Chapter

9

The z Transform

The *two-sided*, or *bilateral*, z transform of a discrete-time sequence $x[n]$ is defined by

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \, z^{-n} \tag{9.1}$$

and the *one-sided*, or *unilateral*, z transform is defined by

$$X(z) = \sum_{n=0}^{\infty} x[n] \, z^{-n} \tag{9.2}$$

Some authors (for example, Rabiner and Gold 1975) use the unqualified term "z transform" to refer to (9.1), while others (for example, Cadzow 1973) use the unqualified term to refer to (9.2). In this book, "z transform" refers to the two-sided transform, and the one-sided transform is explicitly identified as such. For causal sequences (that is, $x[n] = 0$ for $n < 0$) the one-sided and two-sided transforms are equivalent. Some of the material presented in this chapter may seem somewhat abstract, but rest assured that the z transform and its properties play a major role in many of the design and realization methods that appear in later chapters.

9.1 Region of Convergence

For some values of z, the series in (9.1) does not converge to a finite value. The portion of the z plane for which the series does converge is called the *region of convergence* (ROC). Whether or not (9.1) converges depends upon the magnitude of z rather than a specific complex value of z. In other words, for a given sequence $x[n]$, if the series in (9.1) converges for a value of $z = z_1$, then the series will converge for all values of z for which $|z| = |z_1|$. Conversely, if the series diverges for $z = z_2$, then the series will diverge for all values of z for which $|z| = |z_2|$. Because convergence depends on the magni-

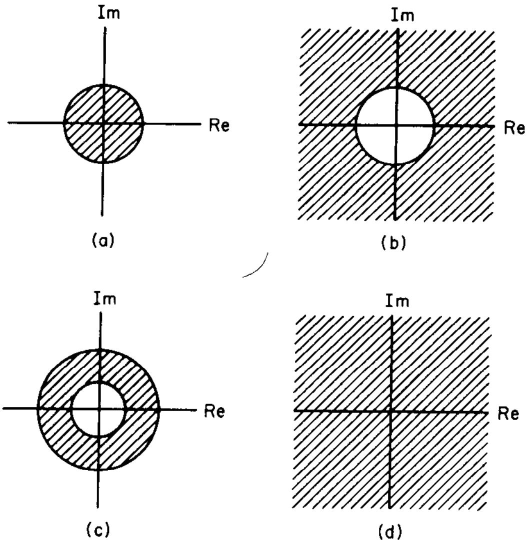

Figure 9.1 Possible configurations of the region of convergence for the z transform.

tude of z, the region or convergence will always be *bounded* by circles centered at the origin of the z plane. This is not to say that the region of convergence is always a circle—it can be the interior of a circle, the exterior of a circle, an annulus, or the entire z plane as shown in Fig. 9.1. Each of these four cases can be loosely viewed as an annulus—a circle's interior being an annulus with an inner radius of zero and a finite outer radius, a circle's exterior being an annulus with nonzero inner radius and infinite outer radius, and the entire z plane being an annulus with an inner radius of zero and an infinite outer radius. In some cases, the ROC has an inner radius of zero, but the origin itself is not part of the region. In other cases, the ROC has an infinite outer radius, but the series diverges *at* $|z| = \infty$.

By definition, the ROC cannot contain any poles since the series becomes infinite at the poles. The ROC for a z transform will always be a simply connected region in the z plane. If we assume that the sequence $x[n]$ has a finite magnitude for all finite values of n, the nature of the ROC can be related to the nature of the sequence in several ways as discussed in the paragraphs that follow and as summarized in Table 9.1.

Finite-duration sequences

If $x[n]$ is nonzero over only a finite range of n, then the z transform can be rewritten as

$$X(z) = \sum_{n=N_1}^{N_2} x[n]\, z^{-n}$$

TABLE 9.1 Properties of the Region of Convergence for the z Transform

$x[n]$	ROC for $X(z)$
All	Includes no poles
All	Simply connected region
Single sample at $n = 0$	Entire z plane
Finite-duration, causal, $x[n] = 0$ for all $n < 0$, $x[n] \neq 0$ for some $n > 0$	z plane except for $z = 0$
Finite-duration, with $x[n] \neq 0$ for some $n < 0$, $x[n] = 0$ for all $n > 0$	z plane except for $z = \infty$
Finite-duration, with $x[n] \neq 0$ for some $n < 0$, $x[n] \neq 0$ for some $n > 0$	z plane except for $z = 0$ and $z = \infty$
Right-sided, $x[n] = 0$ for all $n < 0$	Outward from outermost pole
Right-sided, $x[n] \neq 0$ for some $n < 0$	Outward from outermost pole, $z = \infty$ is excluded
Left-sided, $x[n] = 0$ for all $n > 0$	Inward from innermost pole
Left-sided, $x[n] \neq 0$ for some $n > 0$	Inward from innermost pole, $z = 0$ is excluded
Two-sided	Annulus

This series will converge provided that $|x[n]| < \infty$ for $N_1 \leq n \leq N_2$ and $|z^{-n}| < \infty$ for $N_1 \leq n \leq N_2$. For negative values of n, $|z^{-n}|$ will be infinite for $z = \infty$; and for positive values of n, $|z^{-n}|$ will be infinite for $z = 0$. Therefore, a sequence having nonzero values only for $n = N_1$ through $n = N_2$ will have a z transform that converges everywhere in the z plane except for $z = \infty$ when $N_1 < 0$ and $z = 0$ when $N_2 > 0$. Note that a single sample at $n = 0$ is the only finite-duration sequence defined over the entire z plane.

Infinite-duration sequences

The sequence $x[n]$ is a *right-sided sequence* if $x[n]$ is zero for all n less than some finite value N_1. It can be shown (see Oppenheim and Schafer 1975 or 1989) that the z transform $X(z)$ of a right-sided sequence will have an ROC that extends outward from the outermost finite pole of $X(z)$. In other words, the ROC will be the area outside a circle whose radius equals the magnitude of the pole of $X(z)$ having the largest magnitude (see Fig. 9.2). If $N_1 < 0$, this ROC will not include $z = \infty$.

The sequence $x[n]$ is a *left-sided sequence* if $x[n]$ is zero for all n greater than some finite value N_2. The z transform $X(z)$ of a left-sided sequence will have an ROC that extends inward from the innermost pole of $X(z)$. The ROC will be the interior of a circle whose radius equals the magnitude of the pole of $X(z)$ having the smallest magnitude (see Fig. 9.3). If $N_2 > 0$, this ROC will not include $z = 0$.

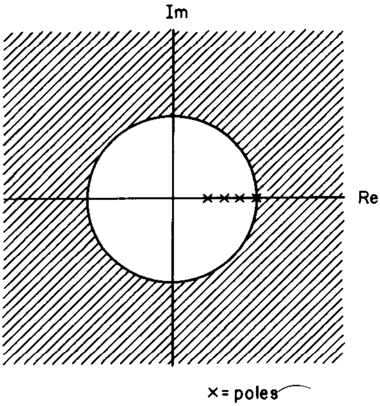

Figure 9.2 Region of convergence for the z transform of a right-sided sequence.

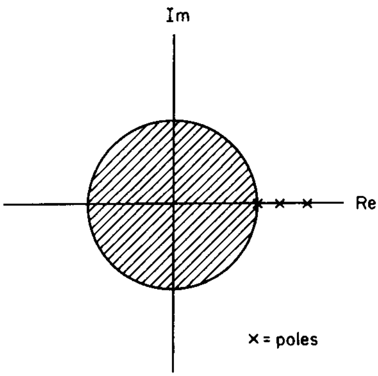

Figure 9.3 Region of convergence for the z transform of a left-sided sequence.

The sequence $x[n]$ is a *two-sided sequence* if $x[n]$ has nonzero values extending to both $-\infty$ and $+\infty$. The ROC for the z transform of a two-sided sequence will be an annulus.

Convergence of the unilateral z transform

Note that all of the properties discussed above are for the two-sided z transform defined by (9.1). Since the one-sided z transform is equivalent to the two-sided transform when $x[n] = 0$ for $n < 0$, the ROC for a one-sided transform will always look like the ROC for the two-sided transform of either a causal finite-duration sequence or a causal right-sided sequence. For all causal systems, the ROC for the bilateral transform always consists of the area outside a circle of radius $R \geq 0$. Therefore, for two-sided transforms of causal sequences and for all one-sided transforms, the ROC can be (and frequently is) specified in terms of a *radius of convergence* R such that the transform converges for $|z| > R$.

9.2 Relationship between the Laplace and z Transforms

The z transform can be related to both the Laplace and Fourier transforms. As noted in Chap. 7, a sequence can be obtained by sampling a function of continuous time. Specifically, for a causal sequence

$$x[n] = \sum_{n=0}^{\infty} x_a(nT)\,\delta(t - nT) \tag{9.3}$$

the Laplace transform is given by

$$X(s) = \sum_{n=0}^{\infty} x_a(nT)\,e^{-nTs} \tag{9.4}$$

Let $X_a(s)$ denote the Laplace transform of $x_a(t)$. The pole-zero pattern for $X(s)$ consists of the pole-zero pattern for $X_a(s)$ replicated at intervals of $\omega_s = 2\pi/T$ along the $j\omega$ axis in the s plane. If we modify (9.4) by substituting

$$z = e^{sT} \tag{9.5}$$

$$x[n] = x_a(nT) \tag{9.6}$$

we obtain the z transform defined by Eq. (9.1).

Relationships between features in the s plane and features in the z plane can be established using (9.5). Since $s = \sigma + j\omega$ with σ and ω real, we can expand (9.5) as

$$z = e^{sT} = e^{\sigma T}\,e^{j\omega T} = e^{\sigma T}(\cos\omega T + j\sin\omega T)$$

Because $|e^{j\omega T}| = (\cos^2\omega T + \sin^2\omega T)^{1/2} = 1$, and $T > 0$, we can conclude that $|z| < 1$ for $\sigma < 0$. Or, in other words, the left half of the s plane maps into the interior of the unit circle in the z plane. Likewise, $|z| = 1$ for $\sigma = 0$, so the $j\omega$ axis of the s plane maps onto the unit circle in the z plane. The "extra" replicated copies of the pole-zero pattern for $X(s)$ will all map into a single pole-zero pattern in the z plane. When evaluated around the unit circle (that is, $z = e^{j\lambda}$), the z transform yields the discrete-time Fourier transform (DTFT) (see Sec. 7.2).

9.3 System Functions

Given the relationships between the Laplace transform and the z transform that were noted in the previous section, we might suspect that the z transform of a discrete-time system's unit sample response (that is, digital impulse response) plays a major role in the analysis of the system in much the same way that the Laplace transform of a continuous-time system's impulse response yields the system's transfer function. This suspicion is indeed correct. The z transform of a discrete-time system's unit sample response is called the *system function*, or *transfer function*, of the system and is denoted by $H(z)$.

The system function can also be derived from the linear difference equation that describes the filter. If we take the z transform of each term in Eq. (7.6), we obtain

$$Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + \cdots + a_k z^{-k} Y(z)$$
$$= b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + b_k z^{-k} X(z)$$

Factoring out $Y(z)$ and $X(z)$ and then solving for $H(z) = Y(z)/X(z)$ yields

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_k z^{-k}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_k z^{-k}}$$

Both the numerator and denominator of $H(z)$ can be factored to yield

$$H(z) = \frac{b_0 (z - q_1)(z - q_2) \cdots (z - q_k)}{(z - p_1)(z - p_2)(z - p_3) \cdots (z - p_k)}$$

The poles of $H(z)$ are $p_1, p_2, \ldots, p_k$, and the zeros are $q_1, q_2, \ldots, q_m$.

9.4 Common z-Transform Pairs and Properties

The use of the unilateral z transform by some authors and the use of the bilateral transform by others does not present as many problems as we might expect, because in the field of digital filters, most of the sequences of interest are causal sequences or sequences that can easily be made causal. As we noted previously, for causal sequences the one-sided and two-sided transforms are equivalent. It really just comes down to a matter of being careful about definitions. An author using the unilateral default (that is, "z transform" means "unilateral z transform") might say that the z transform of $x[n] = a^n$ is given by

$$X(z) = \frac{z}{z - a} \qquad \text{for } |z| > |a| \tag{9.7}$$

On the other hand, an author using the bilateral default might say that (9.7) represents the z transform of $x[n] = a^n\, u[n]$, where $u[n]$ is the unit step sequence. Neither author is concerned with the values of a^n for $n < 0$—the first author is eliminating these values by the way the transform is defined, and the second author is eliminating these values by multiplying them with a unit step sequence that is zero for $n < 0$. There are a few useful bilateral transform pairs that consider values of $x[n]$ for $n < 0$. These pairs are listed in Table 9.2. However, the majority of the most commonly used z-transform pairs involve values of $x[n]$ only for $n \geq 0$. These pairs are most conveniently

TABLE 9.2 Common Bilateral z-Transform Pairs

$x[n]$	$X(z)$	ROC
$\delta[n]$	1	all z
$\delta[n-m],\ m>0$	z^{-m}	$z \neq 0$
$\delta[n-m],\ m<0$	z^{-m}	$z \neq \infty$
$u[n]$	$\dfrac{z}{z-1}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\dfrac{z}{z-1}$	$\lvert z\rvert < 1$
$-a^n\,u[-n-1]$	$\dfrac{z}{z-a}$	$\lvert z\rvert < \lvert a\rvert$
$-na^n\,u[-n-1]$	$\dfrac{az}{(z-a)^2}$	$\lvert z\rvert < \lvert a\rvert$

tabulated as unilateral transforms with the understanding that any unilateral transform pair can be converted into a bilateral transform pair by replacing $x[n]$ with $x[n]\,u[n]$. Some common unilateral z-transform pairs are listed in Table 9.3. Some useful properties exhibited by both the unilateral and bilateral z transforms are listed in Table 9.4.

9.5 Inverse z Transform

The inverse z transform is given by the contour integral

$$x[n] = \frac{1}{2\pi j} \oint_C X(z)\, z^{n-1}\, dz \tag{9.8}$$

where the integral notation indicates a counterclockwise closed contour that encircles the origin of the z plane and that lies within the region of convergence for $X(z)$. If $X(z)$ is rational, the residue theorem can be used to evaluate (9.8). However, direct evaluation of the inversion integral is rarely performed in actual practice. In practical situations, inversion of the z transform is usually performed indirectly, using established transform pairs and transform properties.

9.6 Inverse z Transform via Partial Fraction Expansion

Consider a system function of the general form given by

$$H(z) = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_{m-1} z^1 + b_m}{z_m + a_1 z^{m-1} + \cdots + a_{m-1} z^1 + a_m} \tag{9.9}$$

TABLE 9.3 Common Unilateral *z*-Transform Pairs
(R = radius of convergence)

$x[n]$	$X(z)$	R
1	$\dfrac{z}{z-1}$	1
$u_1[n]$	$\dfrac{z}{z-1}$	1
$\delta[n]$	1	0 ($z = 0$ included)
nT	$\dfrac{Tz}{(z-1)^2}$	1
$(nT)^2$	$\dfrac{T^2 z(z+1)}{(z-1)^3}$	1
$(nT)^3$	$\dfrac{T^3 z(z^2+4z+1)}{(z-1)^4}$	1
a^n	$\dfrac{z}{z-a}$	$\|a\|$
$(n+1)a^n$	$\dfrac{z^2}{(z-a)^2}$	$\|a\|$
$\dfrac{(n+1)(n+2)}{2!}a^n$	$\dfrac{z^3}{(z-a)^3}$	$\|a\|$
$\dfrac{(n+1)(n+2)(n+3)}{3!}a^n$	$\dfrac{z^4}{(z-a)^4}$	$\|a\|$
$\dfrac{(n+1)(n+2)(n+3)(n+4)}{4!}a^n$	$\dfrac{z^5}{(z-a)^5}$	$\|a\|$
na^n	$\dfrac{az}{(z-a)^2}$	$\|a\|$
n^2a^n	$\dfrac{az(z+a)}{(z-a)^3}$	$\|a\|$
n^3a^n	$\dfrac{az(z^2+4az+a^2)}{(z-a)^4}$	$\|a\|$
$\dfrac{a^n}{n!}$	$e^{a/z}$	0
e^{-anT}	$\dfrac{z}{z-e^{-aT}}$	$\|e^{-aT}\|$
$a^n \sin n\omega T$	$\dfrac{az \sin \omega T}{z^2 - 2az \cos \omega T + a^2}$	$\|a\|$
$a^n \cos n\omega T$	$\dfrac{z^2 - za \cos \omega T}{z^2 - 2az \cos \omega T + a^2}$	$\|a\|$
$e^{-anT} \sin \omega_0 nT$	$\dfrac{ze^{-aT} \sin \omega_0 T}{z^2 - 2ze^{-aT} \cos \omega_0 T + e^{-2aT}}$	$\|e^{-aT}\|$
$e^{-anT} \cos \omega_0 nT$	$\dfrac{z^2 - ze^{-aT} \cos \omega_0 T}{z^2 - 2ze^{-aT} \cos \omega_0 T + e^{-2aT}}$	$\|e^{-aT}\|$

TABLE 9.4 Properties of the z Transform

Property no.	Time function	Transform
	$x[n]$	$X(z)$
	$y[n]$	$Y(z)$
1	$a\,x[n]$	$a\,X(z)$
2	$x[n]+y[n]$	$X(z)+Y(z)$
3	$e^{-anT}\,x[n]$	$X(e^{at}\,z)$
4	$\alpha^n\,x[n]$	$X\left(\frac{z}{\alpha}\right)$
5	$x[n-m]$	$z^{-m}\,X(z)$
6	$x[n]*y[n]$	$X(z)\,Y(z)$
7	$n\,x[n]$	$-z\frac{d}{dz}X(z)$
8	$x[-n]$	$X(z^{-1})$
9	$x^*[n]$	$X^*(z^*)$

Such a system function can be expanded into a sum of simpler terms that can be more easily inverse-transformed. Linearity of the z transform allows us to then sum the simpler inverse transforms to obtain the inverse of the original system function. The method for generating the expansion differs slightly depending upon whether the system function's poles are all distinct or if some are multiple poles. Since most practical filter designs involve system functions with distinct poles, the more complicated multiple-pole procedure is not presented. For a discussion of the multiple-pole case, see Cadzow (1973).

Algorithm 9.1 Partial fraction expansion for $H(z)$ having simple poles

Step 1. Factor the denominator of $H(z)$ to produce

$$H(z)=\frac{b_0z^m+b_1z^{m-1}+\cdots+b_{m-1}z^1+b_m}{(z-p_1)(z-p_2)(z-p_3)\cdots(z-p_k)}$$

Step 2. Compute c_0 as given by

$$c_0=H(z)\big|_{z=0}=\frac{b_m}{(-p_1)(-p_2)(-p_3)\cdots(-p_m)}$$

Step 3. Compute c_i for $1 \le i \le m$ using

$$c_i = \frac{z - p_i}{z} H(z)\big|_{z = p_i}$$

Step 4. Formulate the discrete-time function $h[n]$ as given by

$$h(n) = c_0\delta(n) + c_1(p_1)^n + c_2(p_2)^n + \cdots + c_m(p_m)^n \qquad \text{for } n = 0, 1, 2, \ldots$$

The function $h[n]$ is the inverse z transform of $H(z)$.

Example 9.1 Use the partial fraction expansion to determine the inverse z transform of

$$H(z) = \frac{z^2}{z^2 + z - 2}$$

solution

Step 1. Factor the denominator of $H(z)$ to produce

$$H(z) = \frac{z^2}{(z-1)(z+2)}$$

Step 2. Compute c_0 as

$$c_0 = H(z)\big|_{z=0} = 0$$

Step 3. Compute c_1, c_2 as

$$c_1 = \left[\frac{(z-1)}{z} \frac{z^2}{(z-1)(z+2)}\right]\bigg|_{z=1} = \frac{z^2}{z^2+2z}\bigg|_{z=1} = \frac{1}{3}$$

$$c_2 = \left[\frac{(z+2)}{z} \frac{z^2}{(z-1)(z+2)}\right]\bigg|_{z=-2} = \frac{z^2}{z^2-z}\bigg|_{z=-2} = \frac{2}{3}$$

Step 4. The inverse transform $h[n]$ is given by

$$h[n] = \tfrac{1}{3}(1)^n + \tfrac{1}{3}(-2)^n$$

$$= 1 + \tfrac{1}{3}(-2)^n \qquad n = 0, 1, 2, \ldots$$