On the Realization of the Polynomial Wigner–Ville Distribution for Multicomponent Signals

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***Abstract*— A method for the polynomial Wigner–Ville distributions realization, in the case of multicomponent signals, is presented. Using this method, one may theoretically get a sum of the polynomial Wigner–Ville distributions of each component separately. The method is illustrated by a numerical example.**

***Index Terms*— Higher order spectra, instantaneous frequency, time–frequency analysis, time-varying spectra, Wigner distribution.**

I. Introduction

OUT OF THE general Cohen class of quadratic shift covariant distributions, the Wigner distribution (WD) is the only one (with signal-independent kernel) that produces the ideal concentration along the instantaneous frequency for the linear frequency-modulated signals [4], [6], [10]. In order to improve the concentration, when the instantaneous frequency is polynomial function of time, the polynomial Wigner–Ville distributions (PWVD's) are proposed by Boashash *et al.*, [1], [2]. Since these distributions belong to the class of higher order time-varying spectra, they suffer from very emphatic cross-term effects, what makes their application on the multicomponent signals very unsuitable [11]. In this letter, we will show that our recently proposed S-method [5], [6], [9], [10], may be efficiently used for the reduction (removal) of the cross-terms in the PWVD of multicomponent signals. Theoretically, we get a sum of the PWVD of each component separately, which is exactly that we achieved in the example presented in this work.

II. Definitions and Method Derivation

Polynomial Wigner–Ville distributions are derived from the condition that the distribution of a frequency modulated signal $x(t) = A\exp(\phi(t))$, having polynomial phase function $\phi(t) = \Sigma_{i=0}^{p} a_i t^i$, is equal to the ideally concentrated one $W_x(t,\omega) = 2\pi A^r \delta(\omega - \phi'(t))$. It has been shown [1], [2] that such a distribution may be obtained as a Fourier transform of the polynomial kernel $K_x(t,\tau) = \Pi_{k=-q/2}^{q/2} x^{b_k}(t + c_k\tau)$, with respect to τ

$$W_x(t,\omega) = \int_{-\infty}^{\infty} \prod_{k=-q/2}^{q/2} x^{b_k}(t + c_k\tau) e^{-j\omega\tau}\, d\tau \qquad (1)$$

where $q \geq p$ is an even number. Coefficients b_k and c_k should be determined, for a given p and q so that the ideal distribution is achieved ($b_0 \equiv 0$).

In practical and numerical applications, only the PWVD of order $q = 4$ has been used, so we will, without loss of generality, demonstrate the realization procedure on this distribution, since the same technique may be applied to any order distribution. The PWVD with $q = 4$ is defined [1] by

$$W_x(t,\omega) = \int_{-\infty}^{\infty} x^2(t + 0.675\tau) x^{*2}(t - 0.675\tau) \cdot x^*(t + 0.85\tau) x(t - 0.85\tau) e^{-j\omega\tau}\, d\tau. \qquad (2)$$

Rewrite distribution (2) in a frequency scaled form

$$W_x(t,\omega) = \frac{1}{2.7}\int_{-\infty}^{\infty} x^2\Big(t + \frac{\tau}{4}\Big) x^{*2}\Big(t - \frac{\tau}{4}\Big) x^*\Big(t + A\frac{\tau}{2}\Big) \cdot x\Big(t - A\frac{\tau}{2}\Big) e^{-j(\omega/2.7)\tau}\, d\tau \qquad (3)$$

where $A = 0.85/1.35$. The multilinear kernel of the PWVD creates a multiplicity of cross-terms. For example, if we have only a two-component signal, the number of cross-terms in (2) is 13. This illustrates the inapplicability of the original definition for the processing of multicomponent signals. In order to present a procedure for the efficient PWVD realization in the case of multicomponent signals, note that (3) may be expressed as a convolution of the L-Wigner distribution (L-WD) (with $L = 2$), [7], [9], and a scaled WD

$$W_x(t,\omega') = \frac{1}{5.4\pi}\, \mathrm{LWD}_2(t,\omega') *_{\omega'} \mathrm{WD}_A(t,\omega') \qquad (4)$$

where

$$\mathrm{LWD}_2(t,\omega) = \int_{-\infty}^{\infty} x^2\Big(t + \frac{\tau}{4}\Big) x^{*2}\Big(t - \frac{\tau}{4}\Big) e^{-j\omega\tau}\, d\tau, \text{ and}$$

$$\mathrm{WD}_A(t,\omega) = \int_{-\infty}^{\infty} x^*\Big(t + A\frac{\tau}{2}\Big) x\Big(t - A\frac{\tau}{2}\Big) e^{-j\omega\tau}\, d\tau$$

and $\omega' = \omega/2.7$.

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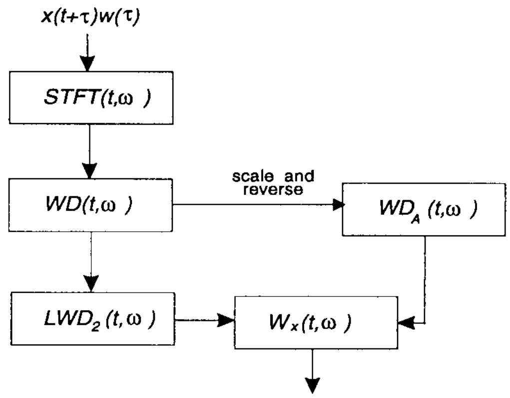

Fig. 1. Block schema for the PWVD realization.

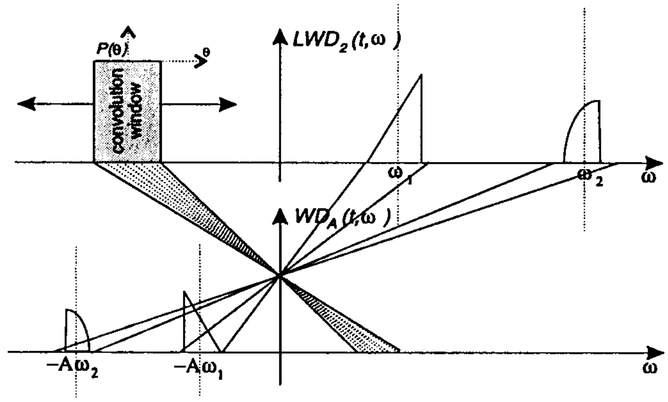

Fig. 2. Illustration of the convolution in the polynomial Wigner–Ville distribution of a two-component signal.

A block schema for the PWVD realization, starting from the signal, via the short-time Fourier transform (STFT), the WD, and the L-WD, is shown in Fig. 1. Since we have already defined the method for the cross-terms free (cross-terms reduced) realization of the L-WD and WD in [5], [6], [8]–[10] (see the Appendix), the only remaining step is to realize the convolution in (4) so that it does not introduce any additional cross-term, as well as produces the auto-terms at their natural positions. In order to examine the convolution in (4), let us consider a two-component signal at an instant t, whose WD and L-WD are shown in Fig. 2. It is obvious that if an auto-term exists at and around the instantaneous frequency ω_1, it will be placed in $\mathrm{LWD}_2(t, \omega')$ at and around the same frequency ω_1. This auto-term is located in $\mathrm{WD}_A(t, \omega')$ at and around $-A\omega_1$. Thus, in order to calculate $W_x(t, \omega')$ at a given frequency ω', we should calculate convolution (4) using only an interval around ω' in $\mathrm{LWD}_2(t, \omega')$ and using an interval around $-A\omega'$ in $\mathrm{WD}_A(t, \omega')$ (see Fig. 2). Theoretically, this interval should be greater or equal to the auto-term width and less than the distance between auto-terms. But, in practical realizations, we usually *a priori* assume its value. Note, if window $P(\theta)$ is too wide, the cross-terms will start appearing, while too narrow a window will degrade auto-terms with respect to their original PWVD form (recently, we derived a technique for variable and self adaptive window $P(\theta)$ width, that may be applied to this case in a straightforward manner [8]). Of course, the position of the convolution value (obtained through a window $P(\theta)$) is kept at the position of ω' in $\mathrm{LWD}_2(t, \omega')$, since this is a true position of the auto-term in the nonscaled frequency axis ω. This way, the auto-terms of the PWVD, at their natural positions are obtained.

In the discrete implementation of the above procedure, the only problem that remains is the evaluation of $\mathrm{WD}_A(t, \omega')$ on the discrete set of points on frequency axis, $\omega' = -k\Delta\omega'$. Since $\mathrm{WD}_A(t, \omega')$ is nothing but a scaled and reversed version of $\mathrm{WD}(t, \omega')$, its values at $-k\Delta\omega'$ are the values of $\mathrm{WD}(t, \omega')$ at $k\Delta\omega'/A$. But, these points do not correspond to any sample location along the frequency axis. Thus, the interpolation has to be done (one way of doing it is in an appropriate zero padding of the signal, as indicated in [1])[1]. A discrete form of convolution (4), including window $P(\theta)$ and the above considerations, is

$$W_x(n, k) = \sum_{i=-L_P}^{L_P} P(i)\mathrm{LWD}_2(n, k+i)\mathrm{WD}(n, k+[i/A]) \tag{5}$$

where L_P is the width of $P(\theta)$ in the discrete domain, while $[i/A]$ is the nearest integer to i/A. The terms in summation in (5), when $k+i$ or $k+[i/A]$ is outside the basic period, are considered as being zero, in order to avoid possible aliasing. Hardware or software implementation of the cross-term free (reduced) PWVD may be easily done according to (5) and systems for the WD and LWD realizations that are presented in [6].

[1]The PWVD distribution, in the case of monocomponent signals, may be realized without interpolation, using the form we proposed, independently from Boashash *et al.* in [7].

III. Numerical Example

Consider a multicomponent signal, with two real FM signals

$$x(t) = \cos(50t^3/3 + 75t) + \cos(30t|t| + 265t) \tag{6}$$

within the interval $-1 \le t \le 1$. Signal is sampled at $2/N$, with $N = 256$. In the realization of the PWVD, an equivalent Hanning window $w(\tau)$ is used, i.e., signal at the input is multiplied by $w^{1/6}(\tau)$. As in [1], the length of $w(\tau)$ is assumed over the entire considered time interval (window length $T = 2$). Although a narrower window would produce more concentrated distribution, we used this width in order to emphasize the artifacts and their reduction by the polynomial distributions. Rectangular windows $P(\theta)$ are used in all convolutions: in the WD of the width $W_P = 35\pi$ (in the analog domain), while in the L-WD and PWVD (since the auto-term widths are significantly reduced as compared to the ones in the STFT) the window $P(\theta)$ width was $W_P = 17.5\pi$. For the reasons described in Section II and in [1], an interpolation with factor four is used, i.e., signal is zero padded up to $4N$. Note that the signal sampling is done according to the Nyquist rate, i.e., twice less than it should be in the "ordinary" Wigner distribution, while the length of sequence

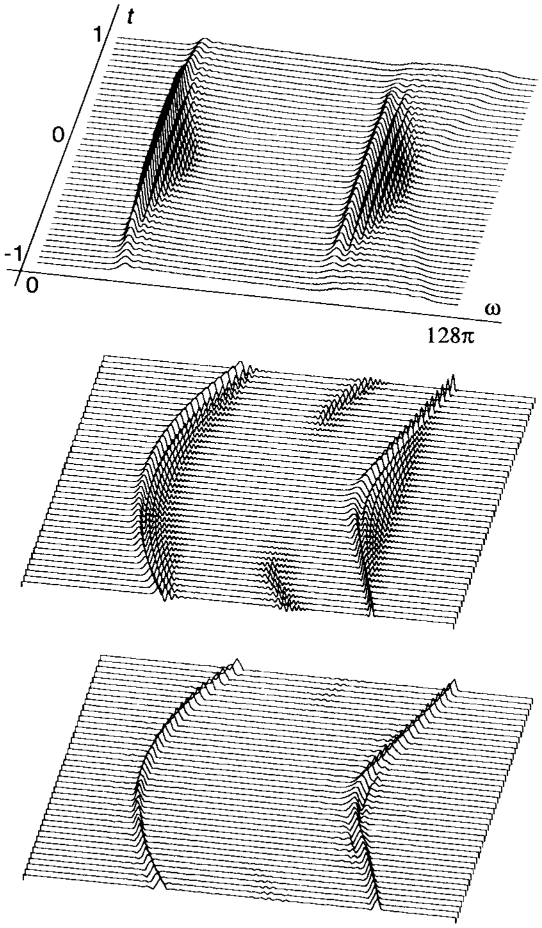

Fig. 3. Top: Spectrogram. Middle: S-method (cross-terms free pseudo-WD). Bottom: Polynomial Wigner–Ville distribution calculated by the proposed algorithm.

for the fast Fourier transform (FFT) calculations, including zero padding, is only twice longer. Also, since the cross-term effects appearing between positive and negative frequencies will be eliminated (reduced) in the same way as the other cross-terms, there is no need for the analytic signal calculation. The results are presented in Fig. 3. The cross-term free PWVD, highly concentrated at the instantaneous frequency (not only in the polynomial phase component, but in the nonpolynomial FM component, as well) is shown in Fig. 3.

IV. Conclusion

The efficient method for a realization of the PWVD, in the case of multicomponent signals, is presented.

Appendix

The S-method is defined by [5], [6], and [8]–[10]

$$\mathrm{SM}(t,\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} P(\theta)\,\mathrm{STFT}(t,\omega{+}\theta)\,\mathrm{STFT}^*(t,\omega{-}\theta)\,d\theta.$$

where $\mathrm{STFT}(t,\omega) = FT[w(\tau)x(t+\tau)]$ is the STFT. Its kernel in the Cohen class of distributions is $c(\theta,\tau) = P(\theta/2) *_\theta A_{ww}(\theta,\tau)/2\pi$, where $A_{ww}(\theta,\tau)$ is the ambiguity function of the lag window $w(\tau)$. Note that $c(\theta,\tau)$ is generally a nonseparable function, meaning that $\mathrm{SM}(t,\omega)$ does not belong to the class of distributions commonly referred to as the smoothed pseudo-WD's. By an appropriate frequency domain window $P(\theta)$ the S-method can, in the case of multicomponent signals $x(t) = \Sigma_{m=1}^{M} x_m(t)$, produce the sum of the pseudo-WD's $\mathrm{WD}_{x_m x_m}(t,\omega)$ of each signal component separately, $\mathrm{SM}(t,\omega) = \Sigma_{m=1}^{M} \mathrm{WD}_{x_m x_m}(t,\omega)$.

Its form for the second-order L-WD [6], [7], [9], [10] reads

$$\mathrm{LWD}_2(t,\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} P(\theta)\,\mathrm{WD}(t,\omega+\theta)\,\mathrm{WD}(t,\omega-\theta)\,d\theta.$$

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