Problem 4.6.8. Let

$$r(t) = \sqrt{2}\, v\, f(t, \mathrm{A}) \cos [\omega_c t + \phi(t, \mathrm{A}) + \theta] + w(t), \qquad 0 \le t \le T,$$

where v is a Rayleigh variable and θ is a uniform variable. The additive noise $w(t)$ is a sample function from a white Gaussian process with spectral height $N_0/2$. The parameter $\mathbf{a}$ is a zero-mean Gaussian vector with a diagonal covariance matrix; $\mathbf{a}$, v, θ, and $w(t)$ are statistically independent. Find the likelihood function as a function of $\mathbf{a}$.

Problem 4.6.9. Let

$$r(t) = \sqrt{2}\, v\, f(t - \tau) \cos [\omega_c t + \phi(t - \tau) + \omega t + \theta] + w(t), \qquad -\infty < t < \infty,$$

where $w(t)$ is a sample function from a zero-mean white Gaussian noise process with spectral height $N_0/2$. The functions $f(t)$ and $\phi(t)$ are deterministic functions that are low-pass compared with ω_c. The random variable v is Rayleigh and the random variable θ is uniform. The parameters τ and ω are nonrandom.

1. Find the likelihood function as a function of τ and ω.
2. Draw the block diagram of a receiver that provides an approximate implementation of the maximum-likelihood estimator.

Problem 4.6.10. A sequence of amplitude modulated signals is transmitted. The signal transmitted in the kth interval is

$$s_k(t, A) = A_k\, s(t), \qquad (k - 1)T \le t \le kT, \qquad k = 1, 2, \ldots.$$

The sequence of random variables is zero-mean Gaussian; the variables are related in the following manner:

$$a_1 \text{ is } N(0, \sigma_a)$$

$$a_2 = \Phi a_1 + u_1$$

$$\vdots$$

$$a_k = \Phi a_{k-1} + u_{k-1}.$$

The multiplier Φ is fixed. The u_i are independent, zero-mean Gaussian random variables, $N(0, \sigma_u)$. The received signal in the kth interval is

$$r(t) = s_k(t, A) + w(t), \qquad (k - 1)T \le t \le kT, \qquad k = 1, 2, \ldots.$$

Find the MAP estimate of a_k, $k = 1, 2, \ldots$. (Note the similarity to Problem 2.6.15.)

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