Because $a(t)$ has a finite mean-square value, (145*a*) and (145*b*) imply that (144) is also valid for $u = 0$ and $u = T$.

The resulting error for the optimum processor follows easily. It is simply the first term in (137).

$$\xi_{P_o}(t) = K_a(t, t) - 2\int_0^T h_o(t, u)\, K_a(t, u)\, du + \int\int_0^T h_o(t, u)\, h_o(t, v)\, K_r(u, v)\, du\, dv \tag{146}$$

or

$$\xi_{P_o}(t) = K_a(t, t) - \int_0^T h_o(t, u)\, K_a(t, u)\, du - \int_0^T h_o(t, u)\left[K_a(t, u) - \int_0^T h_o(t, v)\, K_r(u, v)\, dv\right] du. \tag{147}$$

But (138) implies that the term in brackets is zero. Therefore

$$\xi_{P_o}(t) = K_a(t, t) - \int_0^T h_o(t, u)\, K_a(t, u)\, du. \tag{148}$$

For the white noise case, substitution of (144) into (148) gives

$$\boxed{\xi_{P_o}(t) = \frac{N_0}{2}\, h_o(t, t).} \tag{149}$$

As a final result in our present discussion of optimum linear filters, we demonstrate how to obtain a solution to (144) in terms of the eigenvalues and eigenfunctions of $K_a(t, u)$. We begin by expanding the message covariance function in a series,

$$K_a(t, u) = \sum_{i=1}^{\infty} \lambda_i \phi_i(t)\, \phi_i(u), \tag{150}$$

where λ_i and $\phi_i(t)$ are solutions to (46) when the kernel is $K_a(t, u)$. Using (127), we can expand the white noise component in (143)

$$K_w(t, u) = \frac{N_0}{2}\, \delta(t - u) = \sum_{i=1}^{\infty} \frac{N_0}{2}\, \phi_i(t)\, \phi_i(u). \tag{151}$$

To expand the white noise we need a CON set. If $K_a(t, u)$ is not positive definite we augment its eigenfunctions to obtain a CON set. (See Property 9 on p. 181).
Then

$$K_r(t, u) = \sum_{i=1}^{\infty} \left(\lambda_i + \frac{N_0}{2}\right) \phi_i(t)\, \phi_i(u). \tag{152}$$

Because the $\phi_i(t)$ are a CON set, we try a solution of the form

$$h_o(t, u) = \sum_{i=1}^{\infty} h_i\, \phi_i(t)\, \phi_i(u). \tag{153}$$

Substituting (150), (152), and (153) into (144), we find

$$h_o(t, u) = \sum_{i=1}^{\infty} \frac{\lambda_i}{\lambda_i + N_0/2}\, \phi_i(t)\, \phi_i(u). \tag{154}$$

Thus the optimum linear filter can be expressed in terms of the eigenfunctions and eigenvalues of the message covariance function. A K-term approximation is shown in Fig. 3.15.

The nonrealizability could be eliminated by a T-second delay in the second multiplication. Observe that (154) represents a practical solution only when the number of significant eigenvalues is small. In most cases the solution in terms of eigenfunctions will be useful only for theoretical purposes. When we study filtering and estimation in detail in later chapters, we shall find more practical solutions.

The error can also be expressed easily in terms of eigenvalues and eigenfunctions. Substitution of (154) into (149) gives

$$\boxed{\xi_{P_o}(t) = \frac{N_0}{2} \sum_{i=1}^{\infty} \frac{\lambda_i}{\lambda_i + N_0/2}\, \phi_i^2(t), \qquad 0 \le t \le T} \tag{155}$$

and

$$\boxed{\xi_I = \frac{1}{T}\int_0^T \xi_{P_o}(t)\, dt = \frac{N_0}{2T} \sum_{i=1}^{\infty} \frac{\lambda_i}{\lambda_i + N_0/2}.} \tag{156}$$

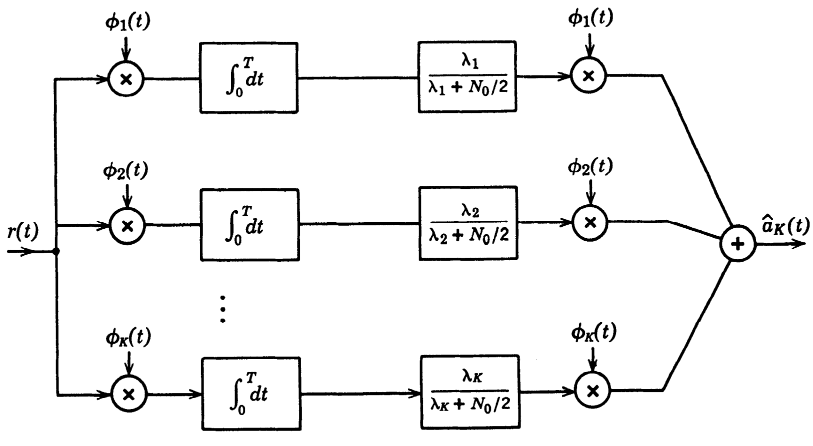

Fig. 3.15 Optimum filter.

In addition to the development of a useful result, this problem has provided an excellent example of the use of eigenfunctions and eigenvalues in finding a series solution to an integral equation. It is worthwhile to re-emphasize that all the results in this section were based on the original constraint of a linear processor and that a Gaussian assumption was not needed. We now return to the general discussion and develop several properties of interest.

3.4.6 Properties of Eigenfunctions and Eigenvalues

In this section we derive two interesting properties that will be useful in the sequel.

Monotonic Property.† Consider the integral equation

$$\lambda_i(T)\, \phi_i(t{:}T) = \int_0^T K_x(t, u)\, \phi_i(u{:}T)\, du, \qquad 0 \le t \le T, \tag{157}$$

where $K_x(t, u)$ is a square-integrable covariance function. [This is just (46) rewritten to emphasize the dependence of the solution on T.] Every eigenvalue $\lambda_i(T)$ is a monotone-increasing function of the length of the interval T.

Proof. Multiplying both sides by $\phi_i(t{:}T)$ and integrating with respect to t over the interval $[0, T]$, we have,

$$\lambda_i(T) = \iint_0^T \phi_i(t{:}T)\, K_x(t, u)\, \phi_i(u{:}T)\, dt\, du. \tag{158}$$

Differentiating with respect to T we have,

$$\frac{\partial \lambda_i(T)}{\partial T} = 2 \int_0^T \frac{\partial \phi_i(t{:}T)}{\partial T}\, dt \int_0^T K_x(t, u)\, \phi_i(u{:}T)\, du$$
$$+\ 2\phi_i(T{:}T) \int_0^T K_x(T, u)\, \phi_i(u{:}T)\, du. \tag{159}$$

Using (157), we obtain

$$\frac{\partial \lambda_i(T)}{\partial T} = 2\lambda_i(T) \int_0^T \frac{\partial \phi_i(t{:}T)}{\partial T}\, \phi_i(t{:}T)\, dt + 2\lambda_i(T)\, \phi_i^2(T{:}T). \quad (160$$

To reduce this equation, recall that

$$\int_0^T \phi_i^2(t{:}T)\, dt = 1. \tag{161}$$

† This result is due to R. Huang [23].

Differentiation of (161) gives

$$2\int_0^T \frac{\partial \phi_i(t:T)}{\partial T}\,\phi_i(t:T)\,dt + \phi_i^2(T:T) = 0. \tag{162}$$

By substituting (162) into (160), we obtain

$$\frac{\partial \lambda_i(T)}{\partial T} = \lambda_i(T)\,\phi_i^2(T:T) \geq 0, \tag{163}$$

which is the desired result.

The second property of interest is the behavior of the eigenfunctions and eigenvalues of stationary processes for *large T*.

Asymptotic Behavior Properties. In many cases we are dealing with stationary processes and are interested in characterizing them over an infinite interval. To study the behavior of the eigenfunctions and eigenvalues we return to (46); we assume that the process is stationary and that the observation interval is infinite. Then (46) becomes

$$\lambda\,\phi(t) = \int_{-\infty}^{\infty} K_x(t-u)\,\phi(u)\,du, \qquad -\infty < t < \infty. \tag{164}$$

In order to complete the solution by inspection, we recall the simple linear filtering problem shown in Figure 3.16. The input is $y(t)$, the impulse response is $h(\tau)$, and the output is $z(t)$. They are related by the convolution integral:

$$z(t) = \int_{-\infty}^{\infty} h(t-u)\,y(u)\,du, \qquad -\infty < t < \infty. \tag{165}$$

In a comparison of (164) and (165) we see that the solution to (164) is simply a function that, when put into a linear system with impulse response $K_x(\tau)$, will come out of the system unaltered except for a gain change. It is well known from elementary linear circuit theory that complex exponentials meet this requirement. Thus

$$\phi(t) = e^{j\omega t}, \qquad -\infty < \omega < \infty, \qquad (166)\dagger$$

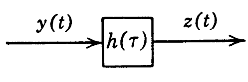

Fig. 3.16 Linear filter.

† The function $e^{(\sigma + j\omega)t}$ also satisfies (165) for values of σ where the exponential transform of $h(\tau)$ exists. The family of exponentials with $\sigma = 0$ is adequate for our purposes. This is our first use of a complex eigenfunction. As indicated at the beginning of Section 3.2, the modifications should be clear. (See Problem 3.4.11)

is an eigenfunction for any ω on the real line. Substituting into (164), we have

$$\lambda e^{+j\omega t} = \int_{-\infty}^{\infty} K_x(t-u)e^{j\omega u}\,du \tag{167}$$

or

$$\lambda = \int_{-\infty}^{\infty} K_x(t-u)e^{-j\omega(t-u)}\,du = S_x(\omega). \tag{168}$$

Thus the eigenvalue for a particular ω is the value of the power density spectrum of the process at that ω.

Now the only difficulty with this discussion is that we no longer have a countable set of eigenvalues and eigenfunctions to deal with and the idea of a series expansion of the sample function loses its meaning. There are two possible ways to get out of this difficulty.

1. Instead of trying to use a series representation of the sample functions, we could try to find some integral representation. The transition would be analogous to the Fourier series–Fourier integral transition for deterministic functions.
2. Instead of starting with the infinite interval, we could consider a finite interval and investigate the behavior as the length increases. This might lead to some simple approximate expressions for large T.

In Sections 3.5 and 3.6 we develop the first approach. It is an approach for dealing with the infinite interval that can be made rigorous. The second, which we now demonstrate, is definitely heuristic but leads to the correct results and is easy to apply.

We start with (46) and assume that the limits are $-T/2$ and $T/2$:

$$\lambda\,\phi(t) = \int_{-T/2}^{+T/2} K_x(t-u)\,\phi(u)\,du, \qquad -\frac{T}{2} \le t \le \frac{T}{2}. \tag{169}$$

We define

$$f_0 = \frac{1}{T}, \tag{170}$$

and try a solution of the form,

$$\phi_n(u) = e^{+j2\pi f_0 n u}, \qquad -\frac{T}{2} \le u \le \frac{T}{2}, \tag{171}$$

where $n = 0, \pm 1, \pm 2, \ldots$. (We index over both positive and negative integers for convenience).

Define

$$f_n = nf_0. \tag{172}$$

Substituting (171) into (169), we have

$$\lambda_n\,\phi_n(t) = \int_{-T/2}^{T/2} K_x(t-u)e^{+j2\pi f_n u}\,du. \tag{173}$$

Now,

$$K_x(t-u) = \int_{-\infty}^{\infty} S_x(f) e^{+j2\pi f(t-u)}\, df. \tag{174}$$

Substituting (174) into (173) and integrating with respect to u, we obtain

$$\lambda_n\, \phi_n(t) = \int_{-\infty}^{\infty} S_x(f) e^{+j2\pi f t} \left[\frac{\sin \pi T(f_n - f)}{\pi(f_n - f)}\right] df. \tag{175}$$

The function in the bracket, shown in Fig. 3.17, is centered at $f = f_n$ where its height is T. Its width is inversely proportional to T and its area equals one for all values of T. We see that for large T the function in the bracket is approximately an impulse at f_n. Thus

$$\lambda_n\, \phi_n(t) \simeq \int_{-\infty}^{\infty} S_x(f) e^{+j2\pi f t}\, \delta(f - f_n)\, df = S_x(f_n) e^{+j2\pi f_n t}. \tag{176}$$

Therefore

$$\lambda_n \simeq S_x(f_n) = S_x(nf_0) \tag{177}$$

and

$$\phi_n(t) \simeq \frac{1}{\sqrt{T}} e^{+j2\pi f_n t}, \qquad -\frac{T}{2} \le t \le \frac{T}{2}, \tag{178}$$

for large T.

From (175) we see that the magnitude of T needed for the approximation to be valid depends on how quickly $S_x(f)$ varies near f_n.

In (156) we encountered the infinite sum of a function of the eigenvalues

$$\xi_I = \frac{N_0}{2T} \sum_{i=1}^{\infty} \frac{\lambda_i}{\lambda_i + N_0/2}. \tag{179}$$

More generally we encounter sums of the form

$$g_\lambda \triangleq \sum_{i=1}^{\infty} g(\lambda_i). \tag{180}$$

An approximate expression for g_λ useful for large T follows directly from the above results. In Fig. 3.18 we sketch a typical spectrum and the approximate eigenvalues based on (177). We see that

$$g_\lambda \simeq \sum_{n=-\infty}^{+\infty} g(S_x(nf_0)) = T \sum_{n=-\infty}^{+\infty} g(S_x(nf_0)) f_0, \tag{181}$$

where the second equality follows from the definition in (170). Now, for large T we can approximate the sum by an integral,

$$\boxed{g_\lambda \cong T \int_{-\infty}^{\infty} g(S_x(f))\, df,} \tag{182}$$

which is the desired result.

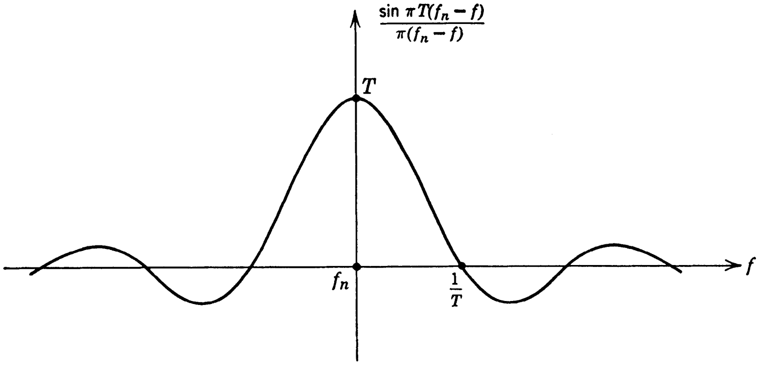

Fig. 3.17 Weighting function in (175).

The next properties concern the size of the largest eigenvalue.

Maximum and Minimum Properties. Let $x(t)$ be a stationary random process represented over an interval of length T. The largest eigenvalue $\lambda_{\max}(T)$ satisfies the inequality

$$\lambda_{\max}(T) \le \max_{f} S_x(f)$$

for any interval T. This result is obtained by combining (177) with the monotonicity property.

Another bound on the maximum eigenvalue follows directly from Property 10 on p. 181.

$$\lambda_{\max}(T) \le \int_{-T/2}^{T/2} K_x(t, t)\, dt = T \int_{-\infty}^{\infty} S_x(f)\, df.$$

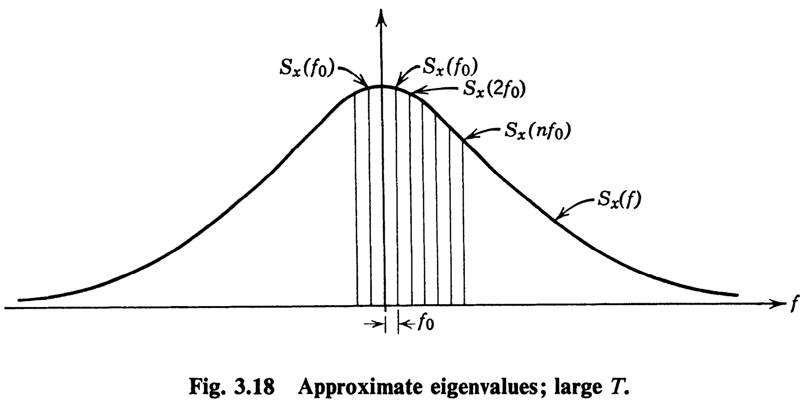

Fig. 3.18 Approximate eigenvalues; large T.

A lower bound is derived in Problem 3.4.4,

$$\lambda_{\max}(T) \geq \iint_{-T/2}^{T/2} f(t)\, K_x(t, u) f(u)\, dt\, du,$$

where $f(t)$ is any function with unit energy in the interval $(-T/2, T/2)$.

The asymptotic properties developed on pp. 205–207 are adequate for most of our work. In many cases, however, we shall want a less heuristic method of dealing with stationary processes and an infinite interval.

In Section 3.6 we develop a method of characterization suitable to stationary processes over an infinite interval. A convenient way to approach this problem is as a limiting case of a periodic process. Therefore in Section 3.5 we digress briefly and develop representations for periodic processes.

3.5 PERIODIC PROCESSES†

Up to this point we have emphasized the representation of processes over a finite time interval. It is often convenient, however, to consider the infinite time interval. We begin our discussion with the definition of a periodic process.

Definition. A periodic process is a stationary random process whose correlation function $R_x(\tau)$ is periodic with period T:

$$R_x(\tau) = R_x(\tau + T), \qquad \text{for all } \tau.$$

It is easy to show that this definition implies that almost every sample function is periodic [i.e., $x(t) = x(t + T)$]. The expectation of the difference is

$$E[(x(t) - x(t + T))^2] = 2R_x(0) - 2R_x(T) = 0.$$

Therefore the probability that $x(t) = x(t + T)$ is one.

We want to represent $x(t)$ in terms of a conventional Fourier series with random coefficients. We first consider a cosine-sine series and assume that the process is zero-mean for notational simplicity.

Cosine-Sine Expansion. The series expansion for the process is

$$x(t) = \underset{N \to \infty}{\text{l.i.m.}} \sum_{i=1}^{N} \left[x_{ci} \cos\left(\frac{2\pi}{T} it\right) + x_{si} \sin\left(\frac{2\pi}{T} it\right)\right], \qquad -\infty < t < \infty, \quad (183)$$

† Sections 3.5 and 3.6 are not essential to most of the discussions in the sequel and may be omitted in the first reading.

where

$$x_{ci} = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos\left(\frac{2\pi}{T}it\right)dt \tag{184}$$

and

$$x_{si} = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin\left(\frac{2\pi}{T}it\right)dt. \tag{185}$$

The covariance function can be expanded as

$$K_x(\tau) = \sum_{i=1}^{\infty} p_i \cos\left(\frac{2\pi}{T}i\tau\right), \tag{186}$$

where

$$p_i = \frac{2}{T}\int_{-T/2}^{T/2} K_x(\tau)\cos\left(\frac{2\pi}{T}i\tau\right)d\tau. \tag{187}$$

It follows easily that

$$\begin{aligned} E(x_{ci}x_{cj}) &= E(x_{si}x_{sj}) = 0, \qquad i \neq j, \\ E(x_{ci}x_{sj}) &= 0, \qquad \text{all } i, j. \end{aligned} \tag{188}$$

Thus the coefficients in the series expansion are uncorrelated random variables. (This means that the eigenfunctions of any periodic process for the interval $(-T/2, T/2)$ are harmonically related cosines and sines.) Similarly,

$$E(x_{ci}^2) = E(x_{si}^2) = p_i. \tag{189}$$

Observe that we have not normalized the coordinate functions. The motivation for this is based on the fact that the *power*, not the energy, at a given frequency is the quantity of interest. By omitting the $\sqrt{T}$ in the coordinate functions the value $(x_{ci}^2 + x_{si}^2)$ represents the power and not the energy. The *expected value* of the power at frequency $\omega_i \triangleq 2\pi i/T \triangleq i\omega_0$ is p_i.

Complex Exponential Expansion. Alternately, we could expand the process by using complex exponentials:

$$K_x(\tau) = \sum_{\substack{i=-\infty \\ i\neq 0}}^{\infty} \frac{p_i}{2}\exp\left(j\frac{2\pi}{T}it\right), \tag{190}$$

and

$$x(t) = \underset{N\to\infty}{\text{l.i.m.}} \sum_{\substack{i=-N \\ i\neq 0}}^{i=N} x_i \exp\left(j\frac{2\pi}{T}it\right), \qquad -\infty < t < \infty. \tag{191}$$

For positive i,

$$x_i = \tfrac{1}{2}(x_{ci} - jx_{si}), \qquad i \geq 1. \tag{192}$$

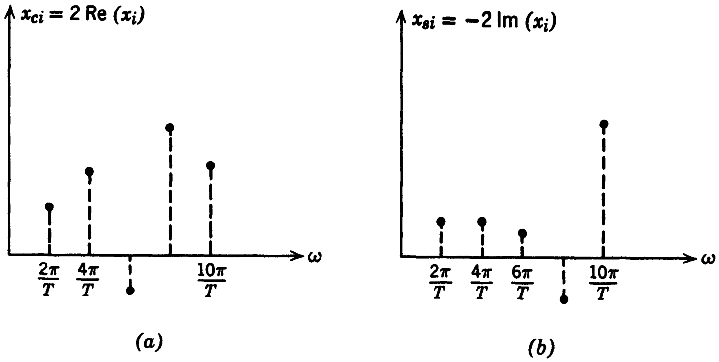

Fig. 3.19 Coefficients for a typical sample function.

The values for negative indices are conjugates of the values for positive indices:

$$x_i = x^*_{-i}, \tag{193}$$

$$E(x_i x_k^*) = \frac{p_i}{2}\,\delta_{ik}, \tag{194}$$

and

$$x_i = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\exp\left(-j\frac{2\pi}{T}it\right)dt. \tag{195}$$

We see that the coefficients are uncorrelated.

Just as in the finite interval case, every sample function is determined in the mean-square sense by its coefficients. We can conveniently catalog these coefficients as a function of ω. In Fig. 3.19 we show them for a *typical* sample function. In Fig. 3.20 we show the *statistical average* of the square of the coefficients (the variance).

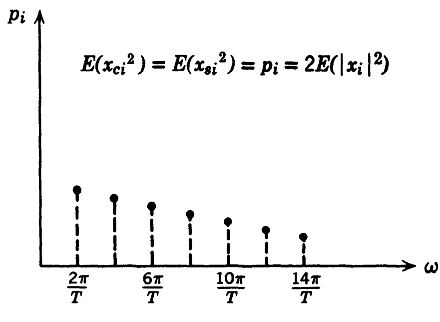

Fig. 3.20 Variance of coefficients: periodic process.

3.6 INFINITE TIME INTERVAL: SPECTRAL DECOMPOSITION

3.6.1 Spectral Decomposition

We now consider the effect of letting T, the period of the process, approach infinity.

Cosine-Sine Representation. Looking at Fig. 3.19, we see that the lines come closer together as T increases. In anticipation of this behavior, a more convenient sketch might be the cumulative amplitude plot shown in Fig. 3.21 for a *typical* sample function. The function $Z_c(\omega_n)$ is the sum of the cosine coefficients from 1 through $\omega_n(\triangleq n\omega_0)$.

$$Z_c(\omega_n) = \sum_{i=1}^{n} x_{ci}. \tag{196}$$

Similarly,

$$Z_s(\omega_n) = \sum_{i=1}^{n} x_{si}. \tag{197}$$

We see that because of the zero-mean assumption,

$$Z_c(0) = Z_s(0) = 0, \tag{198}$$

and

$$Z_c(\omega_n) = \sum_{i=1}^{n} x_{ci} = \sum_{i=1}^{n} \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(i\omega_0 t)\,dt. \tag{199}$$

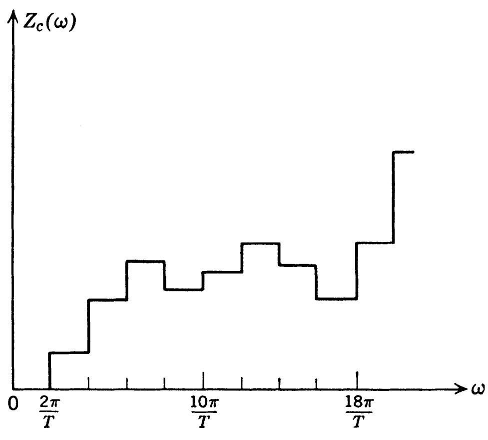

Fig. 3.21 Cumulative voltage function: periodic process.

Looking at (183), we see that we can write

$$x(t) = \underset{N \to \infty}{\text{l.i.m.}} \sum_{n=1}^{N} [Z_c(\omega_n) - Z_c(\omega_{n-1})] \cos \omega_n t$$
$$+ \underset{N \to \infty}{\text{l.i.m.}} \sum_{n=1}^{N} [Z_s(\omega_n) - Z_s(\omega_{n-1})] \sin \omega_n t. \tag{200}$$

From the way it is defined we see that

$$E\{[Z_c(\omega_n) - Z_c(\omega_{n-1})]^2\} = E\{[Z_s(\omega_n) - Z_s(\omega_{n-1})]^2\} = p_n. \tag{201}$$

We can indicate the cumulative mean power by the function $G_c(\omega_n)$, where

$$G_c(\omega_n) = \sum_{i=1}^{n} p_i = G_s(\omega_n), \qquad \omega_n \geq 0. \tag{202}$$

A typical function is shown in Fig. 3.22.

The covariance function can be expressed in terms of $G_c(\omega_n)$ by use of (186) and (202).

$$K_x(\tau) = \sum_{n=1}^{\infty} [G_c(\omega_n) - G_c(\omega_{n-1})] \cos \omega_n \tau. \tag{203}$$

Complex Exponential Representation. Alternately, in complex notation,

$$Z(\omega_n) - Z(\omega_m) = \sum_{i=m+1}^{n} x_i = \sum_{i=m+1}^{n} \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-j(i\omega_0)t}\, dt$$

and

$$x(t) = \underset{N \to \infty}{\text{l.i.m.}} \sum_{n=-N}^{N} [Z(\omega_n) - Z(\omega_{n-1})] e^{j\omega_n t}. \tag{204}$$

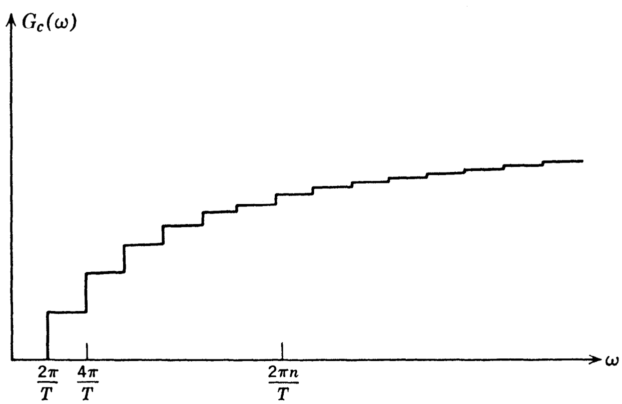

Fig. 3.22 Cumulative power spectrum: periodic process.

The mean-square value of the nth coefficient is,

$$E[|Z(\omega_n) - Z(\omega_{n-1})|^2] = \frac{p_n}{2}, \qquad n = -\infty, \ldots, -1, 1, \ldots, \infty, \tag{205}$$

The cumulative mean power is,

$$G(\omega_n) = \sum_{i=-\infty}^{n} \frac{p_i}{2}, \qquad \omega_n > -\infty, \tag{206}$$

and

$$K_x(\tau) = \sum_{n=-\infty}^{\infty} [G(\omega_n) - G(\omega_{n-1})]e^{j\omega_n\tau}. \tag{207}$$

Cosine-Sine Representation. We now return to the cosine-sine representation and look at the effect of letting $T \to \infty$. First reverse the order of summation and integration in (199). This gives

$$Z_c(\omega_n) = \int_{-T/2}^{T/2} 2x(t) \sum_{i=1}^{n} \cos(i\omega_0 t) \frac{1}{T}\, dt. \tag{208}$$

Now let

$$\frac{\Delta\omega}{2\pi} = \Delta f = \frac{1}{T},$$

and

$$\omega = n\omega_0 = \omega_n.$$

Holding $n\omega_0$ constant and letting $T \to \infty$, we obtain

$$Z_c(\omega) = \int_{-\infty}^{\infty} 2x(t)\, dt \int_0^{\omega} \cos \omega t \frac{d\omega}{2\pi} \tag{209}$$

or

$$Z_c(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2 \frac{\sin \omega t}{t} x(t)\, dt. \tag{210}$$

Similarly,

$$Z_s(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2 \frac{1 - \cos \omega t}{t} x(t)\, dt. \tag{211}$$

The sum representing $x(t)$ also becomes an integral,

$$x(t) \triangleq \int_0^{\infty} dZ_c(\omega) \cos \omega t + \int_0^{\infty} dZ_s(\omega) \sin \omega t. \tag{212}$$

We have written these integrals as Stieltjes integrals. They are defined as the limit of the sum in (200) as $T \to \infty$. It is worthwhile to note that we shall never be interested in evaluating a Stieltjes integral. Typical plots of $Z_c(\omega)$ and $Z_s(\omega)$ are shown in Fig. 3.23. They are zero-mean processes with the following useful properties:

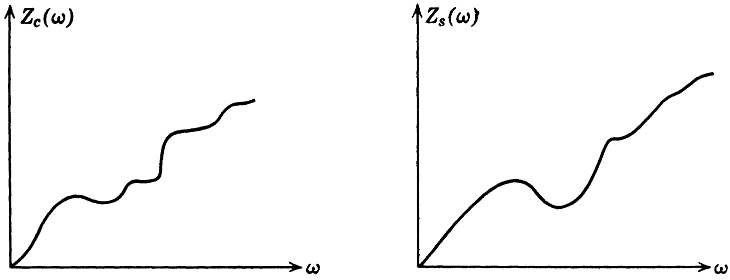

Fig. 3.23 Typical integrated voltage spectrum.

1. The increments in disjoint intervals are uncorrelated; that is,

$$E\{[Z_c(\omega_1) - Z_c(\omega_1 - \Delta\omega_1)][Z_c(\omega_2) - Z_c(\omega_2 - \Delta\omega_2)]\} = 0 \quad (213)$$

and

$$E\{[Z_s(\omega_1) - Z_s(\omega_1 - \Delta\omega_1)][Z_s(\omega_2) - Z_s(\omega_2 - \Delta\omega_2)]\} = 0 \quad (214)$$

if $(\omega_1 - \Delta\omega_1, \omega_1]$ and $(\omega_2 - \Delta\omega_2, \omega_2]$ are disjoint. This result is directly analogous to the coefficients in a series expansion being uncorrelated.

2. The quadrature components are uncorrelated even in the same interval; that is,

$$E\{[Z_c(\omega_1) - Z_c(\omega_1 - \Delta\omega_1)][Z_s(\omega_2) - Z_s(\omega_2 - \Delta\omega_2)]\} = 0 \quad (215)$$

for all ω_1 and ω_2.

3. The mean-square value of the increment variable has a simple physical interpretation,

$$E\{[Z_c(\omega_1) - Z_c(\omega_1 - \Delta\omega)]^2\} = G_c(\omega_1) - G_c(\omega_1 - \Delta\omega). \quad (216)$$

The quantity on the right represents the mean power contained in the frequency interval $(\omega_1 - \Delta\omega, \omega_1]$.

4. In many cases of interest the function $G_c(\omega)$ is differentiable.

$$G_c(\omega_2) - G_c(\omega_1) = \int_{\omega_1}^{\omega_2} 2S_x(\omega)\, \frac{d\omega}{2\pi}. \quad (217)$$

(The "2" inside the integral is present because $S_x(\omega)$ is a double-sided spectrum.)

$$\frac{dG_c(\omega)}{d\omega} \triangleq \frac{2S_x(\omega)}{2\pi}. \quad (218)$$

5. If $x(t)$ contains a periodic component of frequency ω_c, $G_c(\omega)$ will have a step discontinuity at ω_c and $S_x(\omega)$ will contain an impulse at ω_c.

The functions $Z_c(\omega)$ and $Z_s(\omega)$ are referred to as the *integrated Fourier transforms* of $x(t)$. The function $G_c(\omega)$ is the *integrated spectrum* of $x(t)$.

A logical question is: why did we use $Z_c(\omega)$ instead of the usual Fourier transform of $x(t)$?

The difficulty with the ordinary Fourier transform can be shown. We define

$$X_{c,T}(\omega) = \int_{-T/2}^{T/2} x(t) \cos \omega t \, dt \tag{219}$$

and examine the behavior as $T \rightarrow \infty$. Assuming $E[x(t)] = 0$,

$$E[X_{c,T}(\omega)] = 0, \tag{220}$$

and

$$E[|X_{c,T}(\omega)|^2] = \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} du \, R_x(t-u) \cos \omega t \cos \omega u. \tag{221}$$

It is easy to demonstrate that the right side of (221) will become arbitrarily large as $T \rightarrow \infty$. Thus, for every ω, the usual Fourier transform is a random variable with an unbounded variance.

Complex Exponential Representation. A similar result to that in (210) can be obtained for the complex representation.

$$Z(\omega_n) - Z(\omega_m) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{e^{-j\omega_n t} - e^{-j\omega_m t}}{-jt} \right) x(t) \, dt \tag{222}$$

and

$$x(t) = \int_{-\infty}^{\infty} dZ(\omega) e^{j\omega t}. \tag{223}$$

The expression in (222) has a simple physical interpretation. Consider the complex bandpass filter and its transfer function shown in Fig. 3.24. The impulse response is complex:

$$h_g(t) = \frac{1}{2\pi} \frac{e^{j\omega_n t} - e^{j\omega_m t}}{jt}. \tag{224}$$

The output at $t = 0$ is

$$y(0) = \int_{-\infty}^{\infty} \frac{1}{2\pi} \left(\frac{e^{-j\omega_n \tau} - e^{-j\omega_m \tau}}{-j\tau} \right) x(\tau) \, d\tau = Z(\omega_m) - Z(\omega_n). \tag{225}$$

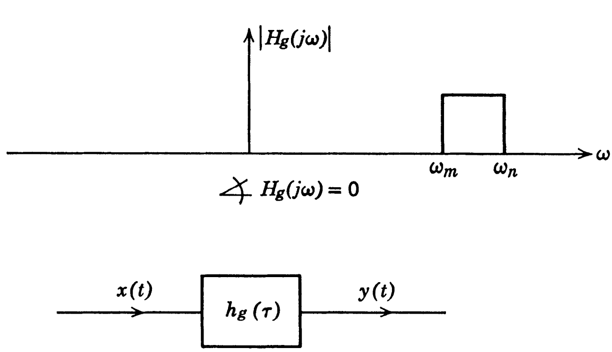

Fig. 3.24 A complex filter.

Thus the increment variables in the $Z(\omega)$ process correspond to the output of a complex linear filter when its input is $x(t)$.

The properties of interest are directly analogous to (213)–(218) and are listed below:

$$E[|Z(\omega) - Z(\omega - \Delta\omega)|^2] = G(\omega) - G(\omega - \Delta\omega). \tag{226}$$

If $G(\omega)$ is differentiable, then

$$\frac{dG(\omega)}{d\omega} \triangleq \frac{S_x(\omega)}{2\pi}. \tag{227}$$

A typical case is shown in Fig. 3.25.

$$E\{|Z(\omega) - Z(\omega - \Delta\omega)|^2\} = \frac{1}{2\pi}\int_{\omega-\Delta\omega}^{\omega} S_x(\omega)\, d\omega. \tag{228}$$

If $\omega_3 > \omega_2 > \omega_1$, then

$$E\{[Z(\omega_3) - Z(\omega_2)][Z^*(\omega_2) - Z^*(\omega_1)]\} = 0. \tag{229}$$

In other words, the increment variables are uncorrelated. These properties can be obtained as limiting relations from the exponential series or directly from (225) using the second-moment relations for a linear system.

Several observations will be useful in the sequel:

1. The quantity $dZ(\omega)$ plays exactly the same role as the Fourier transform of a finite energy signal.

For example, consider the linear system shown in Fig. 3.26. Now,

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t - \tau)\, d\tau \tag{230}$$

or

$$\int_{-\infty}^{\infty} dZ_y(\omega) e^{j\omega t} = \int_{-\infty}^{\infty} d\tau\, h(\tau) \int_{-\infty}^{\infty} dZ_x(\omega) e^{j\omega(t-\tau)} \tag{231}$$

$$= \int_{-\infty}^{\infty} H(j\omega)\, dZ_x(\omega) e^{j\omega t}. \tag{232}$$

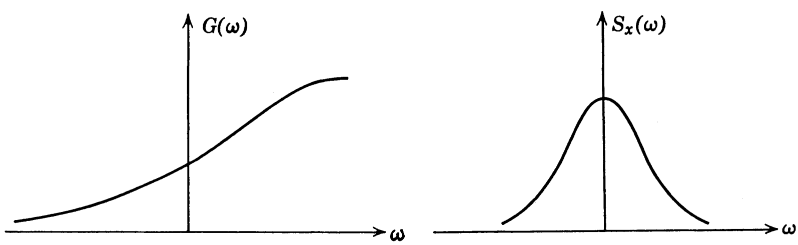

Fig. 3.25 An integrated power spectrum and a power spectrum.

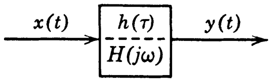

Fig. 3.26 Linear filter.

Thus

$$dZ_y(\omega) = H(j\omega)\, dZ_x(\omega) \tag{233}$$

and

$$S_y(\omega) = |H(j\omega)|^2 S_x(j\omega). \tag{234}$$

2. If the process is Gaussian, the random variables $[Z(\omega_1) - Z(\omega_1 - \Delta\omega)]$ and $[Z(\omega_2) - Z(\omega_2 - \Delta\omega)]$ are *statistically independent* whenever the intervals are disjoint.

We see that the spectral decomposition† of the process accomplishes the same result for *stationary processes* over the *infinite interval* that the Karhunen-Loève decomposition did for the finite interval. It provides us with a function $Z(\omega)$ associated with each sample function. Moreover, we can divide the ω axis into arbitrary nonoverlapping frequency intervals; the resulting increment random variables are uncorrelated (or statistically independent in the Gaussian case).

To illustrate the application of these notions we consider a simple estimation problem.

3.6.2 An Application of Spectral Decomposition: MAP Estimation of a Gaussian Process

Consider the simple system shown in Fig. 3.27:

$$r(t) = a(t) + n(t), \qquad -\infty < t < \infty. \tag{235}$$

We assume that $a(t)$ is a message that we want to estimate. In terms of the integrated transform,

$$Z_r(\omega) = Z_a(\omega) + Z_n(\omega), \qquad -\infty < \omega < \infty. \tag{236}$$

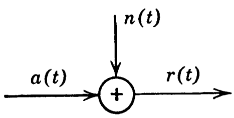

Fig. 3.27 System for estimation example.

† Further discussion of spectral decomposition is available in Gnedenko [27] or Bartlett ([28], Section 6-2).

Assume that $a(t)$ and $n(t)$ are sample functions from uncorrelated zero-mean Gaussian random processes with spectral densities $S_a(\omega)$ and $S_n(\omega)$, respectively. Because $Z_a(\omega)$ and $Z_n(\omega)$ are linear functionals of a Gaussian process, they also are Gaussian processes.

If we divide the frequency axis into a set of disjoint intervals, the increment variables will be independent. (See Fig. 3.28.) Now consider a particular interval $(\omega - d\omega, \omega]$ whose length is $d\omega$. Denote the increment variables for this interval as $dZ_r(\omega)$ and $dZ_n(\omega)$. Because of the statistical independence, we can estimate each increment variable, $dZ_a(\omega)$, separately, and because MAP and MMSE estimation commute over linear transformations it is equivalent to estimating $a(t)$.

The a posteriori probability of $dZ_a(\omega)$, given that $dZ_r(\omega)$ was received, is just

$$p_{dZ_a(\omega)|dZ_r(\omega)}[dZ_a(\omega)|dZ_r(\omega)]$$
$$= k \exp\left(-\frac{1}{2}\frac{|dZ_r(\omega) - dZ_a(\omega)|^2}{S_n(\omega)\, d\omega/2\pi} - \frac{1}{2}\frac{|dZ_a(\omega)|^2}{S_a(\omega)\, d\omega/2\pi}\right). \tag{237}$$

[This is simply (2-141) with $N = 2$ because $dZ_r(\omega)$ is complex.]

Because the a posteriori density is Gaussian, the MAP and MMSE estimates coincide. The solution is easily found by completing the square and recognizing the conditional mean. This gives

$$dZ_{\hat{a}}(\omega) = \widehat{dZ_a(\omega)} = \frac{S_a(\omega)}{S_a(\omega) + S_n(\omega)}\, dZ_r(\omega). \tag{238}$$

Therefore the minimum-mean square error estimate is obtained by passing $r(t)$ through a *linear filter*,

$$H_o(j\omega) = \frac{S_a(\omega)}{S_a(\omega) + S_n(\omega)}. \tag{239}$$

We see that the Gaussian assumption and MMSE criterion lead to a *linear* filter. In the model in Section 3.4.5 we required linearity but did not assume Gaussianness. Clearly, the two filters should be identical. To verify this, we take the limit of the finite time interval result. For the special case of white noise we can modify the result in (154) to take in account the complex eigenfunctions and the doubly infinite sum. The result is,

$$h_o(t, u) = \sum_{i=-\infty}^{\infty} \frac{\lambda_i}{\lambda_i + N_0/2}\, \phi_i(t)\, \phi_i^*(u). \tag{240}$$

Using (177) and (178), we have

$$\lim_{T\to\infty} h_o(t, u) = 2\int_0^{\infty} \frac{S_a(\omega)}{S_a(\omega) + N_0/2} \cos \omega(t - u)\, \frac{d\omega}{2\pi}, \tag{241}$$

which corresponds to (239).

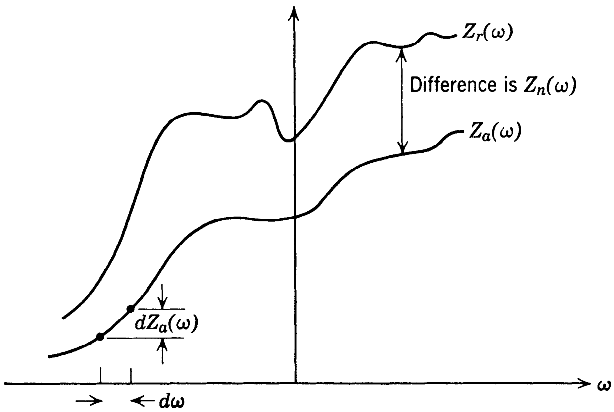

Fig. 3.28 Integrated transforms of $a(t)$ and $r(t)$.

In most of our developments we consider a finite time interval and use the orthogonal series expansion of Section 3.3. Then, to include the infinite interval-stationary process case we use the asymptotic results of Section 3.4.6. This leads us heuristically to the correct answer for infinite time. A rigorous approach for the infinite interval would require the use of the integrated transform technique we have just developed.

Before summarizing the results in this chapter, we discuss briefly how the results of Section 3.3 can be extended to vector random processes.

3.7 VECTOR RANDOM PROCESSES

In many cases of practical importance we are concerned with more than one random process at the same time; for example, in the phased arrays used in radar systems the input at each element must be considered. Analogous problems are present in sonar arrays and seismic arrays in which the received signal has a number of components. In telemetry systems a number of messages are sent simultaneously.

In all of these cases it is convenient to work with a single vector random process $\mathbf{x}(t)$ whose components are the processes of interest. If there are N processes, $x_1(t), x_2(t) \cdots x_N(t)$, we define $\mathbf{x}(t)$ as a column matrix,

$$\mathbf{x}(t) \triangleq \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix}. \tag{242}$$

The dimension N may be finite or countably infinite. Just as in the single process case, the second moment properties are described by the process means and covariance functions. In addition, the cross-covariance functions between the various processes must be known. The mean value function is a vector

$$\mathbf{m_x}(t) \triangleq E\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix} = \begin{bmatrix} m_1(t) \\ m_2(t) \\ \vdots \\ m_N(t) \end{bmatrix}, \tag{243}$$

and the covariances may be described by an $N \times N$ matrix, $\mathbf{K_x}(t, u)$, whose elements are

$$K_{ij}(t, u) \triangleq E\{[x_i(t) - m_i(t)][x_j(u) - m_j(u)]\}. \tag{244}$$

We want to derive a series expansion for the vector random process $\mathbf{x}(t)$. There are several possible representations, but two seem particularly efficient. In the first method we use a set of vector functions as coordinate functions and have scalar coefficients. In the second method we use a set of scalar functions as coordinate functions and have vector coefficients. For the first method and finite N, the modification of the properties on pp. 180–181 is straightforward. For infinite N we must be more careful. A detailed derivation that is valid for infinite N is given in [24]. In the text we go through some of the details for finite N. In Chapter II-5 we use the infinite N result without proof. For the second method, additional restrictions are needed. Once again we consider zero-mean processes.

Method 1. Vector Eigenfunctions, Scalar Eigenvalues. Let

$$\mathbf{x}(t) = \underset{N\to\infty}{\text{l.i.m.}} \sum_{i=1}^{N} x_i \boldsymbol{\phi}_i(t), \tag{245}$$

where

$$x_i = \int_0^T \boldsymbol{\phi}_i^T(t)\, \mathbf{x}(t)\, dt = \int_0^T \mathbf{x}^T(t)\, \boldsymbol{\phi}_i(t)\, dt = \sum_{k=1}^{N} \int_0^T x_k(t)\, \phi_i^k(t)\, dt, \tag{246}$$

and

$$\boldsymbol{\phi}_i(t) \triangleq \begin{bmatrix} \phi_i^1(t) \\ \phi_i^2(t) \\ \vdots \\ \phi_i^N(t) \end{bmatrix} \tag{247}$$

is chosen to satisfy

$$\lambda_i \boldsymbol{\phi}_i(t) = \int_0^T \mathbf{K_x}(t, u)\, \boldsymbol{\phi}_i(u)\, du, \qquad 0 \le t \le T. \tag{248}$$

Observe that the eigenfunctions are *vectors* but that the eigenvalues are still scalars.

Equation 248 can also be written as,

$$\sum_{j=1}^{N} \int_0^T K_{kj}(t, u)\, \phi_i^j(u)\, du = \lambda_i \phi_i^k(t), \qquad k = 1, \ldots, N, \qquad 0 \le t \le T. \quad (249)$$

The scalar properties carry over directly. In particular,

$$E(x_i x_j) = \lambda_i \delta_{ij}, \quad (250)$$

and the coordinate functions are orthonormal; that is,

$$\int_0^T \boldsymbol{\phi}_i^T(t)\, \boldsymbol{\phi}_j(t)\, dt = \delta_{ij}, \quad (251)$$

or

$$\sum_{k=1}^{N} \int_0^T \phi_i^k(t)\, \phi_j^k(t)\, dt = \delta_{ij}. \quad (252)$$

The matrix

$$\begin{aligned} \mathbf{K_x}(t, u) &= E[\mathbf{x}(t)\, \mathbf{x}^T(u)] - \mathbf{m_x}(t)\, \mathbf{m_x}^T(u) \\ &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mathrm{Cov}\,[x_i x_j]\, \boldsymbol{\phi}_i(t)\, \boldsymbol{\phi}_j^T(u) \\ &= \sum_{i=1}^{\infty} \lambda_i \boldsymbol{\phi}_i(t) \boldsymbol{\phi}_i^T(u) \end{aligned} \quad (253)$$

or

$$\mathbf{K}_{\mathbf{x},kj}(t, u) = \sum_{i=1}^{\infty} \lambda_i \phi_i^k(t)\, \phi_i^j(u), \qquad k, j = 1, \ldots, N. \quad (254)$$

This is the multidimensional analog of (50).

One property that makes the expansion useful is that the coefficient is a scalar variable and not a vector. This point is perhaps intuitively troublesome. A trivial example shows how it comes about.

Example. Let

$$\begin{aligned} x_1(t) &= a\, s_1(t), & 0 \le t \le T, \\ x_2(t) &= b\, s_2(t), & 0 \le t \le T, \end{aligned} \quad (255)$$

where a and b are independent, zero-mean random variables and $s_1(t)$ and $s_2(t)$ are orthonormal functions

$$\int_0^T s_i(t)\, s_j(t)\, dt = \delta_{ij}, \qquad i, j = 1, 2, \quad (256)$$

and

$$\begin{aligned} \mathrm{Var}\,(a) &= \sigma_a^2, \\ \mathrm{Var}\,(b) &= \sigma_b^2. \end{aligned} \quad (257)$$

Then

$$\mathbf{K_x}(t, u) = \begin{bmatrix} \sigma_a^2\, s_1(t)\, s_1(u) & 0 \\ 0 & \sigma_b^2\, s_2(t)\, s_2(u) \end{bmatrix}. \tag{258}$$

We can verify that there are two vector eigenfunctions:

$$\boldsymbol{\phi}_1(t) = \begin{bmatrix} s_1(t) \\ 0 \end{bmatrix}; \qquad \lambda_1 = \sigma_a^2, \tag{259}$$

and

$$\boldsymbol{\phi}_2(t) = \begin{bmatrix} 0 \\ s_2(t) \end{bmatrix}; \qquad \lambda_2 = \sigma_b^2. \tag{260}$$

Thus we see that in this degenerate case† we can achieve simplicity in the coefficients by increasing the number of vector eigenfunctions. Clearly, when there is an infinite number of eigenfunctions, this is unimportant.

A second method of representation is obtained by incorporating the complexity into the eigenvalues.

Method 2. Matrix Eigenvalues, Scalar Eigenfunctions

In this approach we let

$$\mathbf{x}(t) = \sum_{i=1}^{\infty} \mathbf{x}_i \psi_i(t), \qquad 0 \le t \le T, \tag{261}$$

and

$$\mathbf{x}_i = \int_0^T \mathbf{x}(t)\, \psi_i(t)\, dt. \tag{262}$$

We would like to find a set of $\boldsymbol{\Lambda}_i$ and $\psi_i(t)$ such that

$$E[\mathbf{x}_i \mathbf{x}_j^T] = \boldsymbol{\Lambda}_i\, \delta_{ij} \tag{263}$$

and

$$\int_0^T \psi_i(t) \psi_j(t)\, dt = \delta_{ij}. \tag{264}$$

These requirements lead to the equation

$$\boldsymbol{\Lambda}_i\, \psi_i(t) = \int_0^T \mathbf{K_x}(t, u) \psi_i(u)\, du, \qquad 0 \le t \le T. \tag{265}$$

For arbitrary time intervals (265) does *not* have a solution except for a few trivial cases. However, if we restrict our attention to stationary processes and *large* time intervals then certain asymptotic results may be obtained. Defining

$$\mathbf{S_x}(\omega) \triangleq \int_{-\infty}^{\infty} \mathbf{K_x}(\tau) e^{j\omega\tau}\, d\tau, \tag{266}$$

† It is important not to be misled by this degenerate example. The useful application is in the case of correlated processes. Here the algebra of calculating the actual eigenfunctions is tedious but the representation is still simple.

and assuming the interval is large, we find

$$\psi_i(t) \simeq \frac{1}{\sqrt{T}}\, e^{j\omega_i t} \tag{267}$$

and

$$\mathbf{\Lambda}_i \simeq \mathbf{S}_{\mathbf{x}}(\omega_i). \tag{268}$$

As before, to treat the infinite time case rigorously we must use the integrated transform

$$\mathbf{Z}_{\mathbf{x}}(\omega_n) - \mathbf{Z}_{\mathbf{x}}(\omega_m) \triangleq \int_{-\infty}^{\infty} \frac{e^{-j\omega_n t} - e^{-j\omega_m t}}{-jt}\, \mathbf{x}(t)\, dt \tag{269}$$

and

$$\mathbf{x}(t) = \int_{-\infty}^{\infty} d\mathbf{Z}_x(\omega) e^{j\omega t}. \tag{270}$$

The second method of representation has a great deal of intuitive appeal in the large time interval case where it is valid, but the first method enables us to treat a more general class of problems. For this reason we shall utilize the first representation in the text and relegate the second to the problems.

It is difficult to appreciate the importance of the first expansion until we get to some applications. We shall then find that it enables us to obtain results for multidimensional problems almost by inspection. The key to the simplicity is that we can still deal with *scalar* statistically independent random variables.

It is worthwhile to re-emphasize that we did not *prove* that the expansions had the desired properties. Specifically, we did not demonstrate that solutions to (248) existed and had the desired properties, that the multidimensional analog for Mercer's theorem was valid, or that the expansion converged in the mean-square sense ([24] does this for the first expansion).

3.8 SUMMARY

In this chapter we developed means of characterizing random processes. The emphasis was on a method of representation that was particularly well suited to solving detection and estimation problems in which the random processes were Gaussian. For non-Gaussian processes the representation provides an adequate second-moment characterization but may not be particularly useful as a complete characterization method.

For finite time intervals the desired representation was a series of orthonormal functions whose coefficients were uncorrelated random variables.

The choice of coordinate functions depended on the covariance function of the process through the integral equation

$$\lambda\,\phi(t) = \int_{T_i}^{T_f} K(t, u)\,\phi(u)\,du, \qquad T_i \le t \le T_f. \tag{271}$$

The eigenvalues λ corresponded physically to the expected value of the energy along a particular coordinate function $\phi(t)$. We indicated that this representation was useful for both theoretical and practical purposes. Several classes of processes for which solutions to (271) could be obtained were discussed in detail. One example, the simple Wiener process, led us logically to the idea of a white noise process. As we proceed, we shall find that this process is a useful tool in many of our studies.

To illustrate a possible application of the expansion techniques we solved the optimum linear filtering problem for a finite interval. The optimum filter for the additive white noise case was the solution to the integral equation

$$\frac{N_0}{2} h_o(t, u) + \int_{T_i}^{T_f} K_a(t, z)\,h_o(z, u)\,dz = K_a(t, u), \qquad T_i \le t, u \le T_f. \tag{272}$$

The solution could be expressed in terms of the eigenfunctions and eigenvalues.

For large time intervals we found that the eigenvalues of a stationary process approached the power spectrum of the process and the eigenfunctions became sinusoids. Thus for this class of problem the expansion could be interpreted in terms of familiar quantities.

For infinite time intervals and stationary processes the eigenvalues were not countable and no longer served a useful purpose. In this case, by starting with a periodic process and letting the period go to infinity, we developed a useful representation. Instead of a series representation for each sample function, there was an integral representation,

$$x(t) = \int_{-\infty}^{\infty} dZ_x(\omega) e^{j\omega t}. \tag{273}$$

The function $Z_x(\omega)$ was the *integrated transform* of $x(t)$. It is a sample function of a random process with *uncorrelated increments* in frequency. For the Gaussian process the increments were statistically independent. Thus the increment variables for the infinite interval played exactly the same role as the series coefficients in the finite interval. A simple example showed one of the possible applications of this property.

Finally, we extended these ideas to vector random processes. The significant result here was the ability to describe the process in terms of *scalar* coefficients.

In Chapter 4 we apply these representation techniques to solve the detection and estimation problem.

3.9 PROBLEMS

Many of the problems in Section P3.3 are of a review nature and may be omitted by the reader with an adequate random process background. Problems 3.3.19–23 present an approach to the continuous problem which is different from that in the text.

Section P3.3 Random Process Characterizations

SECOND MOMENT CHARACTERIZATIONS

Problem 3.3.1. In chapter 1 we formulated the problem of choosing a linear filter to maximize the output signal-to-noise ratio.

$$\left(\frac{S}{N}\right)_o \triangleq \frac{\left[\int_0^T h(T-\tau)\, s(\tau)\, d\tau\right]^2}{N_0/2 \int_0^T h^2(\tau)\, d\tau}.$$

1. Use the Schwarz inequality to find the $h(\tau)$ which maximizes $(S/N)_o$.
2. Sketch $h(\tau)$ for some typical $s(t)$.

Comment. The resulting filter is called a matched filter and was first derived by North [34].

Problem 3.3.2. Verify the result in (3-26).

Problem 3.3.3. [1]. The input to a stable linear system with a transfer function $H(j\omega)$ is a zero-mean process $x(t)$ whose correlation function is

$$R_x(\tau) = \frac{N_0}{2}\, \delta(\tau).$$

1. Find an expression for the variance of the output $y(t)$.
2. The noise bandwidth of a network is defined as

$$B_N \triangleq \frac{\int_{-\infty}^{\infty} |H(j\omega)|^2\, d\omega/2\pi}{|H_{\max}|^2}, \qquad \text{(double-sided in cps).}$$

Verify that

$$\sigma_y^2 = \frac{N_0 B_N |H_{\max}|^2}{2}.$$

Problem 3.3.4. [1]. Consider the fixed-parameter linear system defined by the equation

$$v(t) = x(t-\delta) - x(t)$$

and

$$y(t) = \int_{-\infty}^{t} v(u)\, du.$$

1. Determine the impulse response relating the input $x(t)$ and output $y(t)$.
2. Determine the system function.
3. Determine whether the system is stable.
4. Find B_N.

Problem 3.3.5. [1]. The transfer function of an RC network is

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{1}{1 + RCj\omega}.$$

The input consists of a noise which is a sample function of a stationary random process with a flat spectral density of height $N_0/2$, plus a signal which is a sequence of constant-amplitude rectangular pulses. The pulse duration is δ and the minimum interval between pulses is T, where $\delta \ll T$.

A signal-to-noise ratio at the system output is defined here as the ratio of the maximum amplitude of the output signal with no noise at the input to the rms value of the output noise.

1. Derive an expression relating the output signal-to-noise ratio as defined above to the input pulse duration and the effective noise bandwidth of the network.
2. Determine what relation should exist between the input pulse duration and the effective noise bandwidth of the network to obtain the maximum output signal-to-noise.

ALTERNATE REPRESENTATIONS AND NON-GAUSSIAN PROCESSES

Problem 3.3.6. (sampling representation). When the observation interval is infinite and the processes of concern are bandlimited, it is sometimes convenient to use a sampled representation of the process. Consider the stationary process $x(t)$ with the spectrum shown in Fig. P3.1. Assume that $x(t)$ is sampled every $1/2W$ seconds. Denote the samples as $x(i/2W)$, $i = -\infty, \ldots, 0, \ldots$.

1. Prove

$$x(t) = \operatorname*{l.i.m.}_{K \to \infty} \sum_{-K}^{K} x\left(\frac{i}{2W}\right) \frac{\sin 2\pi W(t - i/2W)}{2\pi W(t - i/2W)}.$$

2. Find $E[x(i/2W)x(j/2W)]$.

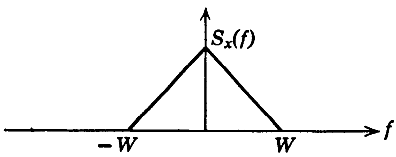

Fig. P3.1

Problem 3.3.7 (continuation). Let

$$\phi_i(t) = \sqrt{2W}\,\frac{\sin 2\pi W(t - i/2W)}{2\pi W(t - i/2W)}, \qquad -\infty < t < \infty.$$

Define

$$x(t) = \operatorname*{l.i.m.}_{K \to \infty} \sum_{i=-K}^{K} x_i\, \phi_i(t).$$

Prove that if

$$E(x_i x_j) = P\delta_{ij} \qquad \text{for all } i, j,$$

then

$$S_x(f) = \frac{P}{2W}, \qquad |f| \le W.$$

Problem 3.3.8. Let $x(t)$ be a bandpass process "centered" around f_c.

$$S_x(f) = 0, \quad \begin{matrix} |f - f_c| > W, & f > 0, \\ |f + f_c| > W, & f < 0. \end{matrix}$$

We want to represent $x(t)$ in terms of two low-pass processes $x_c(t)$ and $x_s(t)$. Define

$$\hat{x}(t) = \sqrt{2}\, x_c(t) \cos(2\pi f_c t) + \sqrt{2}\, x_s(t) \sin(2\pi f_c t),$$

where $x_c(t)$ and $x_s(t)$ are obtained physically as shown in Fig. P3.2.

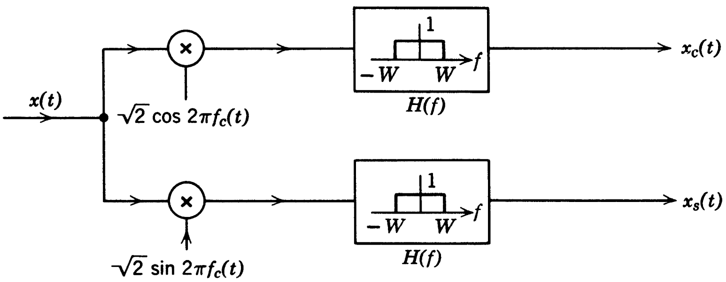

Fig. P3.2

1. Prove

$$E\{[x(t) - \hat{x}(t)]^2\} = 0.$$

2. Find $S_{x_c}(f)$, $S_{x_s}(f)$, and $S_{x_c x_s}(f)$.
3. What is a necessary and sufficient condition for $S_{x_c x_s}(f) = 0$?

Observe that this enables us to replace any bandpass process by *two* low-pass processes or a vector low-pass process.

$$\mathbf{x}(t) = \begin{bmatrix} x_c(t) \\ x_s(t) \end{bmatrix}$$

Problem 3.3.9. Show that the n-dimensional probability density of a Markov process can be expressed as

$$p_{x_{t_1} \cdots x_{t_n}}(X_{t_1}, \cdots, X_{t_n}) = \frac{\prod_{k=2}^{n} p_{x_{t_{k-1}} x_{t_k}}(X_{t_{k-1}}, X_{t_k})}{\prod_{k=2}^{n-1} p_{x_{t_k}}(X_{t_k})}, \qquad n \geq 3.$$

Problem 3.3.10. Consider a Markov process at three ordered time instants, $t_1 < t_2 < t_3$. Show that the conditional density relating the first and third time must satisfy the following equation:

$$p_{x_{t_3}|x_{t_1}}(X_{t_3}|X_{t_1}) = \int dX_{t_2} p_{x_{t_3}|x_{t_2}}(X_{t_3}|X_{t_2}) p_{x_{t_2}|x_{t_1}}(X_{t_2}|X_{t_1}).$$

Problem 3.3.11. A continuous-parameter random process is said to have independent increments if, for all choices of indices $t_0 < t_1 \cdots < t_n$, the n random variables

$$x(t_1) - x(t_0), \ldots, x(t_n) - x(t_{n-1})$$

are independent. Assuming that $x(t_0) = 0$, show that

$$M_{x_{t_1}x_{t_2}\cdots x_{t_n}}(jv_1, \ldots, jv_n) = M_{x_{t_1}}(jv_1 + jv_2 + \cdots + jv_n) \prod_{k=0}^{n} M_{x_{t_k} - x_{t_{k-1}}}(jv_k + \cdots + jv_n).$$

GAUSSIAN PROCESSES

Problem 3.3.12. ***(Factoring of higher order moments).*** Let $x(t)$, $t \in T$ be a Gaussian process with zero mean value function

$$E[x(t)] = 0.$$

1. Show that all odd-order moments of $x(t)$ vanish and that the even-order moments may be expressed in terms of the second-order moments by the following formula:

Let n be an even integer and let $t_1, \ldots, t_n$ be points in T, some of which may coincide. Then

$$E[x(t_1)\cdots x(t_n)] = \sum E[x(t_{i_1})\, x(t_{i_2})]\, E[x(t_{i_3})\, x(t_{i_4})] \cdots E[x(t_{i_{n-1}})\, x(t_{i_n})],$$

in which the sum is taken over all possible ways of dividing the n points into $n/2$ combinations of pairs. The number of terms in the sum is equal to

$$1 \cdot 3 \cdot 5 \cdots (n-3)(n-1);$$

for example,

$$E[x(t_1)\, x(t_2)\, x(t_3)\, x(t_4)] = E[x(t_1)\, x(t_2)]E[x(t_3)\, x(t_4)] + E[x(t_1)\, x(t_3)]E[x(t_2)\, x(t_4)] + E[x(t_1)\, x(t_4)]E[x(t_2)\, x(t_3)].$$

Hint. Differentiate the characteristic function.

2. Use your result to find the fourth-order correlation function

$$R_x(t_1, t_2, t_3, t_4) = E[x(t_1)\, x(t_2)\, x(t_3)\, x(t_4)]$$

of a stationary Gaussian process whose spectral density is

$$S_x(f) = \frac{N_0}{2}, \qquad |f| \le W,$$
$$= 0, \qquad \text{elsewhere.}$$

What is $\lim_{W \to \infty} R_x(t_1, t_2, t_3, t_4)$?

Problem 3.3.13. Let $x(t)$ be a sample function of a stationary real Gaussian random process with a zero mean and finite mean-square value. Let a new random process be defined with the sample functions

$$y(t) = x^2(t).$$

Show that

$$R_y(\tau) = R_x^2(0) + 2R_x^2(\tau).$$

Problem 3.3.14. [1]. Consider the system shown in Fig. P3.3. Let the input $e_0(t)$ be a sample function of stationary real Gaussian process with zero mean and *flat* spectral density at all frequencies of interest; that is, we may assume that

$$S_{e_0}(f) = N_0/2$$

1. Determine the autocorrelation function or the spectral density of $e_2(t)$.
2. Sketch the autocorrelation function or the spectral density of $e_0(t)$, $e_1(t)$, and $e_2(t)$.

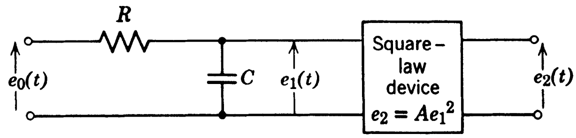

Fig. P3.3

Problem 3.3.15. The system of interest is shown in Fig. P3.4, in which $x(t)$ is a sample function from an ergodic Gaussian random process.

$$R_x(\tau) = \frac{N_0}{2}\,\delta(\tau).$$

The transfer function of the linear system is

$$\begin{aligned} H(f) &= e^{2jf} \qquad & |f| \le W, \\ &= 0, \qquad & \text{elsewhere.} \end{aligned}$$

1. Find the dc power in $z(t)$.
2. Find the ac power in $z(t)$.

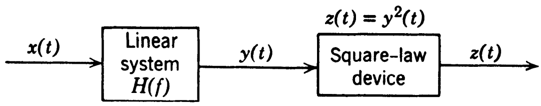

Fig. P3.4

Problem 3.3.16. The output of a linear system is $y(t)$, where

$$y(t) = \int_0^\infty h(\tau)\, x(t - \tau)\, d\tau.$$

The input $x(t)$ is a sample function from a *stationary* Gaussian process with correlation function

$$R_x(\tau) = \delta(\tau).$$

We should like the output at a particular time t_1 to be statistically independent of the input at that time. Find a necessary and sufficient condition on $h(\tau)$ for $x(t_1)$ and $y(t_1)$ to be statistically independent.

Problem 3.3.17. Let $x(t)$ be a real, wide-sense stationary, Gaussian random process with zero mean. The process $x(t)$ is passed through an ideal limiter. The output of the limiter is the process $y(t)$,

$$y(t) = L[x(t)],$$

where

$$L(u) = \begin{cases} +1 & u \ge 0, \\ -1 & u < 0. \end{cases}$$

Show that the autocorrelation functions of the two processes are related by the formula

$$R_y(\tau) = \frac{2}{\pi}\sin^{-1}\left[\frac{R_x(\tau)}{R_x(0)}\right].$$

Problem 3.3.18. Consider the bandlimited Gaussian process whose spectrum is shown in Fig. P3.5.

Write

$$x(t) = v(t)\cos\,[2\pi f_c t + \theta(t)].$$

Find

$$p_{v(t)}(V) \quad \text{and} \quad p_{\theta(t)}(\theta).$$

Are $v(t)$ and $\theta(t)$ independent random variables?

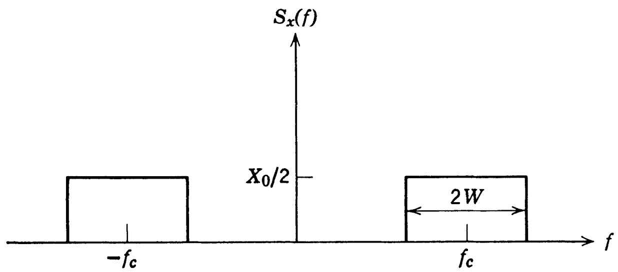

Fig. P3.5

SAMPLING APPROACH TO CONTINUOUS GAUSSIAN PROCESSES†

In Chapters 4, 5, and II-3 we extend the classical results to the waveform case by using the Karhunen-Loève expansion. If, however, we are willing to use a heuristic argument, most of the results that we obtained in Section 2.6 for the general Gaussian problem can be extended easily to the waveform case in the following manner.

The processes and signals are sampled every ϵ seconds as shown in Fig. P3.6*a*. The gain in the sampling device is chosen so that

$$\int_0^T m^2(t)\,dt = \lim_{\epsilon\to 0}\sum_{i=1}^{T/\epsilon} m_i^2. \tag{1}$$

This requires

$$m_i = \sqrt{\epsilon}\, m(t_i). \tag{2}$$

Similarly, for a random process,

$$n_i = \sqrt{\epsilon}\, n(t_i) \tag{3}$$

and

$$E[n_i n_j] = \epsilon E[n(t_i)\, n(t_j)] = \epsilon K_n(t_i, t_j). \tag{4}$$

To illustrate the procedure consider the simple model shown in Fig. P3.6*b*. The continuous waveforms are

$$\begin{aligned} r(t) &= m(t) + n(t), \qquad 0 \le t \le T : H_1,\\ r(t) &= n(t), \qquad 0 \le t \le T : H_0, \end{aligned} \tag{5}$$

where $m(t)$ is a known function and $n(t)$ is a sample function from a Gaussian random process.

The corresponding sampled problem is

$$\begin{aligned} \mathbf{r} &= \mathbf{m} + \mathbf{n} : H_1,\\ \mathbf{r} &= \mathbf{n} : H_0, \end{aligned} \tag{6}$$

† We introduce the sampling approach here because of its widespread use in the literature and the feeling of some instructors that it is easier to understand. The results of these five problems are derived and discussed thoroughly later in the text.

where

$$\mathbf{r} \triangleq \sqrt{\epsilon}\begin{bmatrix} r(t_1) \\ r(t_2) \\ \vdots \\ r(t_N) \end{bmatrix}$$

and $N = T/\epsilon$. Assuming that the noise $n(t)$ is band-limited to $1/\epsilon$ (double-sided, cps) and has a flat spectrum of $N_0/2$, the samples are statistically independent Gaussian variables (Problem 3.3.7).

$$E[\mathbf{n}\mathbf{n}^T] = E[n^2(t)]\mathbf{I} = \frac{N_0}{2}\mathbf{I} \triangleq \sigma^2\mathbf{I}. \quad (7)$$

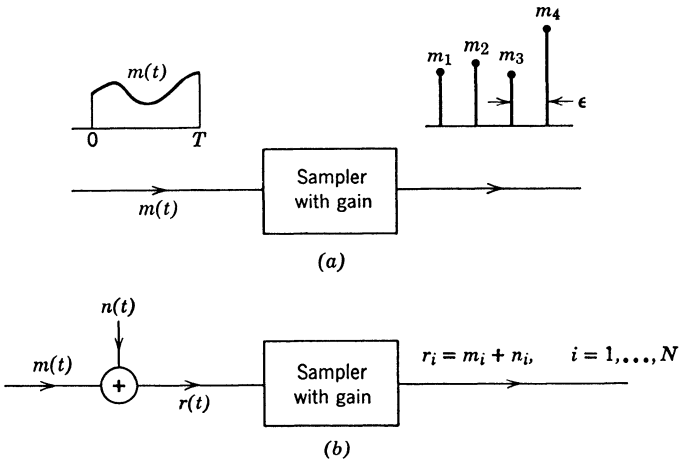

Fig. P3.6

The vector problem in (6) is familiar from Section 2.6 of Chapter 2. From (2.350) the sufficient statistic is

$$l(\mathbf{R}) = \frac{1}{\sigma^2}\sum_{i=1}^{N} m_i R_i. \quad (8)$$

Using (2) and (3) in (8), we have

$$l(\mathbf{R}) = \frac{2}{N_0}\sum_{i=1}^{T/\epsilon} \sqrt{\epsilon}\, m(t_i)\cdot\sqrt{\epsilon}\, r(t_i).$$

As $\epsilon \to 0$, we have (letting $dt = \epsilon$)

$$\underset{\epsilon\to 0}{\text{l.i.m.}}\ l(\mathbf{R}) = \frac{2}{N_0}\int_0^T m(t)\, r(t)\, dt \triangleq l(r(t))$$

which is the desired result. Some typical problems of interest are developed below.

Problem 3.3.19. Consider the simple example described in the introduction.

1. Show that

$$d^2 = \frac{2E}{N_0},$$

where E is the energy in $m(t)$.

2. Draw a block diagram of the receiver and compare it with the result in Problem 3.3.1.

Problem 3.3.20. Consider the discrete case defined by (2.328). Here

$$E[\mathbf{n}\mathbf{n}^T] = \mathbf{K}$$

and

$$\mathbf{Q} = \mathbf{K}^{-1}.$$

1. Sample the bandlimited noise process $n(t)$ every ϵ seconds to obtain $n(t_1)$, $n(t_2), \ldots, n(t_k)$. Verify that in the limit $\mathbf{Q}$ becomes a function with two arguments defined by the equation

$$\int_0^T Q(t, u)\, K(u, z)\, du = \delta(t - z).$$

Hint: Define a function $Q(t_i, t_j) = (1/\epsilon)Q_{ij}$.

2. Use this result to show that

$$l = \iint_0^T m_\Delta(t)\, Q(t, u)\, r(u)\, dt\, du$$

in the limit.

3. What is d^2?

Problem 3.3.21. In the example defined in (2.387) the means are equal but the covariance matrices are different. Consider the continuous waveform analog to this and show that

$$l = \iint_0^T r(t)\, h_\Delta(t, u)\, r(u)\, dt\, du,$$

where

$$h_\Delta(t, u) = Q_0(t, u) - Q_1(t, u).$$

Problem 3.3.22. In the linear estimation problem defined in Problem 2.6.8 the received vector was

$$\mathbf{r} = \mathbf{a} + \mathbf{n}$$

and the MAP estimate was

$$\mathbf{K}_{\mathbf{a}}^{-1}\hat{\mathbf{a}} = \mathbf{K}_{\mathbf{r}}\mathbf{R}.$$

Verify that the continuous analog to this result is

$$\int_0^T K_a^{-1}(t, u)\, \hat{a}(u)\, du = \int_0^T K_r^{-1}(t, u)\, r(u)\, du, \qquad 0 \le t \le T.$$

Problem 3.3.23. Let

$$r(t) = a(t) + n(t), \qquad 0 \le t \le T,$$

where $a(t)$ and $n(t)$ are independent zero-mean Gaussian processes with covariance functions $K_a(t, u)$ and $K_n(t, u)$, respectively. Consider a specific time t_1 in the interval. Find

$$p_{a(t_1)|r(t), 0 \le t \le T}\,[A_{t_1}|r(t),\ 0 \le t \le T].$$

Hint. Sample $r(t)$ every ϵ seconds and then let $\epsilon \to 0$.

Section P3.4 Integral equations

Problem 3.4.1. Consider the integral equation

$$\int_{-T}^{T} du\, P \exp(-\alpha|t - u|)\phi_i(u) = \lambda_i\phi_i(t), \qquad -T \le t \le T.$$

1. Prove that $\lambda = 0$ and $\lambda = 2P/\alpha$ are not eigenvalues.
2. Prove that all values of $\lambda > 2P/\alpha$ cannot be eigenvalues of the above integral equation.

Problem 3.4.2. Plot the behavior of the largest eigenvalue of the integral equation in Problem 3.4.1 as a function of αT.

Problem 3.4.3. Consider the integral equation (114).

$$\lambda\, \phi(t) = \sigma^2 \int_0^t u\, \phi(u)\, du + \sigma^2 t \int_t^T \phi(u)\, du, \qquad 0 \le t \le T.$$

Prove that values of $\lambda \le 0$ are not eigenvalues of the equation.

Problem 3.4.4. Prove that the largest eigenvalue of the integral equation

$$\lambda\phi(t) = \int_{-T}^{T} K_n(t, u)\, \phi(u)\, du, \qquad -T \le t \le T,$$

satisfies the inequality.

$$\lambda_1 \ge \iint_{-T}^{T} f(t)\, K_n(t, u)\, f(u)\, dt\, du,$$

where $f(t)$ is *any* function with unit energy in $[-T, T]$.

Problem 3.4.5. Compare the bound in Problem 3.4.4, using the function

$$f(t) = \frac{1}{\sqrt{2T}}, \qquad -T \le t \le T,$$

with the actual value found in Problem 3.4.2.

Problem 3.4.6. [15]. Consider a function whose total energy in the interval $-\infty < t < \infty$ is E.

$$E = \int_{-\infty}^{\infty} |f(t)|^2\, dt.$$

Now, time-limit $f(t)$, $-T/2 \le t \le T/2$ and then band-limit the result to $(-W, W)$ cps. Call this resulting function $f_{DB}(t)$. Denote the energy in $f_{DB}(t)$ as E_{DB}.

$$E_{DB} = \int_{-\infty}^{\infty} |f_{DB}(t)|^2\, dt.$$

1. Choose $f(t)$ to maximize

$$\gamma \triangleq \frac{E_{DB}}{E}.$$

2. What is the resulting value of γ when $WT = 2.55$?

Problem 3.4.7. [15]. Assume that $f(t)$ is first band-limited.

$$f_B(t) = \int_{-2\pi W}^{2\pi W} F(\omega)\, e^{j\omega t}\, \frac{d\omega}{2\pi}.$$

Now, time-limit $f_B(t)$, $-T/2 \le t \le T/2$ and band-limit the result to $(-W, W)$ to obtain $f_{BDB}(t)$. Repeat Problem 3.4.6 with BDB replacing DB.

Problem 3.4.8 [35]. Consider the triangular correlation function

$$\begin{aligned} K_n(t-u) &= 1 - |t-u|, && |t-u| \le 1, \\ &= 0, && \text{elsewhere}. \end{aligned}$$

Find the eigenfunctions and eigenvalues over the interval $(0, T)$ when $T < 1$.

Problem 3.4.9. Consider the integral equation

$$\lambda\, \phi(t) = \int_{T_i}^{T_f} K_n(t, u)\, \phi(u)\, du, \qquad T_i \le t \le T_f,$$

where

$$K_n(t, u) = \sum_{i=1}^{6} \sigma_i^2 \cos\left(\frac{2\pi i t}{T}\right) \cos\left(\frac{2\pi i u}{T}\right),$$

and

$$T \triangleq T_f - T_i.$$

Find the eigenfunctions and eigenvalues of this equation.

Problem 3.4.10. The input to an unrealizable linear time-invariant system is $x(t)$ and the output is $y(t)$. Thus, we can write

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t - \tau)\, d\tau.$$

We assume that

(i)
$$\int_{-\infty}^{\infty} x^2(t)\, dt = 1.$$

(ii)
$$h(\tau) = \frac{1}{\sqrt{2\pi}\,\sigma_c} \exp\left(-\frac{\tau^2}{2\sigma_c^2}\right), \qquad -\infty < \tau < \infty.$$

(iii)
$$E_y \triangleq \int_{-\infty}^{\infty} y^2(t)\, dt.$$

1. What is the maximum value of E_y that can be obtained by using an $x(t)$ that satisfies the above constraints?
2. Find an $x(t)$ that gives an E_y arbitrarily close to the maximum E_y.
3. Generalize your answers to (1) and (2) to include an arbitrary $H(j\omega)$.

Problem 3.4.11. All of our basic derivations assumed that the coefficients and the coordinate functions in the series expansions of signals and processes were real. In this problem we want to derive the analogous relations for complex coefficients and coordinate functions. We still assume that the signals and processes are real.

Derive the analogous results to those obtained in (12), (15), (18), (20), (40), (44), (46), (50), (128), and (154).

Problem 3.4.12. In (180) we considered a function

$$g_\lambda \triangleq \sum_{i=1}^{\infty} g(\lambda_i)$$

and derived its asymptotic value (181). Now consider the finite energy signal $s(t)$ and define

$$s_i \triangleq \int_{-T/2}^{T/2} s(t)\, \phi_i(t)\, dt,$$

where the $\phi_i(t)$ are the same eigenfunctions used to define (180). The function of interest is

$$g'_\lambda \triangleq \sum_{i=1}^{\infty} s_i^2\, g(\lambda_i).$$

Show that

$$g'_\lambda \simeq \int_{-\infty}^{\infty} |S(j\omega)|^2\, g(S_x(\omega))\, \frac{d\omega}{2\pi}$$

for large T, where the function $S(j\omega)$ is the Fourier transform of $s(t)$.

Section P3.5 Periodic Processes

Problem 3.5.1. Prove that if $R_x(\tau)$ is periodic then almost every sample function is periodic.

Problem 3.5.2. Show that if the autocorrelation $R_x(\tau)$ of a random process is such that

$$R_x(\tau_1) = R_x(0), \qquad \text{for some } \tau_1 \neq 0,$$

then $R_x(\tau)$ is periodic.

Problem 3.5.3. Consider the random process

$$x(t) = \sum_{n=1}^{N} a_n \cos(n\omega t + \theta_n).$$

The a_n $(n = 1, 2, \ldots, N)$ are independent random variables:

$$E(a_1) = E(a_2) = \cdots E(a_N) = 0.$$

Var (a_1), Var $(a_2), \ldots,$Var (a_n) are different. The θ_n $(n = 1, 2, \ldots, N)$ are identically distributed, independent random variables.

$$p_{\theta_n}(\theta) = \frac{1}{2\pi}, \qquad 0 \leq \theta \leq 2\pi.$$

The θ_n and a_n are independent.

1. Find $R_x(t_1, t_2)$.
2. Is the process wide-sense stationary?
3. Can you make any statements regarding the structure of particular sample functions?

Section P3.6 Integrated Transforms

Problem 3.6.1. Consider the feedback system in Fig. P3.7. The random processes $a(t)$ and $n(t)$ are statistically independent and stationary. The spectra $S_a(\omega)$ and $S_n(\omega)$ are known.

1. Find an expression for $Z_x(\omega)$, the integrated Fourier transform of $x(t)$.
2. Express $S_x(\omega)$ in terms of $S_a(\omega)$, $S_n(\omega)$, $G_1(j\omega)$, and $G_2(j\omega)$.

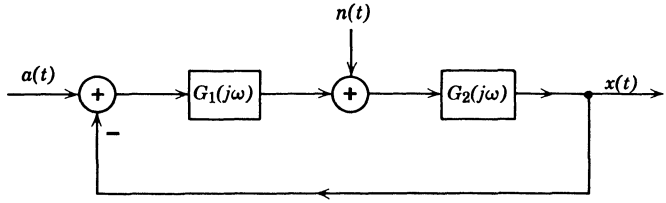

Fig. P3.7

Problem 3.6.2. From (221)

$$E[|X_{c,T}(\omega)|^2] = \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} du\, R_x(t - u) \cos \omega t \cos \omega u.$$

Prove that the right side becomes arbitrarily large as $T \to \infty$.

Section P3.7 Vector Random Processes

Problem 3.7.1. Consider the spectral matrix

$$\mathbf{S}(\omega) = \begin{bmatrix} 2 & \dfrac{\sqrt{2k}}{j\omega + k} \\ \dfrac{\sqrt{2k}}{-j\omega + k} & \dfrac{2k}{\omega^2 + k^2} \end{bmatrix}.$$

Extend the techniques in Section 3.4 to find the vector eigenfunctions and eigenvalues for Method 1.

Problem 3.7.2. Investigate the asymptotic behavior (i.e., as T becomes large) of the eigenvalues and eigenfunctions in Method 1 for an arbitrary stationary matrix kernel.

Problem 3.7.3. Let $x_1(t)$ and $x_2(t)$ be statistically independent zero-mean random processes with covariance functions $K_{x_1}(t, u)$ and $K_{x_2}(t, u)$, respectively. The eigenfunctions and eigenvalues are

$$K_{x_1}(t, u): \lambda_i,\ \phi_i(t), \qquad i = 1, 2, \ldots,$$
$$K_{x_2}(t, u): \mu_i,\ \psi_i(t), \qquad i = 1, 2, \ldots.$$

Prove that the vector eigenfunctions and scalar eigenvalues can always be written as

$$\lambda_1, \begin{bmatrix} \phi_1(t) \\ 0 \end{bmatrix} : \mu_1, \begin{bmatrix} 0 \\ \psi_1(t) \end{bmatrix} : \lambda_2, \begin{bmatrix} \phi_2(t) \\ 0 \end{bmatrix} : \cdots.$$

Problem 3.7.4. Consider the vector process $\mathbf{r}(t)$ in which

$$\mathbf{r}(t) = \mathbf{a}(t) + \mathbf{n}(t), \qquad -\infty < t < \infty.$$

The processes $\mathbf{a}(t)$ and $\mathbf{n}(t)$ are statistically independent, with spectral matrices $\mathbf{S_a}(\omega)$ and $\mathbf{S_n}(\omega)$, respectively. Extend the idea of the integrated transform to the vector case. Use the approach in Section 3.6.2 and the differential operator introduced in Section 2.4.3 (2-239) to find the MAP estimate of $\mathbf{a}(t)$.

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