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Chapter 3

Discrete Signal Representations

In this chapter we discuss the fundamental concepts of discrete signal representations. Such representations are also known as discrete transforms, series expansions, or block transforms. Examples of widely used discrete transforms are given in the next chapter. Moreover, optimal discrete representations will be discussed in Chapter 5. The term "discrete" refers to the fact that the signals are represented by discrete values, whereas the signals themselves may be continuous-time. If the signals that are to be transformed consist of a finite set of values, one also speaks of block transforms. Discrete signal representations are of crucial importance in signal processing. They give a certain insight into the properties of signals, and they allow easy handling of continuous and discrete-time signals on digital signal processors.

3.1 Introduction

We consider a real or complex-valued, continuous or discrete-time signal $\boldsymbol{x}$, assuming that $\boldsymbol{x}$ can be represented in the form

$$\boldsymbol{x} = \sum_{i=1}^{n} \alpha_i \, \boldsymbol{\varphi}_i. \tag{3.1}$$

The signal $\boldsymbol{x}$ is an element of the signal space X spanned by $\{\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_n\}$. The signal space itself is the set of all vectors which can be represented by linear combination of $\{\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_n\}$. For this, the notation

$$X = \operatorname{span}\{\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_n\} \tag{3.2}$$

will be used henceforth. The vectors $\boldsymbol{\varphi}_i$, $i = 1, \ldots, n$ may be linearly dependent or linearly independent of each other. If they are linearly independent, we call them a *basis* for X.

The coefficients α_i, $i = 1, \ldots, n$ can be arranged as a vector

$$\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_n]^T, \tag{3.3}$$

which is referred to as the *representation* of $\boldsymbol{x}$ with respect to the basis $\{\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_n\}$.

One often is interested in finding the best approximation of a given signal $\boldsymbol{x}$ by a signal $\hat{\boldsymbol{x}}$ which has the series expansion

$$\hat{\boldsymbol{x}} = \sum_{i=1}^{m} \beta_i \, \boldsymbol{\varphi}_i \quad \text{with} \quad m < n. \tag{3.4}$$

This problem will be discussed in Sections 3.2 and 3.3 in greater detail. For the present we will confine ourselves to discussing some general concepts of decomposing signal spaces. We start by assuming a decomposition of $\boldsymbol{x}$ into

$$\boldsymbol{x} = \boldsymbol{x}_1 + \boldsymbol{x}_2, \tag{3.5}$$

where

$$\boldsymbol{x}_1 = \sum_{i=1}^{m} \alpha_i \boldsymbol{\varphi}_i, \tag{3.6}$$

$$\boldsymbol{x}_2 = \sum_{i=m+1}^{n} \alpha_i \boldsymbol{\varphi}_i. \tag{3.7}$$

Signal $\boldsymbol{x}_1$ is an element of the *linear subspace*[1] $X_1 = \operatorname{span}\{\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_m\}$ and $\boldsymbol{x}_2$ is an element of the linear subspace $X_2 = \operatorname{span}\{\boldsymbol{\varphi}_{m+1}, \ldots, \boldsymbol{\varphi}_n\}$. The space X is called the *sum of the subspaces* X_1 and X_2. If the decomposition of $\boldsymbol{x} \in X$

[1] Definition of a linear subspace: *let M be a non-empty set of elements of the vector space X. Then M is a linear subspace of X, if M itself is a linear space.* This means that all linear combinations of the elements of M must be elements of M. Hence, X itself is a linear subspace.

into $\boldsymbol{x}_1 \in X_1$ and $\boldsymbol{x}_2 \in X_2$ is unique,[2] we speak of a *direct decomposition* of X into the subspaces X_1 and X_2, and X is called the *direct sum* of X_1 and X_2. The notation for the direct sum is

$$X = X_1 \oplus X_2. \tag{3.8}$$

A direct sum is obtained if the vectors that span X_1 are linearly independent of the vectors that span X_2.

If a space X is the direct sum of two subspaces X_1 and X_2 and $\boldsymbol{x}_1 \in X_1$ and $\boldsymbol{x}_2 \in X_2$ are orthogonal to one another for all signals $\boldsymbol{x} \in X$, that is if $\langle \boldsymbol{x}_1, \boldsymbol{x}_2 \rangle = 0 \; \forall \, \boldsymbol{x} \in X$, then X is the *orthogonal sum* of the subspaces X_1 and X_2. For this we write

$$X = X_1 \overset{\perp}{\oplus} X_2. \tag{3.9}$$

3.2 Orthogonal Series Expansions

3.2.1 Calculation of Coefficients

We consider a signal $\boldsymbol{x}$ that can be represented in the form

$$\boldsymbol{x} = \sum_{i=1}^{n} \alpha_i \, \boldsymbol{u}_i, \tag{3.10}$$

where the vectors $\boldsymbol{u}_i$ satisfy the *orthonormality condition*

$$\langle \boldsymbol{u}_i, \boldsymbol{u}_j \rangle = \delta_{ij}. \tag{3.11}$$

Here, δ_{ij} is the *Kronecker symbol*

$$\delta_{ij} = \begin{cases} 1 & \text{for } \; i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{3.12}$$

For all signals $\boldsymbol{x}$ in (3.10) we have $\boldsymbol{x} \in X$ with $X = \text{span}\,\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$. Because of (3.11), $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ form an *orthonormal basis* for X. Each vector $\boldsymbol{u}_i, \; i = 1, \ldots, n$ spans a one-dimensional subspace, and X is the orthogonal sum of these subspaces.

The question of how the coefficients α_i can be calculated if $\boldsymbol{x}$ and the orthonormal basis $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\}$ are given is easily answered. By taking the inner product of (3.10) with $\boldsymbol{u}_j, \; j = 1, \ldots, n$ and using (3.11) we obtain

$$\alpha_j = \langle \boldsymbol{x}, \boldsymbol{u}_j \rangle \,, \quad j = 1, \ldots, n. \tag{3.13}$$

[2]The decomposition is unique if $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ cannot be represented by means of coefficients $\gamma_i \neq \alpha_i$ as $\boldsymbol{x}_1 = \sum_{i=1}^{m} \gamma_i \boldsymbol{\varphi}_i$ and $\boldsymbol{x}_2 = \sum_{i=m+1}^{n} \gamma_i \boldsymbol{\varphi}_i$.

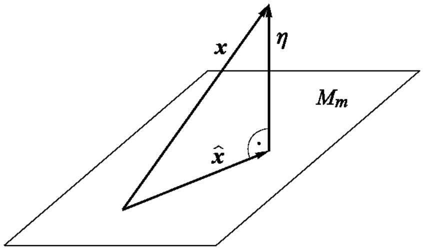

Figure 3.1. Orthogonal projection.

3.2.2 Orthogonal Projection

In (3.10) we assumed that $\boldsymbol{x}$ can be represented by means of n coefficients $\alpha_1, \alpha_1, \ldots, \alpha_n$. Possibly, n is infinitely large, so that for practical applications we are interested in finding the best approximation

$$\hat{\boldsymbol{x}} = \sum_{i=1}^{m} \beta_i \; \boldsymbol{u}_i, \qquad m < n \tag{3.14}$$

in the sense of

$$d(\boldsymbol{x}, \hat{\boldsymbol{x}}) = \|\boldsymbol{x} - \hat{\boldsymbol{x}}\| = \langle \boldsymbol{x} - \hat{\boldsymbol{x}}, \boldsymbol{x} - \hat{\boldsymbol{x}} \rangle^{\frac{1}{2}} \stackrel{!}{=} \min. \tag{3.15}$$

The solution to this problem is[3] $\beta_i = \langle \boldsymbol{x}, \boldsymbol{u}_i \rangle$, which means that

$$\hat{\boldsymbol{x}} = \sum_{i=1}^{m} \langle \boldsymbol{x}, \boldsymbol{u}_i \rangle \;\; \boldsymbol{u}_i. \tag{3.16}$$

This result has a simple geometrical interpretation in terms of an *orthogonal projection.* Each basis vector $\boldsymbol{u}_i$ spans a subspace that is orthogonal to the subspaces spanned by $\boldsymbol{u}_j,\ j \neq i$, which means that the signal space X is decomposed as follows:

$$X = M_m \stackrel{\perp}{\oplus} M_m^{\perp} \tag{3.17}$$

with

$$\boldsymbol{x} = \hat{\boldsymbol{x}} + \boldsymbol{\eta}, \qquad \boldsymbol{x} \in X, \;\; \hat{\boldsymbol{x}} \in M_m, \;\; \boldsymbol{\eta} \in M_m^{\perp}. \tag{3.18}$$

The subspace $M_m^{\perp}$ is orthogonal to M_m, and $\boldsymbol{\eta} = \boldsymbol{x} - \hat{\boldsymbol{x}}$ is orthogonal to $\hat{\boldsymbol{x}}$ (notation: $\boldsymbol{\eta} \perp \hat{\boldsymbol{x}}$). Because of $\boldsymbol{\eta} \perp \hat{\boldsymbol{x}}$ we call $\hat{\boldsymbol{x}}$ the orthogonal projection of $\boldsymbol{x}$ onto M_m. Figure 3.1 gives an illustration.

As can easily be verified, we have the following relationship between the norms of $\boldsymbol{x}, \hat{\boldsymbol{x}}$ and $\boldsymbol{\eta}$

$$\|\boldsymbol{x}\|^2 = \|\hat{\boldsymbol{x}}\|^2 + \|\boldsymbol{\eta}\|^2. \tag{3.19}$$

[3]The proof is given in Section 3.3.2 for general, non-orthogonal bases.

3.2.3 The Gram–Schmidt Orthonormalization Procedure

Given a basis $\{\boldsymbol{\varphi}_i;\ i = 1,\dots,n\}$, we can construct an orthonormal basis $\{\boldsymbol{u}_i;\ i = 1,\dots,n\}$ for the space span $\{\boldsymbol{\varphi}_i;\ i = 1,\dots,n\}$ by using the following scheme:

$$
\begin{aligned}
\boldsymbol{w}_1 &= \boldsymbol{\varphi}_1 \\
\boldsymbol{u}_1 &= \frac{\boldsymbol{w}_1}{\|\boldsymbol{w}_1\|} \\
\boldsymbol{w}_2 &= \boldsymbol{\varphi}_2 - \langle \boldsymbol{\varphi}_2, \boldsymbol{u}_1\rangle\, \boldsymbol{u}_1 \\
\boldsymbol{u}_2 &= \frac{\boldsymbol{w}_2}{\|\boldsymbol{w}_2\|} \\
\boldsymbol{w}_3 &= \boldsymbol{\varphi}_3 - \langle \boldsymbol{\varphi}_3, \boldsymbol{u}_1\rangle\, \boldsymbol{u}_1 - \langle \boldsymbol{\varphi}_3, \boldsymbol{u}_2\rangle\, \boldsymbol{u}_2 \\
\boldsymbol{u}_3 &= \frac{\boldsymbol{w}_3}{\|\boldsymbol{w}_3\|} \\
&\ \vdots \\
\boldsymbol{w}_i &= \boldsymbol{\varphi}_i - \sum_{k=1}^{i-1} \langle \boldsymbol{\varphi}_i, \boldsymbol{u}_k\rangle\, \boldsymbol{u}_k \\
\boldsymbol{u}_i &= \frac{\boldsymbol{w}_i}{\|\boldsymbol{w}_i\|} \\
&\ \vdots
\end{aligned}
\tag{3.20}
$$

This method is known as the *Gram–Schmidt procedure.* It is easily seen that the result is not unique. A re-ordering of the vectors $\boldsymbol{\varphi}_i$ before the application of the Gram–Schmidt procedure results in a different basis.

3.2.4 Parseval's Relation

Parseval's relation states that the inner product of two vectors equals the inner product of their representations with respect to an orthonormal basis. Given

$$\boldsymbol{x} = \sum_{i=1}^{n} \alpha_i\, \boldsymbol{u}_i \tag{3.21}$$

and

$$\boldsymbol{y} = \sum_{i=1}^{n} \beta_i\, \boldsymbol{u}_i \tag{3.22}$$

we have

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \langle \boldsymbol{\alpha}, \boldsymbol{\beta}\rangle \tag{3.23}$$

with

$$\begin{aligned} \boldsymbol{\alpha} &= [\alpha_1, \ldots, \alpha_n]^T, \\ \boldsymbol{\beta} &= [\beta_1, \ldots, \beta_n]^T. \end{aligned} \tag{3.24}$$

This is verified by substituting (3.21) into (3.23) and by making use of the fact that the basis is orthogonal:

$$\begin{aligned} \langle \boldsymbol{x}, \boldsymbol{y} \rangle &= \Big\langle \sum_{i=1}^{n} \alpha_i \boldsymbol{u}_i \, , \, \sum_{j=1}^{n} \beta_j \boldsymbol{u}_j \Big\rangle \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \beta_j^* \, \langle \boldsymbol{u}_i, \boldsymbol{u}_j \rangle \\ &= \sum_{i=1}^{n} \alpha_i \beta_i^* \\ &= \langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle. \end{aligned} \tag{3.25}$$

For $\boldsymbol{x} = \boldsymbol{y}$ we get from (3.23)

$$\|\boldsymbol{x}\| = \|\boldsymbol{\alpha}\|. \tag{3.26}$$

It is important to notice that the inner product of the representations is defined as $\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle = \boldsymbol{\beta}^H \boldsymbol{\alpha}$, whereas the inner product of the signals may have a different definition. The inner product of the signals may even involve a weighting matrix or weighting function.

3.2.5 Complete Orthonormal Sets

It can be shown that the space $L_2(a, b)$ is complete. Thus, any signal $x(t) \in L_2(a, b)$ can be approximated with arbitrary accuracy by means of an orthogonal projection

$$\hat{x}(t) = \sum_{i=1}^{n} \langle \boldsymbol{x}, \boldsymbol{\varphi}_i \rangle \, \varphi_i(t), \tag{3.27}$$

where n is chosen sufficiently large and the basis vectors $\varphi_i(t)$ are taken from a complete orthonormal set.

According to (3.19) and (3.23) we have for the approximation error:

$$\begin{aligned} \|\boldsymbol{x} - \boldsymbol{x}_n\|^2 &= \|\boldsymbol{x}\|^2 - \|\boldsymbol{x}_n\|^2 \\ &= \|\boldsymbol{x}\|^2 - \sum_{i=1}^{n} | \langle \boldsymbol{x}, \boldsymbol{\varphi}_i \rangle |^2. \end{aligned} \tag{3.28}$$

From (3.28) we conclude

$$\sum_{i=1}^{n} |\langle \boldsymbol{x}, \boldsymbol{\varphi}_i \rangle|^2 \leq \|\boldsymbol{x}\|^2 \quad \forall \; n. \tag{3.29}$$

(3.29) is called the *Bessel inequality.* It ensures that the squared sum of the coefficients $\langle \boldsymbol{x}, \boldsymbol{\varphi}_i \rangle$ exists.

An orthonormal set is said to be complete if no additional non-zero orthogonal vector exists which can be added to the set.

When an orthonormal set is complete, the approximation error tends towards zero with $n \to \infty$. The Bessel inequality (3.29) then becomes the *completeness relation*

$$\sum_{i=1}^{\infty} |\langle \boldsymbol{x}, \boldsymbol{\varphi}_i \rangle|^2 = \|\boldsymbol{x}\|^2 \quad \forall \; x \in L_2(a,b). \tag{3.30}$$

Here, Parseval's relation states

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{i=1}^{\infty} \langle \boldsymbol{x}, \boldsymbol{\varphi}_i \rangle \, \langle \boldsymbol{y}, \boldsymbol{\varphi}_i \rangle^*, \tag{3.31}$$

and for $\boldsymbol{x} = \boldsymbol{y}$:

$$\|\boldsymbol{x}\|^2 = \sum_{i=1}^{\infty} |\langle \boldsymbol{x}, \boldsymbol{\varphi}_i \rangle|^2. \tag{3.32}$$

3.2.6 Examples of Complete Orthonormal Sets

Fourier Series. One of the best-known discrete transforms is the *Fourier series expansion.* The basis functions are the complex exponentials

$$\varphi_i(t) = \frac{e^{j\pi i t}}{\sqrt{2}}, \quad i = 0, \pm 1, \pm 2, \ldots, \tag{3.33}$$

which form a complete orthonormal set. The interval considered is $T = [-1, 1]$. The weighting function is $g(t) = 1$. Note that any finite interval can be mapped onto the interval $T = [-1, +1]$.

Legendre Polynomials. The *Legendre polynomials* $P_n(t)$, $n = 0, 1, \ldots$ are defined as

$$P_n(t) = \frac{1}{2^n n!} \frac{d^n}{dt^n} (t^2 - 1)^n \tag{3.34}$$

and can alternatively be computed according to the recursion formula

$$P_n(t) = \frac{1}{n} [(2n-1)t \; P_{n-1}(t) - (n-1) \; P_{n-2}(t)]. \tag{3.35}$$

The first four functions are

$$\begin{aligned} P_0(t) &= 1, \\ P_1(t) &= t, \\ P_2(t) &= \tfrac{3}{2}t^2 - \tfrac{1}{2}, \\ P_3(t) &= \tfrac{5}{2}t^3 - \tfrac{3}{2}t. \end{aligned}$$

A set $\varphi_n(t),\ n = 0, 1, 2, \ldots$ which is orthonormal on the interval $[-1, 1]$ with weighting function $g(t) = 1$ is obtained by

$$\varphi_n(t) = \sqrt{\frac{2n+1}{2}}\, P_n(t). \tag{3.36}$$

Chebyshev Polynomials. The *Chebyshev polynomials* are defined as

$$T_n(t) = \cos(n \arccos t\ ), \qquad n \geq 0, \qquad -1 \leq t \leq 1, \tag{3.37}$$

and can be computed according to the recursion

$$T_n(t) = 2t\, T_{n-1}(t) - T_{n-2}(t). \tag{3.38}$$

The first four polynomials are

$$\begin{aligned} T_0(t) &= 1, \\ T_1(t) &= t, \\ T_2(t) &= 2t^2 - 1, \\ T_3(t) &= 4t^3 - 3t. \end{aligned}$$

Using the normalization

$$\varphi_n(t) = \begin{cases} \sqrt{1/\pi}\ T_0(t) & \text{for } n = 0 \\ \sqrt{2/\pi}\ T_n(t) & \text{for } n > 0 \end{cases} \tag{3.39}$$

we get a set which is orthonormal on the interval $[-1, +1]$ with weighting function $g(t) = (1 - t^2)^{-1/2}$.

Laguerre Polynomials. The *Laguerre polynomials*

$$L_n(t) = e^t \frac{d^n}{dt^n}(t^n e^{-t}), \qquad n = 0, 1, 2, \ldots \tag{3.40}$$

can be calculated by means of the recursion

$$L_n(t) = (2n - 1 - t)\ L_{n-1}(t) - (n-1)^2\ L_{n-2}(t). \tag{3.41}$$

The normalization

$$\varphi_n(t) = \frac{1}{n!} L_n(t) \quad n = 0, 1, 2, \dots \tag{3.42}$$

yields a set which is orthonormal on the interval $[0, \infty]$ with weighting function $g(t) = e^{-t}$. The first four basis vectors are

$$\begin{aligned}
\varphi_0(t) &= 1, \\
\varphi_1(t) &= 1 - t, \\
\varphi_2(t) &= 1 - 2t + \tfrac{1}{2}t^2, \\
\varphi_3(t) &= 1 - 3t + \tfrac{3}{2}t^2 - \tfrac{1}{6}t^3.
\end{aligned}$$

An alternative is to generate the set

$$\psi_n(t) = \frac{e^{-t/2}}{n!} L_n(t), \qquad n = 0, 1, 2, \dots, \tag{3.43}$$

which is orthonormal with weight one on the interval $[0, \infty]$. As will be shown below, the polynomials $\psi_n(t),\ n = 0, 1, 2, \dots$ can be generated by a network. For this, let

$$f_n(t) = \psi_n(2pt) = \frac{e^{-pt}}{n!} L_n(2pt). \tag{3.44}$$

The Laplace transform is given by

$$F_n(s) = L\{f_n(t)\} = \frac{(s-p)^n}{(s+p)^{n+1}}. \tag{3.45}$$

Thus, a function $f_n(t)$ is obtained from a network with the transfer function $F_n(s)$, which is excited by an impulse. The network can be realized as a cascade of a first-order lowpass and n first-order allpass filters.

Hermite Polynomials. The *Hermite polynomials* are defined as

$$H_k(t) = (-1)^k\ e^{t^2}\ \frac{d^k}{dt^k} e^{-t^2}, \qquad k = 0, 1, 2, \dots \tag{3.46}$$

A recursive computation is possible in the form

$$H_k(t) = 2t\ H_{k-1}(t) - 2(k-1)\ H_{k-2}(t). \tag{3.47}$$

With the weighting function $g(t) = e^{-t^2}$ the polynomials

$$\phi_k(t) = (2^k\ k!\ \sqrt{\pi})^{-\frac{1}{2}}\ H_k(t), \qquad k = 0, 1, 2, \ldots \tag{3.48}$$

form an orthonormal basis for $L_2(\mathbb{R})$. Correspondingly, the *Hermite functions*

$$\varphi_k(t) = (2^k\ k!\ \sqrt{\pi})^{-\frac{1}{2}}\ e^{-t^2/2}\ H_k(t), \qquad k = 0, 1, 2, \ldots, \tag{3.49}$$

form an orthonormal basis with weight one. The functions $\varphi_k(t)$ are also obtained by applying the Gram–Schmidt procedure to the basis $\{t^k\ e^{-t^2/2};\ k = 0, 1, \ldots\}$ [57].

Walsh Functions. *Walsh functions* take on the values 1 and −1. Orthogonality is achieved by appropriate zero crossings. The first two functions are given by

$$\begin{aligned} \varphi_0(t) &= \{1 \quad \text{for } 0 \le t \le 1\,, \\ \varphi_1(t) &= \begin{cases} 1 & \text{for } 0 \le t < \frac{1}{2}, \\ -1 & \text{for } \frac{1}{2} < t \le 1. \end{cases} \end{aligned} \tag{3.50}$$

Further functions can be computed by means of the recursion

$$\left.\begin{aligned} \varphi_{m+1}^{(2k-1)}(t) &= \begin{cases} \varphi_m^{(k)}(2t) & \text{for } 0 \le t < \frac{1}{2} \\ (-1)^{k+1}\varphi_m^{(k)}(2t-1) & \text{for } \frac{1}{2} < t \le 1 \end{cases} \\ \varphi_{m+1}^{(2k)}(t) &= \begin{cases} \varphi_m^{(k)}(2t) & \text{for } 0 \le t < \frac{1}{2} \\ (-1)^{k}\varphi_m^{(k)}(2t-1) & \text{for } \frac{1}{2} < t \le 1 \end{cases} \end{aligned}\right\} \quad \begin{aligned} m &= 1, 2, \ldots, \\ k &= 1, \ldots, 2^{m-1} \end{aligned} \tag{3.51}$$

Figure 3.2 shows the first six Walsh functions; they are named according to their number of zero crossings.

3.3 General Series Expansions

If possible, one would choose an orthonormal basis for signal representation. However, in practice, a given basis is often not orthonormal. For example, in data transmission a transmitted signal may have the form $x(t) = \sum_m d(m)\ s(t - mT)$, where $d(m)$ is the data and $s(t)$ is an impulse response that satisfies $\int s(t)s(t - mT)dt = \delta_{m0}$. If we now assume that $x(t)$ is transmitted through a non-ideal channel with impulse response $h(t)$, then we

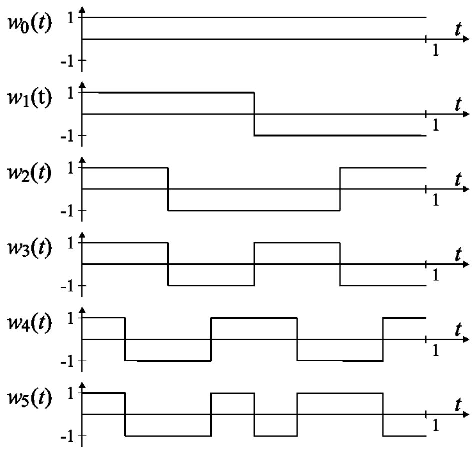

Figure 3.2. Walsh functions.

have a signal $r(t) = \sum_m d(m)g(t-mT)$ with $g(t) = s(t) * h(t)$ on the receiver side. This new basis $\{g(t - mT); m \in \mathbb{Z}\}$ is no longer orthogonal, so that the question arises of how to recover the data if $r(t)$ and $g(t)$ are given.

3.3.1 Calculating the Representation

In the following, signals $\boldsymbol{x} \in X$ with $X = \text{span}\,\{\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_n\}$ will be considered. We assume that the n vectors $\{\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_n\}$ are linearly independent so that all $\boldsymbol{x} \in X$ can be represented uniquely as

$$\boldsymbol{x} = \sum_{i=1}^{n} \alpha_i\, \boldsymbol{\varphi}_i\,, \qquad \boldsymbol{x} \in X. \tag{3.52}$$

As will be shown, the representation

$$\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_n]^T \tag{3.53}$$

with respect to a given basis $\{\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_n\}$ can be computed by solving a linear set of equations and also via the so-called reciprocal basis. The set of equations is obtained by multiplying (inner product) both sides of (3.52) with $\boldsymbol{\varphi}_j$, $j = 1, \ldots, n$:

$$\langle \boldsymbol{x}, \boldsymbol{\varphi}_j \rangle = \sum_{i=1}^{n} \alpha_i \langle \boldsymbol{\varphi}_i, \boldsymbol{\varphi}_j \rangle\,, \qquad j = 1, \ldots, n. \tag{3.54}$$

In matrix notation this is

$$\boldsymbol{\Phi}\, \boldsymbol{\alpha} = \boldsymbol{\beta} \tag{3.55}$$

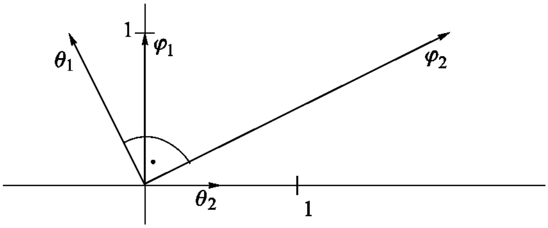

Figure 3.3. Reciprocal basis (The basis is $\varphi_1 = [0,1]^T$, $\varphi_2 = [2,1]^T$; the corresponding reciprocal basis is $\boldsymbol{\theta}_1 = [-0.5,1]^T$, $\boldsymbol{\theta}_2 = [0.5,0]^T$).

with

$$\begin{aligned} \boldsymbol{\Phi} &= \begin{bmatrix} \langle\varphi_1,\varphi_1\rangle & \langle\varphi_2,\varphi_1\rangle & \cdots & \langle\varphi_n,\varphi_1\rangle \\ \langle\varphi_1,\varphi_2\rangle & \langle\varphi_2,\varphi_2\rangle & \cdots & \langle\varphi_n,\varphi_2\rangle \\ \vdots & & & \\ \langle\varphi_1,\varphi_n\rangle & \langle\varphi_2,\varphi_n\rangle & \cdots & \langle\varphi_n,\varphi_n\rangle \end{bmatrix}, \\ \boldsymbol{\beta} &= \begin{bmatrix} \langle\boldsymbol{x},\varphi_1\rangle \\ \langle\boldsymbol{x},\varphi_2\rangle \\ \vdots \\ \langle\boldsymbol{x},\varphi_n\rangle \end{bmatrix}. \end{aligned} \tag{3.56}$$

$\boldsymbol{\Phi}$ is known as the *Grammian matrix.* Due to $\langle\varphi_i,\varphi_k\rangle = \langle\varphi_k,\varphi_i\rangle^*$ it has the property $\boldsymbol{\Phi} = \boldsymbol{\Phi}^H$.

The disadvantage of the method considered above is that for calculating the representation $\boldsymbol{\alpha}$ of a new $\boldsymbol{x}$ we first have to calculate $\boldsymbol{\beta}$ before (3.55) can be solved. Much more interesting is the computation of the representation $\boldsymbol{\alpha}$ by means of the ***reciprocal basis*** $\{\boldsymbol{\theta}_i;\ i = 1,2,3\ldots n\}$, which satisfies the condition

$$\langle\varphi_i,\boldsymbol{\theta}_j\rangle = \delta_{ij} \quad , \quad i,j = 1,\ldots,n, \tag{3.57}$$

which is known as the ***biorthogonality condition***; Figure 3.3 illustrates (3.57) in the two-dimensional plane.

Multiplying both sides of (3.52) with $\boldsymbol{\theta}_j$, $j = 1,\ldots,n$ leads to

$$\langle\boldsymbol{x},\boldsymbol{\theta}_j\rangle = \sum_{i=1}^{n} \alpha_i \underbrace{\langle\varphi_i,\boldsymbol{\theta}_j\rangle}_{\delta_{ij}} = \alpha_i, \quad j = 1,\ldots,n, \tag{3.58}$$

which means that, when using the reciprocal basis, we directly obtain the representation by forming inner products

$$\alpha_j = \langle\boldsymbol{x},\boldsymbol{\theta}_j\rangle, \quad j = 1,\ldots,n. \tag{3.59}$$

A vector $\boldsymbol{x}$ can be represented as

$$\boldsymbol{x} = \sum_{i=1}^{n} \langle \boldsymbol{x}, \boldsymbol{\theta}_i \rangle \ \boldsymbol{\varphi}_i, \tag{3.60}$$

and also as

$$\boldsymbol{x} = \sum_{i=1}^{n} \langle \boldsymbol{x}, \boldsymbol{\varphi}_i \rangle \ \boldsymbol{\theta}_i. \tag{3.61}$$

Parseval's relation holds only for orthonormal bases. However, also for general bases a relationship between the inner product of signals and their representations can be established. For this, one of the signals is represented by means of the basis $\{\boldsymbol{\varphi}_1, \dots, \boldsymbol{\varphi}_n\}$ and a second signal by means of the corresponding reciprocal basis $\{\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_n\}$. For the inner product of two signals

$$\boldsymbol{x} = \sum_{i=1}^{n} \langle \boldsymbol{x}, \boldsymbol{\varphi}_i \rangle \, \boldsymbol{\theta}_i \tag{3.62}$$

and

$$\boldsymbol{y} = \sum_{k=1}^{n} \langle \boldsymbol{y}, \boldsymbol{\theta}_k \rangle \, \boldsymbol{\varphi}_k \tag{3.63}$$

we get

$$\begin{aligned} \langle \boldsymbol{x}, \boldsymbol{y} \rangle &= \left\langle \sum_{i=1}^{n} \langle \boldsymbol{x}, \boldsymbol{\varphi}_i \rangle \, \boldsymbol{\theta}_i, \sum_{k=1}^{n} \langle \boldsymbol{y}, \boldsymbol{\theta}_k \rangle \, \boldsymbol{\varphi}_k \right\rangle \\ &= \sum_{i=1}^{n} \sum_{k=1}^{n} \langle \boldsymbol{x}, \boldsymbol{\varphi}_i \rangle \, \langle \boldsymbol{y}, \boldsymbol{\theta}_k \rangle^* \, \langle \boldsymbol{\theta}_i, \boldsymbol{\varphi}_k \rangle \\ &= \sum_{i=1}^{n} \langle \boldsymbol{x}, \boldsymbol{\varphi}_i \rangle \, \langle \boldsymbol{y}, \boldsymbol{\theta}_i \rangle^* . \end{aligned} \tag{3.64}$$

In the last step, the property $\langle \boldsymbol{\varphi}_i, \boldsymbol{\theta}_k \rangle = \delta_{ik}$ was used.

Calculation of the Reciprocal Basis. Since $\boldsymbol{\varphi}_k, \ k = 1, \dots, n$ as well as $\boldsymbol{\theta}_j, \ j = 1, \dots, n$ are bases for X, the vectors $\boldsymbol{\theta}_j, \ j = 1, \dots, n$ can be written as linear combinations of $\boldsymbol{\varphi}_k, \ k = 1, \dots, n$ with the yet unknown coefficients γ_{jk}:

$$\boldsymbol{\theta}_j = \sum_{k=1}^{n} \gamma_{jk} \ \boldsymbol{\varphi}_k, \quad j = 1, \dots, n. \tag{3.65}$$

Multiplying this equation with φ_i, $i = 1, \ldots, n$ and using (3.57) leads to

$$\left.\begin{aligned} \langle \boldsymbol{\theta}_j, \boldsymbol{\varphi}_i \rangle &= \left\langle \sum_{k=1}^{n} \gamma_{jk}\, \boldsymbol{\varphi}_k \,,\, \boldsymbol{\varphi}_i \right\rangle \\ &= \sum_{k=1}^{n} \langle \gamma_{jk}\, \boldsymbol{\varphi}_k \,, \boldsymbol{\varphi}_i \rangle \\ &= \sum_{k=1}^{n} \gamma_{jk}\, \langle \boldsymbol{\varphi}_k, \boldsymbol{\varphi}_i \rangle \\ &= \delta_{ij} \end{aligned}\right\} \quad i, j = 1, \ldots, n. \tag{3.66}$$

With

$$\boldsymbol{\Gamma} = \begin{bmatrix} \gamma_{11} & \cdots & \gamma_{1n} \\ \vdots & \ddots & \vdots \\ \gamma_{n1} & \cdots & \gamma_{nn} \end{bmatrix} \tag{3.67}$$

and

$$\boldsymbol{\Phi}^T = \begin{bmatrix} \langle \boldsymbol{\varphi}_1, \boldsymbol{\varphi}_1 \rangle & \cdots & \langle \boldsymbol{\varphi}_1, \boldsymbol{\varphi}_n \rangle \\ \vdots & & \vdots \\ \langle \boldsymbol{\varphi}_n, \boldsymbol{\varphi}_1 \rangle & \cdots & \langle \boldsymbol{\varphi}_n, \boldsymbol{\varphi}_n \rangle \end{bmatrix} \tag{3.68}$$

equation (3.66) can be written as

$$\boldsymbol{\Gamma}\boldsymbol{\Phi}^T = \boldsymbol{I}, \tag{3.69}$$

so that

$$\boldsymbol{\Gamma} = \left(\boldsymbol{\Phi}^T\right)^{-1}. \tag{3.70}$$

The reciprocal basis is obtained from (3.65), (3.67) and (3.70).

3.3.2 Orthogonal Projection

We consider the approximation of $\boldsymbol{x} \in X$ by $\hat{\boldsymbol{x}} \in M_m$, where $M_m \subset X$. For the signal spaces let $X = \operatorname{span}\{\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_n\}$ and $M_m = \operatorname{span}\{\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_m\}$ with $m < n$.

As we will see, the best approximation in the sense of

$$||\boldsymbol{x} - \hat{\boldsymbol{x}}|| \stackrel{!}{=} \min \tag{3.71}$$

is obtained for

$$\hat{\boldsymbol{x}} = \sum_{i=1}^{m} \langle \boldsymbol{x}, \boldsymbol{\theta}_i \rangle\, \boldsymbol{\varphi}_i, \tag{3.72}$$

where $\{\boldsymbol{\theta}_i;\ i = 1, \dots, m\}$ is the reciprocal basis to $\{\boldsymbol{\varphi}_i;\ i = 1, \dots, m\}$. Note that the reciprocal basis satisfies

$$M_m = \operatorname{span}\{\boldsymbol{\varphi}_1, \dots, \boldsymbol{\varphi}_m\} = \operatorname{span}\{\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_m\}. \tag{3.73}$$

Requiring only $\langle \boldsymbol{\varphi}_i, \boldsymbol{\theta}_j \rangle = \delta_{ij}$, $i, j = 0, 1, \dots, m$ is not sufficient for $\boldsymbol{\theta}_j$ to form the reciprocal basis.

First we consider the expression $\langle \hat{\boldsymbol{x}}, \boldsymbol{\theta}_j \rangle$ with $\hat{\boldsymbol{x}}$ according to (3.72). Because of $\langle \boldsymbol{\varphi}_i, \boldsymbol{\theta}_j \rangle = \delta_{ij}$ we obtain

$$\langle \hat{\boldsymbol{x}}, \boldsymbol{\theta}_j \rangle = \left\langle \sum_{i=1}^{m} \langle \boldsymbol{x}, \boldsymbol{\theta}_i \rangle \, \boldsymbol{\varphi}_i \ , \quad \boldsymbol{\theta}_j \right\rangle = \langle \boldsymbol{x}, \boldsymbol{\theta}_j \rangle \, , \ j = 1, \dots, m. \tag{3.74}$$

Hence,

$$\langle \boldsymbol{x} - \hat{\boldsymbol{x}}, \boldsymbol{\theta}_j \rangle = 0, \quad j = 1, \dots, m. \tag{3.75}$$

Equation (3.75) shows that

$$\boldsymbol{\eta} = \boldsymbol{x} - \hat{\boldsymbol{x}} \tag{3.76}$$

is orthogonal to all $\boldsymbol{\theta}_j$, $j = 1, \dots, m$. From (3.73) and (3.75) we conclude that $\boldsymbol{\eta}$ is orthogonal to all vectors in M_m:

$$\boldsymbol{\eta} \perp \tilde{\boldsymbol{x}} \quad \text{for all} \quad \tilde{\boldsymbol{x}} \in M_m. \tag{3.77}$$

This also means that X is decomposed into an orthogonal sum

$$X = M_m \overset{\perp}{\oplus} M_m^{\perp}. \tag{3.78}$$

For the vectors we have

$$\boldsymbol{x} = \hat{\boldsymbol{x}} + \boldsymbol{\eta}, \qquad \hat{\boldsymbol{x}} \in M_m, \quad \boldsymbol{\eta} \in M_m^{\perp}, \quad \boldsymbol{x} \in X. \tag{3.79}$$

The approximation $\hat{\boldsymbol{x}}$ according to (3.72) is the *orthogonal projection* of $\boldsymbol{x} \in X$ onto M_m.

In order to show that $\hat{\boldsymbol{x}}$ according to (3.72) is the best approximation to $\boldsymbol{x}$, we consider the distance between $\boldsymbol{x}$ and an arbitrary vector $\tilde{\boldsymbol{x}} \in M_m$ and perform some algebraic manipulations:

$$\begin{aligned} d^2(\boldsymbol{x}, \tilde{\boldsymbol{x}}) &= \|\boldsymbol{x} - \tilde{\boldsymbol{x}}\|^2 \\ &= \|(\boldsymbol{x} - \hat{\boldsymbol{x}}) - (\tilde{\boldsymbol{x}} - \hat{\boldsymbol{x}})\|^2 \\ &= \langle (\boldsymbol{x} - \hat{\boldsymbol{x}}) - (\tilde{\boldsymbol{x}} - \hat{\boldsymbol{x}}), (\boldsymbol{x} - \hat{\boldsymbol{x}}) - (\tilde{\boldsymbol{x}} - \hat{\boldsymbol{x}}) \rangle \\ &= \langle \boldsymbol{x} - \hat{\boldsymbol{x}}, \boldsymbol{x} - \hat{\boldsymbol{x}} \rangle - \langle \boldsymbol{x} - \hat{\boldsymbol{x}}, \tilde{\boldsymbol{x}} - \hat{\boldsymbol{x}} \rangle - \langle \tilde{\boldsymbol{x}} - \hat{\boldsymbol{x}}, \boldsymbol{x} - \hat{\boldsymbol{x}} \rangle + \langle \tilde{\boldsymbol{x}} - \hat{\boldsymbol{x}}, \tilde{\boldsymbol{x}} - \hat{\boldsymbol{x}} \rangle . \end{aligned} \tag{3.80}$$

Because of $(\tilde{\boldsymbol{x}} - \hat{\boldsymbol{x}}) \in M_m$ and (3.75), the second and third terms in (3.80) are zero, such that

$$\|\boldsymbol{x} - \tilde{\boldsymbol{x}}\|^2 = \|\boldsymbol{x} - \hat{\boldsymbol{x}}\|^2 + \|\tilde{\boldsymbol{x}} - \hat{\boldsymbol{x}}\|^2 . \tag{3.81}$$

The minimum is achieved for $\tilde{\boldsymbol{x}} = \hat{\boldsymbol{x}}$, so that (3.72) clearly yields the best approximation.

A relationship between the norms of $\boldsymbol{x}, \hat{\boldsymbol{x}}$ and $\boldsymbol{\eta}$ is obtained from

$$\begin{aligned} \|\boldsymbol{x}\|^2 &= \|\hat{\boldsymbol{x}} + \boldsymbol{\eta}\|^2 \\ &= \langle \hat{\boldsymbol{x}} + \boldsymbol{\eta}, \hat{\boldsymbol{x}} + \boldsymbol{\eta} \rangle \\ &= \langle \hat{\boldsymbol{x}}, \hat{\boldsymbol{x}} \rangle + \langle \hat{\boldsymbol{x}}, \boldsymbol{\eta} \rangle + \langle \boldsymbol{\eta}, \hat{\boldsymbol{x}} \rangle + \langle \boldsymbol{\eta}, \boldsymbol{\eta} \rangle . \end{aligned} \tag{3.82}$$

Because of (3.79) the second and the third term in the last row are zero, and

$$\|\boldsymbol{x}\|^2 = \|\hat{\boldsymbol{x}}\|^2 + \|\boldsymbol{\eta}\|^2 \tag{3.83}$$

remains.

3.3.3 Orthogonal Projection of n-Tuples

The solutions to the projection problem considered so far hold for all vectors, including n-tuples, of course. However, for n-tuples the projection can concisely be described with matrices, and we have a large number of methods at hand for solving the problem.

In the following, we consider the projection of $\boldsymbol{x} = [x_1, \dots, x_n]^T \in \mathbb{C}^n$ onto subspaces $M_m = \text{span}\,\{\boldsymbol{b}_1, \dots, \boldsymbol{b}_m\}$, where $m < n$ and $\boldsymbol{b}_i \in \mathbb{C}^n$. With

$$\boldsymbol{B} = [\boldsymbol{b}_1, \dots, \boldsymbol{b}_m] \qquad n \times m \;\; \text{matrix} \tag{3.84}$$

and

$$\boldsymbol{a} = [a_1, \dots, a_m]^T \qquad m \times 1 \;\; \text{vector} \tag{3.85}$$

the approximation is given by

$$\hat{\boldsymbol{x}} = \boldsymbol{B}\ \boldsymbol{a}. \tag{3.86}$$

Furthermore, the orthogonal projection can be described by a Hermitian matrix $\boldsymbol{P}$ as

$$\hat{\boldsymbol{x}} = \boldsymbol{P}\ \boldsymbol{x}. \tag{3.87}$$

Inner Product without Weighting. To compute the reciprocal basis $\boldsymbol{\Theta} = [\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_n]$ the relationships (3.70), (3.56) and (3.65) are used, which can be written as

$$\begin{aligned} \boldsymbol{\Gamma}^T &= \boldsymbol{\Phi}^{-1}, \\ \boldsymbol{\Phi} &= \boldsymbol{B}^H \boldsymbol{B}, \\ \boldsymbol{\Theta} &= \boldsymbol{B}\boldsymbol{\Gamma}^T. \end{aligned} \tag{3.88}$$

For the reciprocal basis we then get

$$\boldsymbol{\Theta} = \boldsymbol{B}\,[\boldsymbol{B}^H \boldsymbol{B}]^{-1}. \tag{3.89}$$

Observing that the inverse of a Hermitian matrix is Hermitian itself, the representation is calculated according to (3.59) as

$$\boldsymbol{a} = \boldsymbol{\Theta}^H \boldsymbol{x} = [\boldsymbol{B}^H \boldsymbol{B}]^{-1} \boldsymbol{B}^H \boldsymbol{x}. \tag{3.90}$$

With (3.86) the orthogonal projection is

$$\hat{\boldsymbol{x}} = \boldsymbol{B}[\boldsymbol{B}^H \boldsymbol{B}]^{-1} \boldsymbol{B}^H \boldsymbol{x}. \tag{3.91}$$

If $\boldsymbol{B}$ contains an orthonormal basis, we have $\boldsymbol{B}^H \boldsymbol{B} = \boldsymbol{I}$, and the projection problem is simplified.

Note that the representation according to (3.90) is the solution of the equation

$$[\boldsymbol{B}^H \boldsymbol{B}]\,\boldsymbol{a} = \boldsymbol{B}^H \boldsymbol{x}, \tag{3.92}$$

which is known as the *normal equation.*

Inner Product with Weighting. For an inner product with a weighting matrix $\boldsymbol{G}$, equations (3.70), (3.56) and (3.65) give

$$\begin{aligned} \boldsymbol{\Gamma}^T &= \boldsymbol{\Phi}^{-1}, \\ \boldsymbol{\Phi} &= \boldsymbol{B}^H \boldsymbol{G} \boldsymbol{B}, \\ \boldsymbol{\Theta} &= \boldsymbol{B}\boldsymbol{\Gamma}^T. \end{aligned} \tag{3.93}$$

Thus, we obtain

$$\boldsymbol{\Theta} = \boldsymbol{B}\,[\boldsymbol{B}^H \boldsymbol{G} \boldsymbol{B}]^{-1}, \tag{3.94}$$

$$\boldsymbol{a} = \boldsymbol{\Theta}^H \boldsymbol{G} \boldsymbol{x} = [\boldsymbol{B}^H \boldsymbol{G} \boldsymbol{B}]^{-1} \boldsymbol{B}^H \boldsymbol{G} \boldsymbol{x}, \tag{3.95}$$

$$\hat{\boldsymbol{x}} = \boldsymbol{B}[\boldsymbol{B}^H \boldsymbol{G} \boldsymbol{B}]^{-1} \boldsymbol{B}^H \boldsymbol{G} \boldsymbol{x}. \tag{3.96}$$

Alternatively, $\boldsymbol{G}$ can be split into a product $\boldsymbol{G} = \boldsymbol{H}^H \boldsymbol{H}$, and the problem

$$\|\boldsymbol{B\alpha} - \boldsymbol{x}\|_G \Big|_{\boldsymbol{\alpha} = \boldsymbol{a}} \stackrel{!}{=} \min \tag{3.97}$$

can be transformed via

$$\begin{aligned} \boldsymbol{z} &= \boldsymbol{Hx} \\ \boldsymbol{V} &= \boldsymbol{HB} \end{aligned} \tag{3.98}$$

into the equivalent problem

$$\|\boldsymbol{V\alpha} - \boldsymbol{z}\|_I \Big|_{\boldsymbol{\alpha} = \boldsymbol{a}} \stackrel{!}{=} \min. \tag{3.99}$$

The indices of the norms in (3.97) and (3.99) stand for the weighting matrices involved. Thus, the projection problem with weighting can be transformed into one without weighting. Splitting $\boldsymbol{G}$ into $\boldsymbol{G} = \boldsymbol{H}^H \boldsymbol{H}$ can for instance be achieved by applying the *Cholesky decomposition* $\boldsymbol{G} = \boldsymbol{L}\boldsymbol{L}^H$ or by a *singular value decomposition.* Both methods can be applied in all cases since $\boldsymbol{G}$ must be Hermitian and positive definite in order to be a valid weighting matrix.

Note. The computation of the reciprocal basis involves the inversion of the Grammian matrix. If the Grammian matrix is poorly conditioned, numerical problems may occur. Robust methods of handling such cases are the QR decomposition and the Moore–Penrose pseudoinverse, which will be discussed in the next section.

3.4 Mathematical Tools

3.4.1 The QR Decomposition

The methods for solving the projection problem considered so far require an inversion of the Grammian matrix. The inversion does not pose a major problem so long as the vectors that span the subspace in question are linearly independent. However, because of finite-precision arithmetic, a poorly conditioned Grammian matrix may lead to considerable errors, even if the vectors are linearly independent.

A numerically robust solution of

$$\|\boldsymbol{B\alpha} - \boldsymbol{x}\| \Big|_{\boldsymbol{\alpha} = \boldsymbol{a}} \stackrel{!}{=} \min \tag{3.100}$$

is obtained by carrying out a QR decomposition of $\boldsymbol{B}$:

$$\boldsymbol{B} = \boldsymbol{Q}\ \boldsymbol{R}. \tag{3.101}$$

Here, $\boldsymbol{Q}$ is a unitary matrix, and $\boldsymbol{R}$ has the following form:

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & \cdots & r_{1m} \\ & \ddots & \vdots \\ & & r_{mm} \\ & & \\ & & \end{bmatrix}. \tag{3.102}$$

The QR decomposition can, for instance, be computed by using *Householder reflections* or *Givens rotations*; see Sections 3.4.4 and 3.4.5.

In the following we will show how (3.100) can be solved via QR decomposition. Substituting (3.101) in (3.100) yields

$$||\boldsymbol{QR\alpha} - \boldsymbol{x}||\Big|_{\boldsymbol{\alpha}=\boldsymbol{a}} \stackrel{!}{=} \min. \tag{3.103}$$

For (3.103) we can also write

$$\left\|\boldsymbol{Q}^H\boldsymbol{QR\alpha} - \boldsymbol{Q}^H\boldsymbol{x}\right\| = \left\|\boldsymbol{R\alpha} - \boldsymbol{Q}^H\boldsymbol{x}\right\|\Big|_{\boldsymbol{\alpha}=\boldsymbol{a}} \stackrel{!}{=} \min, \tag{3.104}$$

because a multiplication with a unitary matrix does not change the norm of a vector. Using the abbreviation $\boldsymbol{y} = \boldsymbol{Q}^H\boldsymbol{x}$, we get

$$||\boldsymbol{R\alpha} - \boldsymbol{y}|| = \left\| \begin{bmatrix} r_{11} & \cdots & r_{1m} \\ & \ddots & \vdots \\ & & r_{mm} \\ & & \\ & & \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix} - \begin{bmatrix} y_1 \\ \vdots \\ y_m \\ y_{m+1} \\ \vdots \\ y_n \end{bmatrix} \right\|. \tag{3.105}$$

With

$$\boldsymbol{X} = \begin{bmatrix} r_{11} & \cdots & r_{1m} \\ & \ddots & \vdots \\ & & r_{mm} \end{bmatrix}, \quad \boldsymbol{z} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}, \quad \boldsymbol{N} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}, \quad \boldsymbol{f} = \begin{bmatrix} y_{m+1} \\ \vdots \\ y_n \end{bmatrix} \tag{3.106}$$

(3.105) becomes

$$||\boldsymbol{R\alpha} - \boldsymbol{y}|| = \left\| \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{N} \end{bmatrix} \boldsymbol{\alpha} - \begin{bmatrix} \boldsymbol{z} \\ \boldsymbol{f} \end{bmatrix} \right\| = \left\| \begin{bmatrix} \boldsymbol{X}\,\boldsymbol{\alpha} - \boldsymbol{z} \\ \boldsymbol{N\alpha} - \boldsymbol{f} \end{bmatrix} \right\|. \tag{3.107}$$

The norm reaches its minimum if $\boldsymbol{\alpha} = \boldsymbol{a}$ is the solution of

$$\boldsymbol{X}\ \boldsymbol{a} = \boldsymbol{z}. \tag{3.108}$$

Note that $\boldsymbol{X}$ is an upper triangular matrix, so that $\boldsymbol{a}$ is easily computed by using Gaussian elimination. For the norm of the error we have:

$$||\boldsymbol{R}\boldsymbol{a} - \boldsymbol{y}|| = \left\| \begin{bmatrix} \boldsymbol{X}\,\boldsymbol{a} - \boldsymbol{z} \\ \boldsymbol{f} \end{bmatrix} \right\| = ||\boldsymbol{f}||. \tag{3.109}$$

3.4.2 The Moore–Penrose Pseudoinverse

We consider the criterion

$$||\boldsymbol{B}\boldsymbol{\alpha} - \boldsymbol{x}|| \Big|_{\boldsymbol{\alpha} = \boldsymbol{a}} \overset{!}{=} \min. \tag{3.110}$$

The solutions (3.90) and (3.91),

$$\boldsymbol{a} = \left[\boldsymbol{B}^H \boldsymbol{B}\right]^{-1} \boldsymbol{B}^H \boldsymbol{x}, \tag{3.111}$$

$$\hat{\boldsymbol{x}} = \boldsymbol{B}\left[\boldsymbol{B}^H \boldsymbol{B}\right]^{-1} \boldsymbol{B}^H \boldsymbol{x}, \tag{3.112}$$

can only be applied if $\left[\boldsymbol{B}^H \boldsymbol{B}\right]^{-1}$ exists, that is, if the columns of $\boldsymbol{B}$ are linearly independent. However, an orthogonal projection can also be carried out if $\boldsymbol{B}$ contains linearly dependent vectors. A general solution to the projection problem is obtained by introducing a matrix $\boldsymbol{B}^+$ via the following four equations

$$\boldsymbol{B}^+\boldsymbol{B} = (\boldsymbol{B}^+\boldsymbol{B})^H \tag{3.113}$$

$$\boldsymbol{B}\boldsymbol{B}^+ = (\boldsymbol{B}\boldsymbol{B}^+)^H \tag{3.114}$$

$$\boldsymbol{B}\boldsymbol{B}^+\boldsymbol{B} = \boldsymbol{B} \tag{3.115}$$

$$\boldsymbol{B}^+\boldsymbol{B}\boldsymbol{B}^+ = \boldsymbol{B}^+. \tag{3.116}$$

There is only one $\boldsymbol{B}^+$ that satisfies (3.113) – (3.116). This matrix is called the *Moore–Penrose pseudoinverse* [3]. The expressions $\boldsymbol{B}^+\boldsymbol{B}$ and $\boldsymbol{B}\boldsymbol{B}^+$ describe orthogonal projections, since under conditions (3.113) – (3.116) we have

$$\begin{aligned} \left[\boldsymbol{x} - \boldsymbol{B}\boldsymbol{B}^+\boldsymbol{x}\right]^H \boldsymbol{B}\boldsymbol{B}^+\boldsymbol{x} &= 0, \\ \left[\boldsymbol{a} - \boldsymbol{B}^+\boldsymbol{B}\boldsymbol{a}\right]^H \boldsymbol{B}^+\boldsymbol{B}\boldsymbol{a} &= 0. \end{aligned} \tag{3.117}$$

Assuming that $\boldsymbol{B}$ is an $n \times m$ matrix which either has rank $k = m$ or $k = n$, we have

$$\begin{aligned} \boldsymbol{B}^+ &= \left[\boldsymbol{B}^H \boldsymbol{B}\right]^{-1} \boldsymbol{B}^H, & k &= m, \\ \boldsymbol{B}^+ &= \boldsymbol{B}^H \left[\boldsymbol{B}\boldsymbol{B}^H\right]^{-1}, & k &= n, \\ \boldsymbol{B}^+ &= \boldsymbol{B}^{-1}, & k &= n = m. \end{aligned} \tag{3.118}$$

$\boldsymbol{B}^+$ can for instance be computed via the *singular value decomposition*

$$\boldsymbol{B} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^H. \tag{3.119}$$

$\boldsymbol{U}$ and $\boldsymbol{V}$ are unitary. For $m < n$, $\boldsymbol{\Sigma}$ has the following form:

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_m \\ & & \\ & & \end{bmatrix}. \tag{3.120}$$

The non-zero values σ_i are called the singular values of $\boldsymbol{B}$. They satisfy $\sigma_i > 0$. With

$$\boldsymbol{\Sigma}^+ = \begin{bmatrix} \tau_1 & & & \\ & \ddots & & \\ & & \tau_m & \end{bmatrix}, \quad \tau_i = \begin{cases} 1/\sigma_i, & \sigma_i \neq 0, \\ 0, & \sigma_i = 0 \end{cases} \tag{3.121}$$

the pseudoinverse $\boldsymbol{B}^+$ is given by

$$\boldsymbol{B}^+ = \boldsymbol{V}\boldsymbol{\Sigma}^+\boldsymbol{U}^H. \tag{3.122}$$

It can easily be shown that the requirements (3.113) – (3.116) with $\boldsymbol{B}^+$ according to (3.122) are satisfied, so that (3.111) and (3.112) can be replaced by

$$\boldsymbol{a} = \boldsymbol{B}^+\boldsymbol{x}, \tag{3.123}$$

$$\hat{\boldsymbol{x}} = \boldsymbol{B}\boldsymbol{B}^+\boldsymbol{x}. \tag{3.124}$$

Note that (3.123) is not necessarily the only solution to the problem (3.110). We will return to this topic in the next section.

By taking the products $\boldsymbol{B}^H\boldsymbol{B}$ and $\boldsymbol{B}\boldsymbol{B}^H$ we obtain equations for calculating the singular value decomposition. With $\boldsymbol{B}$ according to (3.119) we have

$$\begin{aligned} \boldsymbol{B}^H\boldsymbol{B} = \boldsymbol{V}\boldsymbol{\Sigma}^H\boldsymbol{U}^H\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^H &= \boldsymbol{V}\left[\boldsymbol{\Sigma}^H\boldsymbol{\Sigma}\right]\boldsymbol{V}^H, \\ \boldsymbol{B}\boldsymbol{B}^H = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^H\boldsymbol{V}\boldsymbol{\Sigma}^H\boldsymbol{U}^H &= \boldsymbol{U}\left[\boldsymbol{\Sigma}\boldsymbol{\Sigma}^H\right]\boldsymbol{U}^H. \end{aligned} \tag{3.125}$$

That is, the squares of the singular values of $\boldsymbol{B}$ are the eigenvalues of $\boldsymbol{B}^H\boldsymbol{B}$ and at the same time of $\boldsymbol{B}\boldsymbol{B}^H$. Matrix $\boldsymbol{V}$ contains the orthonormal eigenvectors of $\boldsymbol{B}^H\boldsymbol{B}$. Correspondingly, $\boldsymbol{U}$ contains the eigenvectors of $\boldsymbol{B}\boldsymbol{B}^H$. Further methods of calculating the pseudoinverse are discussed in [3].

Note. The pseudoinverse may be written as

$$\boldsymbol{B}^+ = \left[\boldsymbol{B}^H\boldsymbol{B}\right]^+\boldsymbol{B}^H. \tag{3.126}$$

This property can be applied to continuous functions, and with $\boldsymbol{\Gamma}^T = \boldsymbol{\Phi}^+$ instead of $\boldsymbol{\Gamma}^T = \boldsymbol{\Phi}^{-1}$ we can compute a set of functions $\theta_k(t)$, which is dual to a given set $\varphi_i(t)$; see (3.65) – (3.70).

3.4.3 The Nullspace

Let us consider the problem

$$\boldsymbol{B}\,\boldsymbol{a} = \hat{\boldsymbol{x}}, \tag{3.127}$$

where $\hat{\boldsymbol{x}} = \boldsymbol{B}\,\boldsymbol{B}^+\boldsymbol{x}$ is the orthogonal projection of an arbitrary $\boldsymbol{x}$ onto the column subspace of $\boldsymbol{B}$. It is easily observed that the solution to (3.127) also is the solution to (3.110). Depending on $\boldsymbol{B}$ we either have a unique solution $\boldsymbol{a}$, or we have an infinite number of solutions. Finding all solutions is intimately related to finding the nullspace of matrix $\boldsymbol{B}$.

The nullspace of a matrix $\boldsymbol{B}$ consists of all vectors $\boldsymbol{a}$ such that $\boldsymbol{B}\,\boldsymbol{a} = \boldsymbol{0}$. It is denoted by $\mathcal{N}(\boldsymbol{B})$. In order to describe $\mathcal{N}(\boldsymbol{B})$, let us assume that $\boldsymbol{B}$ is an $n \times m$ matrix that has rank r. If $r = m$ then $\mathcal{N}(\boldsymbol{B})$ is only the null vector, and $\boldsymbol{a} = \boldsymbol{B}^+\hat{\boldsymbol{x}} = \boldsymbol{B}^+\boldsymbol{x}$ is the unique solution to (3.127) and thus also to (3.110). If $r < m$ then $\mathcal{N}(\boldsymbol{B})$ is of dimension $m - r$, which means that $\mathcal{N}(\boldsymbol{B})$ is spanned by $m - r$ linearly independent vectors. These vectors can be chosen to form an orthonormal basis for the nullspace. If we define a matrix $\boldsymbol{N}$ of size $m \times (m - r)$ whose column subspace is the nullspace of $\boldsymbol{B}$ then

$$\boldsymbol{B}\,\boldsymbol{N} = \boldsymbol{0}. \tag{3.128}$$

The set of all solutions to (3.127) is then given by

$$\boldsymbol{a} = \tilde{\boldsymbol{a}} + \boldsymbol{N}\boldsymbol{p}, \qquad \text{where} \qquad \tilde{\boldsymbol{a}} = \boldsymbol{B}^+\hat{\boldsymbol{x}} = \boldsymbol{B}^+\boldsymbol{x}. \tag{3.129}$$

In (3.129) $\boldsymbol{p}$ is an arbitrary vector of length $m - r$. In some applications it is useful to exploit the free design parameters in $\boldsymbol{p}$ in order to find a solution $\boldsymbol{a}$ that optimizes an additional criterion. However, in most cases one will use the solution $\tilde{\boldsymbol{a}}$ given by the pseudoinverse, because this is the solution with minimum Euclidean norm. In order to see this, let us determine the squared norm of $\boldsymbol{a}$:

$$\begin{aligned} \|\boldsymbol{a}\|_{\ell_2}^2 &= \boldsymbol{a}^H \boldsymbol{a} \\ &= \left[\boldsymbol{B}^+\boldsymbol{x} + \boldsymbol{N}\boldsymbol{p}\right]^H \left[\boldsymbol{B}^+\boldsymbol{x} + \boldsymbol{N}\boldsymbol{p}\right] \\ &= \boldsymbol{x}^H(\boldsymbol{B}^+)^H\boldsymbol{B}^+\boldsymbol{x} + \boldsymbol{p}^H\boldsymbol{N}^H\boldsymbol{B}^+\boldsymbol{x} + \boldsymbol{x}^H(\boldsymbol{B}^+)^H\boldsymbol{N}\boldsymbol{p} + \boldsymbol{p}^H\boldsymbol{N}^H\boldsymbol{N}\boldsymbol{p}. \end{aligned} \tag{3.130}$$

The second and third terms vanish, because $\boldsymbol{BN} = \boldsymbol{0}$ implies that $\boldsymbol{N}^H\boldsymbol{B}^+ = \boldsymbol{0}$. Thus, we get the vector $\boldsymbol{a}$ of shortest length for $\boldsymbol{p} = \boldsymbol{0}$, that is for $\boldsymbol{a} = \tilde{\boldsymbol{a}}$.

The matrix $\boldsymbol{N}$ that contains the basis for the nullspace is easily found from the singular value decomposition

$$\boldsymbol{B} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^H. \tag{3.131}$$

Let $\boldsymbol{B}$ have rank r and let the r nonzero singular values be the elements $[\boldsymbol{\Sigma}]_{1,1}, \ldots, [\boldsymbol{\Sigma}]_{r,r}$ of matrix $\boldsymbol{\Sigma}$. Then an orthonormal matrix $\boldsymbol{N}$ is given by the last $m - r$ columns of $\boldsymbol{V}$.

3.4.4 The Householder Transform

Householder transforms allow a simple and numerically robust way of performing QR decompositions, and thus of solving normal equations. The QR decomposition is carried out step by step by reflecting vectors at hyperplanes.

In order to explain the basic idea of the Householder transforms we consider two vectors $\boldsymbol{x}, \boldsymbol{w} \in \mathbb{C}^n$, and we look at the projection of $\boldsymbol{x}$ onto a one-dimensional subspace $W = \text{span}\,\{\boldsymbol{w}\}$:

$$\boldsymbol{P}_w\boldsymbol{x} = \boldsymbol{w}\ \frac{1}{\boldsymbol{w}^H\boldsymbol{w}}\ \boldsymbol{w}^H\boldsymbol{x}. \tag{3.132}$$

Here, $\mathbb{C}^n$ is decomposed into the orthogonal sum

$$\mathbb{C}^n = W \overset{\perp}{\oplus} W^\perp. \tag{3.133}$$

The Householder transform is given by

$$\boldsymbol{H}_w\boldsymbol{x} = \boldsymbol{x} - 2\ \boldsymbol{P}_w\boldsymbol{x}. \tag{3.134}$$

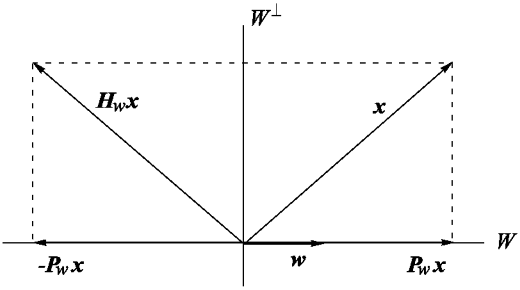

Figure 3.4. Householder reflection.

It is also known as *Householder reflection*, because it is the reflection of $\boldsymbol{x}$ at the hyperplane $W^\perp$, as depicted in Figure 3.4.

With $\boldsymbol{P}_w$ according to (3.132) we get

$$\boldsymbol{H}_w = \boldsymbol{I} - \frac{2}{\boldsymbol{w}^H \boldsymbol{w}}\ \boldsymbol{w}\ \boldsymbol{w}^H \tag{3.135}$$

for the *Householder matrix* $\boldsymbol{H}_w$.

From (3.135) the following property of Householder matrices can be concluded:

$$\begin{aligned} \boldsymbol{H}_w^H \boldsymbol{H}_w &= \boldsymbol{H}_w \boldsymbol{H}_w \\ &= \left[\boldsymbol{I} - \tfrac{2}{\boldsymbol{w}^H\boldsymbol{w}}\ \boldsymbol{w}\boldsymbol{w}^H\right]\ \left[\boldsymbol{I} - \tfrac{2}{\boldsymbol{w}^H\boldsymbol{w}}\ \boldsymbol{w}\boldsymbol{w}^H\right] \\ &= \boldsymbol{I} - \tfrac{4}{\boldsymbol{w}^H\boldsymbol{w}}\ \boldsymbol{w}\boldsymbol{w}^H + \tfrac{4}{\boldsymbol{w}^H\boldsymbol{w}}\ \boldsymbol{w}\boldsymbol{w}^H\boldsymbol{w}\boldsymbol{w}^H \\ &= \boldsymbol{I}. \end{aligned} \tag{3.136}$$

Hence $\boldsymbol{H}_w$ is unitary and Hermitian. Furthermore we have

$$\det\{\boldsymbol{H}_w\} = -1. \tag{3.137}$$

In order to make practical use of the Householder transform, we consider a vector $\boldsymbol{x}$ and try to find that vector $\boldsymbol{w}$ for which only the ith component of $\boldsymbol{H}_w\boldsymbol{x}$ is non-zero. We use the following approach:

$$\boldsymbol{w} = \boldsymbol{x} + \alpha\ \boldsymbol{e}_i, \tag{3.138}$$

where

$$\boldsymbol{e}_i^T = [0,\ldots,0,1,0,\ldots,0]\,. \tag{3.139}$$
$$\uparrow \; i\text{th element}$$

For $\boldsymbol{H}_w\boldsymbol{x}$ we get

$$\begin{aligned}
\boldsymbol{H}_w\boldsymbol{x} &= \boldsymbol{x} - 2\frac{\boldsymbol{w}\boldsymbol{w}^H}{\boldsymbol{w}^H\boldsymbol{w}}\;\boldsymbol{x} \\
&= \boldsymbol{x} - 2\frac{\boldsymbol{w}^H\boldsymbol{x}}{\boldsymbol{w}^H\boldsymbol{w}}\;\boldsymbol{w} \\
&= \boldsymbol{x} - 2\frac{\boldsymbol{w}^H\boldsymbol{x}}{\boldsymbol{w}^H\boldsymbol{w}}\;[\boldsymbol{x} + \alpha\;\boldsymbol{e}_i] \\
&= \left(1 - 2\frac{\boldsymbol{w}^H\boldsymbol{x}}{\boldsymbol{w}^H\boldsymbol{w}}\right)\;\boldsymbol{x} - 2\alpha\;\frac{\boldsymbol{w}^H\boldsymbol{x}}{\boldsymbol{w}^H\boldsymbol{w}}\;\boldsymbol{e}_i.
\end{aligned} \tag{3.140}$$

In order to achieve that only the ith component of $\boldsymbol{H}_w\boldsymbol{x}$ is non-zero, the expression in parentheses in (3.140) must vanish:

$$1 - 2\frac{\boldsymbol{w}^H\boldsymbol{x}}{\boldsymbol{w}^H\boldsymbol{w}} = 1 - 2\frac{||\boldsymbol{x}||^2 + \alpha^* x_i}{||\boldsymbol{x}||^2 + \alpha x_i^* + \alpha^* x_i + |\alpha|^2} = 0, \tag{3.141}$$

where x_i is the ith component of $\boldsymbol{x}$. As can easily be verified, (3.141) is satisfied for

$$\alpha = \pm\frac{x_i}{|x_i|}\;||\boldsymbol{x}||\,. \tag{3.142}$$

In order to avoid $\boldsymbol{w} \approx \boldsymbol{0}$ in the case of $\boldsymbol{x} \approx \beta\boldsymbol{e}_i$ for some $\beta \in \mathbb{R}$ we choose the positive sign in (3.142) and obtain

$$\boldsymbol{w} = \boldsymbol{x} + \frac{x_i}{|x_i|}\;||\boldsymbol{x}||\;\boldsymbol{e}_i. \tag{3.143}$$

By substituting this solution into (3.140) we finally get

$$\boldsymbol{H}_w\boldsymbol{x} = -\frac{x_i}{|x_i|}\;||\boldsymbol{x}||\;\boldsymbol{e}_i. \tag{3.144}$$

Applying the Householder transform to the QR Decomposition. We consider the problem

$$||\boldsymbol{A}\boldsymbol{v} - \boldsymbol{b}|| \stackrel{!}{=} \min \tag{3.145}$$

with

$$\boldsymbol{A} = \begin{bmatrix} a_{11}^{(1)} & \ldots & a_{1m}^{(1)} \\ \vdots & & \vdots \\ a_{n1}^{(1)} & \ldots & a_{nm}^{(1)} \end{bmatrix} \quad \in \mathbb{C}^{n,m}, \qquad n > m \tag{3.146}$$

and try to tackle the problem by applying the QR decomposition. First, we choose $\boldsymbol{x}_1$ to be the first column of $\boldsymbol{A}$

$$\boldsymbol{x}_1 = \left[a_{11}^{(1)}, \ldots, a_{n1}^{(1)}\right]^T. \tag{3.147}$$

Multiplying $\boldsymbol{A}$ with the Householder matrix

$$\boldsymbol{H}_1 = \boldsymbol{I} - 2\frac{\boldsymbol{w}_1 \boldsymbol{w}_1^H}{\boldsymbol{w}_1^H \boldsymbol{w}_1}, \tag{3.148}$$

where $\boldsymbol{w}_1$ is chosen as

$$\boldsymbol{w}_1 = \boldsymbol{x}_1 + \frac{a_{11}^{(1)}}{|a_{11}^{(1)}|}\, \|\boldsymbol{x}_1\|\, \boldsymbol{e}_1 \tag{3.149}$$

yields a matrix where only r_{11} is non-zero in the first column:

$$\boldsymbol{H}_1 \boldsymbol{A} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1m} \\ 0 & a_{22}^{(2)} & \cdots & a_{2m}^{(2)} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n2}^{(2)} & \cdots & a_{nm}^{(2)} \end{bmatrix} \tag{3.150}$$

Then, we choose

$$\begin{aligned} \boldsymbol{x}_2 &= \left[0, a_{22}^{(2)}, \ldots, a_{n2}^{(2)}\right]^T, \\ \boldsymbol{w}_2 &= \boldsymbol{x}_2 + \frac{a_{22}^{(2)}}{|a_{22}^{(2)}|}\, \|\boldsymbol{x}_2\|\, \boldsymbol{e}_2, \\ \boldsymbol{H}_2 &= \boldsymbol{I} - 2\frac{\boldsymbol{w}_2 \boldsymbol{w}_2^H}{\boldsymbol{w}_2^H \boldsymbol{w}_2} \end{aligned}$$

and obtain

$$\boldsymbol{H}_2 \boldsymbol{H}_1 \boldsymbol{A} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & \cdots & r_{1m} \\ 0 & r_{22} & r_{23} & \cdots & r_{2m} \\ 0 & 0 & a_{33}^{(3)} & \cdots & a_{3m}^{(3)} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & a_{n3}^{(3)} & \cdots & a_{nm}^{(3)} \end{bmatrix}$$

After maximally m steps we get

$$\boldsymbol{H}_m \cdots \boldsymbol{H}_2 \boldsymbol{H}_1 \boldsymbol{A} = \boldsymbol{R}, \tag{3.151}$$

where only the upper right-hand triangular matrix of $\boldsymbol{R}$ is non-zero. This means that we have carried out the QR decomposition.

Note. If one of the values $a_{ii}^{(i)}$ becomes zero, $\boldsymbol{w}_i$ is chosen as $\boldsymbol{w}_i = \boldsymbol{x}_i + \|\boldsymbol{x}_i\|\, \boldsymbol{e}_i$. If $\|\boldsymbol{x}_i\| = 0$, the columns must be exchanged.

3.4.5 Givens Rotations

Besides Householder reflections, rotations constitute a further possibility of performing QR decompositions. We first consider the rotation of a real-valued vector $\boldsymbol{x}$ by an angle ϕ through multiplication of $\boldsymbol{x}$ with an orthonormal rotation matrix $\boldsymbol{G}$. For

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} r\ \cos(\alpha) \\ r\ \sin(\alpha) \end{bmatrix} \tag{3.152}$$

and

$$\boldsymbol{G} = \begin{bmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{bmatrix} \tag{3.153}$$

we get

$$\boldsymbol{x}' = \boldsymbol{G}\boldsymbol{x} = \begin{bmatrix} r\ \cos(\alpha-\phi) \\ r\ \sin(\alpha-\phi) \end{bmatrix}. \tag{3.154}$$

We observe that for $\phi = \alpha$ a vector $\boldsymbol{x}'$ is obtained whose second component is zero. This special rotation matrix is

$$\boldsymbol{G} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \tag{3.155}$$

with

$$\begin{aligned} c &= \cos(\alpha) = \frac{x_1}{\sqrt{x_1^2 + x_2^2}} \\ s &= \sin(\alpha) = \frac{x_2}{\sqrt{x_1^2 + x_2^2}}. \end{aligned} \tag{3.156}$$

For the rotated vector we have

$$\boldsymbol{x}' = \boldsymbol{G}\boldsymbol{x} = \begin{bmatrix} r \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{x_1^2 + x_2^2} \\ 0 \end{bmatrix}. \tag{3.157}$$

As can easily be verified, for complex-valued vectors we can apply the rotation matrix

$$\boldsymbol{G} = \begin{bmatrix} c & s \\ -s^* & c \end{bmatrix} \tag{3.158}$$

with

$$c = \frac{x_1}{r}, \qquad s = \frac{x_2^*}{r}, \qquad r = \sqrt{|x_1|^2 + |x_2|^2} \tag{3.159}$$

in order to obtain $\boldsymbol{x}' = [r, 0]^T$. Note that $\boldsymbol{G}$ according to (3.158) is unitary, $\boldsymbol{G}^H\boldsymbol{G} = \boldsymbol{I}$.

We now consider a vector

$$\boldsymbol{x} = [x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n]^T, \tag{3.160}$$

and want to achieve a vector

$$\boldsymbol{x}' = [x_1, \ldots, x_{i-1}, r, x_{i+1}, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_n]^T \tag{3.161}$$

with

$$r = \sqrt{|x_i|^2 + |x_j|^2} \tag{3.162}$$

by carrying out a rotation. The rotation is applied to the elements x_i and x_j only. We have

$$\boldsymbol{x}' = \boldsymbol{G}\ \boldsymbol{x} \tag{3.163}$$

with

$$\boldsymbol{G} = \begin{bmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & c & & & s & & \\ & & & & 1 & & & & \\ & & & & & \ddots & & & \\ & & & & & & 1 & & \\ & & & -s^* & & & c & & \\ & & & & & & & 1 & \\ & & & & & & & & \ddots \\ & & & & & & & & & 1 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow i \\ \\ \\ \\ \leftarrow j \\ \\ \\ \\ \end{matrix} \tag{3.164}$$

$$\qquad\qquad \uparrow i \qquad \uparrow j$$

A QR decomposition of a matrix can be carried out by repeated application of the rotations described above.