Signals and Systems

Projects

Contents

1 Introduction

The last part of the practial works consists in small projects to be performed in groups of 2 students. This document lists the different projects[1] students can choose from. There are two main subjets: signal processing and image processing. Students considering a thesis in the Signal & Image Centre should definitely take a subject in image processing.

Students are expected to provide a written report and a PowerPoint presentation of the main results of their work.

2 Signal processing

2.1 Processing chain of a compact disc

The processing chain of a compact disk was studied during one of the lessons. We recal here that the signals are upsampled to cope with practical realisation difficulties of precise high-performance filters using discrete electronic components while such filters are easy to implement in numerical form.

You are asked to

1. Simulate the numerisation chain of a compact disk. You should be able to handle actual audio signals coming from files. For practical reasons, you can work at lower sampling frequency than the actual $44.1kHz$ sampling frequency, take for instance $4.41kHz$. You should produce graphs of the spectras, illustrating the points.

2. Simulate the reproduction chain of a compact disk.

[1] Input data for the different projects if available on the project's page at `http://www.sic.rma.ac.be/~xne/el401/`.

2.2 Time-frequency representation

In some applications, it might be necessary to characterise a signal both in frequency and in time (e.g. determine when a particular frequency occured such as in DTMF[2] systems).

The Fourier transform allows a precise characterisation of the frequency behaviour of a particular system, but has very poor temporal localisation (the Fourier transform of a DTMF signal could be used to determine which keys were pressed but not in which order).

To solve this problem, the windowed Fourier transform was developped, having both time and frequency localization properties. The windowed Fourier transform is defined as the Fourier transform of a windowed signal $f(t,\tau)$ with

$$f(t,\tau) = f(t)e^{-\frac{(t-\tau)^2}{\sigma}} \tag{1}$$

and hence the windowed Fourier transform of $f(t)$ is

$$F(\tau,\omega) = \int_{-\infty}^{+\infty} f(t)e^{-\frac{(t-\tau)^2}{\sigma}} e^{-j\omega t} dt \tag{2}$$

The windowed Fourier transform $f(t) \longrightarrow F(\tau,\omega)$ is thus a mapping of the 1 dimensional space $\mathcal{R}$ to a two dimensional space $\mathcal{R}^2$.

This can also be seen as the decomposition of the signal $f(t)$ in a space with basis functions[3]

$$e(t,\tau,\omega) = e^{-\frac{(t-\tau)^2}{\sigma}} e^{j\omega t} \tag{3}$$

while the basis functions of the Fourier decomposition were

$$e(t,\omega) = e^{j\omega t} \tag{4}$$

The transform using a gaussian window as described above is called the Gabor transform and is a particular case of the windowed Fourier transforms. Other type of windows (Hamming, Hanning, ...) are sometimes used.

The work consist in the following

1. Display some of the basis functions (real part only) $e(t,\tau,\omega)$ for different values of τ and ω. What is the influence of τ and of ω. What do the basis functions of the Fourier transform look like?

2. Implement the windowed Fourier transform. Take care to be able to vary σ easily and to have a meaningful representation of the result $F(\tau,\omega)$. (only the amplitude of $F(\tau,\omega)$ should be represented).

3. Analyse the signals `debussy.wav` and `dtfm.wav` and compare the result with their Fourier analysis. What is the influence of the parameter σ on the time-frequency localization?

[2]Dual Tone Multi Frequency phone dialing.
[3]notice that these basis functions are not necessarily orthogonal.

3 Image processing

The theory developped for one-dimensional signals $f(t)$ (e.g. time signals, sound, ...) can directly be transposed to two-dimensional signals[4] $f(x,y)$ (e.g. images). The bidimensional convolution[5] is defined by

$$h(x,y) *_{2D} f(x,y) = (h *_{2D} f)(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\tilde{x},\tilde{y})h(x-\tilde{x},y-\tilde{y})d\tilde{x}\,d\tilde{y} \quad (5)$$

The convolution can be computed either directly using[6] the formula or by using the properties of the Fourier transform of a convolution product

$$g(x,y) = h(x,y) *_{2D} f(x,y) \Longleftrightarrow G(\omega_x,\omega_y) = H(\omega_x,\omega_y)F(\omega_x,\omega_y) \quad (6)$$

where ω_x and ω_y denote the spatial frequencies (respectively horizontal and vertical).

3.1 Convolution: Edge-effects

The artefacts that occurs at the first and the last few samples of a sequence are called *edge-effects*. These effects are often neglected in one-dimensional time signal due to the length of the signal. In image processing, where the “length” of the signals are much smaller, these effects are an important issue and hence, image edges should be handled with care.

1. Compute the convolution of an image with the following filters[7] $h(j,i)$

$$\text{a)} = \frac{1}{9}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \text{b)} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\text{c)} = \frac{1}{2}\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{d)} = \begin{pmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{pmatrix}$$

   and explain in words what each filter does. Justify the result from a mathematical point of view.

2. Compute and display the Fourier transform (actually, the DFT) of images consisting in

   - a uniform image
   - an image containing one white point
   - an image containing a vertical line
   - an image containing a white square

   Comment your results and compare them with the theoretical solution.

[4] in particular, the 2D FFT can be computed using the `fft2` command of `matlab`.

[5] the symbol $*_{2D}$ will denote the bidimensional convolution. Where non-ambiguous and to simplify the notation, the $2D$ subscript will be supressed.

[6] the `conv2(A,B)` command computes the bidimensional convolution of matrices A and B.

[7] element h(2,2) correspond to the center of the filter

3. Compute the convolution of an image with the following impulse response

$$h(j,i) = i\,j\,e^{-\frac{i^2+j^2}{\sigma}} \tag{7}$$

(choose a meaningful σ) using a direct implementation of the convolution[8] and an implementation using the Fourier transform eq. (6).

Altough both results should look the same, they are different. Describe the differences and their *fundamental* origin. Can you solve the problem?

Apply filter d) to the result of your solution. Comment.

3.2 Convolution & deconvolution

When performing measures on physical systems, one often only has access to the result of a convolution product (think to the — now fixed — Hubble space telescope whose mirror had an abberation; the images provided by the telescope were a result of the convolution of the real scene with the impulse response of the mirror system, that was different from a dirac impulse). In these cases, it is interesting to be able to compute the original signal/image f starting from the result of the convolution $g = h * f$ (with known h). This process is called *deconvolution*. Real physical processes usually add noise to the convolution and the measure actually is

$$g(t) = h(t) * f(t) + n(t) \quad \text{or} \quad G(\omega) = H(\omega)F(\omega) + N(\omega) \tag{8}$$

where n is the noise signal/image (often at least partially unknown).

A first trivial solution might be to compute

$$F(\omega) = \frac{G(\omega)}{H(\omega)} - \frac{N(\omega)}{H(\omega)} = H_{\text{inv}}(\omega)G(\omega) - H_{\text{inv}}(\omega)N(\omega) \tag{9}$$

this expression will exhibit some artefacts, and H_{inv} is in all practical cases replaced by a more elaborated filter

$$H_w(\omega) = \frac{P_f(\omega)H^*(\omega)}{P_f(\omega)|H(\omega)|^2 + P_n(\omega)} \tag{10}$$

where P_f and P_n are the power spectral densities[9] respectively of the signal f and the noise n.

Perform the analysis of H_{inv} and H_w first in one dimension and then in two dimensions.

1. Compute the convolution of an image with the following impulse response (horizontal line)

$$h(j,i) = \begin{cases} 1 & i = 0,\ j = -5,-4,\ldots,0,1,\ldots,5 \\ 0 & \text{elsewhere} \end{cases} \tag{11}$$

(normalize the filter[10]) using an implementation of the convolution in the Fourier domain (see eq. (6)). Add a random noise of reasonable amplitude.

Compare the original image with the result. What kind of real-world degradation does the filter (11) model?

[8] `conv2` command in `matlab`.

[9] recall that the power spectral density is the square of the Fourier transform of the signal $P_f(\omega) = |F(\omega)|^2$.

[10] `h = h/sum(sum(h));` will do this in `matlab`.

2. Extract a horizontal line in the middle of the image to perform the subsequent analysis. Compare this line with the same line extracted from the original image.

3. Compute the H_{inv} filter. Plot $H(\omega)$, $H_{\text{inv}}(\omega)$ and the product of both. Describe and comment what you see.

4. Compute the H_w filter. Plot $H(\omega)$, $H_w(\omega)$ and the product of both. Describe and comment what you see.

5. Compute the filters $H_{\text{inv}}(\omega_x, \omega_y)$ and $H_w(\omega_x, \omega_y)$ using eq. (9) and (10), and apply the to the image. Discuss the results for different values of the noise amplitude.

3.3 Karhunen-Loève transform

Consider a non-stationary stochastic process $f_\tau(n)$ (τ identifies a particular realisation). The autocorrelation matrix of a stochastic process is defined as

$$R(k,l) = E_\tau\{f_\tau(l) f_\tau^T(k)\} \tag{12}$$

where $E_\tau\{\}$ denotes the expectation on different realisations of the stochastic process $f_\tau(n)$.

The Karhunen-Loève transform Φ is the transform that diagonalizes the autocorrelation matrix R

$$\Phi R \Phi^T = \Lambda \quad \Longleftrightarrow \quad R\Phi^T = \Phi^T \Lambda \tag{13}$$

where Λ is a diagonal matrix. This relation implies

$$R\Phi_n = \lambda_n \Phi_n \tag{14}$$

where Φ_n is the n^{th} column of Φ. Φ_n is an eigenvector of the signal covariance matrix R.

Φ being a unitary transformation, one has

$$\xi_\tau = \Phi f_\tau \tag{15}$$

where ξ_τ is a vector. Φ being a unitary transformation we have from (15)

$$f_\tau = \Phi^T \xi_\tau = \sum_{n=0}^{N-1} \xi_\tau(n) \Phi_n \tag{16}$$

which state that one particular realization τ of the stochastic process $f_\tau(n)$ can be expressed as a linear combination of the eigenvectors Φ_n of the transform Φ.

It is easy to show that the autocorrelation matrix of the ξ_τ is equal to Λ, which means that the Karhunen-Loève transform has transformed the stochastic signal $f_\tau(n)$ into a white noise $\xi_\tau(n)$. One can also show that the Karhunen-Loève transform perform an optimum compaction of the information in the coefficients ξ_τ what means that even only considering $L < N$ terms in (16) yields a close approximation of $f_\tau(n)$.

Consider several images $f_{(\tau_x,\tau_y)}(n)$, where n denotes the number of the image and $\tau = (\tau_x, \tau_y)$ a particular pixel in the image. The random process here is the selection of one particular pixel. Hence the signal $f_\tau(n)$ is constructed by taking the same pixel (same τ) in each image.

1. Load the 19 images available. Have a look at the images.

2. Compute the autocorrelation matrix $R(k.l)$. What are the ranges for k and l? Can you see something characteristic at the matrix R?

3. Compute the eigenvectors[11] Φ_n and the eigenvalues λ_n of R. Normalize each eigenvector.

[11] use the command `[V,D] = eig(R)`.

4. Look at the different decorrelated stochastic variables $\xi_\tau(n)$. For the best visual effect, display them for constant n.

5. Evaluate

$$g_\tau(n) = \sum_{k=0}^{L-1} \xi_\tau(k)\Phi_k \tag{17}$$

   for $L \leq N$. For which values of L does $g_\tau(n)$ closely approximate one particular $f_\tau(n)$ (perform a visual comparison)?

6. Can you see an application of the latter point (provided the Karuhnen-Loève transform could be approximated by a signal-independent transform)?