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Higher-Order Spectral Analysis

Athina P. Petropulu
Drexel University

76.1 Introduction

The past 20 years witnessed an expansion of power spectrum estimation techniques, which have proved essential in many applications, such as communications, sonar, radar, speech/image processing, geophysics, and biomedical signal processing [13, 11, 7]. In power spectrum estimation the process under consideration is treated as a superposition of statistically uncorrelated harmonic components. The distribution of power among these frequency components is the power spectrum. As such, phase relations between frequency components are suppressed. The information in the power spectrum is essentially present in the autocorrelation sequence, which would suffice for the complete statistical description of a Gaussian process of known mean. However, there are applications where one would need to obtain information regarding deviations from the Gaussianity assumption and presence of nonlinearities. In these cases power spectrum is of little help, and one would have to look beyond the power spectrum or autocorrelation domain. Higher-Order Spectra (HOS) (of order greater than 2), which are defined in terms of higher-order cumulants of the data, do contain such information [16]. The third-order spectrum is commonly referred to as bispectrum, the fourth-order one as trispectrum, and in fact, the power spectrum is also a member of the higher-order spectral class; it is the second-order spectrum.

HOS consist of higher-order moment spectra, which are defined for deterministic signals, and cumulant spectra, which are defined for random processes. In general, there are three motivations behind the use of HOS in signal processing: (1) to suppress Gaussian noise of unknown mean and variance; (2) to reconstruct the phase as well as the magnitude response of signals or systems; and (3) to detect and characterize nonlinearities in the data.

The first motivation stems from the property of Gaussian processes to have zero higher-order spectra. Due to this property, HOS are high signal-to-noise ratio domains, in which one can perform detection, parameter estimation, or even signal reconstruction even if the time domain noise is spatially correlated. The same property of cumulant spectra can provide means of detecting and characterizing deviations of the data from the Gaussian model.

The second motivation is based on the ability of cumulant spectra to preserve the Fourier-phase of signals. In the modeling of time series, second-order statistics (autocorrelation) have been heavily used because they are the result of least-squares optimization criteria. However, an accurate phase reconstruction in the autocorrelation domain can be achieved only if the signal is minimum phase. Nonminimum phase signal reconstruction can be achieved only in the HOS domain, due to the HOS ability to preserve phase. Figure 76.1 shows two signals, a nonminimum phase and a minimum phase, with identical magnitude spectra but different phase spectra. Although power spectrum cannot distinguish between the two signals, the bispectrum that uses phase information can.

Being nonlinear functions of the data, HOS are quite natural tools in the analysis of nonlinear systems operating under a random input. General relations for arbitrary stationary random data passing through an arbitrary linear system exist and have been studied extensively. Such expression, however, are not available for nonlinear systems, where each type of nonlinearity must be studied separately. Higher-order correlations between input and output can detect and characterize certain nonlinearities [34], and for this purpose several higher-order spectra-based methods have been developed.

The organization of this chapter is as follows. First the definitions and properties of cumulants and higher-order spectra are introduced. Then two methods for the estimation of HOS from finite length data are outlined and the asymptotic statistics of the obtained estimates are presented. Following that, parametric and nonparametric methods for HOS-based identification of linear systems are described, and the use of HOS in the identification of some particular nonlinear systems is briefly discussed. The chapter concludes with a section on applications of HOS and available software.

76.2 Definitions and Properties of HOS

In this chapter we will consider random one-dimensional processes only. The definitions can be easily extended to the two-dimensional case [15].

The joint moments of order r of the random variables $x_1, \ldots, x_n$ are given by [22]

$$\begin{aligned} Mom\left[x_1^{k_1}, \ldots, x_n^{k_n}\right] &= E\{x_1^{k_1}, \ldots, x_n^{k_n}\} \\ &= (-j)^r \frac{\partial^r \Phi(\omega_1, \ldots, \omega_n)}{\partial\omega_1^{k_1} \ldots \partial\omega_n^{k_n}}|_{\omega_1=\cdots=\omega_n=0} , \end{aligned} \tag{76.1}$$

where $k_1 + \cdots + k_n = r$, and $\Phi()$ is their joint characteristic function. The joint cumulants are defined as

$$Cum[x_1^{k_1}, \ldots, x_n^{k_n}] = (-j)^r \frac{\partial^r ln\Phi(\omega_1, \ldots, \omega_n)}{\partial\omega_1^{k_1} \ldots \partial\omega_n^{k_n}}\}|_{\omega_1=\cdots=\omega_n=0} . \tag{76.2}$$

For a stationary discrete time random process $X(k)$, (k denotes discrete time), the *moments* of order n are given by

$$m_n^x(\tau_1, \tau_2, \ldots, \tau_{n-1}) = E\{X(k)X(k+\tau_1)\cdots X(k+\tau_{n-1})\} , \tag{76.3}$$

where $E\{.\}$ denotes expectation. The nth order *cumulants* are functions of the moments of order up to n, i.e.,

1st order cumulants:

$$c_1^x = m_1^x = E\{X(k)\} \quad \text{(mean)} \tag{76.4}$$

2nd order cumulants:

$$c_2^x(\tau_1) = m_2^x(\tau_1) - \left(m_1^x\right)^2 \quad \text{(covariance)} \tag{76.5}$$

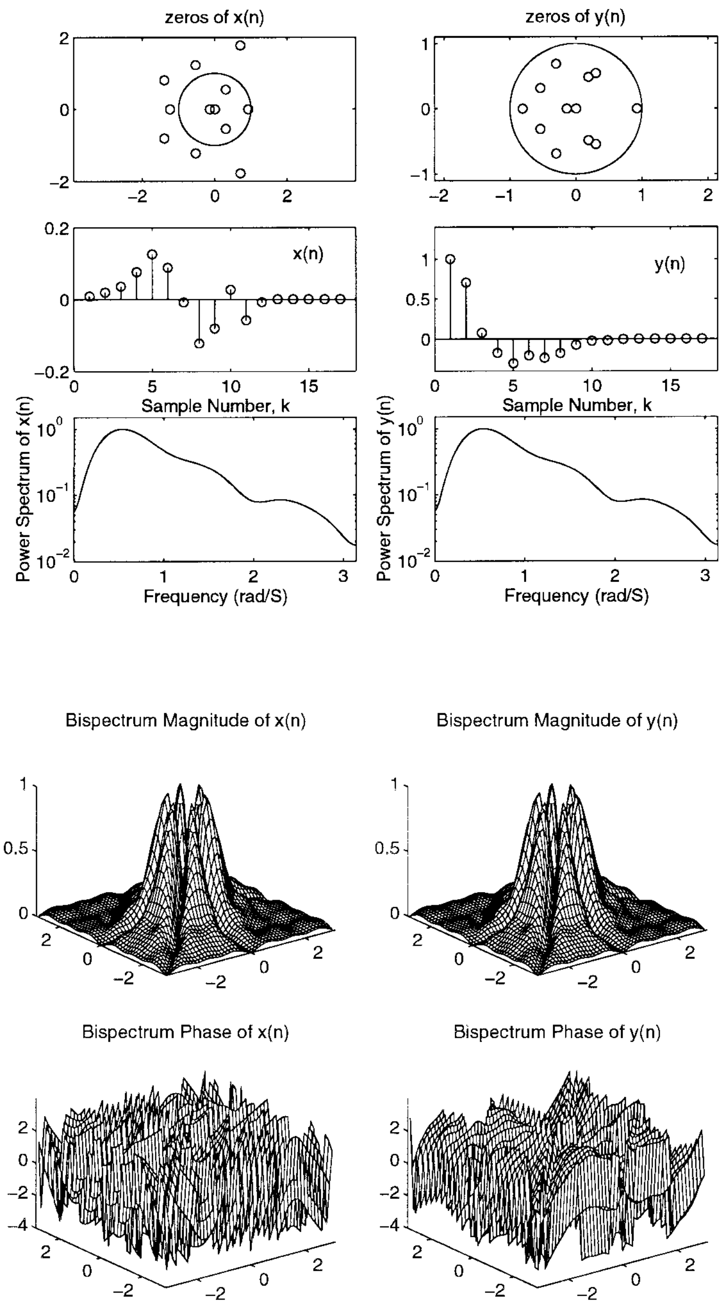

FIGURE 76.1: $x(n)$ is a nonminimum phase signal and $y(n)$ is a minimum phase one. Although their power spectra are identical, their bispectra are different because they contain phase information.

3rd order cumulants:

$$c_3^x(\tau_1, \tau_2) = m_3^x(\tau_1, \tau_2) - \left(m_1^x\right)\left[m_2^x(\tau_1) + m_2^x(\tau_2) + m_2^x(\tau_2 - \tau_1)\right] + 2\left(m_1^x\right)^3 \tag{76.6}$$

4th order cumulants:

$$\begin{aligned} c_4^x(\tau_1, \tau_2, \tau_3) \quad = \quad & m_4^x(\tau_1, \tau_2, \tau_3) - m_2^x(\tau_1)\, m_2^x(\tau_3 - \tau_2) - m_2^x(\tau_2)\, m_2^x(\tau_3 - \tau_1) \\ & - m_2^x(\tau_3)\, m_2^x(\tau_2 - \tau_1) \\ & - m_1^x\left[m_3^x(\tau_2 - \tau_1, \tau_3 - \tau_1) + m_3^x(\tau_2, \tau_3) + m_3^x(\tau_2, \tau_4) + m_3^x(\tau_1, \tau_2)\right] \\ & + \left(m_1^x\right)^2\left[m_2^x(\tau_1) + m_2^x(\tau_2) + m_2^x(\tau_3) + m_2^x(\tau_3 - \tau_1) + m_2^x(\tau_3 - \tau_2)\right. \\ & \left. + \, m_1^x(\tau_2 - \tau_1)\right] - 6\left(m_1^x\right)^4 \end{aligned} \tag{76.7}$$

where $m_3^x(\tau_1, \tau_2)$ is the 3rd order moment sequence, and m_1^x is the mean. The general relationship between cumulants and moments can be found in [16].

Some important properties of moments and cumulants are summarized next.

[P1] If $X(k)$ is Gaussian, the $c_n^x(\tau_1, \tau_2, \ldots, \tau_{n-1}) = 0$ for $n > 2$. In other words, all the information about a Gaussian process is contained in its first and second-order cumulants. This property can be used to suppress Gaussian noise, or as a measure for non-Gaussianity in time series.

[P2] If $X(k)$ is symmetrically distributed, then $c_3^x(\tau_1, \tau_2) = 0$. Third-order cumulants suppress not only Gaussian processes, but also all symmetrically distributed processes, such as uniform, Laplace, and Bernoulli-Gaussian.

[P3] For cumulants additivity holds. If $X(k) = S(k) + W(k)$, where $S(k)$, $W(k)$ are stationary and statistically independent random processes, then $c_n^x(\tau_1, \tau_2, \ldots, \tau_{n-1}) = c_n^s(\tau_1, \tau_2, \ldots, \tau_{n-1}) + c_n^w(\tau_1, \tau_2, \ldots, \tau_{n-1})$. It is important to note that additivity does not hold for moments.

If $W(k)$ is Gaussian representing noise which corrupts the signal of interest, $S(k)$, then by means of (P2) and (P3), we get that $c_n^x(\tau_1, \tau_2, \ldots, \tau_{n-1}) = c_n^s(\tau_1, \tau_2, \ldots, \tau_{n-1})$, for $n > 2$. In other words, in higher-order cumulant domains the signal of interest propagates noise free. Property (P3) can also provide a measure of statistical dependence of two processes.

[P4] if $X(k)$ has zero mean, then $c_n^x(\tau_1, \ldots, \tau_{n-1}) = m_n^x(\tau_1, \ldots, \tau_{n-1})$, for $n \leq 3$.

Higher-order spectra are defined in terms of either cumulants (e.g., cumulant spectra) or moments (e.g., moment spectra).

Assuming that the nth order cumulant sequence is absolutely summable, the nth order **cumulant spectrum** of $X(k)$, $C_n^x(\omega_1, \omega_2, \ldots, \omega_{n-1})$, exists, and is defined to be the $(n-1)$-dimensional Fourier transform of the nth order cumulant sequence. In general, $C_n^x(\omega_1, \omega_2, \ldots, \omega_{n-1})$ is complex, i.e., it has magnitude and phase. In an analogous manner, **moment spectrum** is the multi-dimensional Fourier transform of the moment sequence.

If $v(k)$ is a stationary non-Gaussian process with zero mean and nth order cumulant sequence

$$c_n^v(\tau_1, \ldots, \tau_{n-1}) = \gamma_n^v \delta(\tau_1, \ldots, \tau_{n-1}), \tag{76.8}$$

where $\delta(.)$ is the delta function, $v(k)$ is said to be nth order white. Its nth order cumulant spectrum is then flat and equal to γ_n^v.

Cumulant spectra are more useful in processing random signals than moment spectra since they posses properties that the moment spectra do not share: (1) the cumulants of the sum of two independent random processes equals the sum of the cumulants of the process; (2) cumulant spectra of order > 2 are zero if the underlying process in Gaussian; (3) cumulants quantify the degree of statistical dependence of time series; and (4) cumulants of higher-order white noise are multidimensional impulses, and the corresponding cumulant spectra are flat.

76.3 HOS Computation from Real Data

The definitions of cumulants presented in the previous section are based on expectation operations, and they assume infinite length data. In practice we always deal with data of finite length; therefore, the cumulants can only be approximated. Two methods for cumulants and spectra estimation are presented next for the third-order case.

Indirect Method :

Let $X(k)$, $k = 1, \ldots, N$ be the available data.

1. Segment the data into K records of M samples each. Let $X^i(k)$, $k = 1, \ldots, M$, represent the ith record.
2. Subtract the mean of each record.
3. Estimate the moments of each segments $X^i(k)$ as follows:

$$m_3^{x_i}(\tau_1, \tau_2) = \frac{1}{M} \sum_{l=l_1}^{l_2} X^i(l) X^i(l + \tau_1) X^i(l + \tau_2) ,$$

$$l_1 = max(0, -\tau_1, -\tau_2),\ l_2 = min(M - 1, M - 2), \quad |\tau_1| < L,\ |\tau_2| < L,\ i = 1, 2, \ldots, K . \tag{76.9}$$

   Since each segment has zero mean, its third-order moments and cumulants are identical, i.e., $c_3^{x_i}(\tau_1, \tau_2) = m_3^{x_i}(\tau_1, \tau_2)$.
4. Compute the average cumulants as:

$$\hat{c}_3^x(\tau_1, \tau_2) = \frac{1}{K} \sum_{i=1}^{K} m_3^{x_i}(\tau_1, \tau_2) \tag{76.10}$$

5. Obtain the third-order spectrum (bispectrum) estimate as

$$\hat{C}_3^x(\omega_1, \omega_2) = \sum_{\tau_1=-L}^{L} \sum_{\tau_2=-L}^{L} \hat{c}_3^x(\tau_1, \tau_2) e^{-j(\omega_1\tau_1 + \omega_2\tau_2)} w(\tau_1, \tau_2) , \tag{76.11}$$

   where $L < M - 1$, and $w(\tau_1, \tau_2)$ is a two-dimensional window of bounded support, introduced to smooth out edge effects. The bandwidth of the final bispectrum estimate is $\Delta = 1/L$.

A complete description of appropriate windows that can be used in (76.11) and their properties can be found in [16]. A good choice of cumulant window is:

$$w(\tau_1, \tau_2) = d(\tau_1) d(\tau_2) d(\tau_1 - \tau_2) , \tag{76.12}$$

where

$$d(\tau) = \begin{cases} \frac{1}{\pi}|\sin\frac{\pi\tau}{L}| + (1 - \frac{|\tau|}{L})\cos\frac{\pi\tau}{L} & |\tau| \le L \\ 0 & |\tau| > L \end{cases} \tag{76.13}$$

which is known as the minimum bispectrum bias supremum [17].

Direct Method

Let $X(k)$, $k = 1, \ldots, N$ be the available data.

1. Segment the data into K records of M samples each. Let $X^i(k),\ k = 1, \ldots, M$, represent the ith record.
2. Subtract the mean of each record.
3. Compute the Discrete Fourier Transform $F_x^i(k)$ of each segment, based on M points, i.e.,

$$F_x^i(k) = \sum_{n=0}^{M-1} X^i(n) e^{-j\frac{2\pi}{M}nk},\ \ k = 0, 1, \ldots, M-1,\ i = 1, 2, \ldots, K\ . \tag{76.14}$$

4. The third-order spectrum of each segment is obtained as

$$C_3^{x_i}(k_1, k_2) = \frac{1}{M} F_x^i(k_1) F_x^i(k_2) F_x^{i^*}(k_1 + k_2),\ i = 1, \ldots, K\ . \tag{76.15}$$

   Due to the bispectrum symmetry properties, $C_3^{x_i}(k_1, k_2)$ need to be computed only in the triangular region $0 \leq k_2 \leq k_1,\ k_1 + k_2 < M/2$.
5. In order to reduce the variance of the estimate additional smoothing over a rectangular window of size $(M_3 \times M_3)$ can be performed around each frequency, assuming that the third-order spectrum is smooth enough, i.e.,

$$\tilde{C}_3^{x_i}(k_1, k_2) = \frac{1}{M_3^2} \sum_{n_1=-M_3/2}^{M_3/2-1} \sum_{n_2=-M_3/2}^{M_3/2-1} C_3^{x_i}(k_1 + n_1, k_2 + n_2)\ . \tag{76.16}$$

6. Finally, the third-order spectrum is given as the average over all third-order spectra, i.e.,

$$\hat{C}_3^x(\omega_1, \omega_2) = \frac{1}{K} \sum_{i=1}^{K} \tilde{C}_3^{x_i}(\omega_1, \omega_2)\ ,\ \omega_i = \frac{2\pi}{M} k_i,\ \ i = 1, 2\ . \tag{76.17}$$

   The final bandwidth of this bispectrum estimate is $\Delta = M_3/M$, which is the spacing between frequency samples in the bispectrum domain.

For large N, and as long as

$$\Delta \to 0,\ and\ \Delta^2 N \to \infty \tag{76.18}$$

[32], both the direct and the indirect methods produce asymptotically unbiased and consistent bispectrum estimates, with real and imaginary part variances:

$$\mathrm{var}\left(\mathrm{Re}\left[\hat{C}_3^x(\omega_1, \omega_2)\right]\right) = \mathrm{var}\left(\mathrm{Im}\left[\hat{C}_3^x(\omega_1, \omega_2)\right]\right) \tag{76.19}$$

$$= \frac{1}{\Delta^2 N} C_2^x(\omega_1) C_2^x(\omega_2) C_2^x(\omega_1 + \omega_2) = \begin{cases} \frac{VL^2}{MK} C_2^x(\omega_1) C_2^x(\omega_2) C_2^x(\omega_1 + \omega_2) & \text{indirect} \\ \frac{M}{KM_3^2} C_2^x(\omega_1) C_2^x(\omega_2) C_2^x(\omega_1 + \omega_2) & \text{direct}\ , \end{cases}$$

where V is the energy of the bispectrum window.

From the above expressions, it becomes apparent that the bispectrum estimate variance can be reduced by increasing the number of records, or reducing the size of the region of support of the window in the cumulant domain (L), or increasing the size of the frequency smoothing window (M_3), etc. The relation between the parameters $M,\ K,\ L,\ M_3$ should be such that (76.18) is satisfied.

76.4 Linear Processes

Let $x(k)$ be generated by exciting a linear time-invariant (LTI) system with frequency response $H(\omega)$ with a non-Gaussian process $v(k)$. Its nth order spectrum can be written as

$$C_n^x(\omega_1, \omega_2, \dots, \omega_{n-1}) = C_n^v(\omega_1, \omega_2, \dots, \omega_{n-1}) H(\omega_1) \cdots H(\omega_{n-1}) H^*(\omega_1 + \cdots + \omega_{n-1}) . \tag{76.20}$$

If $v(k)$ is nth order white then (76.20) becomes

$$C_n^x(\omega_1, \omega_2, \dots, \omega_{n-1}) = \gamma_n^v H(\omega_1) \cdots H(\omega_{n-1}) H^*(\omega_1 + \cdots + \omega_{n-1}) , \tag{76.21}$$

where γ_n^v is a scalar constant and equals the nth order spectrum of $v(k)$. For a linear non-Gaussian random process $X(k)$, the nth order spectrum can be factorized as in (76.21) for every order n, while for a nonlinear process such a factorization might be valid for some orders only (it is always valid for $n = 2$).

If we express $H(\omega) = |H(\omega)| \exp\{j\phi_h(\omega)\}$, then (76.21) can be written as

$$\left|C_n^x(\omega_1, \omega_2, \dots, \omega_{n-1})\right| = \gamma_n^v |H(\omega_1)| \cdots |H(\omega_{n-1})| \left|H^*(\omega_1 + \cdots + \omega_{n-1})\right| , \tag{76.22}$$

and

$$\left|\psi_n^x(\omega_1, \omega_2, \dots, \omega_{n-1})\right| = \phi_h(\omega_1) + \cdots + \phi_h(\omega_{n-1}) - \phi_h(\omega_1 + \cdots + \omega_{n-1}) , \tag{76.23}$$

where $\psi_n^x()$ is the phase of the nth order spectrum.

It can be shown easily that the cumulant spectra of successive orders are related as follows:

$$C_n^x(\omega_1, \omega_2, \dots, 0) = C_{n-1}^x(\omega_1, \omega_2, \dots, \omega_{n-2}) H(0) \frac{\gamma_n^v}{\gamma_{n-1}^v} . \tag{76.24}$$

As a result, the power spectrum of a Gaussian linear process can be reconstructed from the bispectrum up to a constant term, i.e.,

$$C_3^x(\omega, 0) = C_2^x(\omega) \frac{\gamma_3^v}{\gamma_2^v} . \tag{76.25}$$

To reconstruct the phase $\phi_h(\omega)$ from the bispectral phase $\psi_3^x(\omega_1, \omega_2)$ several algorithms have been suggested. A description of different phase estimation methods can be found in [14] and also in [16].

76.4.1 Nonparametric Methods

Consider $x(k)$ generated as shown in Fig. 76.2. The system transfer function can be written as

$$H(z) = cz^{-r} I\left(z^{-1}\right) O(z) = cz^{-r} \frac{\Pi_i (1 - a_i z^{-1})}{\Pi_i (1 - b_i z^{-1})} \Pi_i (1 - c_i z), \; |a_i|, |b_i|, |c_i| < 1 , \tag{76.26}$$

where $I(z^{-1})$ and $O(z)$ are the minimum and maximum phase parts of $H(z)$, respectively; c is a constant; and r is an integer. The output nth order cumulant equals [2]

$$\begin{aligned} c_n^x(\tau_1, \dots, \tau_{n-1}) &= c_n^y(\tau_1, \dots, \tau_{n-1}) + c_n^w(\tau_1, \dots, \tau_{n-1}) \\ &= c_n^y(\tau_1, \dots, \tau_{n-1}) \qquad (76.27) \\ &= \gamma_n^v \sum_{k=0}^{\infty} h(k) h(k + \tau_1) \cdots h(k + \tau_{n-1}) , \; n \geq 3 \qquad (76.28) \end{aligned}$$

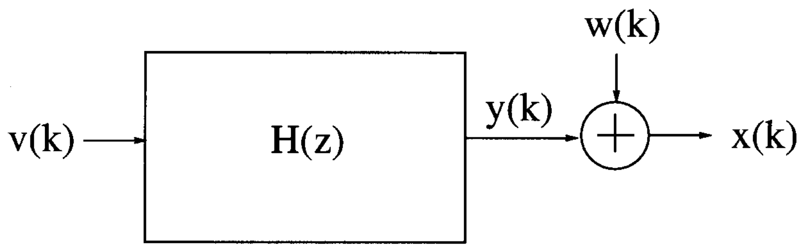

FIGURE 76.2: Single channel model.

where the noise contribution in (76.27) was zero due to the Gaussianity assumption. The Z-domain equivalent of (76.28) for $n = 3$ is

$$C_3^x(z_1, z_2) = \gamma_3^v H(z_1) H(z_2) H\left(z_1^{-1} z_2^{-1}\right) . \tag{76.29}$$

Taking the logarithm of $C_3^x(z_1, z_2)$ followed by an inverse 2-D Z-transform we obtain the output bicepstrum $b_x(m, n)$. The bicepstrum of linear processes is nonzero only along the axes ($m = 0,\ n = 0$) and the diagonal $m = n$ [21]. Along these lines the bicepstrum is equal to the complex cepstrum, i.e.,

$$b_x(m, n) = \begin{cases} \hat{h}(m) & m \neq 0,\ n = 0 \\ \hat{h}(n) & n \neq 0,\ m = 0 \\ \hat{h}(-n) & m = n,\ m \neq 0 \\ \ln(c\gamma_n^v) & m = n = 0, \\ 0 & \text{elsewhere} \end{cases} \tag{76.30}$$

where $\hat{h}(n)$ denotes complex cepstrum [20]. From (76.30), the system impulse response $h(k)$ can be reconstructed from $b_x(m, 0)$ (or $b_x(0, m)$, or $b_x(m, m)$), within a constant and a time delay, via inverse cepstrum operations. The minimum and maximum phase parts of $H(z)$ can be reconstructed by applying inverse cepstrum operations on $b_x(m, 0)u(m)$ and $b_x(m, 0)u(-m)$, respectively, where $u(m)$ is the unit step function.

To avoid phase unwrapping with the logarithm of the bispectrum which is complex, the bicepstrum can be estimated using the group delay approach:

$$b_x(m, n) = \frac{1}{m} F^{-1}\{\frac{F\left[\tau_1 c_3^x(\tau_1, \tau_2)\right]}{C_3^x(\omega_1, \omega_2)}\},\ m \neq 0 \tag{76.31}$$

with $b_x(0, n) = b_x(n, 0)$, and $F\{\cdot\}$ and $F^{-1}\{\cdot\}$ denoting 2-D Fourier transform operator and its inverse, respectively.

The cepstrum of the system can also be computed directly from the cumulants of the system output based on the equation [21]:

$$\sum_{k=1}^{\infty} k\hat{h}(k)\left[c_3^x(m-k, n) - c_3^x(m+k, n+k)\right] + k\hat{h}(-k)\left[c_3^x(m-k, n-k) - c_3^x(m+k, n)\right]$$
$$= mc_3^x(m, n) \tag{76.32}$$

If $H(z)$ has no zeros on the unit circle its cepstrum decays exponentially, thus (76.32) can be truncated to yield an approximate equation. An overdetermined system of truncated equations can be formed for different values of m and n, which can be solved for $\hat{h}(k),\ k = \ldots, -1, 1, \ldots$. The system response $h(k)$ then can be recovered from its cepstrum via inverse cepstrum operations.

The bicepstrum approach for system reconstruction described above led to estimates with smaller bias and variance than other parametric approaches at the expense of higher computational complexity [21]. The analytic performance evaluation of the bicepstrum approach can be found in [25].

The inverse Z-transform of the logarithm of the trispectrum (fourth-order spectrum), or otherwise tricepstrum, $t_x(m, n, l)$, of linear processes is also zero everywhere except along the axes and the diagonal $m = n = l$. Along these lines it equals the complex cepstrum, thus $h(k)$ can be recovered from slices of the tricepstrum based on inverse cepstrum operations.

For the case of nonlinear processes, the bicepstrum will be nonzero everywhere [4]. The distinctly different structure of the bicepstrum corresponding to linear and nonlinear processes has led to tests of linearity [4].

A new nonparametric method has been recently proposed in [1, 26] in which the cepstrum $\hat{h}(k)$ is obtained as:

$$\hat{h}(-k) = \frac{\hat{p}_n^x\left(k;\ e^{j\beta_1}\right) - \hat{p}_n^x\left(k; e^{j\beta_2}\right)}{e^{j(n-2)\beta_1 k} - e^{j(n-2)\beta_2 k}},\ k \neq 0,\ n > 2 \tag{76.33}$$

where $p_n^x\left(k;\ e^{jb_i}\right)$ is the time domain equivalent of the nth order spectrum slice defined as:

$$P_n^x\left(z;\ e^{j\beta_i}\right) = C_n^x\left(z,\ e^{j\beta_i}, \cdots, e^{j\beta_i}\right)\ . \tag{76.34}$$

The denominator of (76.33) is nonzero if

$$|\beta_1 - \beta_2| \neq \frac{2\pi l}{k(n-2)},\ \text{for every integer } k \text{ and } l\ . \tag{76.35}$$

This method reconstructs a complex system using two slices of the nth order spectrum. The slices, defined as shown above, can be selected arbitrarily as long as their distance satisfy (76.35). If the system is real, one slices is sufficient for the reconstruction. It should be noted that the cepstra appearing in (76.33) require phase unwrapping. The main advantage of this method is that the freedom to choose the higher-order spectra areas to be used in the reconstruction allows one to avoid regions dominated by noise or finite data length effects. Also, corresponding to different slice pairs various independent representations of the system can be reconstructed. Averaging out these representations can reduce estimation errors [26].

Along the lines of system reconstruction from selected HOS slices, another method has been proposed in [28, 29] where the log $H(k)$ is obtained as a solution to a linear system of equations. Although logarithimc operation is involved, no phase unwrapping is required and the principal argument can be used instead of real phase. It was also shown that, as long as the grid size and the distance between the slices are coprime, reconstruction is always possible.

76.4.2 Parametric Methods

One of the popular approaches in system identification has been the construction of a white noise driven, linear time invariant model from a given process realization.

Consider the real autoregressive moving average (ARMA) stable process $y(k)$ given by:

$$\sum_{i=0}^{p} a(i)y(k-i) = \sum_{j=0}^{q} b(j)v(k-j) \tag{76.36}$$

$$x(k) = y(k) + w(k) \tag{76.37}$$

where $a(i)$, $b(j)$ represent the AR and MA parameters of the system, $v(k)$ is an independent identically distributed random process, and $w(k)$ represents zero-mean Gaussian noise.

Equations analogous to the Yule-Walker equations can be derived based on third-order cumulants of $x(k)$, i.e.,

$$\sum_{i=0}^{p} a(i) c_3^x(\tau - i, j) = 0, \ \tau > q \ , \tag{76.38}$$

or

$$\sum_{i=1}^{p} a(i) c_3^x(\tau - i, j) = -c_3^x(\tau, j), \ \tau > q \ , \tag{76.39}$$

where it was assumed $a(0) = 1$. Concatenating (76.39) for $\tau = q+1, \ldots, q+M, \ M \geq 0$ and $j = q-p, \ldots, q$, the matrix equation

$$\underline{C}\underline{a} = \underline{c} \tag{76.40}$$

can be formed, where $\underline{C}$ and $\underline{c}$ are a matrix and a vector, respectively, formed by third-order cumulants of the process according to (76.39), and the vector $\underline{a}$ contains the AR parameters. If the AR order p is unknown and (76.40) is formed based on an overestimate of p, the resulting matrix $\underline{C}$ always has rank p. In this case, the AR parameters can be obtained using a low-rank approximation of $\underline{C}$ [5].

Using the estimated AR parameters, $\hat{a}(i), \ i = 1, \ldots, p$, a pth order filter with transfer function $\hat{A}(z) = 1 + \sum_{i=1}^{p} \hat{a}(i) z^{-1}$ can be constructed. Based on the filtered through $\hat{A}(z)$ process $x(k)$, i.e., $\tilde{x}(k)$, or otherwise known as the residual time series [5], the MA parameters can be estimated via any MA method [15], for example:

$$b(k) = \frac{c_3^{\tilde{x}}(q, k)}{c_3^{\tilde{x}}(q, 0)}, \ k = 0, 1, \ldots, q \tag{76.41}$$

known as the $c(q, k)$ formula [6].

Practical problems associated with the described approach are sensitivity to model order mismatch, and AR estimation errors that propagate in the estimation of the MA parameters. A significant amount of research has been devoted to the ARMA parameter estimation problem. A thorough review of existing ARMA system identification methods can be found in [15, 16]; a more recent method can be found in [24].

76.5 Nonlinear Processes

Despite the fact that progress has been established in developing the theoretical properties of nonlinear models, only a few statistical methods exist for detection and characterization of nonlinearities from a finite set of observations. In this section, we will consider nonlinear Volterra systems excited by Gaussian stationary inputs. Let $y(k)$ be the response of a discrete time invariant pth order Volterra filter whose input is $x(k)$. Then,

$$y(k) = h_0 + \sum_i \sum_{\tau_1, \ldots, \tau_i} h_i \left(\tau_1, \ldots, \tau_i\right) x\left(k - \tau_1\right) \cdots x\left(k - \tau_i\right) \ , \tag{76.42}$$

where $h_i(\tau_1, \ldots, \tau_i)$ are the Volterra kernels of the system, which are symmetric functions of their arguments; for causal systems $h_i(\tau_1, \ldots, \tau_i) = 0$ for any $\tau_i < 0$.

The output of a second-order Volterra system when the input is zero-mean stationary is

$$y(k) = h_0 + \sum_{\tau_1} h_1(\tau_1) x(k - \tau_1) + \sum_{\tau_1} \sum_{\tau_2} h_2\left(\tau_1, \tau_2\right) x\left(k - \tau_1\right) x\left(k - \tau_2\right) \ . \tag{76.43}$$

Equation (76.43) can be viewed as a parallel connection of a linear system $h_1(\tau_1)$ and a quadratic system $h_2(\tau_1, \tau_2)$ as illustrated in Fig. 76.3. Let

$$c_2^{xy}(\tau) = E\left\{x(k+\tau)\left[y(k) - m_1^y\right]\right\} \tag{76.44}$$

be the cross-covariance of input and output, and

$$c_3^{xxy}(\tau_1, \tau_2) = E\left\{x\,(k+\tau_1)\,x\,(k+\tau_2)\left[y(k) - m_1^y\right]\right\} \tag{76.45}$$

be the third-order cross-cumulant sequence of input and output.

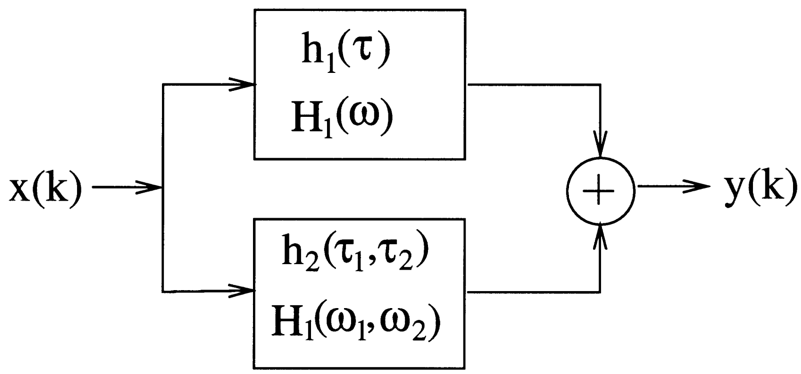

FIGURE 76.3: Second-order Volterra system. Linear and quadratic parts are connected in parallel.

It can be shown that the system's linear part can be identified by

$$H_1(-\omega) = \frac{C_2^{xy}(\omega)}{C_2^x(\omega)}\,, \tag{76.46}$$

and the quadratic part by

$$H_2\,(-\omega_1, -\omega_2) = \frac{C_3^{xxy}\,(\omega_1, \omega_2)}{2C_2^x\,(\omega_1)\,C_2^x\,(\omega_2)}\,, \tag{76.47}$$

where $C_2^{xy}(\omega)$ and $C_3^{xxy}(\omega_1, \omega_2)$ are the Fourier transforms of $c_2^{xy}(\tau)$ and $c_3^{xxy}(\tau_1, \tau_2)$, respectively. It should be noted that the above equations are valid only for Gaussian input signals. More general results assuming non-Gaussian input have been obtained in [9, 27]. Additional results on particular nonlinear systems have been reported in [3, 33].

An interesting phenomenon caused by a second-order nonlinearity is the quadratic phase coupling. There are situations where nonlinear interaction between two harmonic components of a process contribute to the power of the sum and/or difference frequencies. The signal

$$x(k) = A\cos\left(\lambda_1 k + \theta_1\right) + B\cos\left(\lambda_2 k + \theta_2\right) \tag{76.48}$$

after passing through the quadratic system:

$$z(k) = x(k) + \epsilon x^2(k), \quad \epsilon \neq 0\,. \tag{76.49}$$

contains cosinusoidal terms in (λ_1, θ_1), (λ_2, θ_2), $(2\lambda_1, 2\theta_1)$, $(2\lambda_2, 2\theta_2)$, $(\lambda_1 + \lambda_2, \theta_1 + \theta_2)$, $(\lambda_1 - \lambda_2, \theta_1 - \theta_2)$. Such a phenomenon that results in phase relations that are the same as the frequency relations is called quadratic phase coupling [12]. Quadratic phase coupling can arise only among harmonically related components. Three frequencies are harmonically related when one of them is the sum or difference of the other two. Sometimes it is important to find out if peaks at harmonically related positions in the power spectrum are in fact phase coupled. Due to phase suppression, the power spectrum is unable to provide an answer to this problem.

As an example, consider the process [30]

$$X(k) = \sum_{i=1}^{6} \cos(\lambda_i k + \phi_i) \tag{76.50}$$

where $\lambda_1 > \lambda_2 > 0$, $\lambda_4 + \lambda_5 > 0$, $\lambda_3 = \lambda_1 + \lambda_2$, $\lambda_6 = \lambda_4 + \lambda_5$, $\phi_1, \ldots, \phi_5$ are all independent, uniformly distributed random variables over $(0, 2\pi)$, and $\phi_6 = \phi_4 + \phi_5$. Among the six frequencies, $(\lambda_1, \lambda_2, \lambda_3)$ and $(\lambda_4, \lambda_5, \lambda_6)$ are harmonically related, however, only λ_6 is the result of phase coupling between λ_4 and λ_5. The power spectrum of this process consists of six impulses at $\lambda_i, i = 1, \ldots, 6$ (see Fig. 76.4), offering no indication whether each frequency component is independent or result of frequency coupling. On the other hand, the bispectrum of $X(k)$, $C_3^x(\omega_1, \omega_2)$ (evaluate in its principal region) is zero everywhere, except at point (λ_4, λ_5) of the (ω_1, ω_2) plane, where it exhibits an impulse (Fig. 76.4(b)). The peak indicates that only λ_4, λ_5 are phase coupled.

The bicoherence index, defined as

$$P_3^x(\omega_1, \omega_2) = \frac{C_3^x(\omega_1, \omega_2)}{\sqrt{C_2^x(\omega_1)\, C_2^x(\omega_2)\, C_2^x(\omega_1 + \omega_2)}}, \tag{76.51}$$

has been extensively used in practical situations for the detection and quantification of quadratic phase coupling [12]. The value of the bicoherence index at each frequency pair indicates the degree of coupling among the frequencies of that pair. Almost all bispectral estimators can be used in (76.51). However, estimates obtained based on parametric modeling of the bispectrum have been shown to yield superior resolution [30, 31] than the ones obtained with conventional methods.

76.6 Applications/Software Available

Applications of HOS span a wide range of areas [19] such as oceanography (description of wave phenomena), earth sciences (atmospheric pressure, turbulence), crystallography, plasma physics (wave interaction, nonlinear phenomena), mechanical systems (vibration analysis, knock detection), economic time series, biomedical signal analysis (ultrasonic imaging, detection of wave coupling) image processing (texture modeling and characterization, reconstruction, inverse filtering), speech processing (pitch detection, voiced/unvoiced decision), communications (equalization, interference cancellation), array processing (direction of arrival estimation, estimation of number of sources, beamforming, source signal estimation, source classification), harmonic retrieval (frequency estimation), and time delay estimation. Over 500 references can be found in [37]. Additional references can be found in [16, 19, 23].

A software package for signal processing with HOS is the Hi-Spec toolbox, product of Mathworks, Inc. The functions included in Hi-Spec together with a short description are included in Table 76.1.

Acknowledgments

Most of the material presented in this chapter is based on the book *Higher-Order Spectra Analysis: A Non-Linear Signal Processing Framework* [16]. The author wishes to thank Dr. C.L. Nikias for his

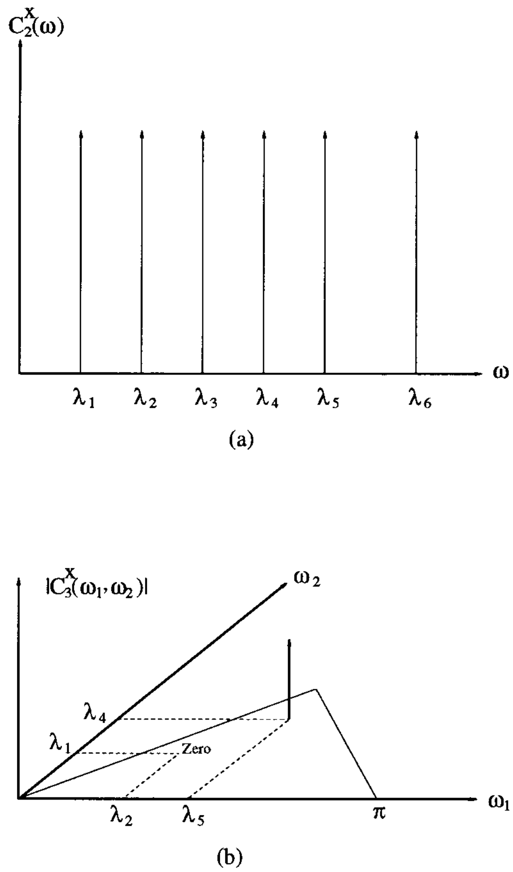

FIGURE 76.4: Quadratic phase coupling. (a) The power spectrum of the process described in Eq. (76.50) cannot determine what frequencies are coupled. (b) The corresponding magnitude bispectrum is zero everywhere in the principle region, except at points corresponding to phase coupled frequencies.

valuable input on the organization of the material. She also thanks U.R. Abeyratne for producing the figures in this chapter. Support for this work came from NSF under grant MIP-9553227 and the Whitaker Foundation.

TABLE 76.1 Functions Included in the Hi-Spec Package

Function name	Description
AR_RCEST	AR parameter estimation based on cumulants
ARMA_QS	ARMA parameter estimation via the Q-slice algorithm
ARMA_RTS	ARMA parameter estimation via the residual time series method
ARMA_SYN	Generates ARMA synthetics
BICEPS	System identification via the bicepstrum approach
BISPEC_D	Bispectrum estimation via the direct method
BISPEC_I	Bispectrum estimation via the indirect method
CUM_EST	Estimates 2nd, 3rd, or 4th order cumulants
CUM_TRUE	Computes the theoretical cumulants of an ARMA model
DOA	Direction-of-arrival estimation
DOA_GEN	Generates synthetics for direction-of-arrival estimation
GL_STAT	Detection statistics for Hinich's Gaussianity and linearity tests
HARM_EST	Estimates frequencies of harmonics in colored noise
HARM_GEN	Generates synthetics for the harmonic retrieval problem
MA_EST	MA parameters estimation
MATUL	System identification via the Matsuoka-Ulrych algorithm
QPC_GEN	Simulation generator for quadratic phase coupling
QPC_TOR	Detects quadratic phase coupling via parametric modeling of bispectrum
RP_IID	Generates samples of an i.i.d. random process
TDE	Estimates time delay between two signals using the parametric cross-cumulant method
TDE_GEN	Synthetics for time delay estimation

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