k=-\infty}^{\infty} \delta(x - k\Delta x).$$

with index period $N = 1$ and with the spacing equal to the period, the comb function is the simplest possible nonzero regular periodic array. By the above discussion,

$$F(\omega) = \mathcal{F}[\text{comb}_{\Delta t}(t)]\big|_{\omega}$$

must also be a regular periodic array,

$$F(\omega) = \sum_{n=-\infty}^{\infty} F_n \delta(\omega - n\Delta\omega),$$

where

$$\Delta\omega = \frac{2\pi}{\Delta t}.$$

Because the index period of $F(\omega)$ must also be $N = 1$,

$$F_n = F_0 = \frac{2\pi}{\Delta t} \sum_{k=0}^{0} f_k e^{-j\frac{2\pi}{N}0\cdot k} = \Delta\omega,$$

for all $n$. Combining the last few equations gives

$$\mathcal{F}\left[\text{comb}_{\Delta t}(t)\right]\Big|_{\omega} = \sum_{n=-\infty}^{\infty} \Delta\omega\delta(\omega - n\Delta\omega) = \Delta\omega \,\text{comb}_{\Delta\omega}(\omega),$$

where

$$\Delta\omega = \frac{2\pi}{\Delta t}.$$

From formulas (2.3.11.6), (2.3.11.7), and (2.3.11.8), the average correlation function and the power spectrum for $\text{comb}_{\Delta t}(t)$ are given by

$$\bar{\rho}(t) = \frac{1}{\Delta t} \sum_{k=-\infty}^{\infty} \delta(t - k\Delta t) = \frac{1}{\Delta t} \text{comb}_{\Delta t}(t)$$

and

$$P(\omega) = \frac{\Delta\omega}{\Delta t} \sum_{n=-\infty}^{\infty} \delta(\omega - n\Delta\omega) = \frac{\Delta\omega}{\Delta t} \text{comb}_{\Delta\omega}(\omega).$$

In addition, using equation (2.3.11.2), the comb function can be expressed as a Fourier series,

$$\text{comb}_{\Delta t}(t) = \frac{\Delta\omega}{2\pi} \sum_{n=-\infty}^{\infty} e^{jn\Delta\omega t} = \frac{1}{\Delta t} \sum_{n=-\infty}^{\infty} e^{jn\Delta\omega t}.$$

## 2.3.12 Powers of Variables and Derivatives of Delta Functions

In Example 2.1.3.1 it was shown that, for any real value of $\alpha$,

$$\mathcal{F}[e^{j\alpha t}]\big|_{\omega} = 2\pi\delta(\omega - \alpha).$$

Letting $\alpha = 0$, this gives

$$\mathcal{F}[1]\big|_{\omega} = 2\pi\delta(\omega),$$

and, by symmetry or near equivalence,

$$\mathcal{F}[\delta(t)]\big|_{\omega} = 1.$$

Now, let $n$ be any nonnegative integer. Because, trivially, $x^n = x^n \cdot 1$, it immediately follows from an application of identities (2.2.11.6) through (2.2.11.9) that

$$\mathcal{F}[t^n]\big|_{\omega} = j^n 2\pi\delta^{(n)}(\omega) \tag{2.3.12.1}$$

$$\mathcal{F}^{-1}[\omega^n]\big|_{t} = (-j)^n \delta^{(n)}(t), \tag{2.3.12.2}$$

$$\mathcal{F}[\delta^{(n)}(t)]\big|_{\omega} = (j\omega)^n, \tag{2.3.12.3}$$

and

$$\mathcal{F}^{-1}\left[\delta^{(n)}(\omega)\right]\Big|_{t} = \frac{(-jt)^n}{2\pi}, \tag{2.3.12.4}$$

where $\delta^{(n)}(x)$ is the $n$th (generalized) derivative of the delta function.

## 2.3.13 Negative Powers and Step Functions

The basic relation between step functions and negative powers is

$$\mathcal{F}[\text{sgn}(t)]\Big|_{\omega} = -j\frac{2}{\omega}, \tag{2.3.13.1}$$

where $\text{sgn}(t)$ is the signum function,

$$\operatorname{sgn}(t) = \begin{cases} -1, & \text{if } t < 0 \\ +1, & \text{if } 0 < t \end{cases} .$$

Because the step function,

$$u(t) = \begin{cases} 0, & \text{if } t < 0 \\ 1, & \text{if } 0 < t \end{cases} ,$$

can be written in terms of the signum function,

$$u(t) = \frac{1}{2}\left[\operatorname{sgn}(t) + 1\right],$$

formula (2.3.13.1) is equivalent to

$$\mathcal{F}\left[u(t)\right]\Big|_{\omega} = \pi\delta(\omega) - j\frac{1}{\omega} . \tag{2.3.13.2}$$

A number of useful formulas can be easily derived from equations (2.3.13.1) and (2.3.13.2) with the aid of various identities from the identities in Subsection 2.2. Some of these formulas are

$$\mathcal{F}\left[\frac{1}{t}\right]\Bigg|_{\omega} = -j\pi \operatorname{sgn}(\omega) , \tag{2.3.13.3}$$

$$\mathcal{F}\left[t^{-n}\right]\Big|_{\omega} = -j\pi\frac{(-j\omega)^{n-1}}{(n-1)!}\operatorname{sgn}(\omega) , \tag{2.3.13.4}$$

$$\mathcal{F}\left[|t|\right]\Big|_{\omega} = -\frac{2}{\omega^2} , \tag{2.3.13.5}$$

$$\mathcal{F}\left[t^{-n}\operatorname{sgn}(t)\right]\Big|_{\omega} = (-j)^{n+1}\frac{2n!}{\omega^{n+1}} , \tag{2.3.13.6}$$

$$\mathcal{F}\left[\operatorname{ramp}(t)\right]\Big|_{\omega} = j\pi\delta'(\omega) - \frac{1}{\omega^2} , \tag{2.3.13.7}$$

and

$$\mathcal{F}\left[t^n u(t)\right]\Big|_{\omega} = j^n\pi\delta^{(n)}(\omega) + n!\left(\frac{-j}{\omega}\right)^{n+1} . \tag{2.3.13.8}$$

In these formulas $n$ denotes an arbitrary positive integer.

Derivations of formulas (2.3.13.1) and (2.3.13.2) are easily obtained. One derivation starts with the observation that, for any $\alpha < 0$,

$$u(t) = \int_{\alpha}^{t} \delta(s)\,ds.$$

By identity (2.2.13.4), with $f(t) = \delta(t)$ and $F(\omega) = \mathcal{F}[\delta(t)]\big|_{\omega} = 1$,

$$\begin{aligned} \mathcal{F}[u(t)]\Big|_{\omega} &= \mathcal{F}\left[\int_{\alpha}^{t} f(s)\,ds\right]\Bigg|_{\omega} \\ &= -j\frac{F(\omega)}{\omega} + c\delta(\omega) \qquad (2.3.13.9) \\ &= -j\frac{1}{\omega} + c\delta(\omega), \end{aligned}$$

where $c$ is some constant. From this

$$\begin{aligned} \mathcal{F}[\operatorname{sgn}(t)]\Big|_{\omega} &= \mathcal{F}[2u(t)-1]\Big|_{\omega} \\ &= 2\left[-j\frac{1}{\omega} + c\delta(\omega)\right] - 2\pi\delta(\omega) \qquad (2.3.13.10) \\ &= -j\frac{2}{\omega} + 2(c-\pi)\delta(\omega). \end{aligned}$$

Because $\operatorname{sgn}(t)$ is an odd function, so is $\mathcal{F}[\operatorname{sgn}(t)]\big|_{\omega}$ and, hence, so is the right-hand side of equation (2.3.13.10). But, because the delta function is even, this is possible only if $c = \pi$. Plugging this only possible choice for $c$ into equations (2.3.13.9) and (2.3.13.10) gives formulas (2.3.13.1) and (2.3.13.2).

**Example 2.3.13.1 Derivation of Formulas (2.3.13.6) and (2.3.13.5)**

Using identity (2.2.11.8),

$$\begin{aligned} \mathcal{F}[t^n \operatorname{sgn}(t)]\Big|_{\omega} &= j^n \frac{d^n}{d\omega^n}\mathcal{F}[\operatorname{sgn}(t)]\Big|_{\omega} \\ &= j^n \frac{d^n}{d\omega^n}\left(-j\frac{2}{\omega}\right) \\ &= (-j)^{n+1}\frac{2n!}{\omega^{n+1}}. \end{aligned}$$

Using this and the observation that

$$|t| = t\,\operatorname{sgn}(t),$$

it immediately follows that

$$\mathcal{F}[|t|]\Big|_{\omega} = \mathcal{F}[t\operatorname{sgn}(t)]\Big|_{\omega} = (-j)^{1+1}\frac{2(1!)}{\omega^{1+1}} = -\frac{2}{\omega^2}.$$

One technical flaw in the above discussion should be noted. If $\phi(x)$ is any function continuous at $x = 0$, and $n \geq 1$, then, from a strict mathematical point of view, the function $x^{-n}\phi(x)$ is not integrable over any interval containing $x = 0$. Because of this, it is not possible to define $\mathcal{F}[t^{-n}]\big|_{\omega}$ or $\mathcal{F}^{-1}[\omega^{-n}]\big|_{t}$ via the classical integral formulas. Neither is it possible for the function $x^{-n}$ to be treated as a generalized function. However, the function $\ln|x|$ is integrable over any finite interval and can be treated as a legitimate generalized function, as can any of its generalized derivatives (as defined in Subsection 2.2.11). It is possible to justify rigorously the formulas given in this subsection, as well as any other standard use of $x^{-n}$, by agreeing that $x^{-1}$ is actually a symbol for the generalized derivative of $\ln|x|$, and that, more generally, for any positive integer $n$, $x^{-n}$ is a symbol for

$$\frac{(-1)^{n-1}}{(n-1)!}\frac{d^n}{dx^n}\ln|x|$$

where the derivatives are taken in the generalized sense as described in Subsection 2.2.11.

### 2.3.14 Rational Functions

Rational functions often turn out to be the transforms of functions of interest. The simplest nontrivial rational function is given by

$$F(\omega) = \frac{1}{(\omega - \lambda)^m},$$

where $m$ is a positive integer and $\lambda$ is some complex constant. Using the elementary identities and the material from the previous subsection, it can be directly verified that

$$\mathcal{F}^{-1}\left[\frac{1}{(\omega-\lambda)^m}\right]\Bigg|_t = j\frac{(jt)^{m-1}}{(m-1)!}e^{j\lambda t}\Gamma_\alpha(t) \tag{2.3.14.1}$$

where $\alpha$ is the imaginary part of $\lambda$ and

$$\Gamma_a(t) = \begin{cases} u(t), & \text{if } 0 < \alpha \\ \frac{1}{2}\operatorname{sgn}(t), & \text{if } \alpha = 0. \\ -u(-t), & \text{if } \alpha < 0 \end{cases}$$

More generally, if $F(\omega)$ is any rational function, then $F(\omega)$ can be written

$$F(\omega) = P(\omega) + R(\omega),$$

where $P(\omega)$ is a polynomial,

$$P(\omega) = \sum_{n=0}^{N} c_n \omega^n,$$

and $R(\omega)$ is the quotient of two polynomials,

$$R(\omega) = \frac{N(\omega)}{D(\omega)},$$

in which the degree of the numerator is strictly less than the degree of the denominator. According to formula (2.3.12.1), the inverse transform of $P(\omega)$ is simply a linear combination of derivatives of delta functions.

$$\mathcal{F}^{-1}\left[P(\omega)\right]\Big|_t = \sum_{n=0}^{N} (-j)^n c_n \delta^{(n)}(t).$$

Letting $\lambda_1, \lambda_2, \ldots, \lambda_K$ be the distinct roots of $D(\omega)$ and $M_1, M_2, \ldots, M_K$ the corresponding multiplicities of the roots, $R(\omega)$, can be written in the partial fraction expansion,

$$R(\omega) = \sum_{k=1}^{K} \sum_{m=1}^{M_k} \frac{a_{k,m}}{\left(\omega - \lambda_k\right)^m}.$$

Thus, applying formula (2.3.14.1),

$$\mathcal{F}^{-1}\left[R(\omega)\right]\Big|_t = j \sum_{k=1}^{K} e^{j\lambda_k t} \Gamma_{\alpha_k}(t) \sum_{m=1}^{M_k} a_{k,m} \frac{(jt)^{m-1}}{(m-1)!}, \tag{2.3.14.2}$$

where, for each $k$, $\alpha_k$ is the imaginary part of $\lambda_k$.

Fourier transforms of rational functions can be computed using the same approach as just described for inverse transforms of rational functions.

**Example 2.3.14.1**

Let

$$F(\omega) = \frac{N(\omega)}{D(\omega)} = \frac{5\omega + 9 - 10j}{\omega^2 - 4j\omega - 13}.$$

Using the quadratic formula, the roots of $D(\omega)$ are found to be

$$\lambda = \frac{4j \pm \sqrt{(4j)^2 + 4(13)}}{2} = \pm 3 + 2j.$$

$F(\omega)$ can then be expanded

$$F(\omega) = \frac{5\omega + 9 - 10j}{\omega^2 - 4j\omega - 13} = \frac{A}{\omega - (3+2j)} + \frac{B}{\omega - (-3+2j)}.$$

Solving for $A$ and $B$ gives

$$F(\omega) = \frac{4}{\omega - (3+2j)} + \frac{1}{\omega - (-3+2j)},$$

whose inverse transform can be computed directly from formula (2.3.14.2),

$$\begin{aligned} f(t) &= j\left[4e^{j(3+2j)t}\Gamma_2(t) + e^{j(-3+2j)t}\Gamma_2(t)\right] \\ &= 4je^{(-2+3j)t}u(t) + je^{(-2-3j)t}u(t) \\ &= j\left[4e^{j3t} + e^{-j3t}\right]e^{-2t}u(t). \end{aligned}$$

## 12.3.15 Causal Functions

A function, $f(t)$, is said to be "causal" if

$$f(t) = 0 \quad \text{whenever } t < 0.$$

Such functions arise in the study of causal systems and are of obvious importance in describing phenomena that have well-defined "starting points."

Let $f(t)$ be a real causal function with Fourier transform $F(\omega)$, and let $R(\omega)$ and $I(\omega)$ be the real and imaginary parts of $F(\omega)$,

$$F(\omega) = R(\omega) + jI(\omega).$$

Then $R(\omega)$ is even, $I(\omega)$ is odd, and, provided the integrals are suitably well defined,

$$f(t) = \frac{2}{\pi}\int_0^\infty R(\omega)\cos(\omega t)\,d\omega \quad \text{for } 0 < t, \tag{2.3.15.1}$$

$$f(t) = -\frac{2}{\pi}\int_0^\infty I(\omega)\sin(\omega t)\,d\omega \quad \text{for } 0 < t, \tag{2.3.15.2}$$

$$\int_0^\infty |f(t)|^2\,dt = \frac{1}{\pi}\int_{-\infty}^\infty |R(\omega)|^2\,d\omega, \tag{2.3.15.3}$$

and

$$\int_0^\infty |f(t)|^2\,dt = \frac{1}{\pi}\int_{-\infty}^\infty |I(\omega)|^2\,d\omega, \tag{2.3.15.4}$$

In addition, if $f(t)$ is bounded at the origin, then provided the integrals exist,

$$R(\omega) = \frac{1}{\pi}\int_{-\infty}^\infty \frac{I(s)}{\omega - s}\,ds \tag{2.3.15.5}$$

and

$$I(\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{R(s)}{\omega - s}\,ds. \qquad (2.3.15.6)$$

The last two integrals are Hilbert transforms and may be defined using Cauchy principal values (see Subsection 2.1.6).

Conversely, it can be shown that $R(\omega)$ and $I(\omega)$ are real-valued functions (with $R(\omega)$ even and $I(\omega)$ odd) satisfying either (2.3.15.1) or (2.3.15.6), then

$$f(t) = \mathcal{F}^{-1}[R(\omega) + jI(\omega)]\big|_t$$

must be a causal function.

Derivations of equations (2.3.15.1) through (2.3.15.6) are quite straightforward. First, observe that because $f(t)$ vanishes for negative values of $t$, then

$$f(t) = 2f_e(t) = 2f_o(t) \quad \text{for } 0 < t,$$

where $f_e(t)$ and $f_o(t)$ are the even and odd components of $f(t)$. Equations (2.3.15.1) and (2.3.15.2) then follow immediately from equations (2.3.1.4) and (2.3.1.5), while equations (2.3.15.3) and (2.3.15.4) are simply Bessel's equality combined with equations from Subsection 2.3.1 and the subsequent observation that

$$\int_0^{\infty} |f(t)|^2\,dt = 4\int_0^{\infty} |f_e(t)|^2\,dt = 2\int_{-\infty}^{\infty} |f_e(t)|^2\,dt = 2\int_{-\infty}^{\infty} |f_o(t)|^2\,dt\,.$$

Finally, for equation (2.3.15.5) observe that

$$f_e(t) = f_o(t)\ \mathrm{sgn}(t) \quad \text{and} \quad f_o = f_e(t)\ \mathrm{sgn}(t).$$

Thus, using results from Subsections 2.3.1, 2.2.9, and 2.3.13,

$$\begin{aligned}
R(\omega) &= \mathcal{F}\left[f_e(t)\right]\Big|_\omega \\
&= \mathcal{F}\left[f_o(t)\,\mathrm{sgn}(t)\right]\Big|_\omega \\
&= \frac{1}{2\pi}\mathcal{F}\left[f_o(t)\right]\Big|_\omega * \mathcal{F}\left[\mathrm{sgn}(t)\right]\Big|_\omega \\
&= \frac{1}{2\pi} jI(\omega) * \left(-j\frac{2}{\omega}\right) \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{I(s)}{\omega - s}\,ds\,,
\end{aligned}$$

which is equation (2.3.15.5). Similar computations yield (2.3.15.6).

### Example 2.3.15.1

Assume $f(t)$ is a causal function whose transform, $F(\omega)$, has real part

$$R(\omega) = \delta(\omega - \alpha) + \delta(\omega + \alpha),$$

for some $\alpha > 0$. Then, according to formula (2.3.15.1), for $t > 0$

$$f(t) = \frac{2}{\pi}\int_0^\infty [\delta(\omega - \alpha) + \delta(\omega + \alpha)]\cos(\omega t)\,d\omega = \frac{2}{\pi}\cos(\alpha t),$$

and by formula (2.3.15.6),

$$\begin{aligned} I(\omega) &= -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{[\delta(s-\alpha) + \delta(s+\alpha)]}{\omega - s}\,ds \\ &= -\left[\frac{1}{\pi(\omega - \alpha)} + \frac{1}{\pi(\omega + \alpha)}\right] \\ &= \frac{2\omega}{\omega(\alpha^2 - \omega^2)}. \end{aligned}$$

Thus,

$$f(t) = \frac{2}{\pi}\cos(\alpha t)u(t)$$

and

$$F(\omega) = \delta(\omega - \alpha) + \delta(\omega + \alpha) + j\frac{2\omega}{\pi(\alpha^2 - \omega^2)}.$$

## 2.3.16 Functions on the Half-Line

Strictly speaking, functions defined only on the half-line, $0 < t < \infty$, do not have Fourier transforms. Fourier analysis in problems involving such functions can be done by first extending the functions (i.e., systematically defining the values of the functions at negative values of $t$), and then taking the Fourier transforms of the extensions. The choice of extension will depend on the problem at hand and the preferences of the individual. Three of the most commonly used extensions are the null extension, the even extension, and the odd extension. Given a function, $f(t)$, defined only for $0 < t$, the null extension is

$$f_{\text{null}}(t) = \begin{cases} f(t), & \text{if } 0 < t \\ 0, & \text{if } t < 0 \end{cases},$$

The even extension is

$$f_{\text{even}}(t) = \begin{cases} f(t), & \text{if } 0 < t \\ f(-t), & \text{if } t < 0 \end{cases},$$

and the odd extension is

$$f_{\text{odd}}(t) = \begin{cases} f(t), & \text{if } 0 < t \\ -f(-t), & \text{if } t < 0 \end{cases}.$$

If $f(t)$ is reasonably well behaved (say, continuous and differentiable) on $0 < t$, then any of the above extensions will be similarly well behaved on both $0 < t$ and $t < 0$. At $t = 0$, however, the extended function is likely to have singularities that must be taken into account, especially if transforms of the derivatives are to be taken. It is recommended that the generalized derivative be explicitly used. Assume, for example, that $f(t)$ and its first two derivatives are continuous on $0 < t$, and that the limits

$$f(0) = \lim_{t \to 0^+} f(t) \quad \text{and} \quad f'(0) = \lim_{t \to 0^+} f'(t)$$

exist. Let $\hat{f}(t)$ be any of the above extensions of $f(t)$, and, for convenience, let $d\hat{f}/dt$ and $D\hat{f}$ denote, respectively, the classical and generalized derivatives of $\hat{f}(t)$. Recalling the relation between the classical and generalized derivatives (see Subsection 2.2.11),

$$D\hat{f} = \frac{d\hat{f}}{dt} + J_0 \delta(t)$$

and

$$D^2 \hat{f} = \frac{d^2 \hat{f}}{dt^2} + J_0 \delta'(t) + J_1 \delta(t),$$

where $J_0$ and $J_1$ are the "jumps" in $\hat{f}(t)$ and $\hat{f}'(t)$ at $t = 0$,

$$J_0 = \lim_{t \to 0^+} \left[ \hat{f}(t) - \hat{f}(-t) \right]$$

and

$$J_1 = \lim_{t \to 0^+} \left[ \hat{f}'(t) - \hat{f}'(-t) \right].$$

Computing these jumps for the extensions yield the following:

$$Df_{\text{null}} = \frac{df_{\text{null}}}{dt} + f(0)\delta(t), \tag{2.3.16.1}$$

$$D^2 f_{\text{null}} = \frac{d^2 f_{\text{null}}}{dt^2} + f(0)\delta'(t) + f'(0)\,\delta(t), \tag{2.3.16.2}$$

$$Df_{\text{even}} = \frac{df_{\text{even}}}{dt}, \tag{2.3.16.3}$$

$$D^2 f_{\text{even}} = \frac{d^2 f_{\text{even}}}{dt^2} + 2 f'(0)\delta(t), \tag{2.3.16.4}$$

$$Df_{\text{odd}} = \frac{df_{\text{odd}}}{dt} + 2f(0)\delta(t), \tag{2.3.16.5}$$

and

$$D^2 f_{\text{odd}} = \frac{d^2 f_{\text{odd}}}{dt^2} + 2f(0)\delta'(t). \tag{2.3.16.6}$$

An example of the use of Fourier transforms in problems on the half-line is given in Subsection 2.8.4. This example also illustrates how the data in the problem determine the appropriate extension for the problem.

## 2.3.17 Functions on Finite Intervals

If a function, $f(t)$, is defined only on a finite interval, $0 < t < L$, then it can be expanded into any of a number of "Fourier series" over the interval. These series equal $f(t)$ over the interval but are defined on the entire real line. Thus, each series corresponds to a particular extension of $f(t)$ to a function defined for all real values of $t$, and, with care, Fourier analysis can be done using the series in place of the original functions. Among the best known "Fourier series" for such functions are the sine series and the cosine series.

The sine series for $f(t)$ over $0 < t < L$ is

$$S[f]\big|_t = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi t}{L}\right),$$

where

$$b_k = \frac{2}{L}\int_0^L f(t)\sin\left(\frac{k\pi t}{L}\right)dt .$$

This series can be viewed as an odd periodic extension of $f(t)$. The Fourier transform of the sine series is

$$\begin{aligned}\mathcal{F}\left[S[f]\big|_t\right]\Big|_\omega &= j\pi\sum_{k=1}^{\infty} b_k\left[\delta\left(\omega + \frac{k\pi}{L}\right) - \delta\left(w - \frac{k\pi}{L}\right)\right] \\ &= \sum_{k=-\infty}^{\infty} B_k \delta\left(\omega - \frac{k\pi}{L}\right),\end{aligned}$$

where

$$B_k = \begin{cases} -j\pi b_k, & \text{if } 0 < k \\ 0, & \text{if } k = 0 \\ j\pi b_{-k}, & \text{if } k < 0 \end{cases}.$$

The cosine series for $f(t)$ over $0 < t < L$ is

$$C[f]\Big|_t = a_0 + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi t}{L}\right),$$

where

$$a_0 = \frac{1}{L}\int_0^L f(t)\,dt$$

and, for $k \neq 0$,

$$a_k = \frac{2}{L}\int_0^L f(t)\cos\left(\frac{k\pi t}{L}\right)dt\,.$$

This series can be viewed as an even periodic extension of $f(t)$. The Fourier transform of the cosine series is

$$\begin{aligned}
\mathscr{F}\left[C[f]\Big|_t\right]\Big|_\omega &= 2\pi a_0 \delta(\omega) + \pi\sum_{k=1}^{\infty} a_k\left[\delta\left(\omega - \frac{k\pi}{L}\right) + \delta\left(\omega + \frac{k\pi}{L}\right)\right] \\
&= \sum_{k=-\infty}^{\infty} A_k \delta\left(\omega - \frac{k\pi}{L}\right),
\end{aligned}$$

where

$$A_k = \begin{cases} \pi a_k, & \text{if } 0 < k \\ 2\pi a_0, & \text{if } k = 0 \\ \pi a_{-k}, & \text{if } k < 0 \end{cases}.$$

The choice of which series to use depends strongly on the actual problem at hand. For example, because the sine functions in the sine series expansion vanish at $t = 0$ and $t = L$, sine series expansions tend to be most useful when the functions of interest are to vanish at both of the end points of the interval. For problems in which the first derivatives are expected to vanish at both end points, the cosine series tends to be a better choice. Other boundary conditions suggest other choices for the appropriate Fourier series. In addition, the equations to be satisfied must be considered in choosing the series to be used. Unfortunately, the development of a reasonably complete criteria for choosing the appropriate “Fourier series” in general goes beyond the scope of this chapter. It is recommended that texts covering eigenfunction expansions and Sturm-Liouville problems be consulted.*

## 2.3.18 Bessel Functions

### Solutions to Bessel’s Equations

For $\nu \geq 0$, the $\nu$th order Bessel equation can be written as

$$t^2 y'' + t y' + (t^2 - \nu^2) y = 0. \qquad (2.3.18.1)$$

*See, for example, *Elementary Differential Equations and Boundary Value Problems* by Boyce and DiPrima, *Applied Analysis by the Hilbert Space Method* by Holland, or *Partial Differential Equations and Boundary-Value Problems with Applications* by Pinsky.

"Power series" solutions to this equation can be found using the method of Frobenius. From these solutions, it can be shown that the general real-valued solution to this equation on $t > 0$ is

$$y(t) = c_1 J_\nu(t) + c_2 y_2(t)$$

where $c_1$ and $c_2$ are arbitrary real constants, $J_\nu$ is the $\nu$th order Bessel function of the first kind (which is a bounded function),* and $y_2$ is any particular real-valued solution to the Bessel equation on $t > 0$ that is unbounded near $t = 0$.

Typically, one is most interested in the bounded function part of the solution to Bessel's equation, $c_1 J_\nu$.

## Zero Order Bessel Functions

For now let $\nu = 0$. Equation (2.3.18.1) then simplifies to

$$ty'' + y' + ty = 0. \tag{2.3.18.2}$$

Its solution on $t > 0$ is

$$y(t) = c_1 J_0(t) + c_2 y_2(t)$$

It is easily verified that the power series formula for $J_0(t)$ actually defines $J_0(t)$ as an even, analytic function on the entire real line, and that $J_0(t)$ satisfies equation (2.3.18.2) everywhere. It is also easily verified from the series formula for $y_2(t)$ on $t > 0$ that $y_2(|t|)$ is an even function satisfying equation (2.3.18.2) for all nonzero values of $t$ and which behaves like $\ln |t|$ near $t = 0$. Consequently, we can seek the Fourier transform of

$$y(t) = c_1 J_0(t) + c_2 y_2(|t|) \tag{2.3.18.3}$$

for any pair $c_1$ and $c_2$ by treating $J_0(t)$ and $y_2(|t|)$ as even, real-valued solutions to the Bessel equation of order zero on the real line.

Taking the Fourier transform of equation (2.3.18.2) and using the differential identities of Subsection 2.2.11 results in the first order linear equation

$$(1 - \omega^2) Y'(\omega) - \omega Y(\omega) = 0 \tag{2.3.18.4}$$

where $Y = \mathcal{F}[y]$. The general classical solution to this equation is easily obtained via standard methods for linear, first order differential equations. Taking into account the possible discontinuities at $\omega = \pm 1$, this general solution is given by

$$Y(\omega) = \begin{cases} A\left(\omega^2 - 1\right)^{-\frac{1}{2}} & \text{if } \omega < -1 \\ B\left(1 - \omega^2\right)^{-\frac{1}{2}} & \text{if } -1 < \omega < 1 \\ C\left(\omega^2 - 1\right)^{-\frac{1}{2}} & \text{if } 1 < \omega \end{cases} \tag{2.3.18.5}$$

where $A$, $B$, and $C$ are "arbitrary" constants. However, here $Y(\omega)$ must be even and real valued since it is the Fourier transform of an even, real-valued function. This forces $A$, $B$, and $C$ to be real constants with $A = C$. Thus,

*An overview of Bessel functions of the first kind is given in the first chapter of this book.

$$Y(\omega) = \begin{cases} B\left(1-\omega^2\right)^{-\frac{1}{2}} & \text{if } |\omega| < 1 \\ C\left(\omega^2-1\right)^{-\frac{1}{2}} & \text{if } 1 < |\omega| \end{cases},$$

or, equivalently,

$$Y(\omega) = BY_1(\omega) + CY_2(\omega)$$

where

$$Y_1(\omega) = \frac{1}{\sqrt{1-\omega^2}}\, p_1(\omega)$$

and

$$Y_2(\omega) = \frac{1}{\sqrt{\omega^2-1}}\left[1 - p_1(\omega)\right].$$

The function $Y_1(\omega)$ is absolutely integrable in addition to being real valued and even. Consequently, $\mathcal{F}^{-1}[Y_1]$ is a bounded, real-valued, even function to Bessel's equation of order zero. Thus,

$$\mathcal{F}^{-1}[Y_1(\omega)]\big|_t = c_1\, J_0(t)$$

for some nonzero constant $c_1$. Conversely, then, there must be a value $B_0$ such that $J_0 = B_0\mathcal{F}^{-1}[Y_1]$. To find this value, first recall that $J_0(0) = 1$ (see the first chapter of this Handbook) and that, by elementary calculus,

$$\mathcal{F}^{-1}\left[Y_1(\omega)\right]\Big|_0 = \frac{1}{2\pi}\int_{-1}^{1} \frac{1}{\sqrt{1-\omega^2}}\, d\omega = \frac{1}{2}.$$

Thus,

$$1 = J_0(0) = B_0\mathcal{F}^{-1}\left[Y_1(\omega)\right]\Big|_0 = \frac{B_0}{2},$$

which, in turn, means that $B_0 = 2$,

$$J_0(t) = \mathcal{F}^{-1}\left[2Y_1(\omega)\right]\Big|_t = \mathcal{F}^{-1}\left[\frac{2}{\sqrt{1-\omega^2}}\, p_1(\omega)\right]\Bigg|_t, \qquad (2.3.18.6)$$

and

$$\mathcal{F}\left[J_0(t)\right]\Big|_\omega = 2Y_1(\omega) \frac{2}{\sqrt{1-\omega^2}}\, p_1(\omega). \qquad (2.3.18.7)$$

The function $Y_2(\omega)$ is not absolutely integrable, but it is the sum of a function that is absolutely integrable,

$$\frac{1}{\sqrt{\omega^2-1}}\left[1-p_1(\omega)\right]p_2(\omega),$$

with a function that is square integrable,

$$\frac{1}{\sqrt{\omega^2-1}}\left[1-p_1(\omega)\right]\left[1-p_2(\omega)\right].$$

From this it follows that $Y_2$ is Fourier transformable in the more general sense described in Subsection 2.1.3 and that its inverse transform is a function in the classical sense. The inverse transform of this function can be used for $y_2$, the unbounded part of equation (2.3.18.3). A more standard choice, however, is to use $y_2 = Y_0$ where

$$Y_0(t) = \mathcal{F}^{-1}\left[-2Y_2(\omega)\right]\Big|_t = \mathcal{F}^{-1}\left[\frac{-2}{\sqrt{\omega^2-1}}\left[1-p_1(\omega)\right]\right]\Bigg|_t \quad \text{for } t>0.$$

This, $Y_0$, is the 0th order Bessel function of the second kind.

### Integral Order Bessel Functions

As with $J_0$, the series formula for each integral order Bessel function of the first kind $J_n$ actually defines $J_n$ as a bounded analytic function on the entire real line whenever $n$ is any positive integer. Consequently, Fourier transforms for these Bessel functions exist and are well defined using, at least, the more general definitions of Subsection 2.1.3. The formulas for these transforms can be obtained using the above formula for $\mathcal{F}[J_0]$, the differentiation identities, and well-known recursion formulas for the Bessel functions (again, see the first chapter of this Handbook).

In particular, since

$$J_1(t) = -J_0{}'(t),$$

we have

$$\mathcal{F}[J_1(t)]\big|_\omega = -\mathcal{F}[J_0{}'(t)]\big|_\omega = -j\omega\mathcal{F}[J_0(t)]\big|_\omega.$$

Combined with equation (2.3.18.7), this gives

$$\mathcal{F}\left[J_1(t)\right]\Big|_\omega = \frac{-2j\omega}{\sqrt{1-\omega^2}}\,p_1(\omega). \tag{2.3.18.8}$$

The Fourier transforms of $J_2(t)$, $J_3(t)$, … can be obtained in a similar fashion using formulas (2.3.18.7) and (2.3.18.8), a differentiation identity, and the recursion formulas

$$J_{\nu+1}(t) = J_{\nu-1}(t) - 2J_\nu{}'(t).$$

The results of these computations can be succinctly described by the formulas

$$\mathcal{F}\left[J_m(t)\right]\Big|_\omega = \frac{2\cos\left[m\arcsin(\omega)\right]}{\sqrt{1-\omega^2}}\,p_1(\omega) \quad \text{for} \quad m = 0, 2, 4, \ldots$$

and

$$\mathcal{F}\left[J_m(t)\right]\Big|_\omega = \frac{-2j\sin\left[m\arcsin(\omega)\right]}{\sqrt{1-\omega^2}}\,p_1(\omega) \quad \text{for} \quad m = 1, 3, 5, \ldots.$$

They can be described even more succinctly by

$$\mathcal{F}\left[J_n(t)\right]\Big|_\omega = \frac{2(-j)^n\,T_n(\omega)}{\sqrt{1-\omega^2}}\,p_1(\omega) \quad \text{for} \quad n = 0, 1, 2, 3, \ldots$$

where $T_n$ is the $n$th Chebyshev polynomial of the first kind.

The derivation of another useful set of identities starts with the observation that

$$\frac{-2j\omega}{\sqrt{1-\omega^2}}\,p_1(\omega) = 2j\frac{d}{d\omega}\left[\sqrt{1-\omega^2}\,p_1(\omega)\right].$$

Combining this with equation (2.3.18.8)

$$\begin{aligned}
J_1(t) &= \mathcal{F}^{-1}\left[\frac{-2j\omega}{\sqrt{1-\omega^2}}\,p_1(\omega)\right]\Bigg|_t \\
&= 2j\mathcal{F}^{-1}\left[\frac{d}{d\omega}\left[\sqrt{1-\omega^2}\,p_1(\omega)\right]\right]\Bigg|_t \\
&= 2j\left(-jt\mathcal{F}^{-1}\left[\sqrt{1-\omega^2}\,p_1(\omega)\right]\Big|_t\right) \\
&= 2t\mathcal{F}^{-1}\left[\sqrt{1-\omega^2}\,p_1(\omega)\right]\Big|_t .
\end{aligned}$$

Dividing by $t$ (which is valid since $J_1(t) \approx \frac{t}{2}$ when $t \approx 0$) then yields

$$t^{-1}J_1(t) = \mathcal{F}^{-1}\left[2\sqrt{1-\omega^2}\,p_1(\omega)\right]\Big|_t .$$

Equivalently,

$$\mathcal{F}\left[t^{-1}J_1(t)\right]\Big|_\omega = 2\sqrt{1-\omega^2}\,p_1(\omega).$$

Continuing these computations eventually leads to the equivalent identities

$$t^{-n} J_n(t) = \mathcal{F}^{-1}\left[ \frac{2\left(1-\omega^2\right)^{n-\frac{1}{2}}}{1\cdot 3\cdot 5\cdots(2n-1)} p_1(\omega) \right]\Bigg|_t \tag{2.3.18.9}$$

and

$$\mathcal{F}\left[t^{-n} J_n(t)\right]\Big|_\omega = \frac{2\left(1-\omega^2\right)^{n-\frac{1}{2}}}{1\cdot 3\cdot 5\cdots(2n-1)} p_1(\omega) . \tag{2.3.18.10}$$

These identities are valid for nonnegative, integral values of $n$ (and reduce to equations (2.3.18.6) and (2.3.18.7) when $n = 0$).

### Nonintegral Order Bessel Functions

Solving equation (2.3.18.9) for $J_n(t)$ and using the fact that, in terms of the gamma function,

$$1\cdot 3\cdot 5\cdots(2n-1) = \frac{2^n}{\sqrt{\pi}} \Gamma\left(n+\frac{1}{2}\right)$$

results in the following formulas:

$$J_n(t) = A_n t^n \mathcal{F}^{-1}\left[ \left(1-\omega^2\right)^{n-\frac{1}{2}} p_1(\omega) \right]\Bigg|_t \tag{2.3.18.11}$$

where

$$A_n = \frac{2^{1-n}\sqrt{\pi}}{\Gamma\left(n+\frac{1}{2}\right)} .$$

This formula for $J_n$ was obtained assuming $n$ is any nonnegative integer. However, for $t > 0$, the right hand side of equation (2.3.18.11) remains well defined when $n$ is any real value greater than –1/2. Moreover, through straightforward but somewhat tedious computations, it can be verified that the formula on the right hand side of equation (2.3.18.11) satisfies Bessel's equation of order $|n|$ on $t > 0$, and is asymptotically identical to $J_n(t)$ when $t \to 0^+$. It thus follows that, for any $\mu > -1/2$,

$$J_\mu(t) = A_\mu t^\mu \mathcal{F}^{-1}\left[ \left(1-\omega^2\right)^{\mu-\frac{1}{2}} p_1(\omega) \right]\Bigg|_t$$

where

$$A_\mu = \frac{2^{1-\mu}\sqrt{\pi}}{\Gamma\left(\mu + \frac{1}{2}\right)} .$$

Consequently, since $\omega^2$ and $p_1(\omega)$ are even functions,

$$\mathcal{F}^{-1}\left[\left(1-\omega^2\right)^{\mu-\frac{1}{2}} p_1(\omega)\right]\Big|_t = \frac{\Gamma\left(\mu+\frac{1}{2}\right)}{2\sqrt{\pi}}\left(\frac{2}{|t|}\right)^{\mu} J_\mu(|t|),$$

and, by near-equivalence,

$$\mathcal{F}\left[\left(1-t^2\right)^{\mu-\frac{1}{2}} p_1(t)\right]\Big|_\omega = \Gamma\left(\mu+\frac{1}{2}\right)\sqrt{\pi}\left(\frac{2}{|\omega|}\right)^{\mu} J_\mu(|\omega|),$$

whenever $\mu > -1/2$.

## 2.4 Extensions of the Fourier transform and Other Closely Related Transforms

A number of applications call for transforms that are closely related to the Fourier transform. This section presents a brief survey and development of some of the transforms having a particularly close relation to the Fourier transform. Many of them, in fact, can be viewed as natural modifications or direct extensions of the transforms defined and developed in the previous sections, or else are special cases of these modifications and extensions.

### 2.4.1 Multidimensional Fourier Transforms

The extension of Fourier analysis to handle functions of more than one variable is quite straightforward. Assuming the functions are suitably integrable, the Fourier transform of $f(x, y)$ is

$$F(\omega, \nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)e^{-j(\omega x + \nu y)}\,dx\,dy$$

and the Fourier transform of $f(x, y, z)$ is

$$F(\omega, \nu, \mu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y, z)e^{-j(\omega x + \nu y + \mu z)}\,dx\,dy\,dz .$$

More generally, using vector notation with $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ and $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots \omega_n)$, the "$n$-dimensional Fourier transform" is defined by

$$\mathcal{F}[f(\mathbf{t})]\Big|_{\boldsymbol{\omega}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(\mathbf{t})e^{-j\boldsymbol{\omega}\cdot\mathbf{t}}\,dt_1 dt_2 \ldots dt_n , \qquad (2.4.1.1)$$

assuming $f(\mathbf{t})$ is sufficiently integrable. The inverse $n$-dimensional Fourier inverse transform given by

$$\mathcal{F}\left[F(\boldsymbol{\omega})\right]\Big|_{\mathbf{t}} = (2\pi)^{-n}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} F(\boldsymbol{\omega})e^{-j\boldsymbol{\omega}\cdot\mathbf{t}}\,d\omega_1 d\omega_2\ldots d\omega_n\,, \tag{2.4.1.2}$$

provided $F(\boldsymbol{\omega})$ is suitably integrable.

For many functions of $\mathbf{t}$ and $\boldsymbol{\omega}$ that are not suitably integrable, the generalized $n$-dimensional Fourier and inverse Fourier transforms can be defined using the $n$-dimensional analogs of the rapidly decreasing test functions described in Subsections 2.1.3 and 2.1.4.

Analogs to the identities discussed in Section 2.2 can be easily derived for the $n$-dimensional transforms. In particular, $\mathcal{F}$ and $\mathcal{F}^{-1}$ are inverses of each other, that is,

$$F(\boldsymbol{\omega}) = \mathcal{F}[f(\mathbf{t})]\big|_{\omega} \Leftrightarrow \mathcal{F}^{-1}[F(\boldsymbol{\omega})]\big|_{\mathbf{t}} = f(\mathbf{t}).$$

The near equivalence (or symmetry) relations for the $n$-dimensional transforms are

$$\mathcal{F}^{-1}[\phi(\mathrm{x})]\big|_{\mathrm{y}} = (2\pi)^{-n}\,\mathcal{F}[\phi(-\mathrm{x})]\big|\mathrm{z}_{\mathrm{y}} = (2\pi)^{-n}\,\mathcal{F}[\phi(\mathrm{x})]\big|_{-\mathrm{y}}$$

and

$$\mathcal{F}[\phi(-\mathrm{x})]\big|_{\mathrm{y}} = (2\pi)^{n}\,\mathcal{F}^{-1}[\phi(-\mathrm{x})]\big|_{\mathrm{y}} = (2\pi)^{n}\,\mathcal{F}^{-1}[\phi(-\mathrm{x})]\big|_{-\mathrm{y}}$$

An abbreviated listing of identities for the $n$-dimensional transforms are given in Table 2.2. In this table, $\phi(\mathbf{x}) * \psi(\mathbf{x})$ and $\phi(\mathbf{x}) \star \psi(\mathbf{x})$ denote the $n$-dimensional convolution and correlation,

**TABLE 2.2** Identities for Multidimensional Transforms

| $h(\mathbf{t})$ | $H(\boldsymbol{\omega}) = \mathcal{F}[h(\mathbf{t})]\big|_{\omega}$ |
|---|---|
| $f(\alpha\mathbf{t})$ | $\dfrac{1}{\lvert\alpha\rvert}F\left(\dfrac{\boldsymbol{\omega}}{\alpha}\right)$ |
| $f(\mathbf{t}\mathbf{A})$ | $\dfrac{1}{\lvert\mathbf{A}\rvert}F\left(\boldsymbol{\omega}\mathbf{A}^{-T}\right)$ |
| $f(\mathbf{t}-\mathbf{t}_0)$ | $e^{-j\mathbf{t}_0\cdot\omega}F(\boldsymbol{\omega})$ |
| $e^{j\omega_0\cdot\mathbf{t}}f(\mathbf{t})$ | $F(\boldsymbol{\omega}-\boldsymbol{\omega}_0)$ |
| $\dfrac{\partial f}{\partial t_k}$ | $j\omega_k F(\omega)$ |
| $t_k f(\mathbf{t})$ | $j\dfrac{\partial F}{\partial \omega_k}$ |
| $f(\mathbf{t})g(\mathbf{t})$ | $(2\pi)^{-n}\,F(\boldsymbol{\omega}) * G(\boldsymbol{\omega})$ |
| $f(\mathbf{t}) * g(\mathbf{t})$ | $F(\boldsymbol{\omega})G(\boldsymbol{\omega})$ |
| $f(\mathbf{t}) \star g(\mathrm{t})$ | $F^*(\omega)G(\omega)$ |
| $f^*(\mathbf{t})g(\mathbf{t})$ | $(2\pi)^{-n}\,F(\boldsymbol{\omega}) \star G(\omega)$ |

*Note*: ($\alpha$ is any nonzero real number, $\mathbf{t}_0$ and $\omega_0$ are fixed $n$-dimensional points, and $F(\boldsymbol{\omega}) = \mathcal{F}[f(\mathbf{t})]\big|_{\omega}$ and $G(\omega) = \mathcal{F}[g(\mathbf{t})]\big|_{\omega}$)

$$\phi(\mathbf{x}) * \psi(\mathbf{x}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\phi(\mathbf{s})\psi(\mathbf{x}-\mathbf{s})\,ds_1 ds_2 \ldots ds_n$$

and

$$\phi(\mathbf{x}) \star \psi(\mathbf{x}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\phi^*(\mathbf{s})\psi(\mathbf{x}+\mathbf{s})\,ds_1 ds_2 \ldots ds_n .$$

There is one particularly useful $n$-dimensional identity that does not have a direct analog to the identities given in Section 2.2 (though it can be viewed as a generalization of the scaling formula). If $\mathbf{A}$ is a real, invertible, $n \times n$ matrix and $F(\boldsymbol{\omega}) = \mathcal{F}[f(\mathbf{t})]\big|_{\omega}$, then

$$\mathcal{F}\big[f(\mathbf{tA})\big]\Big|_{\omega} = \frac{1}{|\mathbf{A}|}F\left(\boldsymbol{\omega}\mathbf{A}^{-\mathrm{T}}\right) \qquad (2.4.1.3)$$

where $|\mathbf{A}|$ is the determinant of $\mathbf{A}$ and $\mathbf{A}^{-\mathrm{T}}$ is the inverse of the transpose of $\mathbf{A}$ (equivalently $\mathbf{A}^{-\mathrm{T}}$ is the transpose of the inverse of $\mathbf{A}$),

$$\mathbf{A}^{-\mathrm{T}} = (\mathbf{A}^{\mathrm{T}})^{-1} = (\mathbf{A}^{-1})^{\mathrm{T}}.$$

Likewise if $f(\mathbf{t}) = \mathcal{F}^{-1}[F(\boldsymbol{\omega})]|_{\mathbf{t}}$, then

$$\mathcal{F}^{-1}\big[F(\boldsymbol{\omega}\mathbf{A})\big]\Big|_{\mathbf{t}} = \frac{1}{|\mathbf{A}|}f\left(\mathbf{t}\mathbf{A}^{-\mathrm{T}}\right). \qquad (2.4.1.4)$$

The derivation of either of these identities is relatively simple. Letting $\mathbf{s} = \mathbf{tA}$ and recalling the change of variables formula for multiple integrals,

$$\begin{aligned}\mathcal{F}\big[f(\mathbf{tA})\big]\Big|_{\boldsymbol{\omega}} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(\mathbf{tA})e^{-j\boldsymbol{\omega}\cdot\mathbf{t}}\,dt_1 dt_2 \ldots dt_n \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(\mathbf{s})e^{-j\boldsymbol{\omega}\cdot\left(\mathbf{s}\mathbf{A}^{-1}\right)}\left|\frac{\partial(t_1, t_2, \ldots, t_n)}{\partial(s_1, s_2, \ldots, s_n)}\right| ds_1 ds_2 \ldots ds_n .\end{aligned} \qquad (2.4.1.5)$$

Now

$$\frac{\partial s_i}{\partial t_j} = A_{j,i}$$

and, so, the Jacobian in (2.4.1.5) is

$$\left|\frac{\partial(t_1, t_2, \ldots, t_n)}{\partial(s_1, s_2, \ldots, s_n)}\right| = \left|\frac{\partial(s_1, s_2, \ldots, s_n)}{\partial(t_1, t_2, \ldots, t_n)}\right|^{-1} = \frac{1}{|\mathbf{A}|}.$$

From linear algebra and the definition of the transpose

$$\boldsymbol{\omega} \cdot (\mathbf{s}\mathbf{A}^{-1}) = (\boldsymbol{\omega}(\mathbf{A}^{-1})^{\mathrm{T}}) \cdot \mathbf{s} = (\boldsymbol{\omega}\mathbf{A}^{-\mathrm{T}}) \cdot \mathbf{s}.$$

Thus, equation (2.4.1.5) can be written

$$\begin{aligned}\mathcal{F}\left[f(\mathbf{t}\mathbf{A})\right]\Big|_{\boldsymbol{\omega}} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(\mathbf{s})e^{-j(\boldsymbol{\omega}\mathbf{A}^{-\mathrm{T}})\cdot\mathbf{s}}\frac{1}{|\mathbf{A}|}ds_1 ds_2 \ldots ds_n \\ &= \frac{1}{|\mathbf{A}|}F\left(\boldsymbol{\omega}\mathbf{A}^{-\mathrm{T}}\right).\end{aligned}$$

An example of how (2.4.1.3) can be used to compute transforms will be given in Section 2.4.2.

## 2.4.2 Multidimensional Transforms of Separable Functions

A function of two variables, $f(x, y)$, is separable if it can be written as the product of two single variable functions.

$$f(x, y) = f_1(x)f_2(y). \tag{2.4.2.1}$$

The transform of such a function is easily computed provided $F_1(\omega) = \mathcal{F}[f_1(x)]\big|_{\omega}$ and $F_2(\nu) = \mathcal{F}[f_2(y)]\big|_{\nu}$ are known. Then

$$\begin{aligned}F(\omega, \nu) &= \mathcal{F}\left[f(x, y)\right]\Big|_{(\omega,\nu)} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(x)f_2(y)e^{-j(\omega x+\nu y)}dx\,dy \\ &= \int_{-\infty}^{\infty} f_1(x)e^{-j\omega x}dx\int_{-\infty}^{\infty} f_2(y)e^{-j\nu y}dy \\ &= F_1(\omega)F_2(\nu).\end{aligned}$$

More generally, $f(\mathbf{t})$ is said to be separable if there are $n$ functions of a single variable, $f_1(t_1)$, $f_2(t_2)$, …, $f_n(t_n)$, such that

$$f(t_1, t_2, \ldots, t_n) = f_1(t_1)f_2(t_2)\cdots f_n(t_n). \tag{2.4.2.2}$$

The Fourier transform of such a function is another separable function

$$F(\omega_1, \omega_2, \ldots, \omega_n) = F_1(\omega_1)F_2(\omega_2)\cdots F_n(\omega_n)$$

where, for each $k$, $F_k(\omega_k)$ is the one-dimensional Fourier transform of $f_k(t_k)$.

Likewise, if

$$F(\omega_1, \omega_2, \ldots\, \omega_n) = F_1(\omega_1)F_2(\omega_2)\cdots F_n(\omega_n),$$

then the $n$-dimensional inverse Fourier transform is

$$f(t_1, t_2, \ldots, t_n) = f_1(t_1)f_2(t_2)\ldots f_n(t_n)$$

where, for each $k$, $f_k(t_k)$ is the one-dimensional inverse Fourier transform of $F_k(\omega_k)$.

**Example 2.4.2.1**

The two-dimensional rectangular aperture function (with half-widths $\alpha$ and $\beta$) is

$$\eta_{\alpha,\beta}(x, y) = \begin{cases} 1, & \text{if} \quad |x| < \alpha \quad \text{and} \quad |y| < \beta \\ 0, & \text{if} \quad \alpha < |x| \quad \text{or} \quad \beta < |y| \end{cases}$$

or, equivalently,

$$\eta_{\alpha,\beta}(x, y) = p_\alpha(x) p_\beta(y).$$

Its Fourier transform is

$$\begin{aligned} N_{\alpha,\beta}(\omega, \nu) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \eta_{\alpha,\beta}(x, y) e^{-j(\omega x + \nu y)} dx\, dy \\ &= \int_{-\infty}^{\infty} p_\alpha(x) e^{-j\omega x} dx \int_{-\infty}^{\infty} p_\beta(y) e^{-j\nu y} dy \\ &= \left( \frac{2\sin(\alpha\omega)}{\omega} \right) \left( \frac{2\sin(\beta\nu)}{\nu} \right) \\ &= \frac{4}{\omega\nu} \sin(\alpha\omega) \sin(\beta\nu). \end{aligned}$$

**Example 2.4.2.2**

The three-dimensional delta function, $\delta(x, y, z)$ is defined as the generalized function such that if $\phi(x, y, z)$ is any function of three variables continuous at the origin,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x, y, z) \phi(x, y, z) dx\, dy\, dz = \phi(0, 0, 0).$$

Because

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x) \delta(y) \delta(z) \phi(x, y, z) dx\, dy\, dz \\ &= \int_{-\infty}^{\infty} \delta(z) \int_{-\infty}^{\infty} \delta(y) \int_{-\infty}^{\infty} \delta(x) \phi(x, y, z) dx\, dy\, dz \\ &= \int_{-\infty}^{\infty} \delta(z) \int_{-\infty}^{\infty} \delta(y) \phi(0, y, z) dy\, dz \\ &= \int_{-\infty}^{\infty} \delta(z) \phi(0, 0, z) dz \\ &= \phi(0, 0, 0), \end{aligned}$$

it is clear that

$$\delta(x, y, z) = \delta(x)\delta(y)\delta(z)$$

and

$$\mathcal{F}[\delta(x, y, z)]\big|_{(\omega,\nu,\mu)} = \mathcal{F}[\delta(x)]\big|_{\omega}\, \mathcal{F}[\delta(y)]\big|_{\nu}\, \mathcal{F}[\delta(z)]\big|_{\mu} = 1\cdot 1\cdot 1 = 1.$$

---

In using formulas (2.4.2.1) or (2.4.2.2) care must be taken to account for all the variables especially if the function depends explicitly on only a small subset of the variables. This can be done by including on the right-hand side of (2.4.2.1) or (2.4.2.2) the unit constant function,

$$1(s) = 1 \quad \text{for all } s,$$

for each variable, $s$, not explicitly involved in the computation of the function.

**Example 2.4.2.3**

The vertical slit aperture of half width $\alpha$ is the function of two variables given by

$$\text{vslit}_{\alpha}(x, y) = \begin{cases} 1, & \text{if } |x| < \alpha \\ 0, & \text{if } \alpha < |x| \end{cases}$$

or, equivalently,

$$\text{vslit}_{\alpha}(x, y) = p_{\alpha}(x) = p_{\alpha}(x)1(y).$$

Its Fourier transform is given by

$$\mathcal{F}\left[\text{vslit}_{\alpha}(x, y)\right]\Big|_{(\omega,\nu)} = \mathcal{F}\left[p_{\alpha}(x)\right]\Big|_{\omega}\, \mathcal{F}[1]\Big|_{\nu} = \frac{2}{\omega}\sin(\alpha\omega)\cdot 2\pi\delta(\nu)$$

and not by

$$\mathcal{F}\left[\text{vslit}_{\alpha}(x, y)\right]\Big|_{(\omega,\nu)} = \mathcal{F}\left[p_{\alpha}(x)\right]\Big|_{\omega} = \frac{2}{\omega}\sin(\alpha\omega).$$

**Example 2.4.2.4**

The three-dimensional vertical line source function is

$$l(x, y, z) = \delta(z).$$

Its Fourier transform is

$$\mathcal{F}[l(x, y, z)]\big|_{(\omega,\nu,\mu)} = \mathcal{F}[1(x)1(y)\delta(z)]\big|_{(\omega,\nu,\mu)} = 2\pi\delta(\omega)2\pi\delta(\nu)\cdot 1 = 4\pi^2\delta(\omega)\delta(\nu).$$

---

Often, functions that are not separable in one set of coordinates are separable in another set of coordinates. In such cases one of the generalized scaling identities (2.4.1.3) and (2.4.1.4), can simplify the computations.

**Example 2.4.2.5**

Let $\mathcal{P}$ be the parallelogram bounded by the lines $y = \pm 1$ and $x = y \pm 1$, and consider the two-dimensional aperture function

$$\eta_{\mathcal{P}}(x, y) = \begin{cases} 1, & \text{if } (x, y) \text{ is in } \mathcal{P} \\ 0, & \text{otherwise} \end{cases}.$$

Note that $\eta_{\mathcal{P}}(x, y) = 1$ if and only if

$$-1 < y < 1 \quad \text{and} \quad -1 < x - y < 1. \tag{2.4.2.3}$$

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

$\mathbf{A}^{\mathrm{T}}$ and the determinant of $\mathbf{A}$ are easily computed,

$$|\mathbf{A}| = 1 \quad \text{and} \quad \mathbf{A}^{-\mathrm{T}} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

For each $\mathbf{x} = (x, y)$ and $\boldsymbol{\omega} = (\omega, \nu)$, let

$$\hat{\mathbf{x}} = (\hat{x}, \hat{y}) = \mathbf{xA} = (x, y)\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = (x - y, y)$$

and

$$\hat{\boldsymbol{\omega}} = (\hat{\omega}, \hat{\nu}) = \boldsymbol{\omega}\mathbf{A}^{-\mathrm{T}} = (\omega, \nu)\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = (\omega, \omega + \nu).$$

It is easily verified that conditions (2.4.2.3) are equivalent to

$$-1 < \hat{y} < 1 \quad \text{and} \quad -1 < \hat{x} < 1.$$

Thus,

$$\eta_{\mathcal{P}}(x, y) = \eta_{1,1}(\hat{x}, \hat{y}) = \eta_{1,1}(\mathbf{xA})$$

where $\eta_{1,1}(\hat{x}, \hat{y})$ is the rectangular aperture function of Example 2.4.2.1. Using these results and the generalized scaling identity, (2.4.1.3),

$$
\begin{aligned}
\mathcal{F}\left[\eta_{\mathcal{P}}(x,y)\right]\Big|_{(\omega,\nu)} &= \mathcal{F}\left[\eta_{1,1}(\mathbf{x}\mathbf{A})\right]\Big|_{\boldsymbol{\omega}} \\
&= \frac{1}{|\mathbf{A}|} N_{1,1}\left(\boldsymbol{\omega}\mathbf{A}^{-\mathrm{T}}\right) \\
&= N_{1,1}(\omega, \omega+\nu) \\
&= \frac{4}{\omega(\omega+\nu)} \sin(\omega)\sin(\omega+\nu) .
\end{aligned}
$$

### 2.4.3 Transforms of Circularly Symmetric Functions and the Hankel Transform

Replacing $(x, y)$ and $(\omega, \nu)$ with their polar equivalents,

$$(x, y) = (r\cos\theta,\ r\sin\theta)$$

and

$$(\omega, \nu) = (\rho\cos\phi,\ \rho\sin\phi),$$

and using a well-known trigonometric identity, the formula for the two-dimensional Fourier transform, $F(\omega, \nu) = \mathcal{F}[f(x, y)]\big|_{(\omega,\nu)}$, becomes

$$
\begin{aligned}
F(\rho\cos\phi, \rho\sin\phi) &= \int_0^\infty \int_{-\pi}^{\pi} f(r\cos\theta, r\sin\theta) e^{-jr\rho(\cos\theta\cos\phi+\sin\theta\sin\phi)} r\,d\theta\,dr \\
&= \int_0^\infty \int_{-\pi}^{\pi} f(r\cos\theta, r\sin\theta) e^{-jr\rho\cos(\theta-\phi)} r\,d\theta\,dr .
\end{aligned}
\tag{2.4.3.1}
$$

Likewise, in polar coordinates, the formula for the two-dimensional inverse Fourier transform, $f(x, y) = \mathcal{F}[F(\omega, \nu)]\big|_{(x,y)}$, is

$$f(r\cos\theta, r\sin\theta) = \frac{1}{4\pi^2}\int_0^\infty \int_{-\pi}^{\pi} F(\rho\cos\theta, \rho\sin\theta) e^{jr\rho\cos(\theta-\phi)} \rho\,d\phi\,d\rho . \tag{2.4.3.2}$$

If $f(r\cos\theta, r\sin\theta)$ is separable with respect to $r$ and $\theta$,

$$f(r\cos\theta,\ r\sin\theta) = f_r(r) f_\theta(\theta),$$

then (2.4.3.1) becomes

$$F(\rho\cos\theta, \rho\sin\theta) = \int_0^\infty f_r(r)\, r K^-(r\rho, \phi)\, dr , \tag{2.4.3.3}$$

where

$$K^-(z, \phi) = \int_{-\pi}^{\pi} f_\theta(\theta) e^{-jz\cos(\theta-\phi)} d\theta .$$

Observe that the integrand for $K^{-}(r, \phi)$ must be periodic with period $2\pi$. Thus, letting $\theta' = \theta - \phi$,

$$K^{-}\left(z,\phi\right)=\int_{-\pi}^{\pi} f_{\theta}\left(\theta'+\phi\right)e^{-jz\cos\left(\theta'\right)}d\theta' . \tag{2.4.3.4}$$

Likewise, if $F(\rho \cos \phi, \rho \sin \phi)$ is separable with respect to $\rho$ and $\phi$,

$$F(\rho\cos\phi,\ \rho\sin\phi) = F_{\rho}(\rho)\ F_{\phi}(\phi)$$

then formula (2.4.3.2) becomes

$$f\left(r\cos\theta,\ r\sin\theta\right)=\frac{1}{4\pi^{2}}\int_{0}^{\infty} F_{\rho}\left(\rho\right)\rho K^{+}\left(r\rho,\theta\right)d\rho , \tag{2.4.3.5}$$

where

$$K^{+}\left(z,\theta\right)=\int_{-\pi}^{\pi} F_{\phi}\left(\theta'+\theta\right)e^{jz\cos\left(\phi'\right)}d\phi' . \tag{2.4.3.6}$$

The above formulas simplify considerably when circular symmetry can be assumed for either $f(x, y)$ or $F(\omega, \nu)$. It follows immediately from (2.4.3.3) through (2.4.3.6) that if either $f(x, y)$ or $F(\omega, \nu)$ is circularly symmetric, that is,

$$f(r\cos\theta,\ r\sin\theta) = f_r(r) \quad \text{or} \quad F(\rho\cos\phi,\ \rho\sin\phi) = F_{\rho}(\rho),$$

then, in fact, both $f(x, y)$ and $F(\omega, \nu)$ are circularly symmetric and can be written

$$f(r\cos\theta,\ r\sin\theta) = f_r(r) \quad \text{and} \quad F(\rho\cos\phi,\ \rho\sin\phi) = F_{\rho}(\rho).$$

In such cases it is convenient to use the Bessel function identity

$$2\pi J_{0}\left(z\right)=\int_{-\pi}^{\pi} \cos\left(z\cos w\right)dw ,$$

where $J_0(z)$ is the 0th order Bessel function of the first kind.* It is easily verified that

$$\int_{-\pi}^{\pi} \sin\left(r\rho\cos w\right)dw=0$$

and so

$$K^{\pm}\left(r\rho,\psi\right)=\int_{-\pi}^{\pi} e^{\pm jr\rho\cos w}dw=\int_{-\pi}^{\pi} \cos\left(r\rho\cos w\right)dw=2\pi J_{0}\left(r\rho\right)$$

and equations (2.4.3.3) and (2.4.3.5) reduce to

*See Chapter 1 for additional information on Bessel functions.

$$F_\rho(\rho) = 2\pi \int_0^\infty f_r(r) J_0(r\rho) r\, dr \tag{2.4.3.7}$$

and

$$f_r(r) = \frac{1}{2\pi} \int_0^\infty F_\rho(\rho) J_0(r\rho) \rho\, d\rho . \tag{2.4.3.8}$$

The 0th order Hankel transform of $g(r)$ is defined to be

$$\hat{g}(\rho) = \int_0^\infty g(r) J_0(r\rho) r\, dr .$$

Such transforms are the topic of Chapter 9 of this Handbook. It should be noted that (2.4.3.7) and (2.4.3.8) can be expressed in terms of 0th order Hankel transforms,

$$F_\rho(\rho) = 2\pi \hat{f}_r(\rho) \quad \text{and} \quad f_r(r) = \frac{1}{2\pi} \hat{F}_\rho(r). \tag{2.4.3.9}$$

From this it should be clear that 0th order Hankel transforms can be viewed as two-dimensional Fourier transforms of circularly symmetric functions. This allows fairly straightforward derivation of many of the properties of these Hankel transforms from corresponding properties of the Fourier transform. For example, letting $g(r) = 2\pi\, f(r)$ in (2.4.3.7) and (2.4.3.8) leads immediately to the inversion formula for the 0th order Hankel transform,

$$g(r) = \int_0^\infty \hat{g}(\rho) J_0(r\rho) \rho\, d\rho .$$

(For further discussion of the Hankel transforms, see Chapter 9 of this Handbook.)

### Example 2.4.3.1

Let $a > 0$ and let $f(x, y)$ be the corresponding circular aperture function,

$$f(x, y) = \begin{cases} 1, & \text{if } x^2 + y^2 < a^2 \\ 0, & \text{otherwise} \end{cases}$$

This function is circularly symmetric with

$$f(x, y) = f_r(r) = \begin{cases} 1, & \text{if } 0 \le r < a \\ 0, & \text{otherwise} \end{cases} .$$

Its Fourier transform must also be circularly symmetric and, using (2.4.3.7), is given by

$$F(\omega, \nu) = F_\rho(\rho) = 2\pi \int_0^a J_0(r\rho) r\, dr . \tag{2.4.3.10}$$

Letting $z = r\rho$ and using the Bessel function identity

$$\frac{d}{dz}\left[zJ_1(z)\right] = zJ_0(z),$$

where $J_1(z)$ is the first-order Bessel function of the first kind, the computation of this transform is easily completed,

$$\begin{aligned} F_\rho(\rho) &= 2\pi\rho^{-2}\int_0^{a\rho} J_0(z)z\,dz \\ &= 2\pi\rho^{-2}\int_0^{a\rho} \frac{d}{dz}\left[zJ_1(z)\right]dz \\ &= 2\pi\rho^{-2}\left[a\rho J_1(a\rho)\right] \\ &= \frac{2\pi a}{\rho}J_1(a\rho). \end{aligned}$$

### 2.4.4 Half-Line Sine and Cosine Transforms

Half-line sine and cosine transforms are usually taken only of functions defined on just the half-line $0 < t < \infty$. For such a function, $f(t)$, the corresponding (half-line) sine transform is

$$F_{\mathcal{S}}(\omega) = \mathcal{S}\left[f(t)\right]\Big|_\omega = \int_0^\infty f(t)\sin(\omega t)\,dt \tag{2.4.4.1}$$

and the corresponding (half-line) cosine transform is

$$F_{\mathcal{C}}(\omega) = \mathcal{C}\left[f(t)\right]\Big|_\omega = \int_0^\infty f(t)\cos(\omega t)\,dt\,. \tag{2.4.4.2}$$

These formulas define $F_{\mathcal{S}}(\omega)$ and $F_{\mathcal{C}}(\omega)$ for all real values of $\omega$, with $F_{\mathcal{S}}(\omega)$ being an odd function of $\omega$, and $F_{\mathcal{C}}(\omega)$ being an even function of $\omega$.

The half-line sine and cosine transforms are directly related to the standard Fourier transforms of the odd and even extensions of $f(t)$,

$$f_{\text{odd}}(t) = \begin{cases} f(t), & \text{if } 0 < t \\ -f(-t), & \text{if } t < 0 \end{cases}$$

and

$$f_{\text{even}}(t) = \begin{cases} f(t), & \text{if } 0 < t \\ f(-t), & \text{if } t < 0 \end{cases},$$

respectively. From the observations made in Subsection 2.3.1,

$$\mathcal{S}\left[f(t)\right]\Big|_\omega = j\frac{1}{2}\mathcal{F}\left[f_{\text{odd}}\right]\Big|_\omega \tag{2.4.4.3}$$

and

$$\mathcal{C}\left[f(t)\right]\Big|_{\omega} = \frac{1}{2}\mathcal{F}\left[f_{\text{even}}(t)\right]\Big|_{\omega} . \tag{2.4.4.4}$$

This shows that the (half-line) sine and cosine transforms can be treated as special cases of the standard Fourier transform. Indeed, by doing so it is possible to extend the class of functions that can be treated by sine and cosine transforms to include functions for which the integrals in (2.4.4.1) and (2.4.4.2) are not defined.

**Example 2.4.4.1**

Let $f(t) = t^2$ for $0 < t$. Formula (2.4.4.2) cannot be used to define $\mathcal{C}[f(t)]\big|_{\omega}$ because

$$\lim_{b\to\infty}\int_0^b t^2\cos(\omega t)\,dt$$

does not converge. However, $f_{\text{even}}(t) = t^2$ for all values of $t$, and, using formula (2.4.4.4),

$$\mathcal{C}\left[f(t)\right]\Big|_{\omega} = \frac{1}{2}\mathcal{F}\left[t^2\right]\Big|_{\omega} = \frac{1}{2}\left[-2\pi\delta''(\omega)\right] = -\pi\delta''(\omega) .$$

---

All the useful identities for the sine and cosine transforms can be derived through relations (2.4.4.3) and (2.4.4.4) from the corresponding identities for the standard Fourier transform.

**Example 2.4.4.2 Inversion Formulas for the Sine and Cosine Transforms**

Let $f(t)$, $F_{\mathcal{S}}(\omega)$, and $f_{\text{odd}}(t)$ be as shown, and let

$$F_{\text{odd}}(\omega) = \mathcal{F}[\,f_{\text{odd}}(t)]\big|_{\omega} .$$

According to equation (2.4.4.3)

$$F_{\text{odd}}(\omega) = -2jF_{\mathcal{S}}(\omega).$$

Thus, for $0 < t$,

$$f(t) = \mathcal{F}^{-1}[F_{\text{odd}}(\omega)]\big|_{t} = -2j\mathcal{F}^{-1}[F_{\mathcal{S}}(\omega)]\big|_{t} .$$

But, because $F_{\mathcal{S}}(\omega)$ is an odd function of $\omega$, the same arguments used in Subsection 2.3.1 yield

$$\mathcal{F}^{-1}\left[F_{\mathcal{S}}(\omega)\right]\Big|_{t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_{\mathcal{S}}(\omega)e^{j\omega t}\,d\omega = \frac{j}{\pi}\int_0^{\infty} F_{\mathcal{S}}(\omega)\sin(\omega t)\,d\omega ,$$

which, combined with the previous equation, gives the inversion formula for the sine transform,

$$f(t) = \frac{2}{\pi}\int_0^{\infty} F_{\mathcal{S}}(\omega)\sin(\omega t)\,d\omega .$$

Precisely the same reasoning shows that the inversion formula for the cosine transform is

$$f(t) = \frac{2}{\pi}\int_0^\infty F_C(\omega)\cos(\omega t)\,d\omega\,.$$

---

In using (2.4.4.3) and (2.4.4.4) to derive identities for the sine and cosine function, it is important to keep in mind that if

$$f(0) = \lim_{t\to 0^+} f(t)$$

exists, then the even extension will be continuous at $t = 0$ with $f_{\text{even}}(0) = f(0)$, but the odd extension will have a jump discontinuity at $t = 0$ with a jump of $2f(0)$. This is why most of the sine and cosine transform analogs to the differentiation formulas of Subsection 2.2.11 include boundary values. Some of these identities are

$$\mathcal{S}[f'(t)]\big|_\omega = -\omega F_C(\omega),$$

$$\mathcal{C}[f'(t)]\big|_\omega = \omega F_S(\omega) - f(0),$$

$$\mathcal{S}[f''(t)]\big|_\omega = \omega\, f(0) - \omega^2 F_S(\omega),$$

and

$$\mathcal{C}[f''(t)]\big|_\omega = -f'(0) - \omega^2 F_C(\omega),$$

where $f'(t)$ and $f''(t)$ denote the generalized first and second derivatives of $f(t)$ for $0 < t$. (See also Subsection 2.3.16.)

## 2.4.5 The Discrete Fourier Transform

The discrete Fourier transform is a computational analog to the Fourier transform and is used when dealing with finite collections of sampled data rather than functions per se. Given an "$N$th order sequence" of values, $\{f_0, f_1, f_2, \ldots, f_{N-1}\}$, the corresponding $N$th order discrete transform is the sequence $\{F_0, F_1, F_2, \ldots, F_{N-1}\}$ given by the formula

$$F_n = \sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}nk} f_k\,. \tag{2.4.5.1}$$

This can also be written in matrix form, $\mathbf{F} = [\mathcal{F}_N]\mathbf{f}$, where

$$\mathbf{F} = \begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ \vdots \\ F_{N-1} \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{N-1} \end{pmatrix},$$

and

$$
[\mathcal{F}_N] = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-j\frac{2\pi}{N}} & e^{-j2\frac{2\pi}{N}} & \cdots & e^{-j(N-1)\frac{2\pi}{N}} \\ 1 & e^{-j2\frac{2\pi}{N}} & e^{-j2\cdot 2\frac{2\pi}{N}} & \cdots & e^{-j2(N-1)\frac{2\pi}{N}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j(N-1)\frac{2\pi}{N}} & e^{-j2(N-1)\frac{2\pi}{N}} & \cdots & e^{-j(N-1)^2\frac{2\pi}{N}} \end{bmatrix}.
$$

On occasion, the matrix $[\mathcal{F}_N]$ is itself referred to as the $N$th order discrete transform.

The inverse to formula (2.4.5.1) is given by

$$
f_k = \frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}kn} F_n . \tag{2.4.5.2}
$$

In matrix form this is $\mathbf{f} = [\mathcal{F}_N]^{-1}\mathbf{F}$, where $[\mathcal{F}_N]^{-1}$ is the matrix

$$
\frac{1}{N}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{j\frac{2\pi}{N}} & e^{j2\frac{2\pi}{N}} & \cdots & e^{j(N-1)\frac{2\pi}{N}} \\ 1 & e^{j2\frac{2\pi}{N}} & e^{j2\cdot 2\frac{2\pi}{N}} & \cdots & e^{j2(N-1)\frac{2\pi}{N}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{j(N-1)\frac{2\pi}{N}} & e^{j2(N-1)\frac{2\pi}{N}} & \cdots & e^{j(N-1)^2\frac{2\pi}{N}} \end{bmatrix}.
$$

The similarity between the definitions for the discrete Fourier transforms, formulas (2.4.5.1) and (2.4.5.2), and formulas (2.3.11.4) and (2.3.11.5) should be noted. The discrete Fourier transforms can be treated as the regular Fourier transforms of corresponding regular periodic arrays generated from the sampled data.

### Example 2.4.5.1

The matrices for the 4-th order discrete Fourier transforms are

$$
[\mathcal{F}_N] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{-j\frac{2\pi}{4}} & e^{-j2\frac{2\pi}{4}} & e^{-j3\frac{2\pi}{4}} \\ 1 & e^{-j2\frac{2\pi}{4}} & e^{-j4\frac{2\pi}{4}} & e^{-j6\frac{2\pi}{4}} \\ 1 & e^{-j3\frac{2\pi}{4}} & e^{-j6\frac{2\pi}{4}} & e^{-j9\frac{2\pi}{4}} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}
$$

and

$$
[\mathcal{F}_N]^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{j\frac{2\pi}{4}} & e^{j2\frac{2\pi}{4}} & e^{j3\frac{2\pi}{4}} \\ 1 & e^{j2\frac{2\pi}{4}} & e^{j4\frac{2\pi}{4}} & e^{j6\frac{2\pi}{4}} \\ 1 & e^{j3\frac{2\pi}{4}} & e^{j6\frac{2\pi}{4}} & e^{j9\frac{2\pi}{4}} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}.
$$

The discrete Fourier transform of $\{f_0, f_1, f_2, f_3\} = \{1, 2, 3, 4\}$ is given by

$$\begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{pmatrix},$$

and the discrete inverse Fourier transform of $\{F_0, F_1, F_2, F_3\} = \{10, -2 + 2j, -2, -2 -2j\}$ is given by

$$\begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{pmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

---

In practice the sample size, $N$, is often quite large and the computations of the discrete transforms directly from formulas (2.4.5.1) and (2.4.5.2) can be a time-consuming process even on fairly fast computers. For this reason it is standard practice to make heavy use of symmetries inherent in the computations of the discrete transforms for certain values of $N$ (e.g., $N = 2^M$) to reduce the total number of calculations. Such implementations of the discrete Fourier transform are called "fast Fourier transforms" (FFTs).

## 2.4.6 Relations Between the Laplace Transform and the Fourier Transform*

Attention in this subsection will be restricted to functions of $t$ (and their transforms) that satisfy all of the following three conditions:

1. $f(t) = 0$ if $t < 0$.
2. $f(t)$ is piecewise continuous on $0 \le t$.
3. For some real value of $\alpha$, $f(t) = O(e^{\alpha t})$ as $\alpha \to \infty$.

It follows from the third condition that there is a minimum value of $\alpha_0$, with $-\infty \le \alpha_0 < \infty$, such that $f(t)e^{-\alpha t}$ is an exponentially decreasing function of $t$ whenever $\alpha_0 < \alpha$. This minimal value of $\alpha_0$ is called the "exponential order" of $f(t)$.

The Laplace transform of $f(t)$ is defined by

$$\mathfrak{L}\left[f(t)\right]\Big|_s = \int_{-\infty}^{\infty} f(t)e^{-st}dt. \tag{2.4.6.1}$$

The variable, $s$, in the transformed function may be any complex number whose real part is greater than the exponential order of $f(t)$.

There is a clear similarity between formula (2.4.6.1) defining the Laplace transform and the integral formula for the Fourier transform (formula [2.1.1.1]). Comparing the two immediately yields the formal relations

$$\mathfrak{L}\left[f(t)\right]\Big|_s = \int_{-\infty}^{\infty} f(t)e^{-j(-js)t}dt = \mathcal{F}\left[f(t)\right]\Big|_{-js}.$$

Another, somewhat more useful relation is found by taking the Fourier transform of $f(t)e^{-xt}$ when $x$ is greater than the order of $f(t)$:

---

*For a more complete discussion of the Laplace transform, see Chapter 5 of this Handbook.

$$\mathcal{F}\left[f(t)e^{-xt}\right]\Big|_{y} = \int_{-\infty}^{\infty} f(t)e^{-(x+jy)t}\,dt = \mathfrak{L}\left[f(t)\right]\Big|_{x+jy} . \tag{2.4.6.2}$$

In particular,

$$\mathcal{F}[f(t)e^{-st}]\big|_0 = \mathfrak{L}[f(t)]\big|_s .$$

The inversion formula for the Laplace transform can be quickly derived using relation (2.4.6.2). Let $\beta$ be any real value greater than the exponential order of $f(t)$ and observe that, letting

$$F_{\mathfrak{L}}(s) = \mathfrak{L}[f(t)]\big|_s ,$$

then, by relation (2.4.6.2)

$$F_{\mathfrak{L}}(\beta + j\omega) = \mathcal{F}[f(t)e^{-\beta t}]\big|_\omega ,$$

and so,

$$\begin{aligned} f(t) &= e^{\beta t} f(t)e^{-\beta t} \\ &= e^{\beta t}\mathcal{F}^{-1}\left[\mathcal{F}\left[f(\tau)e^{-\beta\tau}\right]\Big|_\omega\right]\Big|_t \\ &= e^{\beta t}\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{F}\left[f(\tau)e^{-\beta\tau}\right]\Big|_\omega e^{j\omega t}\,d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F_{\mathfrak{L}}(\beta + j\omega)e^{(\beta + j\omega)t}\,d\omega . \end{aligned}$$

This formula can be expressed in slightly more compact form as a contour integral in the complex plane,

$$f(t) = \mathfrak{L}^{-1}\left[F_{\mathfrak{L}}(s)\right]\Big|_t = \frac{1}{2j\pi}\int_{z=\beta-j\infty}^{\beta+j\infty} F_{\mathfrak{L}}(z)e^{zt}\,dz .$$

Alternatively, it can be left in terms of the Fourier inverse transform,

$$f(t) = \mathfrak{L}^{-1}[F_{\mathfrak{L}}(s)]\big|_t = e^{\beta t}\,\mathcal{F}^{-1}[F_{\mathfrak{L}}(\beta + j\omega)]\big|_t .$$

## 2.5 Reconstruction of Sampled Signals

In practice a function is often known only by a sampling of its values at specific points. The following subsections describe when such a function can be completely reconstructed using its samples and how, using methods based on the Fourier transform, the values of the reconstructed function can be computed at arbitrary points.

### 2.5.1 Sampling Theorem for Band-Limited Functions

Assume $f(t)$ is a band-limited function with Fourier transform $F(\omega)$ (see Subsection 2.3.7). Let $2\Omega_0$ be the minimum bandwidth of $f(t)$, that is, $\Omega_0$ is the smallest nonnegative value such that

$$F(\omega) = 0 \quad \text{whenever } \Omega_0 < |\omega| .$$

The Nyquist interval, $\Delta T$, and the Nyquist rate, $\nu$, for $f(t)$ are defined by

$$\Delta T = \frac{\pi}{\Omega_0} \quad \text{and} \quad \nu = \frac{1}{\Delta T} = \frac{\Omega_0}{\pi} .$$

The sampling theorem for band-limited functions states that $f(t)$ (and hence, also $F(\omega)$ as well as the total energy in $f(t)$) can be completely reconstructed from a uniform sampling taken at the Nyquist rate or greater. More precisely, if $0 < \Delta t \le \Delta T$, then, letting $\Omega = \pi/\Delta t$,

$$f(t) = \sum_{n=-\infty}^{\infty} f(n\Delta t) \frac{\sin(\Omega(t - n\Delta t))}{\Omega(t - n\Delta t)} \tag{2.5.1.1}$$

and, taking the transform

$$F(\omega) = \frac{\pi}{\Omega} \sum_{n=-\infty}^{\infty} f(n\Delta t)\, e^{-jn\Delta t\omega} p_\Omega(\omega), \tag{2.5.1.2}$$

where $p_\Omega(\omega)$ is the pulse function,

$$p_\Omega(\omega) = \begin{cases} 1, & \text{if } |\omega| < \Omega \\ 0, & \text{if } \Omega < |\omega| \end{cases} .$$

The energy in $f(t)$ is easily computed. Using equations (2.3.4.1), (2.5.1.2), and the fact that the exponentials in formula (2.5.1.2) are orthogonal on the interval $-\Omega < \omega < \Omega$ ,

$$\begin{aligned} E &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega \\ &= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \left(\frac{\pi}{\Omega}\right)^2 \int_{-\Omega}^{\Omega} f(n\Delta t) f^*(m\Delta t) e^{-jn\Delta t\omega} e^{jm\Delta t\omega} \, d\omega \\ &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left(\frac{\pi}{\Omega}\right)^2 |f(n\Delta t)|^2 \, 2\Omega \\ &= \Delta t \sum_{n=-\infty}^{\infty} |f(n\Delta t)|^2 . \end{aligned} \tag{2.5.1.3}$$

To see why formulas (2.5.1.1) and (2.5.1.2) are valid, let $\hat{F}(\omega)$ be the periodic extension of $F(\omega)$,

$$\hat{F}(\omega) = \begin{cases} F(\omega), & \text{if } -\Omega < \omega < \Omega \\ \hat{F}(\omega + 2\Omega), & \text{for all } \omega \end{cases} .$$

Observe that $2\Omega$ is a bandwidth for $f(t)$, and so,

$$F(\omega) = \begin{cases} \hat{F}(\omega), & \text{if } |\omega| < \Omega \\ 0, & \text{if } \Omega < |\omega| \end{cases}. \tag{2.5.1.4}$$

This can be written more concisely using the pulse function as

$$F(\omega) = \hat{F}(\omega) p_\Omega(\omega).$$

From this it follows, using convolution, that

$$f(t) = \mathscr{F}^{-1}\left[\hat{F}(\omega) p_\Omega(\omega)\right]\Big|_t = \hat{f}(t) * \left(\frac{\sin(\Omega t)}{\pi t}\right), \tag{2.5.1.5}$$

where $\hat{f}\,(t)$ denotes the inverse transform of $\hat{F}\,(\omega)$. By formula (2.3.9.14),

$$\hat{f}(t) = \sum_{n=-\infty}^{\infty} C_n \delta(t - n\Delta t), \tag{2.5.1.6}$$

where

$$\Delta\tau = \frac{2\pi}{2\Omega} = \Delta t$$

and, using the above,

$$\begin{aligned} C_n &= \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} \hat{F}(\omega) e^{jn\Delta t\omega} d\omega \\ &= \frac{\pi}{\Omega} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{jn\Delta t\omega} d\omega \right) \\ &= \Delta t\, f(n\Delta t). \end{aligned}$$

Combining this last with equations (2.5.1.5) and (2.5.1.6), and using the shifting property of the delta function, yields

$$\begin{aligned} f(t) &= \hat{f}(t) * \left(\frac{\sin(\Omega t)}{\pi t}\right) \\ &= \sum_{n=-\infty}^{\infty} \Delta t\, f(n\Delta t) \delta(t - n\Delta t) * \left(\frac{\sin(\Omega t)}{\pi t}\right) \\ &= \sum_{n=-\infty}^{\infty} \Delta t\, f(n\Delta t) \frac{\sin(\Omega(t - n\Delta t))}{\pi(t - n\Delta t)}, \end{aligned}$$

which is the same as formula (2.5.1.1).

### 2.5.2 Truncated Sampling Reconstruction of Band-Limited Functions

Formula (2.5.1.1) employs an infinite number of samples of $f(t)$. Often this is impractical, and one must approximate $f(t)$ with the truncated version of formula (2.5.1.1),

$$f(t) \approx \sum_{n=-N}^{N} f(n\Delta t) \frac{\sin(\Omega(t-n\Delta t))}{\Omega(t-n\Delta t)}, \tag{2.5.2.1}$$

where $N$ is some positive integer. The pointwise error is

$$\varepsilon_N(t) = f(t) - \sum_{n=-N}^{N} f(n\Delta t) \frac{\sin(\Omega(t-n\Delta t))}{\Omega(t-n\Delta t)}.$$

If $f(t)$ is a band-limited function, then the sampling theorem implies that

$$\varepsilon_N(t) = \sum_{N<|n|} f(n\Delta t) \frac{\sin(\Omega(t-n\Delta t))}{\Omega(t-n\Delta t)},$$

and it can be shown that

$$|\varepsilon_N(t)|^2 \le \frac{E\Omega}{\pi} \sum_{N<|n|} \frac{\sin^2(\Omega(t-n\Delta t))}{\Omega^2(t-n\Delta t)^2}, \tag{2.5.2.2}$$

where $E$ is the energy in $f(t)$. In addition, if the samples are known to vanish sufficiently rapidly, then one can use

$$|\varepsilon_N(t)| \le \sqrt{\sum_{N<|n|} |f(n\Delta t)|^2}. \tag{2.5.2.3}$$

This last bound is a uniform bound directly related to expression (2.5.1.3) for the energy of a band-limited function. It can be derived after observing that $\varepsilon_N(t)$ can be written as a proper integral,

$$\begin{aligned}
\varepsilon_N(t) &= \sum_{N<|n|} f(n\Delta t) \frac{\sin(\Omega(t-n\Delta t))}{\Omega(t-n\Delta t)} \\
&= \sum_{N<|n|} f(n\Delta t) \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} e^{-jn\Delta t\omega} e^{j\omega t} d\omega \\
&= \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} \left( \sum_{N<|n|} f(n\Delta t) e^{-jn\Delta t\omega} \right) e^{jt\omega} d\omega .
\end{aligned}$$

Using the Cauchy–Schwarz inequality,

$$
\begin{aligned}
\left|\varepsilon_N(t)\right|^2 &= \frac{1}{4\Omega^2}\left|\int_{-\Omega}^{\Omega}\left(\sum_{N<|n|} f(n\Delta t)e^{-jn\Delta t\omega}\right)e^{jt\omega}\,d\omega\right|^2 \\
&\le \frac{1}{4\Omega^2}\left(\int_{-\Omega}^{\Omega}\left|\sum_{N<|n|} f(n\Delta t)e^{-jn\Delta t\omega}\right|^2 d\omega\right)\left(\int_{-\Omega}^{\Omega}\left|e^{jt\omega}\right|^2 d\omega\right) \\
&= \frac{1}{4\Omega^2}\left(\sum_{N<|n|}\left|f(n\Delta t)\right|^2 2\Omega\right)(2\Omega) \\
&= \sum_{N<|n|}\left|f(n\Delta t)\right|^2 ,
\end{aligned}
$$

as claimed by equation (2.5.2.3).

**Example 2.5.2.1**

Suppose $f(t)$ is a band-limited function (with bandwidth $2\Omega$) to be approximated on the interval $-L < t < L$. Suppose, further, that an upper bound, $E_0$, is known for the energy of $f(t)$. Let $N$ be any integer such that $L < N\,\Delta t$. Then, for $-L < t < L$, using inequality (2.5.2.2) with well-known bounds,

$$
\begin{aligned}
\left|\varepsilon_N(t)\right|^2 &\le \frac{E\Omega}{\pi}\sum_{N<|n|}\frac{\sin^2\big(\Omega(t-n\Delta t)\big)}{\Omega^2(t-n\Delta t)^2} \\
&\le \frac{2E_0}{\pi\Omega}\sum_{n=N+1}^{\infty}\frac{1}{(L-n\Delta t)^2} \\
&\le \frac{2E_0}{\pi\Omega}\int_{x=N}^{\infty}\frac{1}{(L-x\Delta t)^2}dx \\
&= \frac{2E_0}{\pi\Omega}\frac{1}{\Delta t(N\Delta t-L)} \\
&= \frac{2}{\pi^2(N\Delta t-L)}E_0 .
\end{aligned}
$$

Thus, to ensure an error of less than $0.05\sqrt{E_0}$, it suffices to choose $N$ satisfying

$$
\left(\frac{2}{\pi^2(N\Delta t-L)}\right)^{\frac{1}{2}} < 0.05
$$

or, equivalently,

$$\frac{800}{\pi^2} + L < N\Delta t\,.$$

**Example 2.5.2.2**

Suppose that $f(t)$ is a band-limited function (with bandwidth $2\Omega$) whose transform, $F(\omega)$, is known to be piecewise smooth and continuous. Assume further that, for some $A < \infty$, $|F'(\omega)| < A$ for all values of $\omega$. Then, for each $t$,

$$\begin{aligned}
\left|tf(t)\right| &= \left|\mathcal{F}^{-1}\left[jF'(\omega)\right]\Big|_t\right| \\
&= \left|\frac{1}{2\pi}\int_{-\Omega}^{\Omega} jF'(\omega)e^{j\omega t}\,d\omega\right| \\
&\le \frac{1}{2\pi}\int_{-\Omega}^{\Omega}\left|F'(\omega)\right|d\omega \\
&\le A\frac{\Omega}{\pi}\,.
\end{aligned}$$

So, for each $n$,

$$\left|f(n\Delta t)\right|^2 \le \left(\frac{A\Omega}{n\Delta t\pi}\right)^2 = \frac{1}{n^2}\frac{A^2}{\Delta t^4}\,,$$

and inequality (2.5.2.3) becomes

$$\begin{aligned}
\left|\varepsilon_N(t)\right|^2 &\le \sum_{N<|n|}\frac{1}{n^2}\frac{A^2}{\Delta t^4} \\
&\le \frac{2A^2}{\Delta t^4}\int_{x=N}^{\infty}\frac{1}{x^2}\,dx \\
&= \frac{2A^2}{N\Delta t^4}\,.
\end{aligned}$$

## 2.5.3 Reconstruction of Sampled Nearly Band-Limited Functions

Often, one must deal with a function, $f(t)$, which might not necessarily be band-limited, but is "nearly" band-limited, that is, letting

$$F_\Omega(\omega) = F(\omega)p_\Omega \quad\text{and}\quad f_\Omega(t) = \mathcal{F}^{-1}[F_\Omega(\omega)]\Big|_t\,,$$

one can always choose $\Omega < \infty$ so that

$$\int_{-\infty}^{\infty}\left|F(\omega) - F_\Omega(\omega)\right|d\omega = \int_{\Omega<|\omega|}\left|F(\omega)\right|d\omega \qquad (2.5.3.1)$$

is as small as desired. Because

$$\left|f(t) - f_\Omega(t)\right| = \left|\mathcal{F}^{-1}\left[F(\omega) - F_\Omega(\omega)\right]\right|_t \le \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|F(\omega) - F_\Omega(\omega)\right| d\omega\,,$$

it is clear that $f_\Omega(t)$ can also be made as close to $f(t)$ as desired by a suitable choice of $\Omega$. Any value of $2\Omega$ that makes (2.5.3.1) "sufficiently small" is called an effective bandwidth. For such a function it is reasonable to expect that if $\Omega$ is an effective bandwidth and $\Delta t = \pi/\Omega$, then the interpolation formulas,

$$f_S(t) = \sum_{n=-\infty}^{\infty} f(n\Delta t)\frac{\sin\big(\Omega(t - n\Delta t)\big)}{\Omega(t - n\Delta t)} \tag{2.5.3.2}$$

will be a good approximation to $f(t)$. Starting with the trivial observation that

$$f(t) = f_S(t) + f(t) - f_\Omega(t) + f_\Omega(t) - f_S(t),$$

one can derive

$$f(t) = f_S(t) + \mathcal{E}_0(t) - \mathcal{E}_\Sigma(t), \tag{2.5.3.3}$$

where

$$\mathcal{E}_0(t) = \frac{1}{2\pi}\int_{\Omega<|\omega|} F(\omega)e^{j\omega t}d\omega\,,$$

$$\mathcal{E}_\Sigma(t) = \sum_{n=-\infty}^{\infty} \varepsilon(n\Delta t)\frac{\sin\big(\Omega(t - n\Delta t)\big)}{\Omega(t - n\Delta t)}$$

and

$$\varepsilon(n\Delta t) = \mathcal{F}^{-1}\Big[\big(1 - p_\Omega(\omega)\big)F(\omega)\Big]\Big|_{n\Delta t} = \frac{1}{2\pi}\int_{\Omega<|\omega|} F(\omega)e^{j\omega n\Delta t}d\omega\,.$$

Error estimates can be obtained from equation (2.5.3.3) provided it can be shown that $\varepsilon(n\,\Delta t)$ vanishes sufficiently rapidly as $n \to \infty$. As Example 2.5.3.1 illustrates, finding such error estimates can be quite nontrivial.

### Example 2.5.3.1

Suppose $f(t)$ is an infinitely differentiable, finite duration function with duration $2T$. Since $f(t)$ vanishes whenever $|t| \ge T$,

$$f_S(t) = \sum_{n=-N}^{N} f(n\Delta t)\frac{\sin\big(\Omega(t - n\Delta t)\big)}{\Omega(t - n\Delta t)}\,, \tag{2.5.3.4}$$

where $N$ is the integer satisfying

$$N\,\Delta t < T \le (N+1)\Delta t.$$

To avoid triviality, it may be assumed that $\Delta t < T$ and $N \ge 1$.

By continuity $f(t)$ and each of its derivatives must vanish whenever $|t| \ge T$. Also, for each positive integer, $m$, there is a finite $A_m$ such that

$$\left|f^{(m)}(t)\right| \le A_m$$

for all $t$. Thus, for each nonnegative integer, $m$,

$$\begin{aligned}
\left|\omega^m F(\omega)\right| &= \left|\mathscr{F}\left[f^{(m)}(t)\right]\Big|_{\omega}\right| \\
&= \left|\int_{-\infty}^{\infty} f^{(m)}(t)e^{-j\omega t}\,dt\right| \\
&\le \int_{-T}^{T} A_m\,dt \\
&= 2A_m T.
\end{aligned} \tag{2.5.3.5}$$

Likewise, if $m \ge 2$,

$$\left|\omega^m F'(\omega)\right| = \left|\mathscr{F}\left[\frac{d^m}{dt^m}\big(tf(t)\big)\right]\Big|_{\omega}\right| = \left|\mathscr{F}\left[mf^{(m-1)}(t) + tf^{(m)}(t)\right]\Big|_{\omega}\right| \le B_m \tag{2.5.3.6}$$

where

$$B_m = 2mA_{m-1}T + A_m T^2.$$

It follows from inequality (2.5.3.5) that $F(\omega)$ is absolutely integrable (and hence, nearly band-limited) and that, for $m \ge 2$ and any $t$,

$$\begin{aligned}
\frac{1}{2\pi}\left|\int_{\Omega<|\omega|} F(\omega)e^{j\omega t}\,d\omega\right| &\le \frac{1}{2\pi}\int_{\Omega<|\omega|}\left|F(\omega)\right|d\omega \\
&\le \frac{1}{\pi}\int_{\Omega}^{\infty} 2A_m T\omega^{-m}\,d\omega \\
&= \frac{2A_m T}{(m-1)\pi}\Omega^{1-m} \\
&= C_m \Delta t^{m-1}
\end{aligned} \tag{2.5.3.7}$$

where

$$C_m = \frac{2A_m T}{(m-1)\pi^m}.$$

Thus, in particular, for any positive integer, $k$,

$$\left|\mathcal{E}_0(t)\right| = \frac{1}{2\pi}\left|\int_{\Omega<|\omega|} F(\omega)e^{j\omega t}d\omega\right| \le C_{k+1}\Delta t^k . \tag{2.5.3.8}$$

Two bounds for $\varepsilon(n\,\Delta t)$ can be derived. First, using inequality (2.5.3.7),

$$\left|\varepsilon(n\Delta t)\right| = \frac{1}{2\pi}\left|\int_{\Omega<|\omega|} F(\omega)e^{j\omega n\Delta t}d\omega\right| \le C_m\Delta t^{m-1} \tag{2.5.3.9}$$

provided $m \ge 2$. For $n \ne 0$, observe that

$$\varepsilon(n\,\Delta t) = \varepsilon_+(n\,\Delta t) + \varepsilon_-(n\,\Delta t),$$

where

$$\varepsilon_\pm(n\Delta t) = \pm\frac{1}{2\pi}\int_{\pm\Omega}^{\pm\infty} F(\omega)e^{j\omega n\Delta t}d\omega .$$

Using integration by parts and inequalities (2.5.3.5) and (2.5.3.6),

$$\begin{aligned}
\left|\varepsilon_\pm(n\Delta t)\right| &= \left|\frac{1}{2\pi}\int_{\pm\Omega}^{\pm\infty} F(\omega)e^{j\omega n\Delta t}d\omega\right| \\
&= \frac{1}{2\pi}\left|\frac{j}{n\Delta t}F(\pm\Omega)e^{\pm jn\Delta t\Omega} - \frac{1}{jn\Delta t}\int_{\pm\Omega}^{\pm\infty} F'(\omega)e^{jn\Delta t\omega}d\omega\right| \\
&\le \frac{1}{2\pi}\left(\frac{2}{n\Delta t}A_m T\Omega^{-m} + \frac{1}{n\Delta t}\int_{\Omega}^{\infty} B_m\omega^{-m}d\omega\right) \\
&= \frac{1}{2\pi n}\left(\frac{2}{\Delta t}A_m T\Omega^{-m} + \frac{1}{\Delta t(m-1)}B_m\Omega^{1-m}\right)
\end{aligned}$$

for any integer, $m \ge 2$. Because $\Delta t\,\Omega = \pi$ and $\Delta t \le T$, this reduces to

$$\left|\varepsilon_\pm(n\Delta t)\right| \le \frac{1}{2n}D_m\Delta t^{m-2} ,$$

where

$$D_m = \pi^{-m}\left(\frac{2}{\pi}A_m T^2 + \frac{1}{m-1}B_m\right).$$

Thus, for all $n \neq 0$ and $m \geq 2$,

$$\left|\varepsilon\left(n\Delta t\right)\right| \leq \frac{1}{n} D_m \Delta t^{m-2} . \tag{2.5.3.10}$$

Next, observe that

$$\mathcal{E}_\Sigma(t) = S_1(t) + S_2(t)$$

where

$$S_1\left(t\right) = \sum_{n=-2N-1}^{2N+1} \varepsilon\left(n\Delta t\right) \frac{\sin\left(\Omega\left(t - n\Delta t\right)\right)}{\Omega\left(t - n\Delta t\right)}$$

and

$$S_2\left(t\right) = \sum_{2N+1<|n|} \varepsilon\left(n\Delta t\right) \frac{\sin\left(\Omega\left(t - n\Delta t\right)\right)}{\Omega\left(t - n\Delta t\right)} .$$

Using inequality (2.5.3.9),

$$\begin{aligned}
\left|S_1\left(t\right)\right| &\leq \sum_{n=-2N-1}^{2N+1} \left|\varepsilon\left(n\Delta t\right)\right| \left|\frac{\sin\left(\Omega\left(t - n\Delta t\right)\right)}{\Omega\left(t - n\Delta t\right)}\right| \\
&< \left|\varepsilon\left(0\right)\right| + \left|\varepsilon\left(\left(2N+1\right)\Delta t\right)\right| + \left|\varepsilon\left(-\left(2N+1\right)\Delta t\right)\right| \\
&\quad + \sum_{n=1}^{2N} \left(\left|\varepsilon\left(n\Delta t\right)\right| + \left|\varepsilon\left(-n\Delta t\right)\right|\right) \\
&\leq 3C_{k+2}\Delta t^{k+1} + 2\sum_{n=1}^{2N} C_{k+2}\Delta t^{k+1} \\
&= C_{k+2}\left(3\Delta t + 4N\Delta t\right)\Delta t^k \\
&\leq 7TC_{k+2}\Delta t^k
\end{aligned}$$

for any positive integer, $k$. Next, because of inequality (2.5.3.10) and the fact that $T < (N+1)\Delta t$, it follows that, for $|t| \leq T$ and $k \geq 1$,

$$\left|S_2(t)\right| \le \sum_{2N+1<|n|} \left|\varepsilon(n\Delta t)\right| \left|\frac{\sin\left(\Omega\left(t-n\Delta t\right)\right)}{\Omega\left(t-n\Delta t\right)}\right|$$

$$\le 2\sum_{n=2N+2}^{\infty} \frac{1}{n} D_{k+1}\Delta t^{k-1} \frac{1}{\Omega\left(n\Delta t - |t|\right)}$$

$$\le 2\sum_{n=2N+2}^{\infty} \frac{1}{n} D_{k+1}\Delta t^{k-1} \frac{1}{\Omega\Delta t\left(n-\left(N+1\right)\right)}$$

$$= \frac{2}{\pi} D_{k+1}\Delta t^{k-1} \sum_{n=2N+2}^{\infty} \frac{1}{n\left(n-N-1\right)}.$$

But,

$$\sum_{n=2N+2}^{\infty} \frac{1}{n\left(n-N-1\right)} < \int_{2N+1}^{\infty} \frac{1}{x\left(x-N-1\right)}\,dx$$

$$= \frac{1}{N+1}\ln\left|2+\frac{1}{N}\right|$$

$$< \frac{2}{N+1}.$$

So,

$$\left|S_2(t)\right| < \frac{2}{\pi} D_{k+1}\Delta t^k \frac{1}{\Delta t}\frac{2}{N+1} < \frac{4}{\pi T} D_{k+1}\Delta t^k .$$

Combining the bounds for $|S_1(t)|$ and $|S_2(t)|$ gives

$$\left|\mathcal{E}_\Sigma(t)\right| \le \left|S_1(t)\right| + \left|S_2(t)\right| < E_k\Delta t^k \tag{2.5.3.11}$$

for $|t| \le T$ and $k \ge 1$, where

$$E_k = 7TC_{k+2} + \frac{4}{\pi T} D_{k+1}.$$

Combining (2.5.3.3), (2.5.3.8), and (2.5.3.11) gives an error estimate for using $f_S(t)$ as an approximation for $f(t)$ when $|t| \le T$,

$$\left|f(t) - f_S(t)\right| \le \left|\mathcal{E}_0(t)\right| + \left|\mathcal{E}_\Sigma(t)\right| < [C_{k+1} + E_k]\Delta t^k,$$

where $k$ is any positive integer. In terms of the effective bandwidth, $\Omega = \pi/\Delta t$, this becomes

$$\left|f(t) - f_S(t)\right| = O(\Omega^{-k}),$$

confirming that $f_S(t)$ can be made to approximate $f(t)$ on $-T < t < T$ as accurately as desired by taking the effective bandwidth, $\Omega$, sufficiently large.

### 2.5.4 Sampling Theorem for Finite Duration Functions

Assume $f(t)$ is of finite duration with Fourier transform $F(\omega)$ (see Subsection 2.3.6). Let $2T_0$ be the minimum duration of $f(t)$, that is, $T_0$ is the smallest nonnegative value such that

$$f(t) = 0 \quad \text{whenever} \quad T_0 < |t|.$$

The sampling theorem for functions of finite duration states that $F(\omega)$ (and hence, also $f(t)$ can be reconstructed from a suitable uniform sampling in the frequency domain. More precisely, if $0 < \Delta\omega < \Delta\Omega$ where $\Delta\Omega$ denotes the "frequency Nyquist interval,"

$$\Delta\Omega = \frac{\pi}{T_0},$$

then, letting $T = \pi/\Delta\omega$,

$$F(\omega) = \sum_{n=-\infty}^{\infty} F(n\Delta\omega)\frac{\sin\big(T(\omega - n\Delta\omega)\big)}{T(\omega - n\Delta\omega)},$$

and, taking the inverse transform,

$$f(t) = \sum_{n=-\infty}^{\infty} F(n\Delta\omega)\frac{\Delta\omega}{2\pi}e^{jn\Delta\omega t}p_T(t).$$

The energy in $f(t)$ is

$$E = \frac{\Delta\omega}{2\pi}\sum_{n=-\infty}^{\infty}\left|F(n\Delta\omega)\right|^2.$$

### 2.5.5 Fundamental Sampling Formulas and Poisson's Formula

As long as either $f(t)$ or its Fourier transform, $F(\omega)$, is absolutely integrable and has a bounded derivative, then

$$\Delta t\sum_{n=-\infty}^{\infty} f(t - n\Delta t) = \sum_{n=-\infty}^{\infty} F(n\Delta\omega)\, e^{jn\Delta\omega t} \tag{2.5.5.1}$$

and

$$\frac{\Delta\omega}{2\pi}\sum_{n=-\infty}^{\infty} F(\omega - n\Delta\omega) = \sum_{n=-\infty}^{\infty} f(n\Delta t)\, e^{-jn\Delta t\omega} \tag{2.5.5.2}$$

where $\Delta t$ and $\Delta\omega$ are positive constants with $\Delta t \Delta\omega = 2\pi$. Using these formulas it is possible to derive the sampling theorems for band-limited functions and for finite duration functions. While these formulas are not valid for periodic functions, they can be used to derive the classical Fourier series expansion for periodic functions and hence, can also be viewed as generalizations of the Fourier series expansion for periodic functions. Letting $t = 0$ in formula (2.5.5.1) yields Poisson's formula,

$$\Delta t \sum_{n=-\infty}^{\infty} f(n\Delta t) = \sum_{n=-\infty}^{\infty} F(n\Delta\omega). \qquad (2.5.5.3)$$

These sampling formulas can be derived by a fairly straightforward use of properties of the delta and the comb functions along with the use of the convolution formulas of Subsection 2.2.9. Let

$$\phi(t) = f * \text{comb}_{\Delta t}(t).$$

Because of the properties of the delta functions making up the comb function,

$$\phi(t) = \sum_{n=-\infty}^{\infty} f * \delta(t - n\Delta t) = \sum_{n=-\infty}^{\infty} f(t - n\Delta t).$$

The Fourier transform of $\phi(t)$ is

$$\begin{aligned}
\psi(\omega) &= \mathcal{F}\left[f * \text{comb}_{\Delta t}(t)\right]\Big|_{\omega} \\
&= F(\omega)\Delta\omega\, \text{comb}_{\Delta\omega}(\omega) \\
&= \Delta\omega \sum_{n=-\infty}^{\infty} F(\omega)\delta(\omega - n\Delta\omega) \\
&= \Delta\omega \sum_{n=-\infty}^{\infty} F(n\Delta\omega)\delta(\omega - n\Delta\omega).
\end{aligned}$$

Thus,

$$\begin{aligned}
\Delta t \sum_{n=-\infty}^{\infty} f(t - n\Delta t) &= \Delta t \phi(t) \\
&= \mathcal{F}^{-1}\left[\Delta t \psi(\omega)\right]\Big|_{t} \\
&= \mathcal{F}^{-1}\left[\Delta t \Delta\omega \sum_{n=-\infty}^{\infty} F(n\Delta\omega)\delta(\omega - n\Delta\omega)\right]\Bigg|_{t} \\
&= 2\pi \sum_{n=-\infty}^{\infty} F(n\Delta\omega)\mathcal{F}^{-1}\left[\delta(\omega - n\Delta\omega)\right]\Big|_{t} \\
&= \sum_{n=-\infty}^{\infty} F(n\Delta\omega)\, e^{jn\Delta\omega t},
\end{aligned}$$

which is formula (2.5.5.1)

Similar computations yield formula (2.5.5.2).

**Example 2.5.5.1 Evaluation of an Infinite Series**

To evaluate

$$\sum_{n=-\infty}^{\infty} \frac{1}{1+n^2},$$

observe that

$$\sum_{n=-\infty}^{\infty} \frac{1}{1+n^2} = \sum_{n=-\infty}^{\infty} F(n\Delta\omega),$$

where

$$F(\omega) = \frac{1}{1+\omega^2} \quad \text{and} \quad \Delta\omega = 1.$$

The Fourier inverse transform of $F(\omega)$ is $f(t) = e^{-1|t|}/2$, and so, by Poisson's formula,

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \frac{1}{1+n^2} &= \sum_{n=-\infty}^{\infty} F(n\Delta\omega) \\ &= 2\pi \sum_{n=-\infty}^{\infty} f(n2\pi) \\ &= 2\pi \sum_{n=-\infty}^{\infty} \frac{1}{2} e^{-|n2\pi|} \\ &= 2\pi \left[ \frac{1}{2} + \sum_{n=1}^{\infty} \left(e^{-2\pi}\right)^n \right].\end{aligned}$$

The last summation is simply a geometric series. Using the well-known formula for summing geometric series,

$$\sum_{n=-\infty}^{\infty} \frac{1}{1+n^2} = 2\pi \left[ \frac{1}{2} + \frac{e^{-2\pi}}{1-e^{-2\pi}} \right] = \pi \frac{1+e^{-2\pi}}{1-e^{-2\pi}}.$$

## 2.6 Linear Systems

Much of signal processing can be described in terms of systems that can be readily analyzed using the Fourier transform. This section gives a brief introduction to such systems and how Fourier analysis is employed to study their behavior.

Mathematically, a system, $S$, is an operator that takes, as input, any function, $f_I(t)$, from the set of functions pertinent to the problem at hand (say, finite energy functions) and modifies the inputted function according to some fixed scheme to produce a corresponding function, $f_O(t)$, as output. This is denoted by either

$$S: f_I(t) \to f_O(t)$$

or

$$f_O(t) = S[f_I(t)].$$

As indicated, the input function and corresponding output function will, throughout this section, be denoted via the "$I$" and "$O$" subscripts. The output, $f_O(t)$, is also called the system's response to $f_I(t)$.

### 2.6.1 Linear Shift Invariant Systems

A system, $S$, is said to be linear if every linear combination of inputs leads to the corresponding linear combination of outputs. More precisely, $S$ is linear if, given any pair of inputs, $f_I(t)$ and $g_I(t)$, and any pair of constants, $\alpha$ and $\beta$, then

$$S[\alpha f_I(t) + \beta g_I(t)] = \alpha f_O(t) + \beta g_O(t).$$

A system $S$, is said to be shift invariant if any shift in an input function leads to an identical shift in the output, that is, if

$$S[f_I(t - t_0)] = f_O(t - t_0)$$

for every real value of $t_0$ and every allowed input, $f_I(t)$. Other terms commonly used instead of "shift invariant" include "translation invariant," "time invariant," "stationary," and "fixed."

An LSI system is both linear and shift invariant. If $S$ is an LSI system, then, using both linearity and shift invariance, the following string of equalities can be verified:

$$\begin{aligned} S\left[f_I * g_I(t)\right] &= S\left[\int_{-\infty}^{\infty} f_I(s) g_I(t-s)\,ds\right] \\ &= \int_{-\infty}^{\infty} f_I(s) S\left[g_I(t-s)\right] ds \\ &= \int_{-\infty}^{\infty} f_I(s) g_O(t-s)\,ds \\ &= f_I * g_O(t). \end{aligned} \tag{2.6.1.1}$$

Given an LSI system, $S$, the system's impulse response function, usually denoted by $h(t)$, is the output corresponding to an inputted delta function,

$$h(t) = S[\delta(t)].$$

The transfer function of the system is the Fourier transform of the impulse response function,

$$H(\omega) = \mathcal{F}[h(t)]\big|_{\omega}.$$

Combining equation (2.6.1.1) with the fact that $f_I * \delta(t) = f_I(t)$ leads directly to the following important formula for computing the output of a system from any input:

$$f_O(t) = S[f_I(t)] = f_I * h(t). \qquad (2.6.1.2)$$

Taking the Fourier transform gives the equally important formula

$$F_O(\omega) = F_I(\omega)H(\omega), \qquad (2.6.1.3)$$

where $F_O(\omega)$ and $F_I(\omega)$ are the transforms

$$F_O(\omega) = \mathscr{F}[f_O(t)]\big|_\omega \quad \text{and} \quad F_I(\omega) = \mathscr{F}[f_I(t)]\big|_\omega .$$

Formulas (2.6.1.2) and (2.6.1.3) show that the effect of an LSI system on a signal is completely determined by either the system's impulse response function or the system's transfer function. One advantage of knowing the transfer function is that, in many cases, the transfer function provides better intuition on the effect the system has on inputted signals. Also, in many cases, the actual computations are easier using the transfer function instead of the impulse response function. Both advantages are illustrated in Example 2.6.1.1.

### Example 2.6.1.1 Ideal Low-Pass Filter

An ideal low-pass filter with cutoff frequency $\Omega$ (and zero delay) is an LSI system characterized by the transfer function

$$H(\omega) = p_\Omega(\omega) = \begin{cases} 1, & \text{if} \quad |\omega| < \Omega \\ 0, & \text{if} \quad \Omega < |\omega| \end{cases}.$$

The impulse response function is

$$h(t) = \mathscr{F}^{-1}\left[p_\Omega(\omega)\right]\Big|_t = \frac{\sin(\Omega t)}{\pi t}.$$

Given an input signal, $f_I(t)$, with Fourier transform $F_I(\omega)$, the corresponding output, $f_O(t)$, is the inverse transform of

$$F_I(\omega)H(\omega) = \begin{cases} F(\omega), & \text{if} \quad |\omega| < \Omega \\ 0, & \text{if} \quad \Omega < |\omega| \end{cases}.$$

Clearly, this system passes, unaltered, the frequency components of $f_I(t)$ corresponding to frequencies below the cutoff while completely suppressing the frequency components of $f_I(t)$ corresponding to frequencies above the cutoff. For example, if

$$f_I(t) = \sin(\omega_0 t),$$

then

$$F_I(\omega) = \mathscr{F}[\sin(\omega_0 t)]\big|_\omega = -j\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)].$$

Thus,

$$
\begin{aligned}
F_O(\omega) &= -j\pi\left[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right]p_\Omega(\omega)\\
&= -j\pi\left[p_\Omega(\omega_0)\delta(\omega-\omega_0)-p_\Omega(-\omega_0)\delta(\omega+\omega_0)\right]\\
&= \begin{cases} -j\pi\left[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right] & \text{if} \quad |\omega_0|<\Omega\\ 0, & \text{if} \quad \Omega<|\omega_0| \end{cases}
\end{aligned}
$$

and

$$
f_O(t) = \mathcal{F}^{-1}\left[F_O(\omega)\right]\Big|_t = \begin{cases} \sin(\omega_0 t), & \text{if} \quad |\omega_0|<\Omega\\ 0, & \text{if} \quad \Omega<|\omega_0| \end{cases} .
$$

Alternatively, $f_O(t)$ could have been computed using the impulse response function,

$$
f_O(t) = \sin(\omega_0 t) * \frac{\sin(\Omega t)}{\pi t} = \int_{-\infty}^{\infty} \frac{1}{\pi(t-s)} \sin(\omega_0 s)\sin(\Omega(t-s))\,ds .
$$

---

In many applications it is convenient to write the transfer function in the form

$$
H(\omega) = A(\omega)e^{-j\theta(\omega)}
$$

where $A(\omega)$ and $\theta(\omega)$ are real-valued functions (called, respectively, the amplitude and phase of $H(\omega)$) with $A(\omega)$ often assumed to be nonnegative.

**Example 2.6.1.2**

In the simplest case, $A(\omega)$ is constant and $\phi(\omega)$ is linear,

$$
A(\omega) = A_0 \quad \text{and} \quad \phi(\omega) = \tau_0\omega.
$$

In this case,

$$
f_O(t) = \mathcal{F}^{-1}[F_I(\omega)A_0 e^{-j\tau_0\omega}]_t = A_0 f_I(t-\tau_0)
$$

Thus, a system with transfer function

$$
H(\omega) = A_0 e^{-j\tau_0\omega}
$$

amplifies each inputted signal by $A_0$ and delays it by $\tau_0$.

## 2.6.2 Reality and Stability

An LSI system is a "real" system if the output is a real-valued function whenever the input is a real-valued function. In practice, most physically defined systems can be assumed to be real. An equivalent condition for a system to be real is that the impulse response function, $h(t)$, of the system be real valued. By the discussion in Subsection 2.3.1, if the system is real and the transfer function is given by

$$
H(\omega) = A(\omega)e^{-j\theta(\omega)},
$$

where $A(\omega)$ and $\theta(\omega)$ are the amplitude and phase of $H(\omega)$, then

$$h(t) = \frac{1}{\pi}\int_0^\infty A(\omega)\cos(\omega t - \theta(\omega))\,d\omega\,.$$

An LSI system is stable if there is a finite constant, $B$, such that

$$|f_O(t)| \leq BM \quad \text{for all } t$$

whenever

$$|f_I(t)| \leq M \quad \text{for all } t\,.$$

It can be shown that a system is stable if and only if its impulse response function, $h(t)$, is absolutely integrable and that, in this case,

$$B = \int_{-\infty}^{\infty} |h(t)|\,dt\,.$$

It follows from the discussion in Subsection 2.3.2 that if the transfer function, $H(\omega)$, is not bounded and continuous, then the system cannot be stable.

### Example 2.6.2.1

Let $S$ be the ideal low-pass filter of Example 2.6.1.1. Because the impulse response function,

$$h(t) = \frac{1}{\pi t}\sin(\Omega t)\,,$$

is real, so is the system. This is obvious because, if a given input, $f_I(t)$, is real valued, so must be

$$f_O(t) = f_I * h(t) = \int_{-\infty}^{\infty} f_I(s)\frac{1}{\pi(t-s)}\sin(\Omega(t-s))\,ds\,.$$

Because the transfer function,

$$H(\omega) = p_\Omega(\omega) = \begin{cases} 1, & \text{if} \quad |\omega| < \Omega \\ 0, & \text{if} \quad \Omega < |\omega| \end{cases},$$

is not continuous, the system cannot be stable. This is easily verified using the input function

$$f_I(t) = \begin{cases} +1, & \text{if} \quad 0 \leq \frac{1}{t}\sin(\Omega t) \\ -1, & \text{if} \quad \frac{1}{t}\sin(\Omega t) < 0 \end{cases}.$$

Clearly,

$$|f_I(t)| \leq 1 \quad \text{for all } t,$$

but

$$f_I(s)h(-s) = \left|\frac{1}{s}\sin(\Omega s)\right| \quad \text{for all } s\,.$$

Thus,

$$\begin{aligned} f_O(0) &= f_I * h(0) \\ &= \int_{-\infty}^{\infty} f_I(s)h(0-s)\,ds \\ &= \int_{-\infty}^{\infty} \left|\frac{1}{s}\sin(\Omega s)\right| ds \\ &= \infty\,. \end{aligned}$$

### 2.6.3 System Response to Complex Exponentials and Periodic Functions

Let $S$ be an LSI system with impulse response function $h(t)$ and transfer function $H(\omega)$. Because $\mathcal{F}[e^{j\omega_0 t}]\big|_\omega = 2\pi\delta(\omega - \omega_0)$,

$$\begin{aligned} S\left[e^{j\omega_0 t}\right] &= \mathcal{F}^{-1}\left[2\pi\delta(\omega-\omega_0)H(\omega)\right]\Big|_t \\ &= \mathcal{F}^{-1}\left[2\pi\delta(\omega-\omega_0)H(\omega_0)\right]\Big|_t \\ &= H(\omega_0)e^{j\omega_0 t}\,. \end{aligned} \tag{2.6.3.1}$$

By this it is seen that the complex exponentials are eigenfunctions for $S$ and that the transfer function gives the corresponding eigenvalues.

If $f_I(t)$ is a periodic function with period $p$ and with Fourier series

$$\sum_{n=-\infty}^{\infty} c_n e^{jn\Delta\omega t},$$

where $\Delta\omega = 2\pi/p$, then, from equation (2.6.3.1) and the linearity of the system,

$$S\left[f_I(t)\right] = \sum_{n=-\infty}^{\infty} c_n H(n\Delta\omega)e^{jn\Delta\omega t}\,. \tag{2.6.3.2}$$

In particular, for $\alpha > 0$,

$$S\left[\cos(\alpha t)\right] = \frac{1}{2}\left[H(\alpha)e^{j\alpha t} + H(-\alpha)e^{-j\alpha t}\right] \tag{2.6.3.3}$$

and

$$S[\sin(\alpha t)] = \frac{1}{2j}\left[H(\alpha)e^{j\alpha t} - H(-\alpha)e^{-j\alpha t}\right]. \tag{2.6.3.4}$$

If $S$ is a real LSI system, then the imaginary parts of (2.6.3.3) and (2.6.3.4) must vanish. Using this fact, it can be shown that

$$S[\cos(\alpha t)] = A(\alpha)\cos(\alpha t - \theta(\alpha))$$

and

$$S[\sin(\alpha t)] = A(\alpha)\sin(\alpha t - \theta(\alpha)),$$

where $A(\omega)$ and $\theta(\omega)$ are the amplitude and phase of the transfer function,

$$H(\omega) = A(\omega)e^{-j\theta(\omega)}.$$

**Example 2.6.3.1**

Let $S$ be the ideal low-pass filter from Example 2.6.1.1 with transfer function

$$H(\omega) = p_\Omega(\omega) = \begin{cases} 1, & \text{if } |\omega| < \Omega \\ 0, & \text{if } \Omega < |\omega| \end{cases}.$$

For this example assume $\Omega = 20.5\,\pi$ and let $f_I(t)$ be the sawtooth function from Example 2.3.9.1. As seen in that example, $\Delta\omega = \pi$ and

$$f_I(t) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (-1)^n \frac{j}{n\pi} e^{jn\pi t}.$$

From this and equation (2.6.3.2) it follows that

$$f_O(t) = \sum_{\substack{n=-20 \\ n\neq 0}}^{20} (-1)^n \frac{j}{n\pi} e^{jn\pi t}.$$

The graphs of $f_I(t)$ and $f_O(t)$ are sketched in Figures 2.8 and 2.9, respectively.

## 2.6.4 Casual Systems

A function, $f(t)$, is said to be "casual" if

$$f(t) = 0, \quad \text{whenever } t < 0.$$

An LSI system, $S$, is said to be "casual" if the response of the system to every causal input is a causal output. By shift invariance, this is equivalent to defining $S$ to be casual if

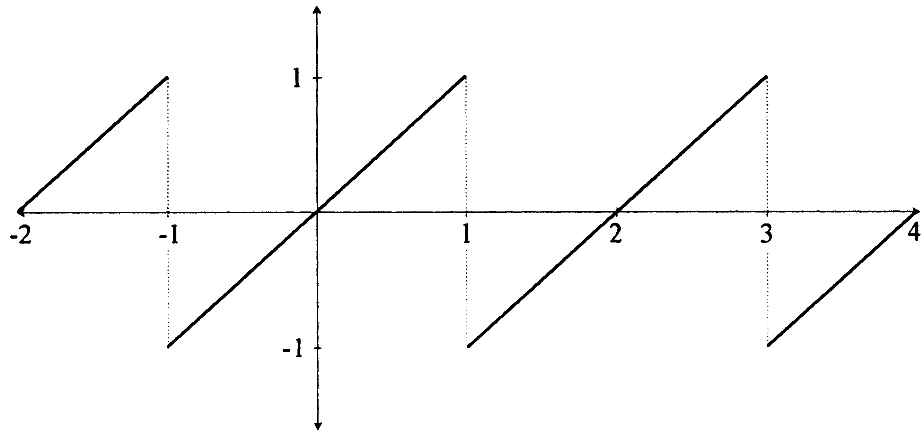



**FIGURE 2.8** The saw function.

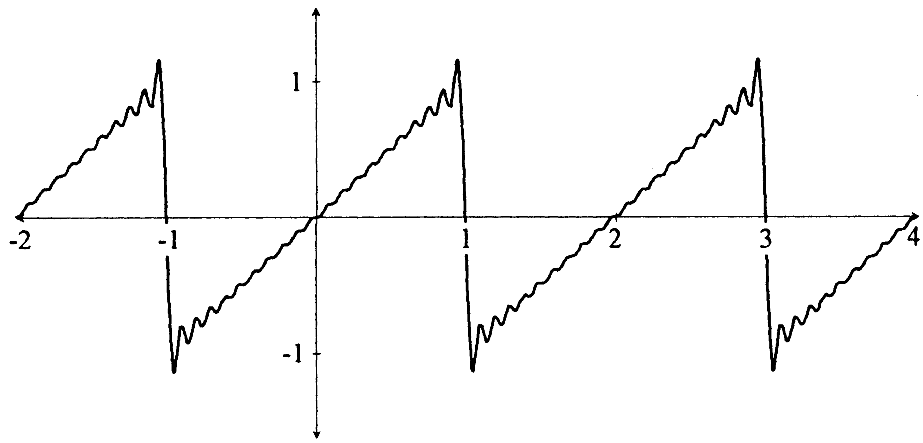



**FIGURE 2.9** Low-pass filter output from a saw function input.

$$f_I(t) = 0, \text{ whenever } t < t_0$$

implies that

$$f_O(t) = S[f_I(t)] = 0, \text{ whenever } t < t_0$$

for any real value of $t_0$.

If $S$ is a causal system, then its impulse response function, $h(t)$, must also be causal and formula (2.6.1.2) for computing the response of a system to an input $f_I(t)$ becomes

$$f_O(t) = \int_{-\infty}^{t} f_I(s) h(t-s) ds$$

or equivalently,

$$f_O(t) = \int_{0}^{\infty} f_I(t-s) h(s) ds .$$

If the input is also causal, then these further reduce to

$$f_O(t) = \int_0^t f_I(s)h(t-s)ds$$

and

$$f_O(t) = \int_0^t f_I(t-s)h(s)ds\,.$$

### 2.6.5 Systems Given by Differential Equations

Often the output, $f_O(t)$, of a system, $S$, is a solution to a nonhomogeneous ordinary differential equation with the input being the nonhomogeneous part of the equation,

$$\sum_{n=0}^{N} A_n \frac{d^n}{dt^n}\left[f_O(t)\right] = f_I(t)\,.$$

As long as the $A_n$'s are constants, it is easily verified that $S$ is an LSI system. The impulse response function, $h(t)$, must satisfy

$$\sum_{n=0}^{N} A_n \frac{d^n h}{dt^n} = \delta(t)\,. \tag{2.6.5.1}$$

The general solution to (2.6.5.1) can be written

$$h_g(t) = h_p(t) + y_c(t),$$

where $y_c(t)$ is the general solution to the corresponding homogeneous equation,

$$\sum_{n=0}^{N} A_n \frac{d^n y}{dt^n} = 0\,, \tag{2.6.5.2}$$

and $h_p(t)$ is any particular solution to (2.6.5.1). After a particular solution is found the undetermined constants in $y_c(t)$ must be determined so that the resulting

$$h(t) = h_p(t) + y_c(t)$$

satisfies any additional constraints on the output (causality, finite energy, etc.).

The particular solution, $h_p(t)$, can be found by taking the Fourier transform of both sides of (2.6.5.1). Using identity (2.2.11.6) gives an equation for $H_p(\omega) = \mathcal{F}[h_p(t)]\big|_\omega$ ,

$$\sum_{n=0}^{N} A_n (j\omega)^n H_p(\omega) = 1\,. \tag{2.6.5.3}$$

Dividing through by

$$D(\omega) = \sum_{n=0}^{N} A_n (j\omega)^n$$

gives

$$H_p(\omega) = \frac{1}{D(\omega)}, \tag{2.6.5.4}$$

which is a rational function of $\omega$. Taking the inverse transform of $H_p(\omega)$ (using, say, the approach described in Subsection 2.3.1.4) then yields $h_p(t)$.

Example 2.6.5.1 illustrates a common situation in which the obtained $h_p(t)$ already satisfies the additional conditions and, thus, can be used directly as the impulse response function. In Example 2.6.5.2, $h_p(t)$ is not a valid output and, so, a nontrivial solution to the corresponding homogeneous equation must be added to obtain the impulse response function.

### Example 2.6.5.1

Let the output $f_O(t)$, corresponding to an input, $f_I(t)$, be given by the finite energy solution to

$$\frac{d^2y}{dt^2} - y = f(t).$$

The solution to the corresponding homogeneous equation,

$$\frac{d^2y}{dt^2} - y = 0,$$

is

$$y_c(t) = c_1e^t + c_2e^{-t},$$

while the impulse response function must satisfy

$$\frac{d^2h}{dt^2} - h = \delta(t). \tag{2.6.5.5}$$

Letting $h_p(t)$ denote a particular solution and taking the Fourier transform of both sides yields

$$-\omega^2 H_p(\omega) - H_p(\omega) = 1,$$

which, after some elementary algebra, reduces to

$$H_p(\omega) = \frac{-1}{1+\omega^2}.$$

The inverse transform of this can be computed directly from tables:

$$h_p(t) = -\frac{1}{2}\mathcal{F}^{-1}\left[\frac{2}{1+\omega^2}\right]\Bigg|_t = -\frac{1}{2}e^{-|t|}.$$

The general solution to (2.6.5.5) is the sum of $h_p(t)$ and $y_c(t)$,

$$h_g(t) = -\frac{1}{2}e^{-|t|} + c_1 e^t + c_2 e^{-t},$$

but, clearly, the only way $h_g(t)$ can be a finite energy function is for $c_1 = c_2 = 0$. Thus, the impulse response function and the transfer function for this system are

$$h(t) = -\frac{1}{2}e^{-|t|}$$

and

$$H(\omega) = \frac{-1}{1+\omega^2}.$$

**Example 2.6.5.2**

Assume $S$: $f_I(t) \to f_O(t)$ is a causal system for which the output satisfies

$$\frac{d^2 f_O}{dt^2} + f_O = f_I(t).$$

The solution to the corresponding homogeneous equation,

$$\frac{d^2 y}{dt^2} + y = 0,$$

is

$$y_c(t) = c_1 \cos(t) + c_2 \sin(t),$$

while the impulse response function must satisfy

$$\frac{d^2 h}{dt^2} + h = \delta(t). \tag{2.6.5.6}$$

Letting $h_p(t)$ denote a particular solution and taking the Fourier transform of both sides yields

$$-\omega^2 H_p(\omega) + H_p(\omega) = 1$$

which, after some elementary algebra, reduces to

$$H_p(\omega) = \frac{1}{1-\omega^2} = \frac{1}{2}\left[\frac{1}{\omega+1} - \frac{1}{\omega-1}\right].$$

Using either the tables or formula (2.3.14.1),

$$h_p(t) = \frac{1}{2}\left[\frac{j}{2}e^{-jt}\operatorname{sgn}(t) - \frac{j}{2}e^{jt}\operatorname{sgn}(t)\right] = \frac{1}{2}\sin(t)\operatorname{sgn}(t).$$

The general solution to (2.6.5.6) is then

$$h_g(t) = h_p(t) + y_c(t) = \frac{1}{2}\sin(t)\operatorname{sgn}(t) + c_1\cos(t) + c_2\sin(t).$$

Because $S$ is a causal system and $\delta(t) = 0$ for $t < 0$, the impulse response function must vanish for negative values of $t$. Thus, $c_1$ and $c_2$ must be chosen so that for $t < 0$,

$$\begin{aligned} h_g(t) &= \frac{1}{2}\sin(t)\operatorname{sgn}(t) + c_1\cos(t) + c_2\sin(t) \\ &= \left(c_2 - \frac{1}{2}\right)\sin(t) + c_1\cos(t) \\ &= 0. \end{aligned}$$

Clearly, $c_1 = 0$, $c_2 = 1/2$, and the impulse response function is

$$h(t) = \frac{1}{2}\sin(t)\operatorname{sgn}(t) + \frac{1}{2}\sin(t) = \sin(t)u(t).$$

The transfer function is

$$\begin{aligned} H(\omega) &= \mathcal{F}\left[\sin(t)u(t)\right]\Big|_{\omega} \\ &= \frac{1}{2j}\left[\left(\pi\delta(\omega-1) - j\frac{1}{\omega-1}\right) - \left(\pi\delta(\omega+1) - j\frac{1}{\omega+1}\right)\right] \\ &= \frac{1}{1-\omega^2} + j\frac{\pi}{2}\left[\delta(\omega+1) - \delta(\omega-1)\right]. \end{aligned}$$

## 2.6.6 RLC Circuits

Consider the electric circuit sketched in Figure 2.10. This circuit consists of

A resistor with a fixed resistance of $R$ ohms,
An inductor with a fixed inductance of $L$ henries,
A capacitor with a fixed capacitance of $C$ farads, and
A time varying voltage supply providing a voltage of $E(t)$ volts.

The charge on the capacitor at time $t$ will be denoted by $q(t)$ and the corresponding current in the circuit by $i(t)$. The charge and current are related by

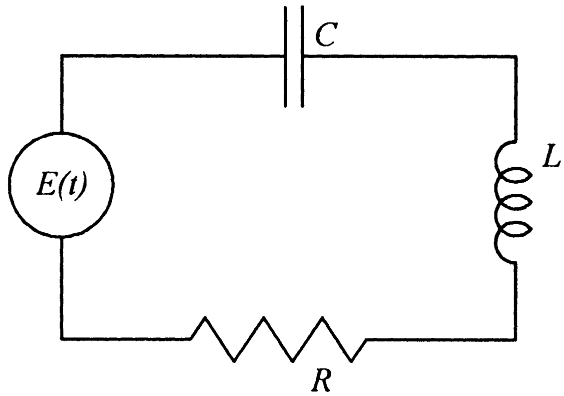



**FIGURE 2.10** A simple RLC circuit.

$$i(t) = \frac{dq}{dt}.$$

By Kirchloff's laws

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t). \tag{2.6.6.1}$$

For a physical circuit $R$ must be positive and $L$ and $C$ cannot be negative. Also, it is reasonable to assume that no charge accumulates on the capacitor if no voltage has previously been provided. Thus, $q(t)$ can be viewed as the output corresponding to an input of $E(t)$ to a causal LSI system. The impulse response function, $h(t)$, to this system satisfies

$$L\frac{d^2h}{dt^2} + R\frac{dh}{dt} + \frac{1}{C}h = \delta(t). \tag{2.6.6.2}$$

If the inductance and capacitance are nonzero, then straightforward computations, similar to those done in the examples of Subsections 2.6.5 and 2.3.14 lead to

$$h(t) = \frac{1}{2L\beta}\left[e^{\beta t} - e^{-\beta t}\right]e^{-\alpha t}u(t)$$

and

$$H(\omega) = \frac{j}{2L\beta}\left[\frac{1}{\omega - j(\alpha+\beta)} - \frac{1}{\omega - j(\alpha-\beta)}\right] = \frac{-C}{LC\omega^2 - jRC\omega - 1},$$

where

$$\alpha = \frac{R}{2L} \quad \text{and} \quad \beta = \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}.$$

It should be noted that, because the real part of $\alpha \pm \beta$ is positive, $h(t)$ is bounded by a decreasing exponential on $0 < t$. Hence, $h(t)$ is absolutely integrable and the system is stable.

In practice, the current, $i(t)$, is often of greater interest than the charge on the capacitor. Because

$$i(t) = \frac{dq}{dt} = \frac{d}{dt}\left(E * h(t)\right) = E * h'(t),$$

it follows that the current is given by a system with impulse response function

$$h_i(t) = h'(t) = \frac{1}{2L\beta}\left[(\beta - \alpha)e^{\beta t} + (\beta + \alpha)e^{-\beta t}\right]e^{-\alpha t}u(t)$$

and transfer function

$$H_i(\omega) = j\omega H(\omega) = \frac{-jC\omega}{LC\omega^2 - jRC\omega - 1}.$$

In either case the response of the system to the impulse function will depend on whether $\beta$ has an imaginary component. If $\beta$ does have a nonzero imaginary component, the unit impulse response will be a sinusoidal function with an exponentially decreasing envelope. If the imaginary part of $\beta$ is zero, then the response is simply a linear combination of decreasing exponentials.

### 2.6.7 Modulation and Demodulation

Let $f(t)$ be any band-limited function with bandwidth $2\Omega$, and let $\omega_c$ and $t_0$ be real constants with $\Omega < \omega_c$. The product

$$g(t) = f(t)\cos(\omega_c t - t_0)$$

is the "modulation of the carrier signal, $\cos(\omega_t - t_0)$, by $f(t)$." The extraction of the modulating signal, $f(t)$, from the modulated signal provides an especially nice example of the application of Fourier analysis in signal processing.

To extract $f(t)$, first multiply $g(t)$ by the carrier signal. This gives

$$\begin{aligned} g(t)\cos(\omega_c t - t_0) &= f(t)\cos^2(\omega_c t - t_0) \\ &= f(t)\left[\frac{1}{2} + \frac{1}{2}\cos(2\omega_c t - 2t_0)\right] \\ &= \frac{1}{2}f(t) + \frac{1}{2}f(t)\cos(2\omega_c t - 2t_0). \end{aligned}$$

Using the basic identities, the Fourier transform of this is found to be

$$\begin{aligned} &\frac{1}{2}F(\omega) + \frac{1}{2}F(\omega) * \left[\pi e^{-j4t_0\omega_c}\delta(\omega - 2\omega_c) + \pi e^{j4t_0\omega_c}\delta(\omega + 2\omega_c)\right] \\ &= \frac{1}{2}F(\omega) + \frac{\pi}{2}\left[e^{-j4t_0\omega_c}F(\omega - 2\omega_c) + e^{j4t_0\omega_c}F(\omega + 2\omega_c)\right]. \end{aligned}$$

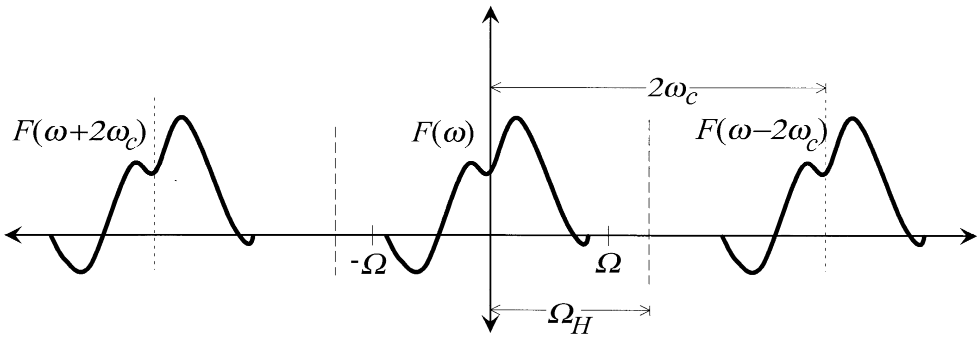



**FIGURE 2.11** Translations of the transform of a band-limited function.

Sketches of $F(\omega)$, $F(\omega - 2\omega_c)$, and $F(\omega + 2\omega_c)$ are given in Figure 2.11. Observe that because $f(t)$ is a band-limited function with bandwidth $2\Omega$ and $\Omega < \omega_c$, if $H(\omega)$ is any band-limited function with bandwidth $2\Omega_H$ satisfying $\Omega_H < 2\omega_c - \Omega$, then

$$\left(\frac{1}{2}F(\omega)+\frac{\pi}{2}\left[e^{-j4t_0\omega_c}F(\omega-2\omega_c)+e^{j4t_0\omega_c}F(\omega+2\omega_c)\right]\right)H(\omega)=\frac{1}{2}F(\omega)H(\omega).$$

In particular if $H(\omega)$ is the perfect low-pass filter of Examples 2.6.1.1 and 2.6.3.1,

$$H(\omega)=\begin{cases}1, & \text{if } |\omega|<\Omega_H \\ 0, & \text{if } |\omega|>\Omega_H\end{cases},$$

with

$$\Omega < \Omega_H < 2\omega_c - \Omega ,$$

then

$$\left(\frac{1}{2}F(\omega)+\frac{\pi}{2}\left[e^{-j4t_0\omega_c}F(\omega-2\omega_c)+e^{j4t_0\omega_c}F(\omega+2\omega_c)\right]\right)H(\omega)=\frac{1}{2}F(\omega).$$

Thus, the signal

$$g(t) = f(t)\cos(\omega_c t - t_0).$$

can be perfectly demodulated (i.e., $f(t)$ can be completely extracted) by first multiplying the modulated signal by the carrier and then passing the result through an appropriate ideal low-pass filter.

## 2.7 Random Variables

Because noise is an intrinsic factor in real-world systems, random variables play an important role in the mathematics of practical engineering problems. As illustrated in this section, the Fourier transform is a useful tool in analyzing signals containing a significant random component and in extracting usable information from these signals.

### 2.7.1 Basic Probability and Statistics

A nonnegative function, $p(x)$, is a probability density function if it satisfies

$$\int_{-\infty}^{\infty} p(x)\,dx = 1\,.$$

Such a function is absolutely integrable and so its Fourier transform, $P(y) = \mathcal{F}[p(x)]\big|_y$, must be continuous and must satisfy $P(0) = 1$ (see Subsection 2.3.2).

If $x$ denotes the outcome of a random process governed by the probability density function $p(x)$ and if $-\infty \le a \le b \le \infty$, then

$$\int_a^b p(x)\,dx$$

is the probability that $x$ is between $a$ and $b$. The "mean" or "expected value" of $x$, is denoted by either $\mu$ or $E[x]$, and is given by the first moment of $p(x)$,

$$\mu = E[x] = \int_{-\infty}^{\infty} x p(x)\,dx\,.$$

The variance of $x$, denoted by either $\sigma^2$ or $\mathrm{Var}[x]$, is the second moment of $p(x)$ about its mean,

$$\sigma^2 = \mathrm{Var}[x] = \int_{-\infty}^{\infty} (x-\mu)^2 p(x)\,dx\,,$$

and the standard deviation, $\sigma$, is the square root of the variance. More generally, if $f(x)$ is any function of $x$ then $f(x)$ is a random variable with expected value

$$\mu = E[f(x)] = \int_{-\infty}^{\infty} f(x) p(x)\,dx$$

and variance

$$\mathrm{Var}[f(x)] = \int_{-\infty}^{\infty} |f(x)-\mu|^2 p(x)\,dx\,.$$

In particular, $\mathrm{Var}[x] = E[|x-\mu|^2]$. It is easy to show that the variance of $x$, $\sigma^2$, is directly related to the first and second moments of $p(x)$,

$$\sigma^2 = E[x^2] - \mu^2 = \int_{-\infty}^{\infty} x^2 p(x)\,dx - \left(\int_{-\infty}^{\infty} x p(x)\,dx\right)^2\,.$$

It follows from the discussion in Subsections 2.2.11 and 2.2.12 that if $p(x)$ is a probability density function, then the corresponding mean, expected value of $x^2$, and variance can be computed from the density function's transform, $P(y) = \mathcal{F}[p(x)]\big|_y$, by

$$\mu = jP'(0), \qquad (2.7.1.1)$$

$$E[x^2] = -P''(0), \tag{2.7.1.2}$$

and

$$\sigma^2 = [P'(0)]^2 - P''(0). \tag{2.7.1.3}$$

**Example 2.7.1.1 The Normal Distribution**

A normal (or Gaussian) probability distribution is given by the density function

$$p(x) = \sqrt{\frac{\alpha}{\pi}} e^{-\alpha(x-x_0)^2},$$

where $\alpha > 0$. Using the tables it is easily verified that

$$P(y) = \mathscr{F}[p(x)]\Big|_y = \exp\left[-\frac{1}{4\alpha}y^2 - jx_0 y\right].$$

Furthermore,

$$P(0) = 1,$$

$$P'(y) = -\left(\frac{1}{2\alpha}y + jx_0\right)P(y),$$

and

$$P''(y) = \frac{1}{4\alpha^2}\left[\left(y + j2\alpha x_0\right)^2 - 2\alpha\right]P(y).$$

Using formulas (2.7.1.1) through (2.7.1.3) to compute the mean and variance,

$$\mu = jP'(0) = -j(0 + jx_0)P(0) = x_0,$$

$$E\left[x^2\right] = -P''(0) = -\frac{1}{4\alpha^2}\left[\left(0 + j2\alpha x_0\right)^2 - 2\alpha\right]P(0) = x_0^2 + \frac{1}{2\alpha},$$

and

$$\sigma^2 = \left[P'(0)\right]^2 - P''(0) = \frac{1}{2\alpha}.$$

Replacing $x_0$ and $\alpha$ in the above formulas for $p(x)$ and $P(y)$ it follows that the normal probability distribution with mean $\mu$ and standard deviation $\sigma$ is given by the density function

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right],$$

and that its Fourier transform is given by

$$P(y) = \exp\left[-\frac{1}{2}(\sigma y)^2 - j\mu y\right].$$

**Example 2.7.1.2 The Binomial Distribution**

Consider a process consisting of $n$ repetitions of an experiment with exactly two outcomes, "success" and "failure." Let $p_0$ be the probability of "success" and $q_0$ the probability of "failure" in one experiment (hence, $p_0 + q_0 = 1$). Such a process is governed by the binomial probability density function

$$p(x) = \sum_{k=0}^{n} \binom{n}{k} p_0^k\, q_0^{n-k}\, \delta(x-k)$$

with

$$\int_a^b p(x)\, dx = \sum_{a<k<b} \binom{n}{k} p_0^k\, q_0^{n-k}$$

being the probability that the number of "successes," $x$, satisfies $a < x < b$.

In example 2.3.10.3 the Fourier transform of this function was found to be

$$P(y) = (p_0 e^{-jy} + q_0)^n.$$

Assuming that $n > 1$,

$$P'(y) = -jnp_0 e^{-jy}\,(p_0 e^{-jy} + q_0)^{n-1}$$

and

$$P''(y) = -np_0 e^{-jy}\,(np_0 e^{-jy} + q_0)\,(p_0 e^{-jy} + q_0)^{n-2}.$$

Thus,

$$P'(0) = -jnp_0 \quad\text{and}\quad P''(0) = -np_0(np_0 + q_0).$$

So, using formulas (2.7.1.1) through (2.7.1.3) to compute the mean and variance,

$$\mu = jP'(0) = np_0,$$

$$E[x^2] = -P''(0) = np_0(np_0 + q_0),$$

and

$$\sigma^2 = [P'(0)]^2 - P''(0) = np_0 q_0.$$

## 2.7.2 Multiple Random Processes and Independence

Let $x_1$ and $x_2$ denote the outcomes of two random processes governed, respectively, by probability density functions $p_1(x_1)$ and $p_2(x_2)$, and with corresponding means $\mu_1$ and $\mu_2$, and corresponding standard deviations $\sigma_1$ and $\sigma_2$. Taken as a single pair, $(x_1, x_2)$ can be viewed as the outcome of a single two-dimensional random process. This process will be governed by a probability density function of two variables, $q(x_1, x_2)$. Given any $-\infty \le a_1 \le b_1 \le \infty$ and $-\infty \le a_2 \le b_2 \le \infty$, the probability that both

$$a_1 < x_1 < b_1 \quad \text{and} \quad a_2 < x_2 < b_2$$

is

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} q(x_1, x_2) dx_1 dx_2 .$$

In general, the relationship between the joint density function $q(x_1, x_2)$, and the individual density functions, $p_1(x_1)$ and $p_2(x_2)$, depends strongly on the relationship that exists between the two random processes. If $x_1$ and $x_2$ are, in fact, the same, then

$$q(x_1, x_2) = p_1(x_1)\delta(x_2 - x_1). \tag{2.7.2.1}$$

If the two random processes are completely independent of each other, then, for all values of $x_1$ and $x_2$,

$$q(x_1, x_2) = p_1(x_1)p_2(x_2). \tag{2.7.2.2}$$

### Example 2.7.2.1

Let $x$ denote the number of heads resulting from a single toss of a fair coin, and let $x_1$ and $x_2$ be the number of heads reported by two perfectly accurate observers, each observing the single toss of a fair coin. The probability density function for $x$, $p(x)$, is well known to be

$$p(x) = \frac{1}{2}\delta(x) + \frac{1}{2}\delta(x-1) .$$

If both observers are observing the same coin toss, then $x_1 = x_2$ and, according to formula (2.7.2.1), the joint probability density function is

$$q_{\text{same}}(x_1, x_2) = \left[\frac{1}{2}\delta(x_1) + \frac{1}{2}\delta(x_1 - 1)\right]\delta(x_2 - x_1) . \tag{2.7.2.3}$$

Note that if $\alpha$ is any real number and $\phi(x_1, x_2)$ is any two-dimensional test function, then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x_1 - \alpha)\delta(x_1 - x_2)\phi(x_1, x_2) dx_1 dx_2 = \phi(\alpha, \alpha)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x_1 - \alpha, x_2 - \alpha)\phi(x_1, x_2) dx_1 dx_2 .$$

This shows that, in general,

$$\delta(x_1 - \alpha)\delta(x_1 - x_2) = \delta(x_1 - \alpha, x_2 - \alpha),$$

which, in turn, verifies that formula (2.7.2.3) is completely equivalent to the formula

$$q_{\text{same}}(x_1, x_2) = \frac{1}{2}\delta(x_1, x_2) + \frac{1}{2}\delta(x_1 - 1, x_2 - 1),$$

obtained by elementary probability theory.

On the other hand, if the two observers are observing two different tosses of the coin, then the value of $x_1$ and $x_2$ are independent of each other and the joint density function is

$$\begin{aligned} q_{\text{indep}}(x_1, x_2) &= \left[\frac{1}{2}\delta(x_1) + \frac{1}{2}\delta(x_1 - 1)\right]\left[\frac{1}{2}\delta(x_2) + \frac{1}{2}\delta(x_2 - 1)\right] \\ &= \frac{1}{4}\delta(x_1)\delta(x_2) + \frac{1}{4}\delta(x_1)\delta(x_2 - 1) \\ &\quad + \frac{1}{4}\delta(x_1 - 1)\delta(x_2) + \frac{1}{4}\delta(x_1 - 1)\delta(x_2 - 1), \end{aligned}$$

which agrees with the formula

$$q_{\text{indep}}(x_1, x_2) = \frac{1}{4}\left[\delta(x_1, x_2) + \delta(x_1, x_2 - 1) + \delta(x_1 - 1, x_2) + \delta(x_1 - 1, x_2 - 1)\right]$$

obtained by elementary probability theory.

---

Formula (2.7.2.2) gives the mathematical definition for $x_1$ and $x_2$ being independent random variables. Assuming $x_1$ and $x_2$ are independent, the mean of the product is

$$\begin{aligned} E[x_1 x_2] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 p_1(x_1) p_2(x_2)\, dx_1 dx_2 \\ &= \int_{-\infty}^{\infty} x_1 p_1(x_1)\, dx_1 \int_{-\infty}^{\infty} x_2 p_2(x_2)\, dx_2 \qquad (2.7.2.4) \\ &= \mu_1 \mu_2 . \end{aligned}$$

Similar computations show that

$$\text{Var}[x_1 x_2] = \sigma_1^2 \sigma_2^2 .$$

It should also be noted that the two-dimensional transform of the joint probability density is

$$\begin{aligned} Q(y_1 y_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_1(x_1) p_2(x_2) e^{-j(x_1 y_1 + x_2 y_2)}\, dx_1 dx_2 \\ &= \int_{-\infty}^{\infty} p_1(x_1) e^{-jx_1 y_1}\, dx_1 \int_{-\infty}^{\infty} p_2(x_2) e^{-jx_2 y_2}\, dx_2 \\ &= P_1(y_1) P_2(y_2) \end{aligned}$$

where $P_1(y_1)$ and $P_2(y_2)$ and the Fourier transforms of $p_1(x_1)$ and $p_2(x_2)$, respectively.

More generally, any number of random variables — $x_1, x_2, \ldots, x_n$ — are considered to be independent if the probability density function for the vector $(x_1, x_2, \ldots, x_n)$ is the product of the density functions of the individual variables,

$$q(x_1, x_2, \ldots, x_n) = p_1(x_1)p_2(x_2) \cdots p_n(x_n).$$

If $x_1, x_2, \ldots, x_n$ are independent, then the $n$-dimensional Fourier transform of the joint density function is simply the product of the one-dimensional Fourier transforms of the individual density functions,

$$Q(x_1, x_2, \ldots, x_n) = P_1(x_1)P_2(x_2) \cdots P_n(x_n),$$

and the mean of the product of the variables and the corresponding variance are merely the products of the means and variances of the individual variables

$$E[x_1 x_2 \cdots x_n] = \mu_1 \mu_2 \cdots \mu_n$$

and

$$\mathrm{Var}\left[x_1 x_2 \cdots x_n\right] = \sigma_1^2 \sigma_2^2 \cdots \sigma_n^2 .$$

### 2.7.3 Sums of Random Processes

Let $x_1$ and $x_2$ denote the outcome of two independent random processes governed, respectively, by probability density functions $p_1(x)$ and $p_2(x)$, and with corresponding means $\mu_1$ and $\mu_2$, and corresponding standard deviations $\sigma_1$ and $\sigma_2$. The sum of these two outcomes,

$$x_S = x_1 + x_2,$$

can be viewed as the outcome of another random process, which is governed by the probability density function

$$p_S(x) = \int_{-\infty}^{\infty} p_1(x - \xi) p_2(\xi)\, d\xi = p_1 * p_2(x).$$

If $P_1(y)$, $P_2(y)$, and $P_S(y)$ are the Fourier transforms of $p_1(x)$, $p_2(x)$, and $p_S(x)$, then, by identity (2.2.9.3),

$$P_S(y) = P_1(y)P_2(y).$$

Thus,

$$P_S(0) = P_1(0)P_2(0) = 1$$

and

$$P_S'(0) = P_1'(0)P_2(0) + P_1(0)P_2'(0) = P_1'(0) + P_2'(0).$$

From this last equation and equation (2.7.1), it immediately follows that the mean of $x_S$ is the sum of the means of $x_1$ and $x_2$,

$$\mu_S = \mu_1 + \mu_2.$$

Likewise, computing $P_S''(0)$ and using equations (2.7.1.2) and (2.7.1.3) leads to

$$E\left[x_S^2\right] = E\left[x_1^2\right] + E\left[x_2^2\right] + 2\mu_1\mu_2$$

and

$$\sigma_S^2 = \sigma_1^2 + \sigma_2^2 .$$

More generally, if

$$x_S = x_1 + x_2 + \cdots + x_N,$$

where each $x_n$ denotes the outcome of an independent random process governed by a probability density function, $p_n(x)$, and with corresponding mean and standard deviation, $\mu_n$ and $\sigma_n$, then $x_s$ is governed by the probability density function

$$p_S(x) = p_1 * p_2 * \cdots * p_N(x)$$

and has mean and variance

$$\mu_S = \mu_1 + \mu_2 + \cdots + \mu_N,$$

and

$$\sigma_S^2 = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_N^2 .$$

If $N$ is fairly large, the central limit theorem of probability theory states that under very general conditions,

$$p_S(x) \approx \frac{1}{\sigma_S\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu_S}{\sigma_S}\right)^2\right]$$

or, equivalently, that

$$P_S(y) = \mathscr{F}\left[p_S(x)\right]\Big|_y \approx \exp\left[-\frac{1}{2}(\sigma_S y)^2 - j\mu_S y\right].$$

In practice, the "noise" in a system is often the result of a large number of random processes each of which contributes a term to the total noise. According to the above discussion, it is not necessary to describe each source of noise accurately. Instead, the aggregate can be treated as a random process governed by a normal distribution.

## 2.7.4 Random Signals and Stationary Random Signals

A signal, $x(t)$, is "deterministic" if it can be treated, mathematically, as a well-defined function of $t$, that is, if for each value of $t$ there is a single fixed value for $x(t)$. The signal is "random" if, instead, for each value of $t$, $x(t)$ must be treated as the outcome of a nontrivial random process.

Assume $x(t)$ is a random signal. For each value of $t$ there is a corresponding probability density function, $p(x, t)$, with

$$\int_a^b p(x, t)dx$$

being the probability of $a < x(t) < b$. The corresponding mean and variance,

$$E[x(t)] = \mu(t) = \int_{-\infty}^{\infty} xp(x, t)dx$$

and

$$\mathrm{Var}[x(t)] = \sigma^2(t) = \int_{-\infty}^{\infty} (x - \mu(t))^2 p(x, t)dx ,$$

are deterministic functions of $t$. The Fourier transform of the density function is

$$P(y, t) = \int_{-\infty}^{\infty} p(x, t)e^{-jxy}dx .$$

If the statistical properties of the process generating $x(t)$ do not vary with $t$, then the process is said to be a "stationary" random process. The corresponding signal will also be called "stationary" though its value will certainly depend — in a random manner — on $t$. For a stationary random signal, $x(t)$, it is reasonable to expect that the long-term time average of $x(t)$ will equal its mean, $E[x] = \mu$,

$$\mu = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)dt . \tag{2.7.4.1}$$

Mathematically, it can be shown that, under fairly broad conditions, the probability that equation (2.7.4.1) is not correct for a given stationary random signal is vanishingly small. Likewise,

$$E[x^2(t)] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2 dt$$

and

$$\sigma^2 = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t) - \mu|^2 dt .$$

Thus, stationary random signals (with finite mean and variances) can be treated as finite power functions (see Subsection 2.3.8). Given a stationary random signal, $x(t)$, the corresponding average autocorrelation function is

$$\bar{\rho}_x(t) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x^*(s)x(t+s)ds .$$

The average power is

$$\bar{\rho}_x(0) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(s)|^2 ds$$

and the power spectrum is

$$P_x(\omega) = \mathscr{F}\left[\bar{\rho}_x(t)\right]\Big|_\omega = \int_{-\infty}^{\infty} \bar{\rho}_x(t) e^{-j\omega t} dt$$

or, equivalently,

$$P_x(\omega) = \lim_{T\to\infty} \frac{1}{2T} \left| \int_{-T}^{T} x(t) e^{-j\omega t} dt \right|^2 .$$

It should be recalled that one property of the average autocorrelation function is that

$$\left(\bar{\rho}_x(t)\right)^* = \bar{\rho}_x(-t). \qquad (2.7.4.2)$$

Thus, if $x(t)$ is a real random signal, then the average autocorrelation will be an even real-valued function. So, also, will the power spectrum.

### 2.7.5 Correlation of Stationary Random Signals and Independence

The average cross-correlation of two stationary random signals, $x(t)$ and $y(t)$, is

$$\bar{\rho}_{xy}(t) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x^*(s) y(t+s) ds,$$

or, equivalently,

$$\bar{\rho}_{xy}(t) = E\left[x^* \Theta_{-t} y\right]$$

where, for any $\alpha$, $\Theta_\alpha y$ denotes the translation of $y$ by $\alpha$,

$$\Theta_\alpha y(s) = y(s - \alpha).$$

The corresponding cross-power spectrum is

$$P_{xy}(\omega) = \mathscr{F}\left[\bar{\rho}_{xy}(t)\right]\Big|_\omega = \int_{-\infty}^{\infty} \bar{\rho}_{xy}(t) e^{-j\omega t} dt.$$

It should be noted that

$$\begin{aligned}
\left[\bar{\rho}_{xy}(t)\right]^* &= \lim_{T\to\infty} \frac{1}{2T} \left[\int_{-T}^{T} x^*(s) y(s+t) ds\right]^* \\
&= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(s) y^*(s+t) ds \\
&= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(\sigma - t) y^*(\sigma) d\sigma \\
&= \bar{\rho}_{yx}(-t).
\end{aligned} \qquad (2.7.5.1)$$

From Schwarz's inequality, it follows that

$$\left|\bar{\rho}_{xy}(t)\right|^2 \le \bar{\rho}_x(0)\bar{\rho}_y(0).$$

A somewhat more general statement, namely, that for any $-\infty \le a < b \le \infty$,

$$\left|\int_a^b P_{xy}(\omega)e^{j\omega t}d\omega\right|^2 \le \int_a^b P_x(\omega)d\omega\int_a^b P_y(\omega)d\omega$$

can also be proven. This shows that if the power spectrum of a signal vanishes on an interval, then so does cross-power spectrum of that signal with any other signal.

The average cross-correlation indicates the extent to which the two processes are independent of each other. If, for example, $x(t)$ and $y(t)$ are generated by two completely independent stationary processes, then, following the discussion in Subsection 2.7.2,

$$\bar{\rho}_{xy}(t) = \mu_x^*\mu_y \quad \text{for all } t$$

and

$$P_{xy}(\omega) = 2\pi\mu_x^*\mu_y\delta(\omega).$$

In particular, if one of the two independent processes has mean zero, then

$$\bar{\rho}_{xy}(t) = 0 \quad \text{for all } t$$

and

$$P_{xy}(\omega) = 0.$$

On the other hand, if $y(t) = \gamma x(t - \tau)$ for some pair, $\gamma$ and $\tau$, of real values, then

$$\bar{\rho}_{xy}(t) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x^*(s)\gamma x(s-\tau+t)ds.$$

Thus, even if the expected value of $x(t)$ is zero,

$$\bar{\rho}_{xy}(\tau) = \gamma\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left|x(s)\right|^2 ds = \gamma\bar{\rho}_x(0)$$

and the cross-power spectrum, $P_{xy}(\omega)$, will not vanish.

Of particular interest is the average cross-correlation of a random signal, $x(t)$, with itself. This is the same as the average autocorrelation of $x(t)$ and indicates the extent to which the value of $x(t + \alpha)$ can be predicted from the value of $x(\alpha)$. If the expected value of $x(t)$ is zero and, for every $\alpha$ and nonzero value of $t$, $x(t)$ and $x(t + \alpha)$ are outcomes of completely independent random processes, then $x(t)$ is called "white noise." For such a signal there is a constant, $P_0$, such that

$$\bar{\rho}_x(t) = P_0 \delta(t)$$

and

$$P_x(\omega) = P_0.$$

### 2.7.6 Systems and Random Signals

Let $S$ be a linear shift invariant system with impulse response function $h(t)$ and transfer function $H(\omega)$. Assume the input, $x(t)$, is a stationary random signal with mean $\mu_x$, and let $y(t)$ be the corresponding output,

$$y(t) = S[x(t)].$$

The output is also a stationary random signal. It is related to the input by

$$y(t) = h * x(t)$$

or, equivalently, by

$$Y(\omega) = H(\omega) X(\omega),$$

where $X(\omega)$ and $Y(\omega)$ are, respectively, the Fourier transforms of $x(t)$ and $y(t)$. It should be easy to see that the expected value of the output is directly related to the expected value of the input,

$$\mu_y = S[\mu_x] = h * \mu_x = \mu_x \int_{-\infty}^{\infty} h(s)\, ds\,.$$

The auto- and cross-correlations of the input and output signals are related by

$$\bar{\rho}_y(t) = h * \bar{\rho}_{yx}(t), \tag{2.7.6.1}$$

$$\bar{\rho}_{xy}(t) = h * \bar{\rho}_x(t), \tag{2.7.6.2}$$

and

$$\bar{\rho}_y(t) = (h \star \bar{\rho}_x)(t). \tag{2.7.6.3}$$

Taking the Fourier transforms of these relations gives the corresponding relations for the power spectra,

$$P_y(\omega) = H(\omega) P_{yx}(\omega), \tag{2.7.6.4}$$

$$P_{xy}(\omega) = H(\omega) P_x(\omega), \tag{2.7.6.5}$$

and

$$P_y(\omega) = |H(\omega)|^2 P_x(\omega). \tag{2.7.6.6}$$

Derivations of identities (2.7.6.1) through (2.7.6.3) are relatively straightforward. For identity (2.7.6.1),

$$
\begin{aligned}
\bar{\rho}_y(t) &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} y^*(s)y(s+t)\,ds \\
&= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} y^*(s)\left[h * x(s+t)\right]ds \\
&= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} y^*(s)\int_{-\infty}^{\infty} h(\lambda)x(s+t-\lambda)\,d\lambda\, ds \\
&= \int_{-\infty}^{\infty} h(\lambda)\left(\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} y^*(s)x(s+t-\lambda)\,ds\right)d\lambda \\
&= \int_{-\infty}^{\infty} h(\lambda)\bar{\rho}_{yx}(t-\lambda)\,d\lambda \\
&= h * \bar{\rho}_{yx}(t).
\end{aligned}
$$

Derivations of (2.7.6.2) and (2.7.6.3) are similar with the derivation of equation (2.7.6.3) aided by identities (2.7.5.1) and (2.7.4.2).

### Example 2.7.6.1

Let $S$ be an LSI system with transfer function

$$H(\omega) = \frac{1}{j+\omega}.$$

Assume the input, $x(t)$, is white noise, that is, the power spectrum of $x(t)$ is a constant,

$$P_x(\omega) = P_0.$$

The corresponding output of the system, $y(t)$, will have power spectrum

$$P_y(\omega) = \left|H(\omega)\right|^2 P_x(\omega) = \frac{P_0}{1+\omega^2}$$

and autocorrelation

$$\bar{\rho}_y(t) = \mathscr{F}^{-1}\left[P_y(\omega)\right]\Big|_t = \frac{1}{2}P_0 e^{-|t|}.$$

The mean squared output is then

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left|y(t)\right|^2 dt = \bar{\rho}_y(0) = \frac{1}{2}P_0.$$

### Example 2.7.6.2

Let $S$ be the ideal low-pass filter from Example 2.6.1.1 with cutoff frequency $\Omega$. The transfer function is

$$H(\omega) = p_\Omega(\omega) = \begin{cases} 1, & \text{if } |\omega| < \Omega \\ 0, & \text{if } \Omega < |\omega| \end{cases}.$$

The power spectrum of the output, $y(t)$, resulting from a white noise input, $x(t)$, with power spectrum $P_x(\omega) = P_0$ is

$$P_y(\omega) = |H(\omega)|^2 P_x(\omega) = P_0 p_\Omega(\omega)$$

and the autocorrelation of the output is

$$\bar{\rho}_y(t) = \mathscr{F}^{-1}\left[P_y(\omega)\right]\Big|_t = P_0 \frac{1}{\pi t}\sin(\Omega t).$$

The mean squared output of the white noise is then

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |y(t)|^2 \, dt = \bar{\rho}_y(0) = P_0 \lim_{t\to 0} \frac{\sin(\Omega t)}{\pi t} = \frac{\Omega}{\pi} P_0 .$$

Consider, now, an input of $f(t) + x(t)$ where $x(t)$ is the above white noise and $f(t)$ is a deterministic band-limited signal with bandwidth less than $2\Omega$. The output of this low-pass filter is then $f(t) + y(t)$. The expected "intensity" of the output is

$$\begin{aligned} E\left[|f(t)+y(t)|^2\right] &= E\left[|f(t)|^2 + f^*(t)y(t) + f(t)y^*(t) + |y(t)|^2\right] \\ &= |f(t)|^2 + f^*(t)E[y(t)] + f(t)E[y^*(t)] + E\left[|y(t)|^2\right] \end{aligned}$$

Because $x(t)$ comes from white noise,

$$E[y(t)] = E[S[x(t)]] = S[E[x(t)]] = S[0] = 0,$$

and, by the above,

$$E\left[|y(t)|^2\right] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |y(t)|^2 \, dt = \frac{\Omega}{\pi} P_0 .$$

Thus, the expected intensity of the output is

$$|f(t)|^2 + \frac{\Omega}{\pi} P_0 ,$$

and the ratio of the intensity of the deterministic signal to the intensity of the outputted noise (signal-to-noise ratio) is

$$\frac{\pi |f(t)|^2}{\Omega P_0} .$$

## 2.8 Partial Differential Equations

The Fourier transform is an especially useful tool for solving problems involving partial differential equations. To illustrate how the Fourier transform can be used in a variety of such problems, three different problems involving the partial differential equation describing heat flow are examined below.

### 2.8.1 The One-Dimensional Heat Equation

The next few sections concern a uniform rod of some heat conducting material positioned on the $X$-axis between $x = \alpha$ and $x = \beta$. It is assumed that the sides of the rod are thermally insulated from the surroundings. The relevant material constants are

$c$ = specific heat of the material,
$\rho$ = the linear density of the material,

and

$k$ = the thermal diffusivity.

Any heat sources (and sinks) are described by a density function, $f(x, t)$, where, for any $0 \le t_1 < t_2$ and $\alpha \le x_1 < x_2 \le \beta$,

$$\int_{x_1}^{x_2} \int_{t_1}^{t_2} c\rho f(x, t)\,dt\,dx$$

is the total heat (in calories) generated in the rod between $x = x_1$ and $x = x_2$ during the period of time between $t = t_1$ and $t = t_2$.

The temperature distribution throughout the rod is described by

$$\upsilon(x, t) = \text{the temperature at time } t \text{ and position } x \text{ in the rod.}$$

Using basic thermodynamics it can be shown that $\upsilon(x, t)$ must satisfy the "one-dimensional heat equation,"

$$\frac{\partial \upsilon}{\partial t} - k\frac{\partial^2 \upsilon}{\partial x^2} = f(x, t) \tag{2.8.1.1}$$

for $\alpha < x < \beta$.

In Sections 2.8.2 through 2.8.4, the above equation is solved under various conditions. In each case the Fourier transform is taken with respect to the spatial variable, $x$, with (assuming an infinite rod, $\alpha = -\infty$ and $\beta = \infty$)

$$V = V(\xi, t) = \mathscr{F}_x\left[\upsilon(x, t)\right]\Big|_{\xi} = \int_{-\infty}^{\infty} \upsilon(x, t) e^{-j\xi x}\,dx$$

and

$$F = F(\xi, t) = \mathscr{F}_x\left[f(x, t)\right]\Big|_{\xi} = \int_{-\infty}^{\infty} f(x, t) e^{-j\xi x}\,dx .$$

Observe that

$$\mathcal{F}_x\left[\frac{\partial \upsilon}{\partial t}\right]\bigg|_\xi = \int_{-\infty}^{\infty} \frac{\partial \upsilon}{\partial t}\bigg|_{(x,t)} e^{-j\xi x}\,dx = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} \upsilon(x,t)e^{-j\xi x}\,dx = \frac{\partial V}{\partial t}.$$

On the other hand, it is appropriate to use identity (2.2.11.6) to compute the transform (with respect to $x$) of any partial derivatives with respect to $x$. In particular,

$$\mathcal{F}_x\left[\frac{\partial^2 \upsilon}{\partial x^2}\right]\bigg|_\xi = (j\xi)^2 V(\xi,t) = -\xi^2 V(\xi,t).$$

Thus, taking the Fourier transform of equation (2.8.1.1) with respect to $x$ yields

$$\frac{\partial V}{\partial t} + k\xi^2 V = F, \tag{2.8.1.2}$$

which can be treated as an ordinary first-order linear differential equation. From the elementary theory of ordinary differential equations, the general solution to equation (2.8.1.2) is

$$V(\xi,t) = e^{-k\xi^2 t}\int_a^t e^{k\xi^2\tau}F(\xi,\tau)\,d\tau + G(\xi)e^{-k\xi^2 t} \tag{2.8.1.3}$$

where $a$ is any convenient value and $G(\xi)$ is an "arbitrary" function of $\xi$. The temperature distribution $\upsilon(x, t)$, is then found by taking the inverse Fourier transform with respect to the "spatial frequency" variable, $\xi$.

## 2.8.2 The Initial Value Problem for Heat Flow on an Infinite Rod

If the rod is infinite, there are no heat sources or sinks in the rod, and the initial temperature distribution is known to be given by $\upsilon_0(x)$, then $\upsilon(x, t)$ is the solution to the following system of equations:

$$\frac{\partial \upsilon}{\partial t} - k\frac{\partial^2 \upsilon}{\partial x^2} = 0$$

$$\upsilon(x,0) = \upsilon_0(x).$$

Because $\upsilon_0(x)$ is the initial temperature distribution, it suffices to find $\upsilon(x, t)$ for $0 < t$.

Taking the Fourier transform with respect to $x$ of each of the above equations yields the system of two equations,

$$\frac{\partial V}{\partial t} + k\xi^2 V = 0$$

$$V(\xi,0) = V_0(\xi),$$

where

$$V_0(\xi) = \mathcal{F}[\upsilon_0(x)]\big|_\xi.$$

From formula (2.8.1.3) the general solution to the differential equation is

$$V(\xi, t) = G(\xi)e^{-k\xi^2 t}.$$

Plugging in the initial values,

$$G(\xi) = V(\xi, 0) = V_0(\xi),$$

shows that

$$V(\xi, t) = V_0(\xi)e^{-k\xi^2 t}.$$

The temperature distribution for all time is then found by taking the inverse transform (with respect to the spatial variable),

$$\begin{aligned}
\upsilon(x, t) &= \mathcal{F}_{\xi}^{-1}\left[V_0(\xi)e^{-kt\xi^2}\right]\Big|_x \\
&= \mathcal{F}_{\xi}^{-1}\left[V_0(\xi)\right]\Big|_x * \mathcal{F}_{\xi}^{-1}\left[e^{-kt\xi^2}\right]\Big|_x \\
&= \upsilon_0(x) * \left(\frac{1}{\sqrt{4\pi kt}}\exp\left[-\frac{1}{4kt}x^2\right]\right) \\
&= \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty}\upsilon_0(s)\exp\left[-\frac{1}{4kt}(x-s)^2\right]ds.
\end{aligned}$$

### 2.8.3 An Infinite Rod with Heat Sources and Sinks

For this problem it is assumed that the rod is infinite, and that the initial temperature distribution is $\upsilon(x, 0) = 0$. The source term, $f(x, t)$, is assumed to be nonzero for $0 < t$. Because the initial temperature distribution is a constant zero and it is only necessary to find $\upsilon(x, t)$ for $t > 0$, the following two assumptions may be made:

1. For $t \le 0$, $\upsilon(x, t) = 0$.
2. For $t < 0$, $f(x, t) = 0$.

The heat flow problem is then one of solving the heat equation,

$$\frac{\partial \upsilon}{\partial t} - k\frac{\partial^2 \upsilon}{\partial x^2} = f(x, t) \tag{2.8.3.1}$$

for all real values of $x$ and $t$, subject to conditions (1) and (2).

This problem is similar to that of finding the output to a causal LSI system. This suggests that it is convenient to find first the solution to

$$\frac{\partial h}{\partial t} - k\frac{\partial^2 h}{\partial x^2} = \delta(x)\delta(t) \tag{2.8.3.2}$$

where $h(x, t)$ is assumed to vanish if $t < 0$. It is then relatively easy to verify that the solution to (2.8.3.2) is given by the two-dimensional convolution of $h(x, t)$ with $f(x, t)$. Because $f(s, \tau)$ vanishes for $\tau < 0$, this can be written

$$\upsilon(x,t) = f * h(x,t) = \int_{s=-\infty}^{\infty}\int_{\tau=0}^{\infty} f(s,\tau)h(x-s,t-\tau)\,d\tau\,ds. \tag{2.8.3.3}$$

Taking the Fourier transform of equation (2.8.3.2) (with respect to the spatial variable) yields

$$\frac{\partial H}{\partial t} + k\xi^2 H = \delta(t) \tag{2.8.3.4}$$

where, by the assumptions on $h(x, t)$,

$$H(\xi, t) = \mathcal{F}_\xi[h(x, t)]\big|_x$$

must vanish when $t < 0$. From (2.8.13) it follows that the solution to equation (2.8.3.4) is given by

$$H(\xi,t) = e^{-k\xi^2 t}\int_a^t e^{k\xi^2\tau}\delta(\tau)\,d\tau + G(\xi)e^{-k\xi^2 t}.$$

It is convenient to take $a = -1$. Observe, then, that

$$\int_a^t e^{k\xi^2\tau}\delta(\tau)\,d\tau = \begin{Bmatrix} 1, & \text{if } 0 < t \\ 0, & \text{if } t < 0 \end{Bmatrix} = u(t).$$

Combining this with the fact that $H(\xi, t)$ vanishes for negative values of $t$ gives

$$0 = H(\xi, -1) = \mathrm{e}^{-k\xi^2(-1)}\,u(-1) + G(\xi)\mathrm{e}^{-k\xi^2(-1)} = G(\xi)\mathrm{e}^{k\xi^2},$$

implying that $G(\xi)$ vanishes and

$$H(\xi, t) = \mathrm{e}^{-kt\xi^2}u(t).$$

Taking the inverse transform

$$h(x,t) = \mathcal{F}_\xi^{-1}\left[e^{-kt\xi^2}u(t)\right]\Big|_x = \frac{1}{\sqrt{4\pi kt}}\exp\left[\frac{-x^2}{4kt}\right]u(t).$$

Formula (2.8.3.3) for the solution to the heat equation then becomes

$$\begin{aligned}\upsilon(x,t) &= \int_{s=-\infty}^{\infty}\int_{\tau=0}^{\infty} f(s,\tau)\frac{1}{\sqrt{4\pi k(t-\tau)}}\exp\left[\frac{-(x-s)^2}{4k(t-\tau)}\right]u(t-\tau)\,d\tau\,ds \\ &= \int_{s=-\infty}^{\infty}\int_{\tau=0}^{t} f(s,\tau)\frac{1}{\sqrt{4\pi k(t-\tau)}}\exp\left[\frac{-(x-s)^2}{4k(t-\tau)}\right]d\tau\,ds.\end{aligned}$$

## 2.8.4 A Boundary Value Problem for Heat Flow on a Half-Infinite Rod

For this problem it is assumed that the rod occupies the positive $X$-axis, $0 < x$, that there are no sources or sinks of heat in the rod, and that the initial temperature throughout the rod is zero. At $x = 0$ the temperature is known to be given by some function, $\Theta(t)$, for $0 < t$. The temperature distribution function, $\upsilon(x, t)$, then must satisfy the following system of equations:

$$\begin{aligned} \frac{\partial \upsilon}{\partial t} - k\frac{\partial^2 \upsilon}{\partial x^2} &= 0, && 0 < x \text{ and } 0 < t \\ \upsilon(x, 0) &= 0, && 0 < x \\ \upsilon(0, t) &= \Theta(t), && 0 < t. \end{aligned}$$

To apply the Fourier transform with respect to the spatial variable, $\upsilon(x, t)$ must be extended to a function on all of $x$. A review of relations (2.3.16.1) through (2.3.16.6) along with the observation that $\upsilon(0, t)$ is known, suggests that the odd extension,

$$\hat{\upsilon}(x, t) = \begin{cases} \upsilon(x, t), & \text{if } 0 < t \\ -\upsilon(-x, t), & \text{if } t < 0 \end{cases},$$

is appropriate. It is easily verified that $\hat{\upsilon}(x, t)$ satisfies

$$\frac{\partial \hat{\upsilon}}{\partial t} - k\frac{\partial^2 \hat{\upsilon}}{\partial x^2} = 0$$

for all $0 < t$ and $x \neq 0$. Combining this with relation (2.3.16.6),

$$D_{xx}\hat{\upsilon} = \frac{\partial^2 \hat{\upsilon}}{\partial x^2} + 2\upsilon(0, t)\delta'(x) = \frac{1}{k}\frac{\partial \hat{\upsilon}}{\partial t} + 2\Theta(t)\delta'(x),$$

gives the equation

$$\frac{\partial \hat{\upsilon}}{\partial t} - kD_{xx}\hat{\upsilon} = -2k\Theta(t)\delta'(x), \tag{2.8.4.1}$$

where $D_{xx}\,\hat{\upsilon}$ explicitly denotes the second generalized derivative of $\hat{\upsilon}(x, t)$ with respect to $x$. Equation (2.8.4) is valid for all $x$ and is the same as equation (2.8.3.1) with

$$f(x, t) = -2k\Theta(t)\delta'(x).$$

Because $\hat{\upsilon}(x, t)$ also satisfies the same initial condition as assumed in Section 2.8.3, the solution derived in that section applies here,

$$\hat{\upsilon}(x, t) = \int_{s=-\infty}^{\infty}\int_{\tau=0}^{t}\left[-2k\Theta(\tau)\delta'(s)\right]\frac{1}{\sqrt{4\pi k(t-\tau)}}\exp\left[\frac{-(x-s)^2}{4k(t-\tau)}\right]d\tau\, ds.$$

Now,

$$\int_{-\infty}^{\infty} \delta'(s) e^{-\gamma(x-s)^2} ds = -\int_{-\infty}^{\infty} \delta(s) \frac{d}{ds}\left[ e^{-\gamma(x-s)^2} \right] ds$$

$$= -\int_{-\infty}^{\infty} \delta(s) 2\gamma(x-s) e^{-\gamma(x-s)^2} ds$$

$$= -2\gamma x e^{-\gamma x^2} \, .$$

Thus, for $0 < x$ and $0 < t$,

$$\begin{aligned}
\upsilon(x, t) &= \hat{\upsilon}(x, t) \\
&= \int_{\tau=0}^{t} -2k\Theta(\tau) \frac{1}{\sqrt{4\pi k(t-\tau)}} \left( \int_{s=-\infty}^{\infty} \delta'(s) \exp\left[ -\frac{(x-s)^2}{4k(t-\tau)} \right] ds \right) d\tau \\
&= \int_{\tau=0}^{t} -2k\Theta(\tau) \frac{1}{\sqrt{4\pi k(t-\tau)}} \left( \frac{-2}{4k(t-\tau)} x \exp\left[ \frac{-x^2}{4k(t-\tau)} \right] \right) d\tau \\
&= \frac{x}{2\sqrt{\pi k}} \int_{\tau=0}^{t} \Theta(\tau)(t-\tau)^{-3/2} \exp\left[ \frac{-x^2}{4k(t-\tau)} \right] d\tau \, .
\end{aligned}$$

Letting $\sigma = \dfrac{x}{2\sqrt{\pi k}}(t-\tau)^{-1/2}$, this simplifies somewhat to

$$\upsilon(x, t) = 2\int_{\sigma_0}^{\infty} \Theta\left( t - \frac{x^2}{4\pi k \sigma^2} \right) e^{-\pi\sigma^2} d\sigma \, ,$$

where

$$\sigma_0 = \frac{x}{2\sqrt{\pi k t}} \, .$$

In particular, if the boundary temperature is constant, $\Theta(t) = \Theta_0$, then

$$\upsilon(x, t) = 2\Theta_0 \int_{\sigma_0}^{\infty} e^{-\pi\sigma^2} d\sigma = \Theta_0 \operatorname{erfc}\left( \frac{x}{2\sqrt{kt}} \right).$$

## 2.9 Tables

**TABLE 2.3** Fundamental Fourier Identities

If $f(t)$ and $G(\omega)$ are suitably integrable:

Integral Definitions:

$$F(\omega) = \mathcal{F}\left[f(t)\right]\Big|_{\omega} = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

$$g(t) = \mathcal{F}^{-1}\left[G(\omega)\right]\Big|_{t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega t}\,d\omega$$

Parseval's Equality:

$$\int_{-\infty}^{\infty} f(t)g^*(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)G^*(\omega)\,d\omega$$

Bessel's Equality:

$$\int_{-\infty}^{\infty} \left|f(t)\right|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|F(\omega)\right|^2 d\omega$$

For all transformable functions:

Linearity:

$$\mathcal{F}[\alpha f(t) + \beta g(t)]\big|_{\omega} = \alpha\mathcal{F}[f(t)]\big|_{\omega} + \beta\mathcal{F}[g(t)]\big|_{\omega}$$

$$\mathcal{F}^{-1}[\alpha F(\omega) + \beta G(\omega)]\big|_{t} = \alpha\mathcal{F}^{-1}[F(\omega)]\big|_{t} + \beta\mathcal{F}^{-1}[G(\omega)]\big|_{t}$$

Near Equivalence (Symmetry of Transforms):

$$\mathcal{F}^{-1}\left[\phi(x)\right]\Big|_{y} = \frac{1}{2\pi}\mathcal{F}\left[\phi(-x)\right]\Big|_{y} = \frac{1}{2\pi}\mathcal{F}\left[\phi(x)\right]\Big|_{-y}$$

$$\mathcal{F}\left[\phi(x)\right]\Big|_{y} = 2\pi\mathcal{F}^{-1}\left[\phi(-x)\right]\Big|_{y} = 2\pi\mathcal{F}^{-1}\left[\phi(x)\right]\Big|_{-y}$$

**TABLE 2.4** Commonly Used Fourier Identities

| $h(t)$ | $H(\omega) = \mathcal{F}[h(t)]\big|_{\omega}$ |
|---|---|
| $f(\alpha t)$ | $\frac{1}{\lvert\alpha\rvert}F\left(\frac{\omega}{\alpha}\right)$ |
| $f(t-\alpha)$ | $e^{-j\alpha\omega}F(\omega)$ |
| $e^{j\alpha t}f(t)$ | $F(\omega-\alpha)$ |
| $\cos(\alpha t)f(t)$ | $\frac{1}{2}[F(\omega-\alpha)+F(\omega+\alpha)]$ |

**TABLE 2.4** Commonly Used Fourier Identities (continued)

| $h(t)$ | $H(\omega) = \mathcal{F}\,[h(t)]\vert_\omega$ |
|---|---|
| $\sin(\alpha t) f(t)$ | $\frac{1}{2j}[F(\omega-\alpha) - F(\omega+\alpha)]$ |
| $\frac{df}{dt}$ | $j\omega F(\omega)$ |
| $\frac{d^n f}{dt^n}$ | $(j\omega)^n F(\omega)$ |
| $tf(t)$ | $j\frac{dF}{d\omega}$ |
| $t^n f(t)$ | $j^n \frac{d^n F}{d\omega^n}$ |
| $\frac{f(t)}{t}$ | $-j\int_\alpha^\omega F(s)\,ds + c_\alpha$ |
| $\int_\alpha^t f(s)\,ds$ | $-j\frac{F(\omega)}{\omega} + c_\alpha \delta(\omega)$ |
| $f(t)g(t)$ | $\frac{1}{2\pi} F(\omega) * G(\omega)$ |
| $f(t) * g(t)$ | $F(\omega)G(\omega)$ |
| $f(t) \star g(t)$ | $F^*(\omega)G(\omega)$ |
| $f^*(t)g(t)$ | $\frac{1}{2\pi} F(\omega) \star G(\omega)$ |

*Note*: ($\alpha$ is any nonzero real number, $F(\omega) = \mathcal{F}[f(t)]\big|_\omega$ and $G(\omega) = \mathcal{F}[g(t)]\big|_\omega$ )

**TABLE 2.5** Fourier Transforms of Some Common Functions
($\alpha$, $\beta$, $\gamma$, $\lambda$, $\nu$, and $n$ denote real numbers with $\alpha > 0$, $0 < \lambda < 1$, $\nu > 0$, and $n = 1, 2, 3, \ldots$)

| $f(t)$ | $F(w) = \mathcal{F}[f(t)]\big\vert_\omega$ |
|---|---|
| $p_\alpha(t)$ | $\frac{2}{\omega}\sin(\alpha\omega)$ |
| $(\alpha - \lvert t\rvert)\, p_\alpha(t)$ | $\left(\frac{2\sin\left(\frac{\alpha}{2}\omega\right)}{\omega}\right)^2$ |
| $\sqrt{1-t^2}\, p_1(t)$ | $\frac{\pi}{\omega} J_1(\omega)$ |
| $(1-t^2)^{\alpha-1}p_1(t)$ | $\Gamma(\alpha)\sqrt{\pi}\left(\frac{2}{\lvert\omega\rvert}\right)^{\alpha-\frac{1}{2}} J_{\alpha-\frac{1}{2}}(\lvert\omega\rvert)$ |
| $\operatorname{sgn}(t)p_\alpha(t)$ | $-2j\frac{1-\cos(\alpha\omega)}{\omega}$ |
| $\cos\left(\frac{\pi}{2\alpha}t\right)p_\alpha(t)$ | $\frac{4\pi\alpha}{\pi^2 - 4\alpha^2\omega^2}\cos(\alpha\omega)$ |
| $\operatorname{Rect}_{(\beta,\gamma)}(t)$ | $\frac{j}{\omega}\left[e^{-j\gamma\omega} - e^{-j\beta\omega}\right]$ |
| $e^{-(\alpha+j\beta)t}u(t)$ | $\frac{1}{\alpha + j\beta + j\omega}$ |
| $t^{\upsilon-1}e^{-(\alpha+j\beta)t}u(t)$ | $\frac{\Gamma(\upsilon)}{(\alpha + j\beta + j\omega)^\upsilon}$ |
| $e^{(\alpha+j\beta)t}u(-t)$ | $\frac{1}{\alpha + j\beta - j\omega}$ |
| $(-1)^{\upsilon-1}e^{(\alpha+j\beta)t}u(-t)$ | $\frac{\Gamma(\upsilon)}{(\alpha + j\beta - j\omega)^\upsilon}$ |
| $e^{-\alpha\lvert t\rvert}$ | $\frac{2\alpha}{\alpha^2+\omega^2}$ |
| $\operatorname{sgn}(t)\, e^{-\alpha\lvert t\rvert}$ | $\frac{-2j\omega}{\alpha^2+\omega^2}$ |

**TABLE 2.5** Fourier Transforms of Some Common Functions ($\alpha$, $\beta$, $\gamma$, $\lambda$, $\nu$, and $n$ denote real numbers with $\alpha > 0$, $0 < \lambda < 1$, $\nu > 0$, and $n = 1, 2, 3, \ldots$) (continued)

| $f(t)$ | $F(w) = \mathscr{F}[f(t)]\big\vert_\omega$ |
|---|---|
| $e^{-\alpha t^2}$ | $\sqrt{\dfrac{\pi}{\alpha}}\exp\left[-\dfrac{1}{4\alpha}\omega^2\right]$ |
| $e^{-\alpha t^2+\beta t}$ | $\sqrt{\dfrac{\pi}{\alpha}}\exp\left[-\dfrac{1}{4\alpha}\omega^2 - j\dfrac{\beta}{2\alpha}\omega + \dfrac{\beta^2}{4\alpha}\right]$ |
| $\dfrac{e^{-\lambda t}}{\alpha + j\beta + e^{-t}}$ | $\pi\,(\alpha + j\beta)^{\lambda-1+j\omega}\csc(\pi\lambda + j\pi\omega)$ |
| $\operatorname{sech}(\alpha t)$ | $\dfrac{\pi}{\alpha}\operatorname{sech}\left(\dfrac{\pi}{2\alpha}\omega\right)$ |
| $e^{\pm j\alpha t^2}$ | $\sqrt{\dfrac{\pi}{\alpha}}\exp\left[\mp j\dfrac{1}{4\alpha}\left(\omega^2 - \alpha\pi\right)\right]$ |
| $e^{-\theta t^2}$ $\left(w/\theta = \alpha + j\beta\right)$ | $\left[\sqrt{\vert\theta\vert + \alpha} - j\,\operatorname{sgn}(\beta)\sqrt{\vert\theta\vert - \alpha}\right] \times \dfrac{1}{\vert\theta\vert}\sqrt{\dfrac{\pi}{2}}\exp\left(-\dfrac{1}{4\theta}\omega^2\right)$ |
| $\dfrac{1}{t}\sin(\alpha t)$ | $\pi\, p_\alpha(\omega)$ |
| $\left(\dfrac{1}{t}\sin(\alpha t)\right)^2$ | $\dfrac{\pi}{2}\left(2\alpha - \vert\omega\vert\right)p_{2\alpha}(\omega)$ |
| $\dfrac{1}{\vert t\vert}\sin(\alpha t)$ | $-j\,\operatorname{sgn}(\omega)\ln\left\vert\dfrac{\vert\omega\vert + \alpha}{\vert\omega\vert - \alpha}\right\vert$ |
| $1$ | $2\pi\delta(\omega)$ |
| $t^n$ | $j^n 2\pi\delta^{(n)}(\omega)$ |
| $e^{j\beta t}$ | $2\pi\delta(\omega - \beta)$ |
| $\delta(t - \beta)$ | $e^{-j\beta\omega}$ |
| $\delta^{(n)}(t)$ | $(j\omega)^n$ |
| $\sin(\alpha t)$ | $-j\pi[\delta(\omega - \alpha) - \delta(\omega + \alpha)]$ |
| $\cos(\alpha t)$ | $\pi[\delta(\omega - \alpha) + \delta(\omega + \alpha)]$ |
| $\sin(\alpha t^2)$ | $-\sqrt{\dfrac{\pi}{\alpha}}\sin\left[\dfrac{1}{4\alpha}\left(\omega^2 - \alpha\pi\right)\right]$ |

**TABLE 2.5** Fourier Transforms of Some Common Functions ($\alpha$, $\beta$, $\gamma$, $\lambda$, $\nu$, and $n$ denote real numbers with $\alpha > 0$, $0 < \lambda < 1$, $\nu > 0$, and $n = 1, 2, 3, \ldots$) (continued)

| $f(t)$ | $F(w) = \mathcal{F}[f(t)]\big\vert_{\omega}$ |
|---|---|
| $\cos(\alpha t^2)$ | $\sqrt{\dfrac{\pi}{\alpha}}\cos\left[\dfrac{1}{4\alpha}\left(\omega^2-\alpha\pi\right)\right]$ |
| $e^{-\alpha t^2}\cos\left(\nu t^2\right)$ $\left(w/\theta=\alpha+j\nu\right)$ | $\dfrac{1}{\lvert\theta\rvert}\sqrt{\dfrac{\pi}{2}}\exp\left(-\dfrac{\alpha}{4\lvert\theta\rvert^2}\omega^2\right)\times$ $\left[\sqrt{\lvert\theta\rvert+\alpha}\cos\left(\dfrac{\nu\omega^2}{4\lvert\theta\rvert^2}\right)+\sqrt{\lvert\theta\rvert-\alpha}\sin\left(\dfrac{\nu\omega^2}{4\lvert\theta\rvert^2}\right)\right]$ |
| $e^{-\alpha t^2}\sin\left(\nu t^2\right)$ $\left(w/\theta=\alpha+j\nu\right)$ | $\dfrac{1}{\lvert\theta\rvert}\sqrt{\dfrac{\pi}{2}}\exp\left(-\dfrac{\alpha}{4\lvert\theta\rvert^2}\omega^2\right)\times$ $\left[\sqrt{\lvert\theta\rvert+\alpha}\cos\left(\dfrac{\nu\omega^2}{4\lvert\theta\rvert^2}\right)-\sqrt{\lvert\theta\rvert-\alpha}\sin\left(\dfrac{\nu\omega^2}{4\lvert\theta\rvert^2}\right)\right]$ |
| $\text{comb}_{\alpha}(t)$ | $\dfrac{2\pi}{\alpha}\text{comb}_{\frac{2\pi}{\alpha}}(\omega)$ |
| $\lvert\sin(\alpha t)\rvert$ | $\displaystyle\sum_{k=-\infty}^{\infty}\frac{4}{1-4k^2}\delta\left(\omega-2\alpha k\right)$ |
| $\lvert\cos(\alpha t)\rvert$ | $\displaystyle\sum_{k=-\infty}^{\infty}\left(-1\right)^k\frac{4}{1-4k^2}\delta\left(\omega-2\alpha k\right)$ |
| $\text{saw}(t)$ | $\displaystyle j\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-1\right)^n\frac{2}{n}\delta\left(\omega-n\pi\right)$ |
| $\displaystyle\sum_{m=-\infty}^{\infty}p_{\alpha}\left(t-m\nu\right)$ $\left(w/2\alpha\le\nu\right)$ | $\displaystyle\frac{4\pi\alpha}{\nu}\delta\left(\omega\right)+\sum_{\substack{k=-\infty\\ k\neq 0}}^{\infty}\frac{2}{k}\sin\left(\frac{2\pi k\alpha}{\nu}\right)\delta\left(\omega-\frac{2\pi k}{\nu}\right)$ |
| $\text{sgn}(t)$ | $-j\dfrac{2}{\omega}$ |
| $u(t)$ | $\pi\delta\left(\omega\right)-j\dfrac{1}{\omega}$ |

**TABLE 2.5** Fourier Transforms of Some Common Functions
($\alpha$, $\beta$, $\gamma$, $\lambda$, $\nu$, and $n$ denote real numbers with $\alpha > 0$, $0 < \lambda < 1$, $\nu > 0$, and $n = 1, 2, 3, \ldots$) (continued)

| $f(t)$ | $F(w) = \mathcal{F}[f(t)]\big\|_{\omega}$ |
|---|---|
| $\dfrac{1}{t}$ | $-j\pi\,\mathrm{sgn}(\omega)$ |
| $t^{-n}$ | $-j\pi\dfrac{(-j\omega)^{n-1}}{(n-1)!}\mathrm{sgn}(\omega)$ |
| $\|t\|$ | $-\dfrac{2}{\omega^2}$ |
| $t^n\,\mathrm{sgn}(t)$ | $(-j)^{n+1}\dfrac{2(n!)}{\omega^{n+1}}$ |
| $\mathrm{ramp}(t)$ | $j\pi\delta'(\omega)-\dfrac{1}{\omega^2}$ |
| $t^n\,u(t)$ | $j^n\pi\delta^{(n)}(\omega)+n!\left(\dfrac{-j}{\omega}\right)^{n+1}$ |
| $\|t\|^{-1/2}$ | $\sqrt{2\pi}\,\|\omega\|^{-1/2}$ |
| $\|t\|^{\lambda-1}$ | $2\Gamma(\lambda)\cos\left(\dfrac{\lambda\pi}{2}\right)\|\omega\|^{-\lambda}$ |
| $J_0(t)$ | $\dfrac{2}{\sqrt{1-\omega^2}}\,p_1(\omega)$ |
| $Y_0(\|t\|)$ | $\dfrac{-2}{\sqrt{\omega^2-1}}\left[1-p_1(\omega)\right]$ |
| $J_{2n}(t)$ | $\dfrac{2\cos\left[2n\arcsin(\omega)\right]}{\sqrt{1-\omega^2}}\,p_1(\omega)$ |
| $J_{2n+1}(t)$ | $\dfrac{-2j\sin\left[(2n+1)\arcsin(\omega)\right]}{\sqrt{1-\omega^2}}\,p_1(\omega)$ |
| $J_n(t)$ | $\dfrac{2(-j)^n T_n(\omega)}{\sqrt{1-\omega^2}}\,p_1(\omega)$ |
| $\dfrac{1}{t^n}J_n(t)$ | $\dfrac{2\left(1-\omega^2\right)^{n-\frac{1}{2}}}{1\cdot 3\cdot 5\cdots(2n-1)}\,p_1(\omega)$ |

**TABLE 2.5** Fourier Transforms of Some Common Functions
($\alpha$, $\beta$, $\gamma$, $\lambda$, $\nu$, and $n$ denote real numbers with $\alpha > 0$, $0 < \lambda < 1$, $\nu > 0$, and $n = 1, 2, 3, \ldots$) (continued)

| $f(t)$ | $F(w) = \mathscr{F}[f(t)]\big\vert_{\omega}$ |
|---|---|
| $\lvert t\rvert^{-\alpha+\frac{1}{2}} J_{\alpha-\frac{1}{2}}\left(\lvert t\rvert\right)$ | $\dfrac{\sqrt{2\pi}}{\Gamma(\alpha)}\left(\dfrac{1-\omega^2}{2}\right)^{\alpha-1} p_1(\omega)$ |
| $\dfrac{1}{t} J_n(t)$ | $(-j)^n \dfrac{2j}{n}\sqrt{1-\omega^2}\, U_{n-1}(\omega) p_1(\omega)$ |
| $t^{-1/2} J_{n+\frac{1}{2}}(t)$ | $(-j)^n \sqrt{2\pi}\, P_n(\omega) p_1(\omega)$ |
| $\mathrm{sgn}(t) J_0(t)$ | $j\dfrac{2}{\sqrt{\omega^2-1}}\mathrm{sgn}(\omega)\left[1-p_1(\omega)\right]$ |
| $J_0(t)u(t)$ | $\dfrac{p_1(\omega)+j\,\mathrm{sgn}(\omega)\left[1-p_1(\omega)\right]}{\sqrt{\lvert 1-\omega^2\rvert}}$ |

*Notes*: $\Gamma(x)$ = the Gamma function; $P_n(x)$ = the $n$th Legendre polynomial; $J_\nu$ = the Bessel function of the first kind of order $\nu$; $Y_\nu$ = the Bessel function of the second kind of order $\nu$; $T_n(x)$ = the $n$th Chebyshev polynomial of the first kind; $U_n(x)$ = the $n$th Chebyshev polynomial of the second kind; saw($t$) = the saw function of Example 2.3.9.1.

**TABLE 2.6** Graphical Representations of Some Fourier Transforms

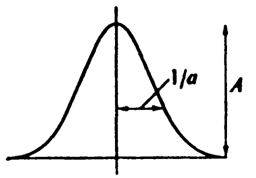
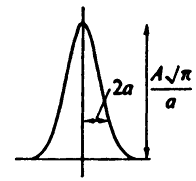
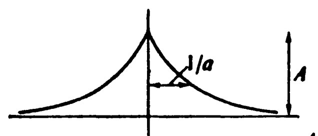
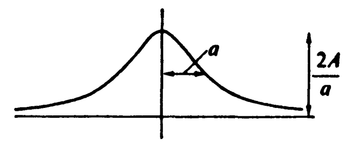
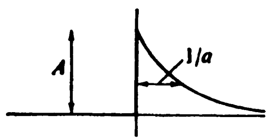
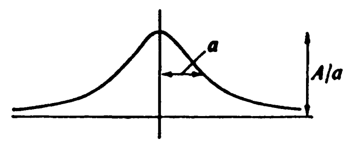
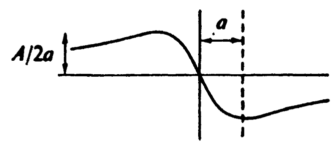
| $f(x)$<br>$\left[f(x) = (1/2\pi)\int_{-\infty}^{+\infty} F(y)e^{+ixy}\,dy = FT^{+}\{F(y)\}\right]$ | $F(y)$<br>$\left[F(y) = \int_{-\infty}^{+\infty} f(x)e^{-ixy}\,dx = FT^{-}\{f(x)\}\right]$ | |
|---|---|---|
| <br>$A\exp(-a^2x^2)$<br>[Gaussian] | <br>$\frac{A\sqrt{\pi}}{a}\exp(-y^2/4a^2)$<br>[Gaussian] | (2.38) |
| <br>$A\exp(-a\lvert x\rvert)$ | $\frac{2A}{a}\,\frac{a^2}{a^2+y^2}$<br>[Lorentzian]<br> | (2.39) |
| <br>$A\exp(-ax)$ [$x > 0$]<br>0 [$x < 0$] | <br><br>$A\left\{\frac{a-iy}{a^2+y^2}\right\}$ | (2.40) |

**TABLE 2.6** Graphical Representations of Some Fourier Transforms (Continued)

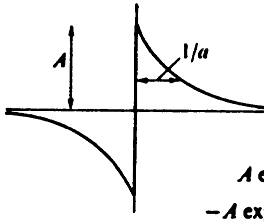
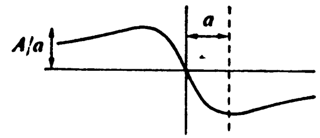
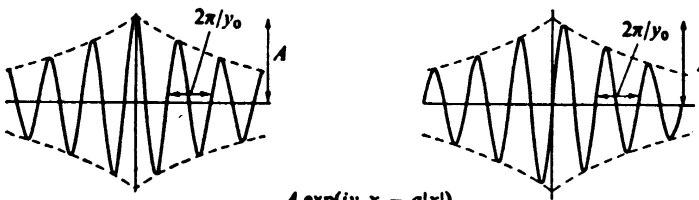
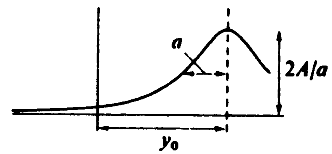
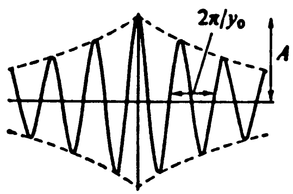
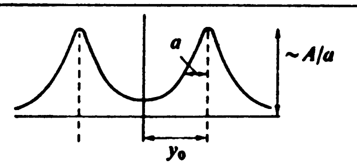
| $f(x)$ | $F(y)$ | |
|---|---|---|
| <br>$A\exp(-ax)$ $[x > 0]$<br>$-A\exp(-a\|x\|)$ $[x < 0]$ | <br>$-2iA\dfrac{y}{a^2 + y^2}$ | (2.41) |
| <br>$A\exp(iy_0x - a\|x\|)$ | <br>$\dfrac{2A}{a}\dfrac{a^2}{a^2 + (y - y_0)^2}$ | (2.42) |
| <br>$A\cos y_0 x\exp(-a\|x\|)$ | <br>$\dfrac{A}{a}\left\{\dfrac{a^2}{a^2 + (y - y_0)^2} + \dfrac{a^2}{a^2 + (y + y_0)^2}\right\}$<br>$= \dfrac{A}{a}\left\{\dfrac{2a^2(a^2 + y_0^2 + y^2)}{(a^2 + y_0^2 - y^2)^2 + 4a^2y^2}\right\}$ | (2.43) |

**TABLE 2.6** Graphical Representations of Some Fourier Transforms (Continued)

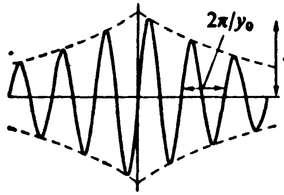
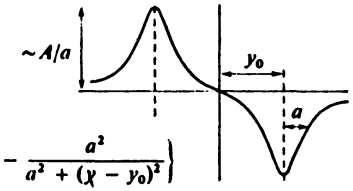
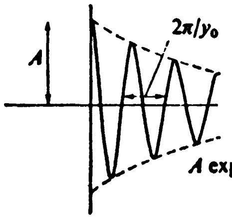
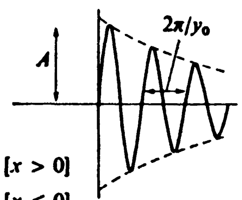
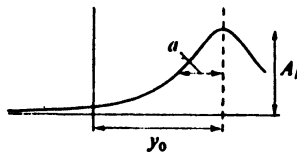
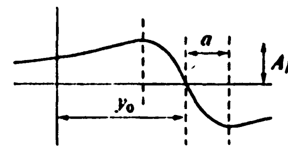
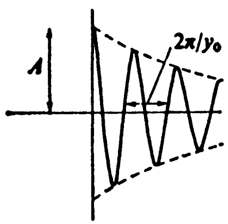
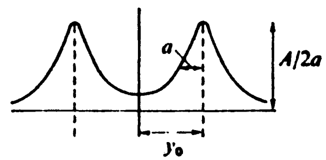
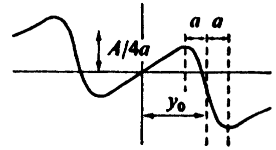
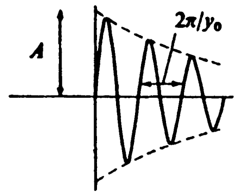
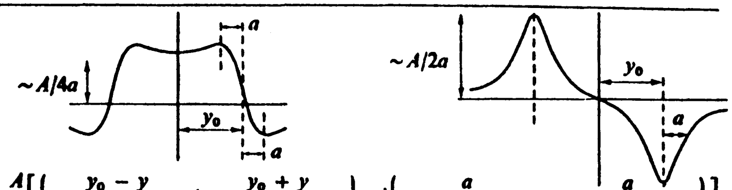
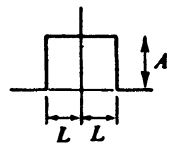
| $f(x)$ | $F(y)$ |
|---|---|
| <br>$A \sin y_0 x \exp(-a\|x\|)$ | <br>$\frac{iA}{a}\left\{\frac{a^2}{a^2+(y+y_0)^2}-\frac{a^2}{a^2+(y-y_0)^2}\right\}$<br>$=\frac{iA}{a}\left\{\frac{-4a^2yy_0}{(a^2+y_0^2-y^2)^2+4a^2y^2}\right\}$ (2.44) |
| <br>$A \exp(iy_0x - ax)$ $[x > 0]$<br>$0$ $[x < 0]$ | <br>$A\left\{\frac{a+i(y_0-y)}{a^2+(y_0-y)^2}\right\} = A\left\{\frac{1}{a+i(y-y_0)}\right\}$ (2.45) |
| <br>$A \cos y_0x \exp(-ax)$ $[x > 0]$<br>$0$ $[x < 0]$ | <br>$\frac{A}{2}\left[\left\{\frac{a}{a^2+(y+y_0)^2}+\frac{a}{a^2+(y-y_0)^2}\right\}+i\left\{\frac{y_0-y}{a^2+(y_0-y)^2}-\frac{y_0+y}{a^2+(y_0+y)^2}\right\}\right]$<br>$=A\left\{\frac{a(a^2+y_0^2+y^2)-iy(a^2+y^2-y_0^2)}{(a^2+y_0^2-y^2)^2+4a^2y^2}\right\}$ (2.46) |

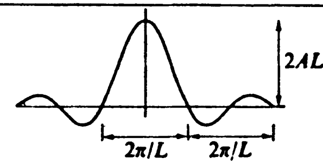
**TABLE 2.6** Graphical Representations of Some Fourier Transforms (Continued)

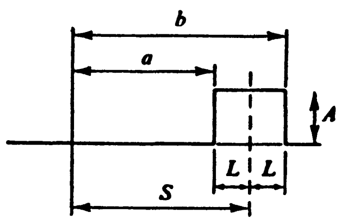
<table>
<tr><th>f(x)</th><th>F(y)</th></tr>
<tr><td>


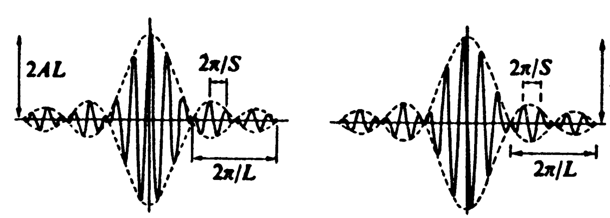

$A \sin y_0 x \exp(-ax)$ [$x > 0$]<br>
0 [$x < 0$]
</td><td>



$$\frac{A}{2}\left[\left\{\frac{y_0 - y}{a^2 + (y_0 - y)^2} + \frac{y_0 + y}{a^2 + (y_0 + y)^2}\right\} + i\left\{\frac{a}{a^2 + (y_0 + y)^2} - \frac{a}{a^2 + (y_0 - y)^2}\right\}\right]$$

$$= Ay_0\left\{\frac{1}{(a^2 + y_0^2 - y^2) + i2ay}\right\} \quad (2.47)$$
</td></tr>
<tr><td>



$A$ [$|x| < L$]<br>
0 [$|x| > L$]
</td><td>



$$2A\frac{\sin Ly}{y} \quad (2.48)$$
</td></tr>
<tr><td>



$A$ [$a < x < b$]<br>
0 [$x < a$; $x > b$]
</td><td>



$$2A\frac{\sin Ly}{y}\exp(-iSy) = A\left[\frac{(\sin by - \sin ay) - i(\cos ay - \cos by)}{y}\right]$$

$$= 2A\left[\frac{(\sin Ly \cos Sy) - i(\sin Ly \sin Sy)}{y}\right] = \frac{iA}{y}[\exp(-iby) - \exp(-iay)] \quad (2.49)$$
</td></tr>
</table>

**TABLE 2.6** Graphical Representations of Some Fourier Transforms (Continued)

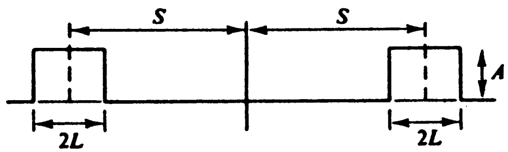
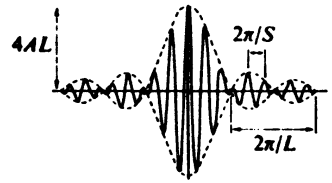
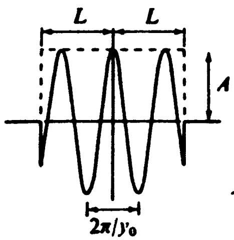
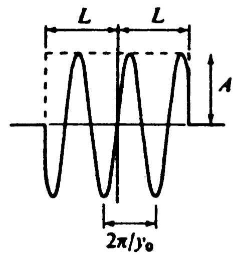
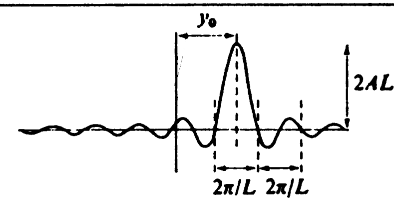
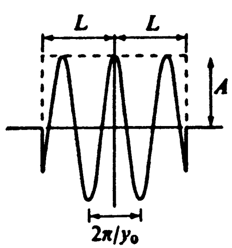
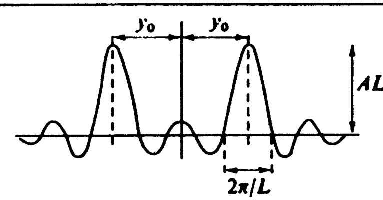
| $f(x)$ | $F(y)$ | |
|---|---|---|
| <br>$A \quad [(S-L) < \lvert x\rvert < (S+L)]$<br>$0 \quad$ [otherwise] | <br>$4A\dfrac{\cos Sy \sin Ly}{y}$ | (2.50) |
|  <br>$A\exp(iy_0x) \quad [\lvert x\rvert < L]$<br>$0 \quad [\lvert x\rvert > L]$ | <br>$2A\dfrac{\sin\{L(y_0-y)\}}{(y_0-y)}$ | (2.51) |
| <br>$A\cos y_0x \quad [\lvert x\rvert < L]$<br>$0 \quad [\lvert x\rvert > L]$ | <br>$A\left[\dfrac{\sin L(y-y_0)}{(y-y_0)} + \dfrac{\sin L(y+y_0)}{(y+y_0)}\right]$ | (2.52) |

**TABLE 2.6** Graphical Representations of Some Fourier Transforms (Continued)

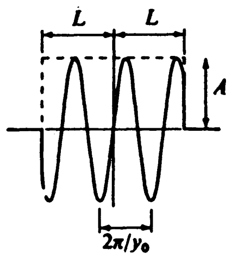
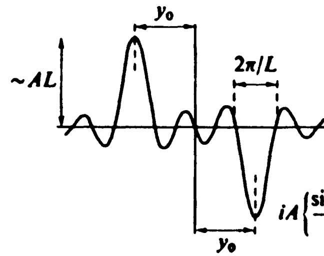
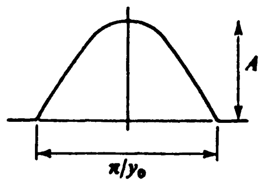
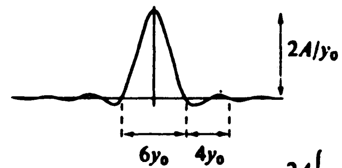
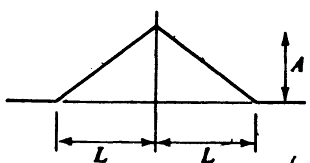
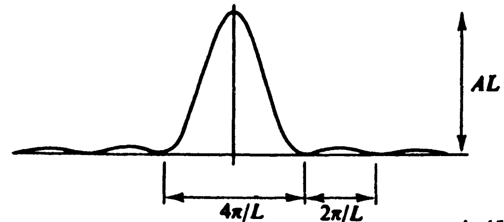
| $f(x)$ | $F(y)$ |
|---|---|
|   $A \sin y_0 x \quad [\lvert x\rvert < L]$<br>$0 \quad [\lvert x\rvert > L]$ | (2.53)   $iA\left\{\frac{\sin L(y+y_0)}{(y+y_0)} - \frac{\sin L(y-y_0)}{(y-y_0)}\right\}$ |
|   $A \cos y_0 x \quad [\lvert x\rvert < (\pi/2y_0)]$<br>$0 \quad [\lvert x\rvert > (\pi/2y_0)]$ | (2.54)   $2A\left\{\frac{y_0}{y_0^2 - y^2}\right\} \cos\left(\frac{\pi y}{2y_0}\right)$<br>[See (2.52) with $L = \pi/2y_0$]. |
|   $A\left(1 - \frac{\lvert x\rvert}{L}\right) \quad [\lvert x\rvert < L]$<br>$0 \quad [\lvert x\rvert > L]$ | (2.55)   $AL\left\{\frac{\sin(Ly/2)}{(Ly/2)}\right\}^2$ |

**TABLE 2.6** Graphical Representations of Some Fourier Transforms (Continued)

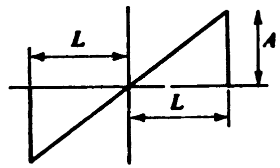
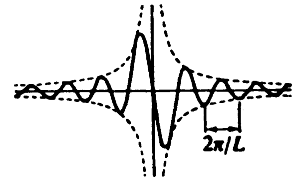
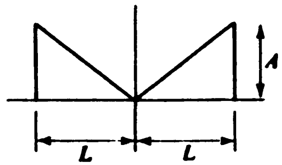
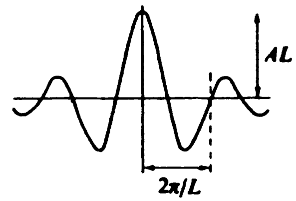
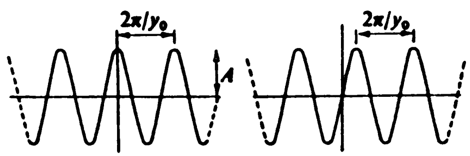
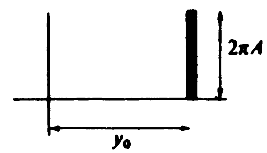
| $f(x)$ | $F(y)$ | |
|---|---|---|
|  $\frac{Ax}{L}$ $[\lvert x\rvert < L]$; $0$ $[\lvert x\rvert > L]$ |  $\frac{2iA}{y}\left\{\cos Ly - \frac{\sin Ly}{Ly}\right\}$ | (2.56) |
|  $\frac{A\lvert x\rvert}{L}$ $[\lvert x\rvert < L]$; $0$ $[\lvert x\rvert > L]$ |  $2AL\left\{\frac{\sin Ly}{Ly} - 2\left(\frac{\sin (Ly/2)}{Ly}\right)^2\right\}$ | (2.57) |
|  $A\exp(iy_0x)$ |  $2\pi A\delta(y - y_0)$. | (2.58) |

**TABLE 2.6** Graphical Representations of Some Fourier Transforms (Continued)

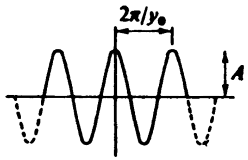
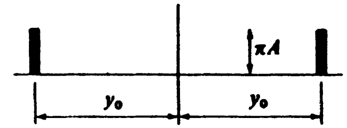
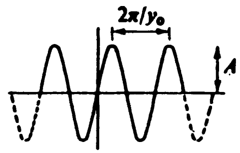
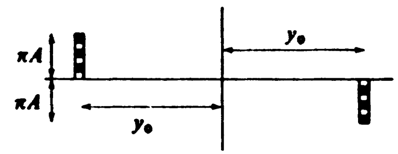
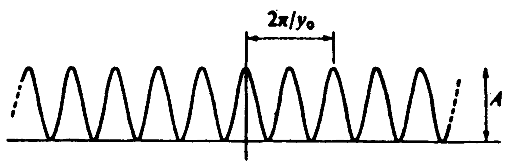
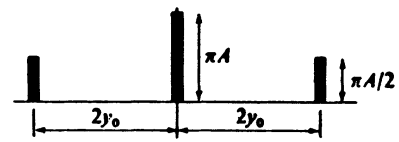
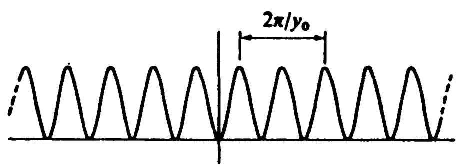
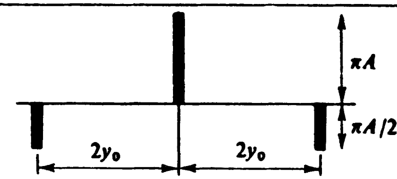
| $f(x)$ | $F(y)$ | |
|---|---|---|
| <br>$A\cos y_0x$ | <br>$\pi A\{\delta(y - y_0) + \delta(y + y_0)\}$ | (2.59) |
| <br>$A\sin y_0x$ | <br>$\pi iA\{\delta(y + y_0) - \delta(y - y_0)\}$ | (2.60) |
| <br>$A\cos^2 y_0x$ | <br>$\pi A\{\frac{1}{2}\delta(y + 2y_0) + \delta(y) + \frac{1}{2}\delta(y - 2y_0)\}$ | (2.61) |
| <br>$A\sin^2 y_0x$ | <br>$\pi A\{-\frac{1}{2}\delta(y + 2y_0) + \delta(y) - \frac{1}{2}\delta(y - 2y_0)\}$ | (2.62) |

**TABLE 2.6** Graphical Representations of Some Fourier Transforms (Continued)

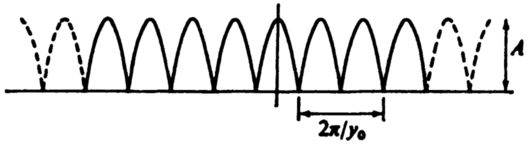
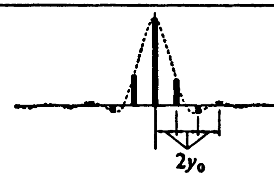
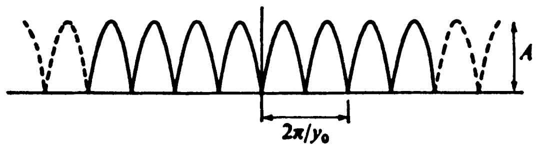
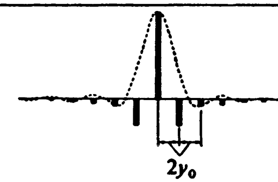
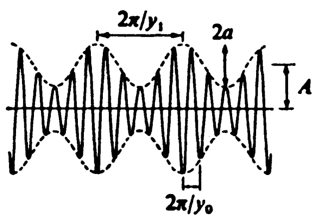
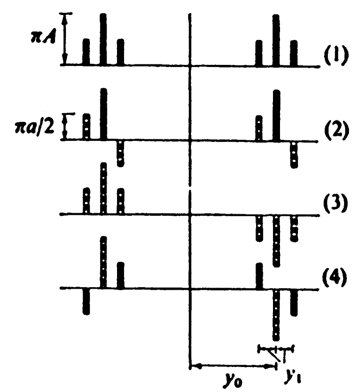
| $f(x)$ | $F(y)$ |
|---|---|
|   $A\lvert\cos y_0 x\rvert$ |   $\sum_{n=-\infty}^{n=+\infty} 4A\left\{\frac{y_0^2}{y_0^2-y^2}\right\}\cos\left(\frac{\pi y}{2y_0}\right)\delta(y-2ny_0)$ $[n = 0, \pm 1, \pm 2, \ldots]$ (2.63) |
|   $A\lvert\sin y_0 x\rvert$ |   $\sum_{n=-\infty}^{n=+\infty} (-1)^n 4A\left\{\frac{y_0^2}{y_0^2-y^2}\right\}\cos\left(\frac{\pi y}{2y_0}\right)\delta(y-2ny_0)$ $[n = 0, \pm 1, \pm 2, \ldots]$ (2.64) |
|   $\cos y_0 x\{A + a\cos y_1 x\}$ ... (1)<br>$\cos y_0 x\{A + a\sin y_1 x\}$ ... (2)<br>$\sin y_0 x\{A + a\cos y_1 x\}$ ... (3)<br>$\sin y_0 x\{A + a\sin y_1 x\}$ ... (4) |   $F(y)$ consists of delta functions as shown (2.65) |

**TABLE 2.6** Graphical Representations of Some Fourier Transforms (Continued)

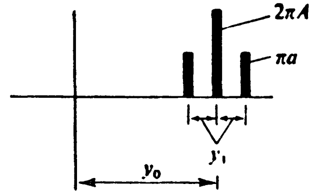
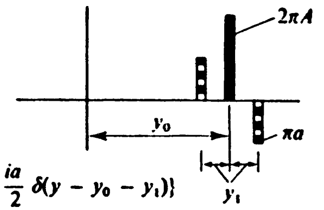
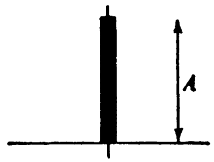
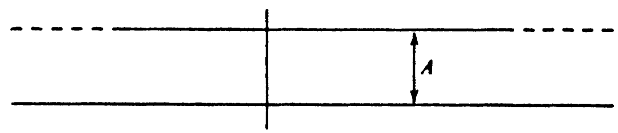
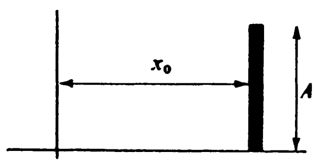
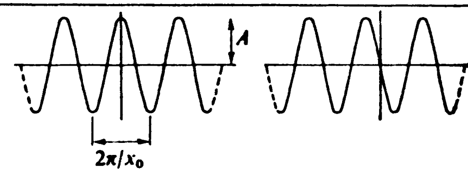
| $f(x)$ | $F(y)$ | |
|---|---|---|
| $\exp(iy_0x)(A + a\cos y_1x)$ | $2\pi\left\{A\delta(y-y_0)+\frac{a}{2}\delta(y-y_0+y_1)+\frac{a}{2}\delta(y-y_0-y_1)\right\}$   | (2.66) |
| $\exp(iy_0x)(A + a\sin y_1x)$ |   $2\pi\{A\delta(y-y_0)+\frac{ia}{2}\delta(y-y_0+y_1)-\frac{ia}{2}\delta(y-y_0-y_1)\}$ | (2.67) |
|   $A\delta(x)$ |   $A$ | (2.68) |
|   $A\delta(x-x_0)$ |   $A\exp(-ix_0y)$ | (2.69) |

**TABLE 2.6** Graphical Representations of Some Fourier Transforms (Continued)

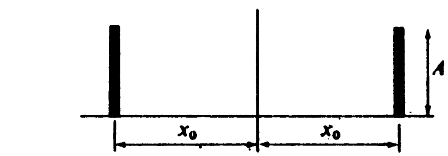
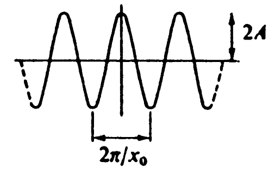
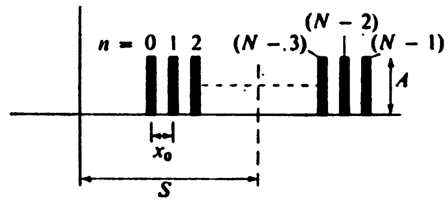
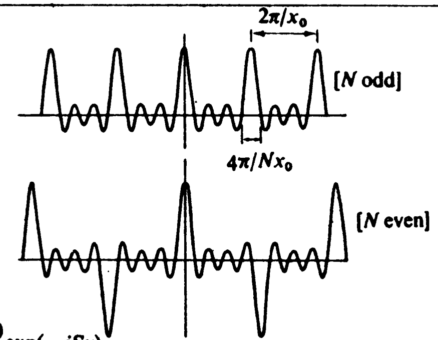
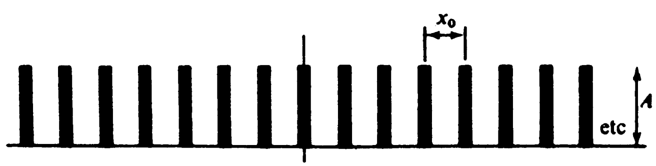
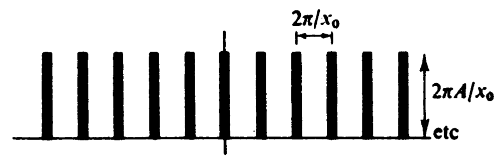
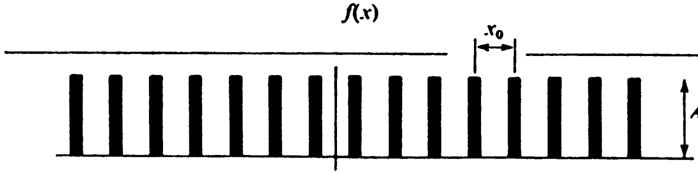
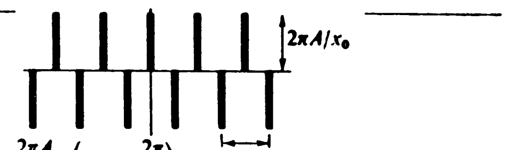
| $f(x)$ | $F(y)$ | |
|---|---|---|
|  <br>$A\{\delta(x - x_0) + \delta(x + x_0)\}$ |  <br>$2A \cos x_0 y$ | (2.70) |
|  <br>$\sum_{n=0}^{N-1} A\delta\left\{x - nx_0 - S + \frac{(N-1)x_0}{2}\right\}$<br>Set of $N$ delta functions symmetrically placed about $x = S$. |  <br>$A\,\frac{\sin(Nyx_0/2)}{\sin(yx_0/2)}\exp(-iSy)$ [Drawn for $S = 0$; $N = 7$ and $N = 8$] | (2.71) |
|  <br>$\sum_{n=-\infty}^{+\infty} A\delta(x - nx_0)$ |  <br>$\sum_{n=-\infty}^{+\infty} \frac{2\pi A}{x_0}\delta\left(y - n\frac{2\pi}{x_0}\right)$ | (2.72) |

**TABLE 2.6** Graphical Representations of Some Fourier Transforms (Continued)

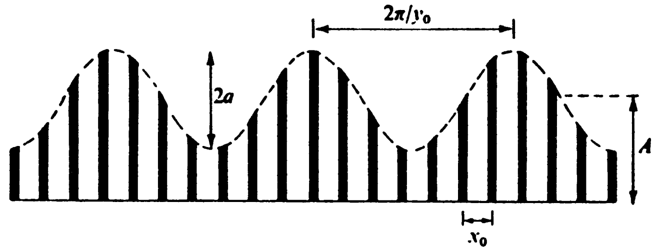
| $f(x)$ | $F(y)$ | |
|---|---|---|




$$\sum_{n=-\infty}^{+\infty} A\delta\left(x - \frac{x_0}{2} - nx_0\right)$$



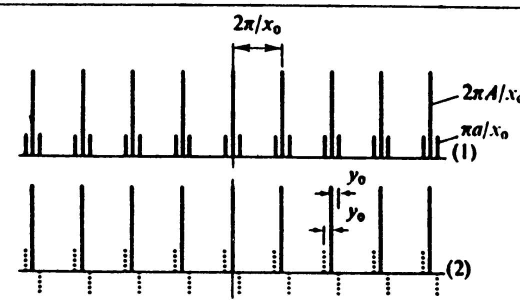

$$\sum_{n=-\infty}^{+\infty} (-1)^n \frac{2\pi A}{x_0}\,\delta\left(y - n\frac{2\pi}{x_0}\right)$$ (2.73)




$\sum \delta(x - nx_0)\{A + a\cos y_0x\}$ (1)

$\sum \delta(x - nx_0)\{A + a\sin y_0x\}$ (2)

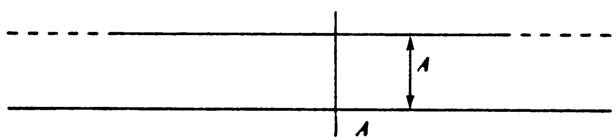
$[n = 0, \pm 1, \pm 2, \ldots]$



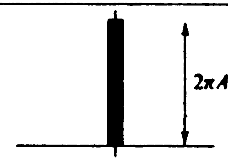

$$\sum_n \frac{2\pi}{x_0}\left\{A\delta\left(y - n\frac{2\pi}{x_0}\right) + \frac{a}{2}\delta\left(y - n\frac{2\pi}{x_0} + y_0\right) + \frac{a}{2}\delta\left(y - \frac{2\pi}{x_0} - y_0\right)\right\}$$ (1)

$$\sum_n \frac{2\pi}{x_0}\left\{A\delta\left(y - n\frac{2\pi}{x_0}\right) + \frac{ia}{2}\delta\left(y - n\frac{2\pi}{x_0} + y_0\right) - \frac{ia}{2}\delta\left(y - n\frac{2\pi}{x_0} - y_0\right)\right\}$$ (2)

$[n = 0, \pm 1, \pm 2, \ldots]$ (2.74)




$A$




$2\pi A\delta(y)$ (2.75)

**TABLE 2.6** Graphical Representations of Some Fourier Transforms (Continued)

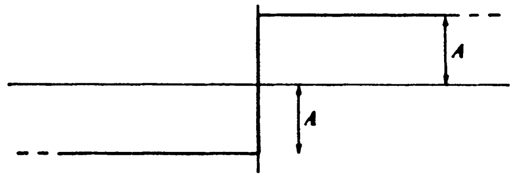
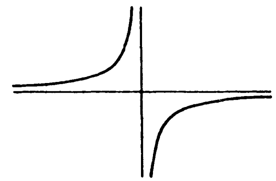
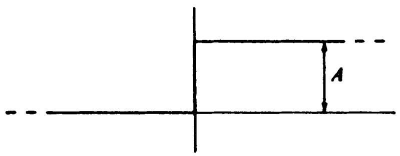
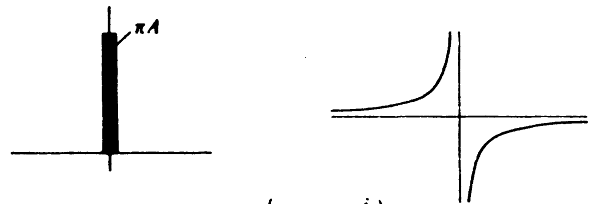
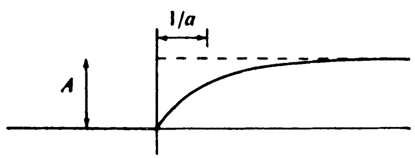
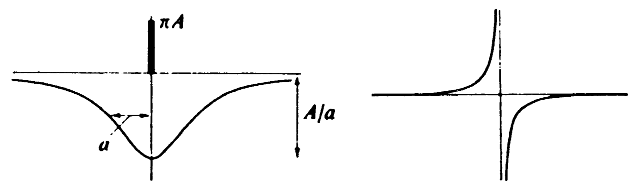
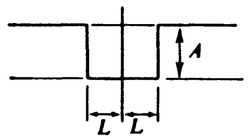
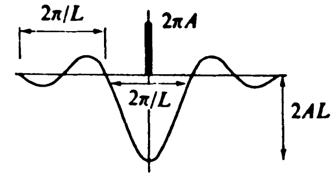
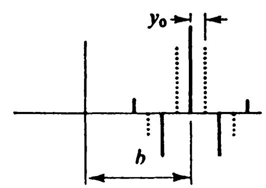
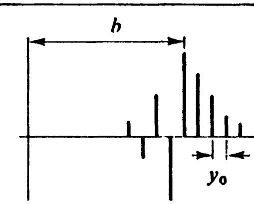
| $f(x)$ | $F(y)$ | |
|---|---|---|
|   $+A \quad [x > 0]$ <br> $-A \quad [x < 0]$ <br> $[f(x) = A\,\mathrm{sgn}(x)]$ |  $-2iA\dfrac{1}{y}$ | (2.76) |
|   $A \quad [x > 0]$ <br> $0 \quad [x < 0]$ <br> $[f(x) = AU(x)]$ |   $A\left\{\pi\delta(y) - \dfrac{i}{y}\right\}$ | (2.77) |
|   $A\{1-\exp(-ax)\} \quad [x > 0]$ <br> $0 \quad [x < 0]$ |   $\pi A\delta(y) - A\left\{\dfrac{a}{a^2+y^2} + i\dfrac{a^2}{y(a^2+y^2)}\right\}$ | (2.78) |

**TABLE 2.6** Graphical Representations of Some Fourier Transforms (Continued)

| $f(x)$ | $F(y)$ | |
|---|---|---|
| <br>$A \quad [\lvert x\rvert > L]$<br>$0 \quad [\lvert x\rvert < L]$ | <br>$2\pi A\delta(y) - 2A\,\dfrac{\sin Ly}{y}$ | (2.79) |
| $A\exp\{i(a\cos y_0 x + bx)\}$ | <br>$2\pi A \sum_{n=-\infty}^{+\infty} (i)^n J_n(a)\delta(y - b - ny_0)$ | (2.80) |
| $A\exp\{i(a\sin y_0 x + bx)\}$ | <br>$2\pi A \sum_{n=-\infty}^{+\infty} J_n(a)\delta(y - b - ny_0)\}$ | (2.81) |

Note: $J_n(-a) = J_{-n}(a) = (-1)^n J_n(a)$. See Appendix H for some properties of Bessel functions.

**TABLE 2.6** Graphical Representations of Some Fourier Transforms (Continued)

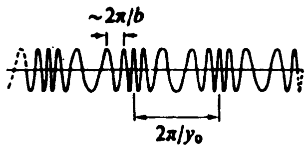
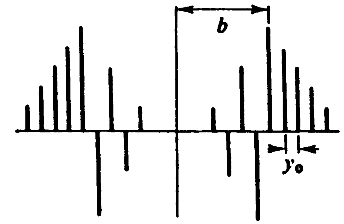
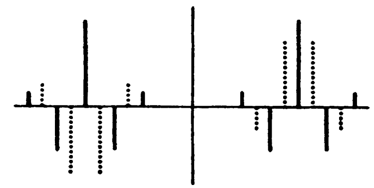
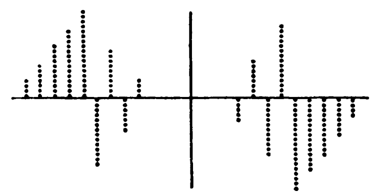
| $f(x)$ | $F(y)$ |
|---|---|
|   $A\cos(a\sin y_0x + bx)$ |   $\pi A \sum_{n=-\infty}^{+\infty} \{J_n(a)\delta(y - b - ny_0) + J_n(a)\delta(y + b + ny_0)\}$ (2.82) |
| $A\cos(a\cos y_0x + bx)$ |  $\pi A \sum_{n=-\infty}^{+\infty} \{(+i)^n J_n(a)\delta(y - b - ny_0) + (-i)^n J_n(a)\delta(y + b + ny_0)\}$ (2.83) |
| $A\sin(a\sin y_0x + bx)$ |  $i\pi A \sum_{n=-\infty}^{+\infty} \{-J_n(a)\delta(y - b - ny_0) + J_n(a)\delta(y + b + ny_0)\}$ (2.84) |

**TABLE 2.6** Graphical Representations of Some Fourier Transforms (Continued)

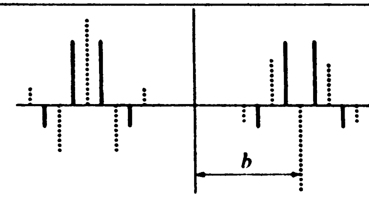
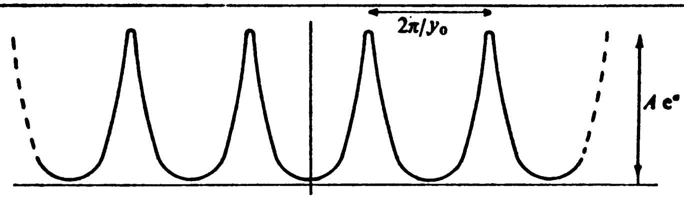
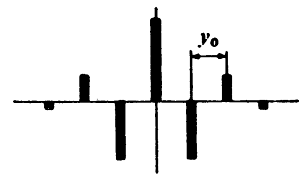
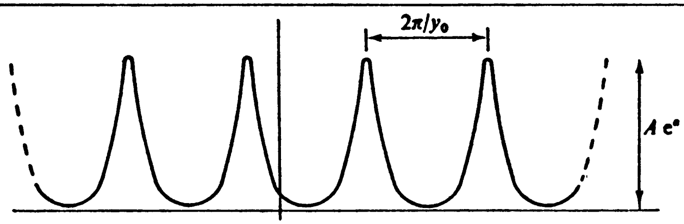
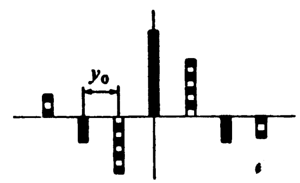
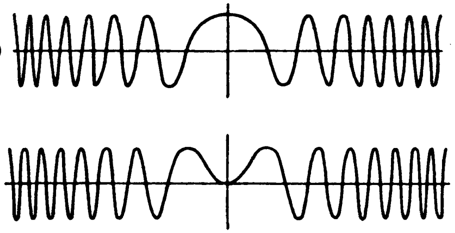
| $f(x)$ | $F(y)$ |
|---|---|
| $A \sin(a \cos y_0 x + bx)$ |  $i\pi A \sum_{n=-\infty}^{+\infty} \{-(i)^n J_n(a)\delta(y - b - ny_0) + (-i)^n J_n(a)\delta(y + b + ny_0)\}$ (2.85) |
|  $A \exp(-a \cos y_0 x)$ |  $2\pi A \sum_{n=-\infty}^{+\infty} (-1)^n I_n(a)\delta(y - ny_0)$ (2.86) |
|  $A \exp(-a \sin y_0 x)$ |  $2\pi A \sum_{n=-\infty}^{+\infty} (i)^n I_n(a)\delta(y - ny_0)$ (2.87) |

**TABLE 2.6** Graphical Representations of Some Fourier Transforms (Continued)

| $f(x)$ | $F(y)$ |
|---|---|

Re $f(x)$

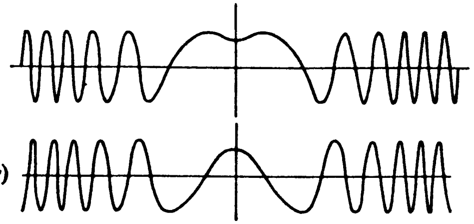
Im $f(x)$



$A\exp(\pm i a^2 x^2)$

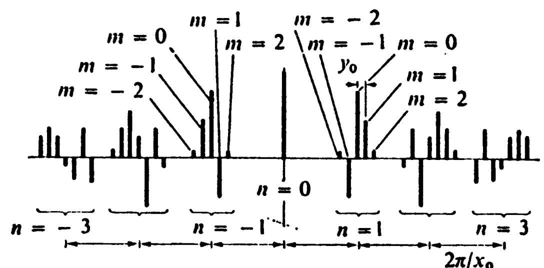
Re $F(y)$

Im $F(y)$



$$\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{A(1 \pm i)}{a} \exp(\mp i y^2/4a^2) \qquad (2.88)$$


$$f(x) = A \sum_{n} \delta(x - n x_0 + a \sin y_0 x)$$

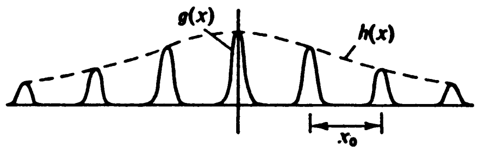
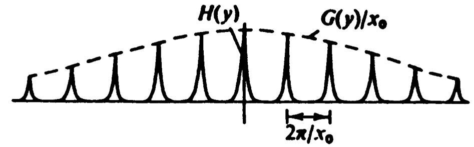



$$F(y) = \frac{2\pi A}{x_0} \sum_{m,n} J_m\left(n\,\frac{2\pi a}{x_0}\right) \delta\left(y - n\,\frac{2\pi}{x_0} - m y_0\right)$$

$(m = 0, \pm 1, \pm 2, \pm 3, \ldots)$

$(n = 0, \pm 1, \pm 2, \pm 3, \ldots)$ (2.89)

**TABLE 2.6** Graphical Representations of Some Fourier Transforms (Continued)

<table>
<tr><th>$f(x)$</th><th>$F(y)$</th><th></th></tr>
<tr><td>
</td><td>
</td><td></td></tr>
<tr><td>$f(x) = h(x) \sum_{n=-\infty}^{+\infty} g(x - nx_0)$</td><td>$F(y) = \frac{1}{x_0} \sum_{n=-\infty}^{+\infty} \left\{ G\left(\frac{n2\pi}{x_0}\right) H\left(y - \frac{n2\pi}{x_0}\right) \right\}$</td><td>(2.9</td></tr>
<tr><td>$f(x) = \sum_{n=-\infty}^{+\infty} h(nx_0) g(x - nx_0)$</td><td>$F(y) = \frac{1}{x_0} G(y) \sum_{n=-\infty}^{+\infty} H\left(y - \frac{n2\pi}{x_0}\right)$</td><td>(2.9</td></tr>
</table>

*Source*: Champeney, D.C., *Fourier Transforms and Their Physical Applications*, Academic Press, New York, 1973. With permission.

## References

Abramowitz, M. and Stegun, I. 1972. *Handbook of Mathematical Functions.* New York: Dover Publications.

Arsac, J. 1966. *Fourier Transforms and the Theory of Distributions.* Englewood Cliffs, N.J.: Prentice-Hall.

Boyce, W. and DiPrima, R. 1977. *Elementary Differential Equations and Boundary Value Problems.* New York: John Wiley & Sons.

Bracewell, R. 1965. *The Fourier Transforms and Its Applications.* New York: McGraw-Hill.

Campbell, G.A. and Foster, R.M. 1948. *Fourier Integrals for Practical Applications.* New York: D. Van Nostrand.

Champeney, D.C. 1973. *Fourier Transforms and Their Physical Applications*, New York: Academic Press.

Champeney, D.C. 1987. *A Handbook of Fourier Theorems.* Cambridge, U.K.: Cambridge University Press.

Erdélyi, A. (Ed.) 1954. *Tables of Integral Transforms (Bateman Manuscript Project).* New York: McGraw-Hill.

Holland, S. 1990. *Applied Analysis by the Hilbert Space Method.* New York: Marcel Dekker.

Körner, T.W. 1988. *Fourier Analysis.* Cambridge, U.K.: Cambridge University Press.

Papoulis, A. 1962. *The Fourier Transforms and Its Applications.* New York: McGraw-Hill.

Papoulis, A. [1968]. 1986. *Systems and Transforms with Applications in Optics.* New York: McGraw-Hill. Reprinted, Malabar, FL: Robert E. Krieger Publishing Company.

Pinsky, M. 1991. *Partial Differential Equations and Boundary-Value Problems with Applications.* New York: McGraw-Hill.

Strichartz, R. 1994. *A Guide to Distribution Theory and Fourier Transforms.* Boca Raton, FL: CRC Press LLC.

Walker, J.S. 1988. *Fourier Analysis.* New York: Oxford University Press.

Walker, J.S. 1996. *Fast Fourier Transforms (2nd ed.).* Boca Raton, FL: CRC Press LLC.