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9
The Hankel Transform

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ABSTRACT Hankel transforms are integral transformations whose kernels are Bessel functions. They are sometimes referred to as Bessel transforms. When we are dealing with problems that show circular symmetry, Hankel transforms may be very useful. Laplace's partial differential equation in cylindrical coordinates can be transformed into an ordinary differential equation by using the Hankel transform. Because the Hankel transform is the two-dimensional Fourier transform of a circularly symmetric function, it plays an important role in optical data processing.

9.1 Introductory Definitions and Properties

Bessel functions are solutions of the differential equation

$$x^2 y'' + xy' + (x^2 - p^2)y = 0 \tag{9.1}$$

where p is a parameter.

Equation (9.1) can be solved using series expansions. The Bessel function $J_p(x)$ of the first kind and of order p is defined by

$$J_p(x)=\left(\frac{1}{2}x\right)^p\sum_{k=0}^{\infty}\frac{\left(-\frac{1}{4}x^2\right)^k}{k!\Gamma(p+k+1)} \quad . \tag{9.2}$$

The Bessel function $Y_p(x)$ of the second kind and of order p is another solution that satisfies

$$W(x)=\det\begin{bmatrix} J_p(x) & Y_p(x) \\ J'_p(x) & Y'_p(x)\end{bmatrix}=\frac{2}{\pi x} \quad .$$

Properties of Bessel function have been studies extensively (see References 7, 22, and 26).

Elementary properties of the Bessel functions are

1. *Asymptotic forms.*

$$J_p(x)\sim\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{1}{2}p\pi-\frac{1}{4}\pi\right), \qquad x\to\infty\,. \tag{9.3}$$

2. *Zeros.* $J_p(x)$ and $Y_p(x)$ have an infinite number of real zeros, all of which are simple, with the possible exception of $x = 0$. For nonnegative p the sth positive zero of $J_p(x)$ is denoted by $j_{p,s}$. The distance between two consecutive zeros tends to π: $\lim_{s\to\infty}(j_{p,s+1} - j_{p,s}) = \pi$.
3. *Integral representations.*

$$J_p(x)=\frac{\left(\frac{1}{2}x\right)^p}{\pi^{1/2}\Gamma(p+1/2)}\int_0^{\pi}\cos(x\cos\theta)\sin^{2p}\theta\,d\theta \quad . \tag{9.4}$$

If p is a positive integer or zero, then

$$\begin{aligned} J_p(x) &= \frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta-p\theta)d\theta \\ &= \frac{j^{-n}}{\pi}\int_0^{\pi}e^{jx\cos\theta}\cos(p\theta)d\theta \quad . \end{aligned} \tag{9.5}$$

4. *Recurrence relations.*

$$J_{p-1}(x)-\frac{2p}{x}J_p(x)+J_{p+1}(x)=0 \tag{9.6}$$

$$J_{p-1}(x)-J_{p+1}(x)=2J'_p(x) \tag{9.7}$$

$$J'_p(x)=J_{p-1}(x)-\frac{p}{x}J_p(x) \tag{9.8}$$

$$J'_p(x)=-J_{p+1}(x)+\frac{p}{x}J_p(x) \quad . \tag{9.9}$$

5. *Hankel's repeated integral.* Let $f(r)$ be an arbitrary function of the real variable r, subject to the condition that

$$\int_0^\infty f(r)\sqrt{r}\,dr$$

is absolutely convergent. Then for $p \geq -1/2$

$$\int_0^\infty s\,ds \int_0^\infty f(r) J_p(sr) J_p(su) r\,dr = \frac{1}{2}[f(u+) + f(u-)] \tag{9.10}$$

provided that $f(r)$ satisfies certain Dirichlet condtions.

For a proof, see Reference 26. The reader should also refer to Chapter 1, Section 5.6 for more information regarding Bessel functions.

9.2 Definition of the Hankel Transform

Let $f(r)$ be a function defined for $r \geq 0$. The νth order Hankel transform of $f(r)$ is defined as

$$F_\nu(s) \equiv \mathcal{H}_\nu\{f(r)\} \equiv \int_0^\infty r f(r) J_\nu(sr) dr \quad . \tag{9.11}$$

If $\nu > -1/2$, Hankel's repeated integral immediately gives the inversion formula

$$f(r) = \mathcal{H}_\nu^{-1}\{F_\nu(s)\} \equiv \int_0^\infty s F_\nu(s) J_\nu(sr) ds \quad . \tag{9.12}$$

The most important special cases of the Hankel transform correspond to $\nu = 0$ and $\nu = 1$. Sufficient but not necessary conditions for the validity of (9.11) and (9.12) are

1. $f(r) = O(r^{-k})$, $r \to \infty$ where $k > 3/2$.
2. $f'(r)$ is piecewise continuous over each bounded subinterval of $[0, \infty)$.
3. $f(r)$ is defined as $[f(r+) + f(r-)]/2$.

These conditions can be relaxed.

9.3 Connection with the Fourier Transform

We consider the two-dimensional Fourier transform of a function $\varphi(x,y)$, which shows a circular symmetry. This means that $\varphi(r\cos\theta, r\sin\theta) \equiv f(r,\theta)$ is independent of θ.

The Fourier transform of φ is

$$\Phi(\zeta,\eta) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(x,y) e^{-j(x\zeta, y\eta)} dx\,dy \quad . \tag{9.13}$$

We introduce the polar coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta$$

and

$$\zeta = s\cos\varphi, \qquad \eta = s\sin\varphi \quad .$$

We have then

$$\phi(s\cos\varphi, s\sin\varphi) \equiv F(s,\varphi) = \frac{1}{2\pi}\int_0^\infty r\,dr\int_0^{2\pi} e^{-jrs\cos(\theta-\varphi)} f(r)d\theta$$

$$= \frac{1}{2\pi}\int_0^\infty rf(r)dr\int_0^{2\pi} e^{-jrs\cos\alpha}d\alpha$$

$$= \int_0^\infty rf(r)J_0(rs)dr.$$

This result shows that $F(s,\varphi)$ is independent of φ, so that we can write $F(s)$ instead of $F(s, \varphi)$. Thus, the two-dimensional Fourier transform of a circularly symmetric function is, in fact, a Hankel transform of order zero.

This result can be generalized: the N-dimensional Fourier transform of a circularly symmetric function of N variables is related to the Hankel transform of order $N/2 - 1$. If $f(r,\theta)$ depends on θ, we can expand it into a Fourier series

$$f(r,\theta) = \sum_{n=-\infty}^{\infty} f_n(r)e^{jn\theta} \tag{9.14}$$

and, similarly

$$F(s,\varphi) = \frac{1}{2\pi}\int_0^\infty r\,dr\int_0^\infty e^{-jrs\cos(\theta-\varphi)} f(r,\theta)d\theta = \sum_{n=-\infty}^{\infty} F_n(s)e^{jn\varphi} \tag{9.15}$$

where

$$f_n(r) = \frac{1}{2\pi}\int_0^{2\pi} f(r,\theta)e^{-jn\theta}d\theta \tag{9.16}$$

and

$$F_n(s) = \frac{1}{2\pi}\int_0^{2\pi} F(s,\varphi)e^{-jn\varphi}d\varphi \quad . \tag{9.17}$$

Substituting (9.15) into (9.17) and using (9.14), we obtain

$$F_n(s) = \frac{1}{(2\pi)^2}\int_0^{2\pi} e^{-jn\varphi}d\varphi\int_0^{2\pi} d\theta\int_0^\infty f(r,\theta)e^{jsr\cos(\theta-\varphi)}r\,dr$$

$$= \frac{1}{(2\pi)^2}\int_0^{2\pi} e^{-jn\varphi}d\varphi\int_0^\infty r\,dr\int_0^{2\pi} e^{jsr\cos(\theta-\varphi)}d\theta \times \sum_{m=-\infty}^{\infty} f_m(r)e^{jm\theta}$$

$$= \frac{1}{(2\pi)}\int_0^\infty r\,dr\int_0^{2\pi} e^{-jn\alpha}e^{jsr\cos\alpha} f_n(r)d\alpha$$

$$= \int_0^\infty rf_n(r)J_n(sr)dr$$

$$= \mathscr{H}_n\{f_n(r)\}.$$

In a similar way, we can derive

$$f_n(r) = \mathcal{H}_n\{F_n(s)\} \quad . \tag{9.18}$$

9.4 Properties and Examples

Hankel transforms do not have as many elementary properties as do the Laplace or the Fourier transforms.
For example, because there is no simple addition formula for Bessel functions, the Hankel transform does not satisfy any simple convolution relation.

1. *Derivatives.* Let

$$F_\nu(s) = \mathcal{H}_\nu\{f(x)\} .$$

Then

$$G_\nu(s) = \mathcal{H}_\nu\{f'(x)\} = s\left[\frac{\nu+1}{2\nu}F_{\nu-1}(s) - \frac{\nu-1}{2\nu}F_{\nu+1}(s)\right]. \tag{9.19}$$

Proof

$$G_\nu(s) = \int_0^\infty xf'(x)\,J_\nu(sx)\,dx$$

$$= [xf(x)\,J_\nu(sx)]_0^\infty - \int_0^\infty f(x)\frac{d}{dx}[xJ_\nu(sx)]dx \ \cdot$$

In general, the expression between the brackets is zero, and

$$\frac{d}{dx}[xJ_\nu(sx)] = \frac{sx}{2\nu}[(\nu+1)\,J_{\nu-1}(sx) - (\nu-1)\,J_{\nu+1}(sx)] \ .$$

Hence, we have (9.19).

2. *The Hankel transform of the Bessel differential operator.* The Bessel differential operator

$$\Delta_\nu \equiv \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \left(\frac{\nu}{r}\right)^2 = \frac{1}{r}\frac{d}{dr}r\frac{d}{dr} - \left(\frac{\nu}{r}\right)^2$$

is derived from the Laplacian operator

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

after separation of variables in cylindrical coordinates (r, θ, z).

Let $f(r)$ be an arbitrary function with the property that $\lim_{r\to\infty} f(r) = 0$. Then

$$\mathcal{H}_\nu\{\Delta_\nu f(r)\} = -s^2\,\mathcal{H}_\nu\{f(r)\} \quad . \tag{9.20}$$

This result shows that the Hankel transform may be a useful tool in solving problems with cylindrical symmetry and involving the Laplacian operator.

Proof Integrating by parts, we have

$$\mathcal{H}_\nu\{\Delta_\nu f(r)\} = \int_0^\infty \left[\frac{d}{dr} r \frac{df}{dr} - \frac{\nu^2}{r} f(r)\right] J_\nu(sr)dr$$

$$= \int_0^\infty \left[s^2 J_\nu''(sr) + \frac{s}{x} J_\nu'(sr) - \frac{\nu^2}{r^2} J_\nu(sr)\right] f(r) r\, dr$$

$$= -s^2 \int_0^\infty rf(r) J_\nu(rs) dr$$

$$= -s^2 \mathcal{H}_\nu \{f(r)\}.$$

This property is the principal one for applications of the Hankel transforms to solving differential equations. See References 2, 3, 24, 25, and 28.

3. *Similarity.*

$$\mathcal{H}_\nu\{f(ar)\} = \frac{1}{a^2} F_\nu\left(\frac{s}{a}\right). \tag{9.21}$$

4. *Division by r.*

$$\mathcal{H}_\nu\{r^{-1} f(r)\} = \frac{s}{2\nu}[F_{\nu-1}(s) + F_{\nu+1}(s)]. \tag{9.22}$$

5.

$$\mathcal{H}_\nu\left\{r^{\nu-1} \frac{d}{dr}[r^{1-\nu} f(r)]\right\} = -sF_{\nu-1}(s). \tag{9.23}$$

6.

$$\mathcal{H}_\nu\left\{r^{-\nu-1} \frac{d}{dr}[r^{\nu+1} f(r)]\right\} = sF_{\nu+1}(s). \tag{9.24}$$

7. *Parseval's theorem.* Let

$$F_\nu(s) = \mathcal{H}_\nu\{f(r)\}$$

and

$$G_\nu(s) = \mathcal{H}_\nu\{g(r)\}.$$

Then

$$\int_0^\infty F_\nu(s)G_\nu(s)s\,ds = \int_0^\infty F_\nu(s)s\,ds\int_0^\infty r\,g(r)J_\nu(sr)dr$$
$$= \int_0^\infty r\,g(r)dr\int_0^\infty sF_\nu(s)J_\nu(sx)ds \tag{9.25}$$
$$= \int_0^\infty r\,g(r)f(r)dr\,.$$

Example 9.1

From the Fourier pair (see Chapter 2) $\mathcal{F}\{e^{-a(x^2+y^2)}\} = (\pi/a)e^{-(\zeta^2+\eta^2)/4a}$ and the Fourier transform relationship $\mathcal{F}\{f(\sqrt{x^2+y^2})\} = 2\pi F_0(\sqrt{\zeta^2+\eta^2}) \equiv 2\pi F_0(s)$, we obtain the Hankel transform

$$\mathcal{H}\{e^{-ar^2}\} = \frac{1}{2a}e^{-s^2/4a}, \qquad a>0\,.$$

Example 9.2

From the relationship $\int_0^a r\,J_0(sr)\,dr = \int_0^a (1/s)(d/dr)[rJ_1(sr)] = [aJ_1(as)]/s$ (see Chapter 1, Section 5.6), we conclude that

$$\mathcal{H}_0\{p_a(r)\} = \frac{aJ_1(as)}{s}$$

where $p_a(r) = 1$ for $|r| < a$ and zero otherwise.

Example 9.3

From the identity $\int_0^\infty J_0(sr)\,dr = 1/s,\ s>0$ (see Chapter 1, Section 5.6), we obtain

$$\mathcal{H}_0\left\{\frac{1}{r}\right\} = \frac{1}{s}\ .$$

Example 9.4

Since $\int_0^\infty r\,\delta(r-a)J_0(sr)\,dr = aJ_0(as)$ (see Chapter 1, Section 2.4), we obtain

$$\mathcal{H}_0\{\delta(r-a)\} = aJ_0(as), \qquad a>0$$

and because of symmetry

$$\mathcal{H}_0\{aJ_0(ar)\} = \delta(s-a), \qquad a>0\ .$$

Convolution Identity

Let $f_1(r)$ and $f_2(r)$ have Hankel transforms $F_1(s)$ and $F_2(s)$, respectively. From Section 2.4.1 above, we have

$$\mathfrak{F}\left\{\int\int_{-\infty}^{\infty} f_1(\sqrt{x_1^2+y_1^2})f_2(\sqrt{(x-x_1)^2+(y-y_1)^2})dx_1dy_1\right\} = 4\pi^2 F_1(s)F_2(s) .$$

Hence, we have

$$\mathcal{H}_0\{f_1(r)\star\star f_2(r)\} = \frac{1}{2\pi}\mathfrak{F}_{(2)}\{f_1(r)\star\star f_2(r)\} = 2\pi F_1(s)F_2(s) .$$

Therefore, to find the inverse Hankel transform of $2\pi F_1(s)F_2(s)$, we convolve $f_1(\sqrt{x^2+y^2})$ with $f_2(\sqrt{x^2+y^2})$, and in the answer we replace $\sqrt{x^2+y^2}$ by r. We can also write the above relationship in the form

$$\mathcal{H}_0\{2\pi f_1(r)f_2(r)\} = F_1(s)\star\star F_2(s) .$$

Example 9.5

If $f_1(r) = f_2(r) = [J_1(ar)]/r$ then from the convolution identity above, we obtain

$$\mathcal{H}_0\left\{2\pi\frac{J_1^2(ar)}{r^2}\right\} = \frac{1}{a^2}p_a(s)\star\star p_a(s)$$

where

$$p_a(s)\star\star p_a(s) = \left(2\cos^{-1}\frac{s}{2a} - \frac{s}{a}\sqrt{1-\frac{s^2}{4a^2}}\right)a^2 .$$

Hence,

$$\mathcal{H}_0\left\{2\pi\frac{J_1^2(ar)}{r^2}\right\} = \left(2\cos^{-1}\frac{s}{2a} - \frac{s}{a}\sqrt{1-\frac{s^2}{4a^2}}\right)p_{2a}(s)$$

$$p_{2a}(s) = \begin{cases}1 & |s|\le 2a\\ 0 & \text{otherwise}\end{cases} .$$

Example 9.6

From the definition, the Hankel transform of $r^\nu h(a-r)$, $a > 0$ is given by

$$\mathcal{H}_\nu\{r^\nu h(a-r)\} = \int_0^a r^{\nu+1}J_\nu(sr)dr = \frac{1}{s^{\nu+2}}\int_0^{as} x^{\nu+1}J_\nu(x)dx$$

since $h(a-r)$ is the unit step function with value equal to 1 for $r \le a$ and 0 for $r > a$. But $\int t^\nu J_{\nu-1}(t)\, dt = t^\nu J_\nu(t) + C$ (see Chapter 1, Section 5.6, Table 5.6.1) and hence,

$$\mathcal{H}_\nu\{r^\nu h(a-r)\} = \frac{(as)^{\nu+1}}{s^{\nu+2}}J_{\nu+1}(as) = \frac{a^{\nu+1}}{s}J_{\nu+1}(as), \qquad a>0,\ \nu>-\frac{1}{2} .$$

Example 9.7

The Hankel transform of $r^{\nu-1}e^{-ar}$, $a > 0$ is given by

$$\mathcal{H}_\nu\{r^{\nu-1}e^{-ar}\} = \int_0^\infty r^\nu e^{-ar} J_\nu(sr)dr = \frac{1}{s^{\nu+1}}\int_0^\infty t^\nu J_\nu(t) e^{-\frac{a}{s}t} dt$$

$$= \frac{1}{s^{\nu+1}} \mathcal{L}\left\{t^\nu J_\nu(t);\ p = \frac{a}{s}\right\}$$

where we set $t = rs$ and $\mathcal{L}$ is the Laplace transform operator (see also Chapter 5). But

$$t^\nu J_\nu(t) = \sum_{n=0}^\infty \frac{(-1)^n t^{2n+2\nu}}{n!\Gamma(n+\nu+1)2^{2n+\nu}}$$

and, hence,

$$\mathcal{L}\{t^\nu J_\nu(t);\ p\} = \sum_{n=0}^\infty \frac{(-1)^n}{n!\Gamma(n+\nu+1)2^{2n+\nu}} \mathrm{L}\{t^{2n+2\nu};\ p\}$$

$$= \sum_{n=0}^\infty \frac{(-1)^n \Gamma(2n+2\nu+1)}{n!\Gamma(n+\nu+1)2^{2n+\nu} p^{2n+2\nu+1}}\ .$$

From Chapter 1, Section 2.5, the duplication formula of the gamma function gives the relationship

$$\frac{\Gamma(2n+2\nu+1)}{\Gamma(n+\nu+1)} = \frac{1}{\sqrt{\pi}} 2^{2n+2\nu}\Gamma\left(n+\nu+\frac{1}{2}\right)$$

and, therefore, the Laplace transform relation becomes

$$\mathcal{L}\{t^\nu J_\nu(t);\ p\} = \frac{2^\nu}{\sqrt{\pi}p^{2\nu+1}} \sum_{n=0}^\infty \frac{(-1)^n\Gamma\left(n+\nu+\frac{1}{2}\right)}{n!}\left(\frac{1}{p^2}\right)^n .$$

The last series can be summed by using properties of the binomial series

$$(1+x)^{-b} = \sum_{n=0}^\infty \binom{-b}{n} x^n = \sum_{n=0}^\infty \frac{(-1)^n\Gamma(n+b)}{n!\Gamma(b)} x^n, \qquad |x| < 1$$

where the relation

$$\binom{-b}{n} = \frac{(-1)^n b(b+1)\cdots(b+n-1)}{n!} = \frac{(-1)^n\Gamma(n+b)}{n!\Gamma(b)}$$

was used. The Laplace transform now becomes

$$\mathcal{L}\{t^\nu J_\nu(t);\; p\} = \frac{2^\nu \Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}(p^2+1)^{\nu+\frac{1}{2}}} = \frac{2^\nu \Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\left[\left(\frac{a}{s}\right)^2+1\right]^{\nu+\frac{1}{2}}}, \quad \mathrm{Re}(p) > 1$$

and, hence,

$$\mathcal{H}_\nu\{r^{\nu-1}e^{-ar}\} = \frac{1}{s^{\nu+1}} \frac{2^\nu \Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\left[\left(\frac{a}{s}\right)^2+1\right]^{\nu+\frac{1}{2}}} = \frac{s^\nu 2^\nu \Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}(a^2+s^2)^{\nu+\frac{1}{2}}}, \qquad \nu > -\frac{1}{2}\ .$$

If we set $\nu = 0$ and $a = 0$ in the above equation, we obtain the results of Example 9.3. If we set $\nu = 0$, we obtain

$$\mathcal{H}_0\{r^{-1}e^{-ar}\} = \frac{1}{\sqrt{a^2+s^2}}, \qquad a > 0\ .$$

Example 9.8

The Hankel transform $\mathcal{H}_0\{e^{-ar}\}$ is given by

$$\mathcal{H}_0\{e^{-ar}\} = \mathcal{L}\{r\ \ J_0(sr);\; r \to a\} = -\frac{d}{da}\left[(s^2+a^2)^{-\frac{1}{2}}\right]$$

$$= \frac{a}{[s^2+a^2]^{3/2}}, \qquad a > 0$$

since multiplication by r corresponds to differentiation in the Laplace transform domain.

Example 9.9

From Chapter 1, Section 5.6, we have the following identity:

$$\frac{d^2 r_n(x)}{dx^2} + \frac{1}{x}\frac{dr_n(x)}{dx} + r_n(x) = 2nr_{n+1}(x)\ ,$$

where $r_n(x) = J_n(x)/x^n$.

Using the Hankel transform property of the Bessel operator, we obtain the relationship

$$(1-s^2)R_n(s) = 2nR_{n+1}(s)$$

or

$$R_{n+1}(s) = \frac{1-s^2}{2n}R_n(s) = \cdots = \frac{(1-s^2)^n}{2^n n!}R_1(s)\ .$$

But from Example 9.2, $\mathcal{H}_0\left\{\frac{J_1(r)}{r}\right\} = p_1(s)$ and, hence,

$$\mathcal{H}_0\left\{\frac{J_n(r)}{r^n}\right\} = \frac{(1-s^2)^{n-1}}{2^{n-1}(n-1)!}p_1(s)$$

where $p_1(s)$ is a pulse of width 2 centered at $s = 0$.

Example 9.10

If the impulse response of a linear space invariant system is $h(r)$ and the input to the system is $f(r)$, then its output is $g(r) = f(r)\star\star h(r)$ and, hence,

$$G(s) = 2\pi F(s)H(s) .$$

Since $\mathcal{H}_0\{J_0(ar)\} = [\delta(s-a)]/a$ (see Example 9.4) and $\varphi(s)\delta(s-a) = \varphi(a)\delta(s-a)$, we conclude that if the input is $f(r) = J_0(ar)$, then

$$G(s) = \frac{2\pi}{a}\delta(s-a)H(s) = \frac{2\pi H(a)}{a}\delta(s-a) .$$

Therefore, the output is

$$g(r) = 2\pi H(a)\, J_0(ar) .$$

9.5 Applications

9.5.1 The Electrified Disc

Let υ be the electric potential due to a flat circular electrified disc, with radius $R = 1$, the center of the disc being at the origin of the three-dimensional space and its axis along the z-axis.

In polar coordinates, the potential satisfies Laplace's equation

$$\nabla^2\upsilon \equiv \frac{\partial^2\upsilon}{\partial r^2} + \frac{1}{r}\frac{\partial\upsilon}{\partial r} + \frac{\partial^2\upsilon}{\partial z^2} = 0 . \tag{9.26}$$

The boundary conditions are

$$\upsilon(r,0) = \upsilon_0, \qquad 0 \le r < 1 \tag{9.27}$$

$$\frac{\partial\upsilon}{\partial z}(r,0) = 0, \qquad r > 1 . \tag{9.28}$$

In (9.27), υ_0 is the potential of the disc. Condition (9.28) arises from the symmetry about the plane $z = 0$.

Let

$$V(s,z) = \mathcal{H}_0\{\upsilon(r,z)\}$$

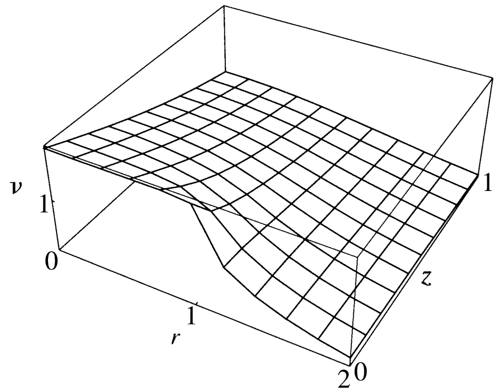

FIGURE 9.1 Electrical potential due to an electrified disc.

so that

$$\mathcal{H}_0\{\nabla^2 \upsilon\} = -s^2 V(s,z) + \frac{\partial^2 V}{\partial z^2}(s,z) = 0 \ .$$

The solution of this differential equation is

$$V(s,z) = A(s)e^{-sz} + B(s)e^{sz}$$

where A and B are functions that we have to determine using the boundary conditions.

Because the potential vanishes as z tends to infinity, we have $B(s) \equiv 0$. By inverting the Hankel transform, we have

$$\upsilon(r,z) = \int_0^\infty sA(s)e^{-sz} J_0(sr)ds \ . \qquad (9.29)$$

The boundary conditions are now

$$\upsilon(r,0) = \int_0^\infty sA(s)J_0(rs)ds = \upsilon_0, \qquad 0 \le r < 1 \qquad (9.30)$$

$$\frac{\partial \upsilon}{\partial z}(r,0) = \int_0^\infty s^2 A(s)J_0(rs)ds = 0, \qquad r > 1 \ . \qquad (9.31)$$

Using entries (9.8) and (9.9) of Table 9.1 (see Section 9.11), we see that $A(s) = \sin s/s^2$ so that

$$\upsilon(r,z) = \frac{2\upsilon_0}{\pi}\int_0^\infty \frac{\sin s}{s} e^{-sz} J_0(sr)ds \ . \qquad (9.32)$$

In Figure 9.1, the graphical representation of $\upsilon(r,z)$ for $\upsilon_0 = 1$ is depicted on the domain $0 \le r \le 2$, $0 \le z \le 1$. The evaluation of $\upsilon(r,z)$ requires numerical integration.

Equations (9.30) and (9.31) are special cases of the more general pair of equations

$$\int_0^\infty f(t)t^{2\alpha} J_\nu(xt)dt = a(x), \qquad 0 \le x < 1 \tag{9.33}$$

$$\int_0^\infty f(t) J_\nu(xt)dt = 0, \qquad x > 1 \tag{9.34}$$

where $a(x)$ is given and $f(x)$ is to be determined.

The solution of (9.29) can be expressed as a repeated integral:[12]

$$f(x) = \frac{2^{-\alpha} x^{1-\alpha}}{\Gamma(\alpha+1)} \int_0^1 s^{-\nu-\alpha} J_{\nu+\alpha}(xs) \frac{d}{ds} \int_0^s a(t) t^{\nu+1} (s^2 - t^2)^\alpha \, dt\, ds, \quad -1 < \alpha < 0 \tag{9.35}$$

$$f(x) = \frac{(2x)^{1-\alpha}}{\Gamma(\alpha)} \int_0^1 s^{-\nu-\alpha+1} J_{\nu+\alpha}(xs) \int_0^s a(t) t^{\nu+1} (s^2 - t^2)^{\alpha-1} \, dt\, ds, \quad 0 < \alpha < 1\ . \tag{9.36}$$

If $a(x) = x^\beta$, and $\alpha < 1$, $2\alpha + \beta > -3/2$, $\alpha + \nu > -1$, $\nu > -1$, then

$$f(x) = \frac{\Gamma\left(1 + \frac{\beta+\nu}{2}\right) x^{-(2\alpha+\beta+1)}}{2^\alpha \Gamma\left(1 + \alpha + \frac{\beta+\nu}{2}\right)} \int_0^x t^{\alpha+\beta+1} J_{\nu+\alpha}(t) dt\ . \tag{9.37}$$

With $\beta = \nu$ and $\alpha < 1$, $\alpha + \nu > -1$, $\nu > -1$ further simplification is possible:

$$f(x) = \frac{\Gamma(\nu+1)}{(2x)^\alpha \Gamma(\nu+\alpha+1)} J_{\nu+\alpha+1}(x)\ . \tag{9.38}$$

9.5.2 Heat Conduction

Heat is supplied at a constant rate Q per unit area and per unit time through a circular disc of radius a in the plane $z = 0$, to the semi-infinite space $z > 0$. The thermal conductivity of the space is K. The plane $z = 0$ outside the disc is insulated. The mathematical model of this problem is very similar to that of Section 5.1. The temperature is denoted by $\upsilon(r,z)$. We have again the Laplace Equation (9.26) in polar coordinates, but the boundary conditions are now

$$\begin{aligned} -K\frac{\partial \upsilon(r,z)}{\partial z} &= Q, \qquad r < a,\ \ z = 0 \\ &= 0, \qquad r > a,\ \ z = 0\ . \end{aligned} \tag{9.39}$$

The Hankel transform of the differential equation is again

$$\frac{\partial^2 V}{\partial z^2}(s,z) - s^2 V(s,z) = 0\ . \tag{9.40}$$

We can now transform also the boundary condition, using formula (3) in Table 9.1:

$$-K\frac{\partial V}{\partial z}(s,0)=Qa\ \ J_1(as)/\,s\ . \tag{9.41}$$

The solution of (9.39) must remain finite as z tends to infinity. We have

$$V(s,z)=A(s)e^{-sz}\ .$$

Using condition (9.41) we can determine

$$A(s)=Qa\ \ J_1(as)/(Ks^2)\ .$$

Consequently, the temperature is given by

$$\upsilon(r,z)=\frac{Qa}{K}\int_0^\infty e^{-sz}J_1(as)J_0(rs)s^{-1}ds\ . \tag{9.42}$$

9.5.3 The Laplace Equation in the Halfspace z > 0, with a Circularly Symmetric Dirichlet Condition at z = 0

We try to find the solution $\upsilon(r,z)$ of the boundary value problem

$$\begin{cases}\dfrac{\partial^2\upsilon}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial\upsilon}{\partial r}+\dfrac{\partial^2\upsilon}{\partial z^2}=0, \qquad z>0,\ \ 0<r<\infty\\ \upsilon(r,0)=f(r).\end{cases} \tag{9.43}$$

Taking the Hankel transform of order 0 yields

$$\frac{\partial^2 V}{\partial z^2}(s,z)-s^2V(s,z)=0$$

and

$$V(s,0)=\int_0^\infty rf(r)\,J_0(sr)dr\ .$$

The solution is

$$V(s,z)=e^{-sz}\int_0^\infty rf(r)\,J_0(sr)dr$$

so that

$$\upsilon(r,z)=\int_0^\infty s\,e^{-sz}J_0(sr)ds\int_0^\infty p\,f(p)J_0(sp)dp\ . \tag{9.44}$$

For the special case

$$f(r)=h(a-r)$$

where $h(r)$ is the unit step function, we have the solution

$$\upsilon(r,z) = a\int_0^\infty e^{-sz} J_0(sr) J_1(as) ds \, . \tag{9.45}$$

9.5.4 An Electrostatic Problem

The electrostatic potential $Q(r,z)$ generated in the space between two grounded horizontal plates at $z = \pm \ell$ by a point charge q at $r = 0$, $z = 0$ shows a singular behavior at the origin. It is given by

$$\upsilon(r,z) = \varphi(r,z) + q(r^2 + z^2)^{-1/2} \tag{9.46}$$

where $\varphi(r,z)$ satisfies Laplace's Equation (9.26). The boundary conditions are

$$\varphi(r, \pm\ell) + q(r^2 + \ell^2)^{-1/2} = 0 \, . \tag{9.47}$$

Taking the Hankel transform of order 0, we obtain

$$\frac{\partial^2 \Phi}{\partial z^2}(s,z) - s^2 \Phi(s,z) = 0 \tag{9.48}$$

$$\Phi(s, \pm\ell) = -\frac{qe^{-s\ell}}{s} \tag{9.49}$$

(see formula (18) in Table 9.1).

The solution is

$$A(s)e^{-sz} + B(s)e^{sz}$$

where $A(s)$ and $B(s)$ must satisfy

$$A(s)e^{+s\ell} + B(s)e^{-s\ell} = -\frac{q \; e^{-s\ell}}{s}$$

$$A(s)e^{-s\ell} + B(s)e^{s\ell} = -\frac{q \; e^{-s\ell}}{s} \, .$$

Hence,

$$A(s) = B(s) = -\frac{q \; e^{-s\ell}}{2s \cosh(s\ell)}$$

and

$$\Phi(s,z) = -\frac{q \; e^{-s\ell}}{s} \frac{\cosh(sz)}{\cosh(s\ell)} \, .$$

Hence,

$$\varphi(r,z)=\frac{q}{\sqrt{r^2+z^2}}-q\int_0^\infty e^{-s\ell}\frac{\cosh(sz)}{\cosh(s\ell)}J_0(sr)\,ds\ . \tag{9.50}$$

9.6 The Finite Hankel Transform

We consider the integral transformation

$$F_\nu(\alpha)=H_\nu\{f,\alpha\}=\int_0^1 rf(r)\,J_\nu(\alpha r)\,dr\ . \tag{9.51}$$

A property of this transformation is that

$$H_\nu(\Delta_\nu f,\alpha)=-\alpha^2F_\nu(\alpha)+[J_\nu(\alpha)f'(1)-\alpha J'_\nu(\alpha)f(1)]$$

where $\triangle_\nu$ is the Bessel differential operator.

If α is equal to the sth positive zero $j_{\nu,s}$ *of* $J_\nu(x)$, we have

$$H_\nu(\Delta_\nu f,j_{\nu,s})=-j_{\nu,s}^2H_\nu(f,j_{\nu,s})+j_{\nu,s}J_{\nu+1}(j_{\nu,s})f(1)\ .$$

If α is equal to the sth positive root $\beta_{\nu,s}$ of

$$hJ_\nu(x)+xJ'_\nu(x)=0$$

where h is a nonnegative constant, we have

$$H_\nu(\Delta_\nu f,\beta_{\nu,s})=-\beta_{\nu,s}^2H_\nu(f,\beta_{\nu,s})+J_\nu(\beta_{\nu,s})[hf(1)+f'(1)]\ .$$

The transformation (9.51) with $\alpha=j_{\nu,s}$, $s=1,2,\ldots$ is the finite Hankel transform. It maps the function $f(r)$ into the vector $(F_\nu(j_{\nu,1}), F_\nu(j_{\nu,2}), F_\nu(j_{\nu,3})\ \ldots)$. The inversion formula can be obtained from the well-known theory of Fourier-Bessel series

$$f(r)=2\sum_{s=1}^\infty\frac{F_\nu(j_{\nu,s})}{J_{\nu+1}^2(j_{\nu,s})}J_\nu(j_{\nu,s}r)\ . \tag{9.52}$$

The transformation (9.51) with $\alpha=\beta_{\nu,s}$, $s=1,2,\ldots$ is the modified finite Hankel transform. The inversion formula is

$$f(r)=2\sum_{s=1}^\infty\frac{\beta_{\nu,s}^2F_\nu(\beta_{\nu,s})}{h^2+\beta_{\nu,s}^2-\nu^2}\frac{J_\nu(\beta_{\nu,s}r)}{J_\nu^2(\beta_{\nu,s})}\ . \tag{9.53}$$

If $h=0$, $\beta_{\nu,s}$ is the sth positive zero of $J'_\nu(x)$, denoted by $j'_{\nu,s}$.

Formulas for the computation of $j_{\nu,s}$ and $j'_{\nu,s}$ are given by Olver.[15] Values of $j_{\nu,s}$ and $j'_{\nu,s}$ are tabulated in Reference 1. A Fortran program for the computation of $j_{\nu,s}$ and $j'_{\nu,s}$ is given in Reference 18.

Application

We calculate the temperature $\upsilon(r,t)$ at time t of a long solid cylinder of unit radius. The initial temperature is unity and radiation takes place at the surface into the surrounding medium maintained at zero temperature.

The mathematical model of this problem is the diffusion equation in polar coordinates

$$\frac{\partial^2 \upsilon}{\partial r^2} + \frac{1}{r}\frac{\partial \upsilon}{\partial r} = \frac{\partial \upsilon}{\partial t}, \qquad 0 \le r < 1, \quad t > 0 \tag{9.54}$$

The initial condition is

$$\upsilon(r,0) = 1, \qquad 0 \le r \le 1 \ . \tag{9.55}$$

The radiation at the surface of the cylinder is described by the mixed boundary condition

$$\frac{\partial \upsilon}{\partial r}(1,t) = -h\upsilon(1,t) \tag{9.56}$$

where h is a positive constant.

Transformation of (9.54) by the modified finite Hankel transform yields

$$\frac{dV}{dt}(\beta_{0,s},t) = -\beta_{0,s}^2 V(\beta_{0,s},t) \tag{9.57}$$

where

$$V(\alpha,t) = \int_0^1 r\upsilon(r,t) J_0(\alpha r)dr$$

so that

$$V(\alpha,0) = \int_0^1 rJ_0(\alpha r)dr = \frac{J_1(\alpha)}{\alpha} \ . \tag{9.58}$$

The solution of (9.57), with the initial condition (9.58), is

$$V(\beta_{0,s},t) = \frac{J_1(\beta_{0,s})}{\beta_{0,s}} e^{-\beta_{0,s}^2 t} \ .$$

Using the inversion formula, we obtain

$$\upsilon(r,t) = 2\sum_{j=1}^{\infty} e^{-\beta_{0,s}^2 t} \frac{\beta_{0,s} J_1(\beta_{0,s})}{h^2 + \beta_{0,s}^2} \frac{J_0(\beta_{0,s} r)}{J_0^2(\beta_{0,s})} \ . \tag{9.59}$$

9.7 Related Transforms

For some applications, Hankel transforms with a more general kernel may be useful. We give one example.

We consider the cylinder function

$$Z_\nu(s,r) = J_\nu(sr)Y_\nu(s) - Y_\nu(sr)J_\nu(s)\ . \tag{9.60}$$

Using this function as a kernel, we can construct the following transform pair:

$$F_\nu(s) = \int_1^\infty rf(r)Z_\nu(s,r)dr \tag{9.61}$$

$$f(r) = \int_0^\infty sF_\nu(s)\frac{Z_\nu(s,r)}{J_\nu^2(s)+Y_\nu^2(s)}ds \tag{9.62}$$

The inversion formula follows immediately from Weber's integral theorem (see Watson[26]):

$$\int_1^\infty u\ du\int_0^\infty f(s)Z_\nu(r,u)Z_\nu(s,u)s\ \ ds = \frac{1}{2}[J_\nu^2(r)+Y_\nu^2(r)][f(r+)+f(r-)]\ . \tag{9.63}$$

For this reason, we will refer to (9.61) and (9.62) as the Weber transform. This transform has the following important property:

If

$$f(x) = g''(x) + \frac{1}{x}g'(x) - \frac{\nu^2}{x^2}g(x) \tag{9.64}$$

then

$$F_\nu(s) = -s^2G_\nu(s) - \frac{2}{\pi}g(1)\ . \tag{9.65}$$

We may expect that this transform is useful for solving Laplace's equation in cylindrical coordinates, with a boundary condition at $r = 1$.

Example

We want to compute the steady-state temperature $u(r,z)$ in a horizontal infinite homogeneous slab of thickness 2ℓ, through which there is a vertical circular hole of radius 1. The horizontal faces are held at temperature zero and the circular surface in the hole is at temperature T_0.

The mathematical model is

$$\begin{aligned} \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} &= 0 \\ u(r,\ell) = u(r,-\ell) &= 0 \\ u(1,z) &= T_0\ . \end{aligned} \tag{9.66}$$

Taking the Weber transform of order zero, we have

$$\frac{\partial^2 U_0}{\partial z^2}(s,z) - s^2U_0(s,z) = \frac{2}{\pi}T_0\ .$$

The solution of this ordinary differential equation, satisfying the boundary condition, is

$$U_0(s, z) = \frac{2T_0}{\pi s^2}\left[\frac{\cosh sz}{\cosh s\ell} - 1\right].$$

Consequently, we have

$$u(r,z) = \frac{2T_0}{\pi}\int_0^\infty \frac{1}{s}\left[\frac{\cosh sz}{\cosh s\ell} - 1\right]\frac{Z_0(s,r)}{J_0^2(s)+Y_0^2(s)}ds \qquad (r > 1). \tag{9.67}$$

9.8 Need of Numerical Integration Methods

When using the Hankel transform for solving partial differential equations, the solution is found as an integral of the form

$$I(a,\ p,\ \nu) = \int_0^a J_\nu(px) f(x) dx \tag{9.68}$$

where a is a positive real number or infinite. In most cases, analytical integration of (9.68) is impossible, and numerical integration is necessary. But integrals of type (9.68) are difficult to evaluate numerically if

1. The product ap is large.
2. a is infinite.
3. $f(x)$ shows a singular or oscillatory behavior.

In cases 1 and 2, the difficulties arise from the oscillatory behavior of $J_\nu(x)$ and they grow when the oscillations become stronger.

We give here a survey of numerical methods that are especially suited for the evaluation of $I(a, p, \nu)$ when ap is large or a is infinite. We restrict ourselves to cases where $f(x)$ is smooth, or where $f(x) = x^\alpha g(x)$ where $g(x)$ is smooth and α is a real number.

9.9 Computation of Bessel Function Integrals Over a Finite Interval

Integral (9.68) can be written as

$$I(a,\ p,\ \nu) = a^{\alpha+1}\int_0^1 x^\alpha J_\nu(apx) g(ax) dx. \tag{9.69}$$

We assume that $\alpha + \nu > -1$. If $(\alpha+\nu)$ is not an integer, then there is an algebraic singularity of the integrand at $x = 0$. If ap is large, then the integrand is strongly oscillatory.

If $(\alpha + \nu)$ is an integer and ap is small, classical numerical integration methods, such as Romberg integration, Clenshaw-Curtis integration of Gauss-Legendre integration (see Davis and Rabinowitz[4]) are applicable. If $(\alpha + \nu)$ is not an integer and ap is small, the only difficulty is the algebraic singularity at $x = 0$, and Gauss-Jacobi quadrature or Iri-Moriguti-Takesawa (IMT) integration[4] can be used. If ap is large, special methods should be applied that take into account the oscillatory behavior of the integrand. We describe two methods here.

9.9.1 Integration between the Zeros of $J_\nu(x)$

We denote the sth positive zero of $J_\nu(x)$ by $j_{\nu,s}$ and we set $j_{\nu,0} = 0$. Then

$$I(a,\ p,\ \nu) = \sum_{k=1}^{N} (-1)^{k+1} I_k + \int_{j_{\nu,N}/p}^{a} J_\nu(px) f(x) dx \tag{9.70}$$

where

$$I_k = \int_{j_{\nu,k-1}/p}^{j_{\nu,k}/p} |J_\nu(px)| f(x) dx \tag{9.71}$$

and where N is the largest natural number for which $j_{\nu,N} \le ap$. This means that N is large when ap is large.

Using a transformation attributed to Longman,[10] the summation in Equation (9.70) can be written as

$$\begin{aligned} S = \sum_{k=1}^{N} (-1)^{k+1} I_k &= \frac{1}{2} I_1 - \frac{1}{4} \Delta I_1 + \frac{1}{8} \Delta^2 I_1 + \cdots + (-1)^{p-1} 2^{-p} \Delta^{p-1} I_1 \\ &+ (-1)^{N-1} \left[\frac{1}{2} I_N + \frac{1}{4} \Delta I_{N-1} + \frac{1}{8} \Delta^2 I_{N-2} + \cdots + 2^{-p} \Delta^{p-1} I_{n-p+1} \right] \\ &+ 2^{-p} (-1)^p [\Delta^p I_1 - \Delta^p I_2 + \Delta^p I_3 - \cdots + (-1)^{N-1-p} \Delta^p I_{N-p}] . \end{aligned}$$

Assuming now that N and p are large and that high-order differences are small, the last bracket may be neglected and

$$\begin{aligned} S &\simeq \frac{1}{2} I_1 - \frac{1}{4} \Delta I_1 + \frac{1}{8} \Delta^2 I_1 - \cdots \\ &+ (-1)^{N-1} \left[\frac{1}{2} I_N + \frac{1}{4} \Delta I_{N-1} + \frac{1}{8} \Delta^2 I_{N-2} + \cdots \right]. \end{aligned} \tag{9.72}$$

The summations in (9.72) may be truncated as soon as the terms are small enough. For the evaluation of I_k, $k = 1, 2, \ldots$, classical integration methods (e.g., Lobatto's rule) can be used, but special Gauss quadrature formulas (see Piessens[16]) are more efficient. If the integral I_1 has an algebraic singularity at $x = 0$, then the Gauss-Jacobi rules or the IMT rule are recommended.

9.9.2 Modified Clenshaw-Curtis Quadrature

The Clenshaw-Curtis quadrature method is a well-known and efficient method for the numerical evaluation of an integral I with a smooth integrand. This method is based on a truncated Chebyshev series approximation of the integrand. However, when the integrand shows a singular or strongly oscillatory behavior, the classical Clenshaw-Curtis method is not efficient or even applicable, unless it is modified in an appropriate way, taking into account the type of difficulty of the integrand. We call this method then a modified Clenshaw-Curtis method (MCC method). The principle of the MCC method is the following: the integration interval is mapped onto [–1, +1] and the integrand is written as the product of a smooth function $g(x)$ and a weight function $w(x)$ containing the singularities or the oscillating factors of the integrand; that is,

$$I = \int_{-1}^{+1} w(x) g(x) dx . \tag{9.73}$$

The smooth function is then approximated by a truncated series of Chebyshev polynomials

$$g(x) \simeq \sum_{k=0}^{N}{}' c_k T_k(x), \qquad -1 \le x \le 1 \; . \tag{9.74}$$

Here the symbol Σ' indicates that the first term in the sum must be halved. For the computation of the coefficients c_k in Equation (9.74) several good algorithms, based on the Fast Fourier Transform, are available.

The integral in (9.73) can now be approximated by

$$I \simeq \sum_{k=0}^{N}{}' c_k M_k \tag{9.75}$$

where

$$M_k = \int_{-1}^{+1} w(x) T_k(x) dx$$

are called modified moments.

The integration interval may also be mapped onto [0, 1] instead of [–1, 1], but then the shifted Chebyshev polynomials $T_k^\star(x)$ are to be used.

We now consider the computation of the integral (9.69),

$$I = \int_0^1 x^\alpha J_\nu(\omega x) g(x) dx \; . \tag{9.76}$$

If

$$g(x) \simeq \sum_{k=0}^{N}{}' c_k T_k'(x) \tag{9.77}$$

then

$$I \simeq \sum_{k=0}^{N}{}' c_k M_k(\omega,\ \nu,\ \alpha) \tag{9.78}$$

where

$$M_k(\omega,\ \nu,\ \alpha) = \int_0^1 x^\alpha J_\nu(\omega x) T_k^\star(x) dx \; . \tag{9.79}$$

These modified moments satisfy the following homogeneous, linear, nine-term recurrence relation:

$$
\begin{aligned}
&\frac{\omega^2}{16}M_{k+4}+\left[(k+3)(k+3+2\alpha)+\alpha^2-\nu^2-\frac{\omega^2}{4}\right]M_{k+2}\\
&\quad+[4(\nu^2-\alpha^2)-2(k+2)(2\alpha-1)]M_{k+1}\\
&\quad-\left[2(k^2-4)+6(\nu^2-\alpha^2)-2(2\alpha-1)-\frac{3\omega^2}{8}\right]M_k\\
&\quad+[4(\nu^2-\alpha^2)+2(k-2)(2\alpha-1)]M_{k-1}\\
&\quad+\left[(k-3)(k-3-2\alpha)+\left(\alpha^2-\nu^2-\frac{\omega^2}{4}\right)\right]M_{k-2}+\frac{\omega^2}{16}M_{k-4}=0\,.
\end{aligned}
\tag{9.80}
$$

Because of the symmetry of the recurrence relation of the shifted Chebyshev polynomials, it is convenient to define

$$T^{\star}_{-k}(x)=T^{\star}_{k}(x),\qquad k=1,\ 2,\ 3,\ \dots$$

and consequently

$$M_{-k}(\omega,\ \nu,\ \alpha)=M_k(\omega,\ \nu,\ \alpha)\,.$$

To start the recurrence relation with k = 0, 1, 2, 3, … we need only M_0, M_1, M_2, and M_3. Using the explicit expressions of the shifted Chebyshev polynomials, we obtain

$$
\begin{aligned}
M_0&=G(\omega,\ \nu,\ \alpha)\\
M_1&=2G(\omega,\ \nu,\ \alpha+1)-G(\omega,\ \nu,\ \alpha)\\
M_2&=8G(\omega,\ \nu,\ \alpha+2)-8G(\omega,\ \nu,\ \alpha+1)+G(\omega,\ \nu,\ \alpha)\\
M_3&=32G(\omega,\ \nu,\ \alpha+3)-48G(\omega,\ \nu,\ \alpha+2)+18G(\omega,\ \nu,\ \alpha+1)-G(\omega,\ \nu,\ \alpha)
\end{aligned}
\tag{9.81}
$$

where

$$G(\omega,\ \nu,\ \alpha)=\int_0^1 x^{\alpha}J_{\nu}(\omega x)\,dx\,. \tag{9.82}$$

Because

$$
\begin{aligned}
\omega^2G(\omega,\ \nu,\ \alpha+2)&=[\nu^2-(\alpha+1)^2]G(\omega,\ \nu,\ \alpha)\\
&\quad+(\alpha+\nu+1)\,J_{\nu}(\omega)-\omega J_{\nu-1}(\omega)
\end{aligned}
\tag{9.83}
$$

we need only $G(\omega, \nu, \alpha)$ and $G(\omega, \nu, \alpha + 1)$.

Luke[12] has given the following formulas:

1. A Neumann series expansion that is suitable for small ω

$$G(\omega,\ \nu,\ \alpha)=\frac{2}{\omega(\alpha+\nu+1)}\sum_{k=0}^{\infty}\frac{(\nu+2k+1)\left(\dfrac{\nu-\alpha+1}{2}\right)_k}{\left(\dfrac{\nu+\alpha+3}{2}\right)_k}J_{\nu+2k+1}(\omega) \tag{9.84}$$

2. An asymptotic expansion that is suitable for large ω

$$G(\omega,\ \nu,\ \alpha)=\frac{2^{\alpha}}{\omega^{\alpha+1}}\frac{\Gamma\left(\dfrac{\nu+\alpha+1}{2}\right)}{\Gamma\left(\dfrac{\nu-\alpha+1}{2}\right)}-\sqrt{\frac{2}{\pi\omega^3}}\,(g_1\cos\theta+g_2\sin\theta) \tag{9.85}$$

where

$$\theta=\omega-\nu\pi/2+\pi/4$$

and

$$g_1\sim\sum_{k=0}^{\infty}(-1)^k a_{2k}\omega^{-2k},\qquad \omega\to\infty$$

$$g_2\sim\sum_{k=0}^{\infty}(-1)^k a_{2k+1}\omega^{-2k-1},\qquad \omega\to\infty$$

$$a_k=\frac{(1/2-\nu)_k(1/2+\nu)_k}{2^k k!}b_k$$

$$b_0=1$$

$$b_{k+1}=1+\frac{2(k+1)(\alpha-k-1/2)}{(\nu-k-1/2)(\nu+k+1/2)}b_k\,.$$

If α and ν are integers, the following formulas are useful[1]:

$$\int_0^1 J_{2\nu}(\omega x)dx=\int_0^1 J_0(\omega x)dx-\frac{2}{\omega}\sum_{k=0}^{\nu-1}J_{2k+1}(\omega)$$

$$\int_0^1 J_{2\nu+1}(\omega x)dx=\frac{1-J_0(\omega)}{\omega}-\frac{2}{\omega}\sum_{k=1}^{\nu}J_{2k}(\omega)\,.$$

For the evaluation of

$$\int_0^1 J_0(\omega x)\,dx$$

Chebyshev series approximations are given by Luke.[11] We now discuss the numerical aspect of the recurrence formula (9.80). The numerical stability of forward recursion depends on the asymptotic behavior of $M_k(\omega, \upsilon, \alpha)$ and of eight linearly independent solutions $y_{i,k}$, $i = 1, 2, \ldots, 8$, $k \to \infty$.

Using the asymptotic theory of Fourier integrals, we find

$$\begin{aligned}
|y_{1,k}| &\sim k^{-2} \\
|y_{2,k}| &\sim k^{-4} \\
|y_{3,k}| &\sim k^{-2(\alpha+1)-2\nu} \\
|y_{4,k}| &\sim k^{-2(\alpha+1)+2\nu}, \qquad \text{if } \nu \neq 0 \\
&\sim \mathrm{k}^{-2(\alpha+1)} \ell nk, \qquad \text{if } \nu = 0 \\
|y_{5,k}| \sim |y_{6,k}| &\sim \left(\frac{\omega}{4k}\right)^k e^k k^\alpha \\
|y_{7,k}| \sim |y_{8,k}| &\sim \left(\frac{4k}{\omega}\right)^k e^{-k} k^\alpha
\end{aligned} \tag{9.86}$$

and

$$\begin{aligned}
M_k(\omega, \nu, \alpha) &\sim -\frac{1}{2} J_\nu(\omega) k^{-2} \\
&+ (-1)^k 2^{-3\nu-2\alpha-1} \frac{\omega^\nu}{\Gamma(\nu+1)} \cos\left[\pi(\alpha+1)\right] \Gamma(2\alpha+2) k^{-2\alpha-2\nu-2}.
\end{aligned} \tag{9.87}$$

The asymptotically dominant solutions are $y_{7,k}$ and $y_{8,k}$. The asymptotically minimal solutions are $y_{5,k}$ and $y_{6,k}$. We may conclude that forward and backward recursion are asymptotically unstable. However, the instability of forward recursion is less pronounced if $k \leq \omega/2$. Indeed, practical experiments demonstrate that $M_k(\omega, \nu, \alpha)$ can be computed accurately using forward recursion for $k \leq \omega/2$. For $k > \omega/2$ the loss of significant figures increases and forward recursion is no longer applicable. In that case, Oliver's algorithm[14] has to be used. This means that (9.80) has to be solved as a boundary value problem with six initial values and two end values. The solution of this boundary value problem requires the solution of a linear system of equations having a band structure.

An important advantage of the MCC method is that the function evaluations of g, needed for the computation of the coefficients c_k of the Chebyshev series expansion, are independent of the value of ω. Consequently, the same function evaluations may be used for different values of ω, and have to be computed only once.

Numerical examples can be found in References 19 and 20.

9.10 Computation of Bessel Function Integrals over an Infinite Interval

In this section we consider methods for the computation of

$$I(p, \nu) = \int_0^\infty J_\nu(px) f(x) dx\,. \tag{9.88}$$

9.10.1 Integration between the Zeros of $J_\nu(x)$ and Convergence Acceleration

We have

$$I(p,\ \nu) = \sum_{k=1}^{\infty} (-1)^{k+1} I_k \tag{9.89}$$

where

$$I_k = \int_{j_{\nu,k-1}/p}^{j_{\nu,k}/p} \left| J_\nu(px) \right| f(x) dx \ . \tag{9.90}$$

Using Euler's transformation,[4] the convergence of series (9.89) can be accelerated

$$I(p,\ \nu) = \frac{1}{2} I_1 - \frac{1}{4} \Delta I_1 + \frac{1}{8} \Delta^2 I_1 - \frac{1}{16} \Delta^3 I_1 + \cdots \ . \tag{9.91}$$

It is not always desirable to start the convergence acceleration with I_1, but with some later term, say I_m, so that

$$I(p,\ \nu) = \int_0^{j_{\nu,m-1}/p} J_\nu(px) f(x) dx + (-1)^{m-1} \left[\frac{1}{2} I_m - \frac{1}{4} \Delta I_m + \frac{1}{8} \Delta^2 I_m - \cdots \right] .$$

Other convergence accelerating methods, for example the ϵ-algorithm,[23] are also applicable (for an example, see Reference 21).

9.10.2 Transformation into a Double Integral

Substituting the integral expression

$$J_\nu(x) = 2 \frac{(x/2)^\nu}{\Gamma(\nu + 1/2)\sqrt{\pi}} \int_0^1 (1 - t^2)^{\nu - 1/2} \cos(xt) dt \tag{9.92}$$

into (9.88) and changing the order of integration, we obtain

$$I(p,\ \nu) = \frac{2(p/2)^\nu}{\Gamma(\nu + 1/2)\sqrt{\pi}} \int_0^1 (1 - t^2)^{\nu - 1/2} F(t) dt \tag{9.93}$$

where

$$F(t) = \int_0^\infty x^\nu f(x) \cos(pxt) dx \ . \tag{9.94}$$

We assume that the integral in (9.94) is convergent. If we want to evaluate (9.93) using an N-point Gauss-Jacobi rule, then we have to compute the Fourier integral (9.94) for N values of t. Because $F(t)$ shows a peaked or even a singular behavior especially when $f(x)$ is slowly decaying, a large enough N has to be chosen.

This method is closely related to Linz's method,[8] which is based on the Abel transformation of $I(p,\nu)$.

9.10.3 Truncation of the Infinite Interval

If a is an arbitrary positive real number, we can write

$$I(p,\ \nu)=\int_0^a J_\nu(px)f(x)dx+R(p,\ a) \tag{9.95}$$

where

$$R(p,\ a)=\int_a^\infty J_\nu(px)f(x)dx\ . \tag{9.96}$$

The first integral in the right side of (9.95) can be computed using the methods of Section 9.9.

If a is sufficiently large and f is strongly decaying, then we may neglect $R(p,a)$.

If

$$f(x)\sim\frac{c_1}{x}+\frac{c_2}{x^2}+\cdots \tag{9.97}$$

is an asymptotic series approximation which is sufficiently accurate in the interval $[a, \infty)$, then

$$R(p,\ a)\simeq\sum_k c_k\int_a^\infty\frac{J_\nu(px)}{x^k}dx\ . \tag{9.98}$$

Longman[9] has tabulated the values of the integrals in (9.98) for some values of ν and ap.

Using Hankel's asymptotic expansion,[1] for $x\to\infty$

$$J_\nu(px)\sim\sqrt{\frac{2}{\pi px}}[P_\nu(px)\cos\chi-Q_\nu(px)\sin\chi] \tag{9.99}$$

where $\chi = px - (\nu/2+1/4)\pi$, and where $P_\nu(x)$ and $Q_\nu(x)$ can be expressed as a well-known asymptotic series, $R(p,a)$ can be written as the sum of two Fourier integrals.

Especially, if a = $(8 + \nu/2 + 1/4)\pi/p$, we have

$$\begin{aligned}R(p,\ a)&=\int_0^\infty\sqrt{\frac{2}{\pi p(u+a)}}P_\nu(p(u+a))f(u+a)\cos pu\ \ du\\&-\int_0^\infty\sqrt{\frac{2}{\pi p(u+a)}}Q_\nu(p(u+a))f(u+a)\sin pu\ \ du\ .\end{aligned} \tag{9.100}$$

For the computation of the Fourier integrals in (9.100), tailored methods are available.[4]

9.11 Tables of Hankel Transforms

Table 9.1 lists the Hankel transform of some particular functions for the important special case $\nu = 0$. Table 9.2 lists Hankel transforms of general order ν. In these tables, $h(x)$ is the unit step function, I_ν and K_ν are modified Bessel functions, $\mathbf{L}_0$ and $\mathbf{H}_0$ are Struve functions, and Ker and Kei are Kelvin functions as defined in Abramowitz and Stegun.[1] Extensive tables are given by Erdélyi et al.,[6] Ditkin and Prudnikov,[5] Luke,[11] Wheelon,[27] Sneddon,[24] and Oberhettinger.[13]

TABLE 9.1 Hankel Transforms of Order 0.

	$f(r)$	$F_0(s) = \mathcal{H}_0\{f(r)\}$
(1)	$\frac{1}{r}$	$\frac{1}{s}$
(2)	$r^{-\mu}, \quad 1/2 < \mu < 2$	$2^{1-\mu} \frac{\Gamma(1-\frac{\mu}{2})}{\Gamma(\frac{\mu}{2})} \frac{1}{s^{2-\mu}}$
(3)	$h(a-r)$	$\frac{a}{s} J_1(as)$
(4)	e^{-ar}	$\frac{a}{(s^2+a^2)^{3/2}}$
(5)	$\frac{e^{-ar}}{r}$	$\frac{1}{\sqrt{s^2+a^2}}$
(6)	$\frac{1-e^{-ar}}{r^2}$	$\log\left(\frac{a+\sqrt{a^2+s^2}}{s}\right)$
(7)	$\log\left(1+\frac{a^2}{r^2}\right)$	$\frac{2}{s}\left[\frac{1}{s} - a\, K_1(as)\right]$
(8)	$\frac{\sin r}{r}$	$\frac{1}{\sqrt{1-s^2}}, \quad s < 1$ $0, \quad s > 1$
(9)	$\frac{\sin r}{r^2}$	$\frac{\pi}{2}, \quad s \le 1$ $\arcsin \frac{1}{s}, \quad s > 1$
(10)	$\frac{\sin(ar)}{r^2+b^2}$	$\frac{\pi}{2} e^{-ab} I_0(bs), \quad o < s < a$
(11)	$\frac{\cos(ar)}{r^2+b^2}$	$\cosh(ab) K_0(bs), \quad a < s < \infty$
(12)	$e^{-a^2 r^2}$	$\frac{e^{-s^2/4a^2}}{2a^2}$
(13)	$\frac{1}{r(r+a)}$	$\frac{\pi}{2}\left[\mathbf{H}_0(as) - Y_0(as)\right]$
(14)	$\frac{1}{r^2+a^2}$	$K_0(as)$
(15)	$\frac{1}{r(r^2+a^2)}$	$\frac{\pi}{2a}\left[I_0(as) - \mathbf{L}_0(as)\right]$
(16)	$\frac{1}{1+r^4}$	$-\operatorname{Kei}(s)$
(17)	$\frac{r^3}{1+r^4}$	$\operatorname{Ker}(s)$
(18)	$\frac{1}{\sqrt{r^2+a^2}}$	$\frac{e^{-sa}}{s}$
(19)	$\frac{1}{\sqrt{r^4+a^4}}$	$K_0(as/\sqrt{2})\, J_0(as/\sqrt{2})$
(20)	$\frac{1-J_0(ar)}{r^2}$	$\log \frac{a}{s}, \quad s \le a$ $0, \quad s \ge a$
(21)	$\frac{a}{r} J_1(ar)$	$1, \quad \text{if } 0 < s < a$ $0, \quad \text{if } s > a$
(22)	$\frac{1}{r} J_0(2\sqrt{ar})$	$\frac{1}{s} J_0\left(\frac{a}{s}\right)$

TABLE 9.2 Hankel Transforms of General Order ν.

	$f(r)$	$F_\nu(s) = \mathcal{H}_\nu\{f(r)\}$
(1)	$\frac{1}{r}$	$\frac{1}{s}$
(2)	$r^{-\mu},\ \frac{1}{2} < \mu < \nu + 2$	$\frac{2^{1-\mu}}{s^{2-\mu}} \frac{\Gamma(\frac{\nu+2-\mu}{2})}{\Gamma(\frac{\nu+\mu}{2})}$
(3)	$x^\nu (a^2 - r^2)^\mu h(a-r),$ $\mu > -1$	$2^\mu a^{\mu+\nu+1} s^{-\mu-1} \Gamma(\mu+1) J_{\nu+\mu+1}(as)$
(4)	$\frac{\sin ar}{r}$	$\frac{1}{(s^2 - a^2)^{1/2}} \sin\left(\nu \arcsin\left(\frac{a}{s}\right)\right) \quad s > a$ $\cos\left(\frac{\mathrm{n}\nu}{2}\right) \frac{1}{(a^2 - s^2)^{1/2}} \frac{s^\nu}{(a + (a^2 - s^2)^{1/2})^\nu} \quad s < a$
(5)	$\frac{\sin ar}{r^2}$	$\frac{\nu^{-1} s^\nu}{\left(a + \sqrt{a^2 - s^2}\right)^\nu} \sin \frac{\nu\pi}{2} \quad s \le a$ $\nu^{-1} \sin\left(\nu \arcsin\left(\frac{a}{s}\right)\right) \quad s > a$
(6)	$\frac{e^{-ar}}{r}$	$\frac{(\sqrt{s^2 + a^2} - a)^\nu}{s^\nu \sqrt{s^2 + a^2}}$
(7)	$\frac{e^{-ar}}{r^2}$	$\frac{(\sqrt{s^2 + a^2} - a)^\nu}{\nu s^\nu}$
(8)	$r^{\nu-1} e^{-ar}$	$\frac{(2s)^\nu \Gamma(\nu + 1/2)}{(s^2 + a^2)^{\nu+1/2} \sqrt{\pi}}$
(9)	$r^\nu e^{-ar}$	$\frac{2a(2s)^\nu \Gamma(\nu + 3/2)}{(s^2 + a^2)^{\nu+3/2} \sqrt{\pi}}$
(10)	$e^{-a^2 r^2} r^\nu$	$\frac{s^\nu}{(2a^2)^{\nu+1}} \exp\left(-\frac{s^2}{4a^2}\right)$
(11)	$e^{-a^2 r^2} r^\mu$	$\frac{\Gamma((\nu + \mu + 2)/2) \left(\frac{1}{2}\frac{s}{a}\right)^\nu}{2a^{\mu+2} \Gamma(\nu+1)} \times {}_1F_1\left(\frac{\nu + \mu + 2}{2}; \nu + 1; -\frac{s^2}{4a^2}\right)$
(12)	$\frac{r^\nu}{(r^2 + a^2)^{\mu+1}}$	$\frac{s^\mu a^{\nu-\mu}}{2^\mu \Gamma(\mu+1)} K_{\nu-\mu}(as)$
(13)	$\frac{r^\nu}{(r^4 + 4a^4)^{\nu+\frac{1}{2}}}$	$\frac{\left(\frac{1}{2}s\right)^\nu \sqrt{\pi}}{(2a)^{2\nu} \Gamma\left(\nu + \frac{1}{2}\right)} J_\nu(as) K_\nu(as)$
(14)	$\frac{r^{\nu+2}}{(r^4 + 4a^4)^{\nu+\frac{1}{2}}}$	$\frac{\left(\frac{1}{2}s\right)^\nu \sqrt{\pi}}{2(2a)^{2\nu-2} \Gamma\left(\nu + \frac{1}{2}\right)} J_{\nu-1}(as) K_{\nu-1}(as)$
(15)	$r^{\mu-\nu} J_\mu(ar)$	$0 \quad 0 < s < a$ $\frac{2^{\mu-\nu+1} a^\mu (s^2 - a^2)^{\nu-\mu-1}}{s^\nu \Gamma(\nu - \mu)} \quad a < s$

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