4

Other constructions of wavelets

We have constructed several types of wavelets in the previous chapters. Among them, the Haar wavelet, that appeared as an example of a multiresolution analysis (see Examples A and B of Chapter 2). This wavelet is obtained by taking V_0 to be the space of all functions in $L^2(\mathbb{R})$ which are constant on intervals of the form $[k, k+1]$, $k \in \mathbb{Z}$. It is natural to generalize this construction by allowing a greater degree of smoothness for the functions of V_0. By doing so we obtain the Franklin and more general spline wavelets in the next two sections. It is important to indicate that these functions, that have been used in applications for more than 25 years, have found a natural and most useful place in the recent theory of wavelets.

The previous chapters treat the case of orthonormal wavelets in $L^2(\mathbb{R})$. There is a procedure to periodize the wavelets defined on the real line and obtain orthonormal, periodic wavelets for $L^2(\mathbb{T})$. This is presented successively in section 4.3 (for the Franklin wavelets), in section 4.4 (for the more general spline wavelets) and in section 4.5 (for very general wavelets arising from multiresolution analyses).

4.1 Franklin wavelets on the real line

We shall show how to construct a piecewise linear continuous function on $\mathbb{R}$ which is an orthonormal wavelet, and we will show that it is unique in a sense that is described below. This will be called a Franklin wavelet. We will construct this wavelet using the notion of a multiresolution analysis (MRA) introduced in Chapter 2.

Let V_0 be the space of all functions $f \in L^2(\mathbb{R})$ which are continuous on $\mathbb{R}$ and linear when restricted to each interval of the form $[k, k+1]$, $k \in \mathbb{Z}$. Define V_j, $j \in \mathbb{Z}$, as the space of all functions $f \in L^2(\mathbb{R})$ such that $f(2^{-j}\cdot) \in V_0$; the functions in V_j are continuous on $\mathbb{R}$ and linear on each interval of the form $[2^{-j}k, 2^{-j}(k+1)]$, $k \in \mathbb{Z}$. It is clear that properties (1.1), (1.2), (1.3) and (1.4) of the definition of a multiresolution analysis (see section 2.1) are satisfied by this sequence of subspaces. In order to show property (1.5) (on page 44) of an MRA we need to find a scaling function; that is, a function φ in V_0 whose integer translates form an orthonormal basis of V_0. We begin by studying the functions in V_0.

It is clear that the sequence of values $\{f(k)\}_{k\in\mathbb{Z}}$ completely determines the general function $f \in V_0$. In fact, this sequence $\{f(k)\}_{k\in\mathbb{Z}}$ must belong to $\ell^2(\mathbb{Z})$ and, conversely, any such sequence determines a unique function $f \in V_0$. We shall show, in fact,

$$\tfrac{1}{3}\big\|\{f(k)\}\big\|^2_{\ell^2(\mathbb{Z})} \le \big\|f\big\|^2_{L^2(\mathbb{R})} \le \big\|\{f(k)\}\big\|^2_{\ell^2(\mathbb{Z})} \qquad \text{for all } \; f \in V_0. \tag{1.1}$$

The linearity of f on $[k, k+1]$ allows us to write

$$\begin{aligned}\int_k^{k+1} |f(x)|^2\,dx &= \int_0^1 \big[(1-t)f(k) + tf(k+1)\big]^2\,dt\\ &= \frac{[f(k)]^2 + [f(k+1)]^2}{3} + \frac{f(k)f(k+1)}{3}.\end{aligned}$$

Now observe that

$$\begin{aligned}\frac{[f(k)]^2 + [f(k+1)]^2}{6} &\le \frac{[f(k)]^2 + [f(k+1)]^2}{3} + \frac{f(k)f(k+1)}{3}\\ &\le \frac{[f(k)]^2 + [f(k+1)]^2}{2}.\end{aligned}$$

But

$$\big\|f\big\|^2_{L^2(\mathbb{R})} = \sum_{k=-\infty}^{\infty} \int_k^{k+1} |f(x)|^2\,dx,$$

and, therefore, the above inequalities give us (1.1), since

$$\sum_{k\in\mathbb{Z}} [f(k)]^2 = \big\|\{f(k)\}\big\|^2_{\ell^2(\mathbb{Z})}.$$

Let

$$\Delta(x) = \begin{cases} x & \text{if } \ 0 \le x \le 1, \\ 2 - x & \text{if } \ 1 < x \le 2, \\ 0 & \text{otherwise,} \end{cases}$$

be the "triangle" function depicted in Figure 4.1. It is clear that every $f \in V_0$ has the representation

$$f(x) = \sum_{k \in \mathbb{Z}} f(k)\Delta(x - k + 1). \tag{1.2}$$

If $\chi = \chi_{[0,1]}$ we have $\hat{\chi}(\xi) = e^{-i\frac{\xi}{2}} \dfrac{\sin(\xi/2)}{\xi/2}$ and $\Delta = \chi * \chi$; hence

$$\widehat{\Delta}(\xi) = \left[\hat{\chi}(\xi)\right]^2 = e^{-i\xi}\left(\frac{\sin(\xi/2)}{\xi/2}\right)^2.$$

From (1.2) we have

$$\begin{aligned} \hat{f}(\xi) &= e^{-i\xi}\left(\frac{\sin(\xi/2)}{\xi/2}\right)^2 \sum_{k \in \mathbb{Z}} f(k) e^{-i(k-1)\xi} \\ &= \left(\frac{\sin(\xi/2)}{\xi/2}\right)^2 \sum_{k \in \mathbb{Z}} f(k) e^{-ik\xi} \equiv \left(\frac{\sin(\xi/2)}{\xi/2}\right)^2 m_f(\xi), \end{aligned}$$

where $m_f(\xi)$ is a 2π-periodic function on $\mathbb{R}$ belonging to $L^2(\mathbb{T})$. In fact,

$$\|m_f\|^2_{L^2(\mathbb{T})} = 2\pi \|\{f(k)\}\|^2_{\ell^2(\mathbb{Z})}.$$

We have proved the following result (compare this result with the characterization of V_0 given in Lemma 2.6 of Chapter 2 for a general MRA).

Theorem 1.3 *A function f in $L^2(\mathbb{R})$ belongs to V_0 if and only if*

$$\hat{f}(\xi) = \left(\frac{\sin(\xi/2)}{\xi/2}\right)^2 m_f(\xi),$$

where m_f is a 2π-periodic function in $L^2(\mathbb{T})$. Moreover,

$$\|f\|_{L^2(\mathbb{R})} \approx \|m_f\|_{L^2(\mathbb{T})}.$$

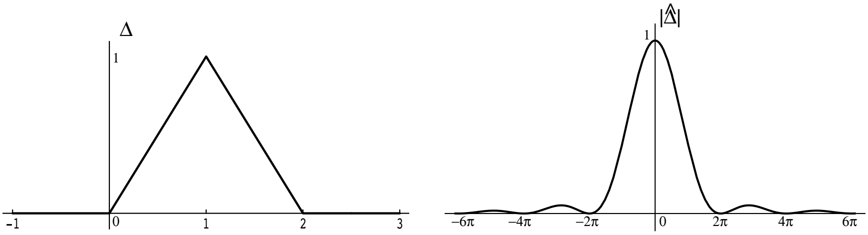

Figure 4.1: Graphs of Δ *and* $|\widehat{\Delta}|$.

There is another way of characterizing the elements of V_0:

THEOREM 1.4 *A function* f *in* $L^2(\mathbb{R})$ *belongs to* V_0 *if and only if* $\xi^2\hat{f}(\xi)$ *is a* 2π*-periodic function on* $\mathbb{R}$.

PROOF : Theorem 1.3 immediately implies that, if $f \in V_0$, then

$$\xi^2\hat{f}(\xi) = 4\sin^2(\tfrac{1}{2}\xi)m_f(\xi)$$

is a 2π-periodic function on $\mathbb{R}$.

Assume, now, that $\xi^2\hat{f}(\xi)$ is a 2π-periodic function on $\mathbb{R}$. Define

$$\mu(\xi) = \frac{\xi^2\hat{f}(\xi)}{4\sin^2(\frac{1}{2}\xi)}\,.$$

The 2π-periodicity of the functions involved and the boundedness of the function $\dfrac{\xi^4}{16\sin^4(\xi/2)}$ on $[-\pi,\pi]$ allow us to obtain

$$\int_0^{2\pi}|\mu(\xi)|^2\,d\xi = \int_{-\pi}^{\pi}\frac{\xi^4|\hat{f}(\xi)|^2}{16\sin^4(\xi/2)}\,d\xi \le C\int_{-\pi}^{\pi}|\hat{f}(\xi)|^2\,d\xi \le C_o\|f\|^2_{L^2(\mathbb{R})}.$$

Thus $\mu \in L^2(\mathbb{T})$ and we can write its Fourier series in the form

$$\mu(\xi) \sim \sum_{k\in\mathbb{Z}}\hat{\mu}(k)e^{-ik\xi}$$

with $\sum_{k\in\mathbb{Z}}|\hat{\mu}(k)|^2 < \infty$. Since $\widehat{\Delta}(\xi) = e^{-i\xi}\dfrac{\sin^2(\xi/2)}{(\xi/2)^2}$ we obtain

$$\hat{f}(\xi) = \frac{\sin^2(\xi/2)}{(\xi/2)^2}\mu(\xi) = e^{-i\xi}\frac{\sin^2(\xi/2)}{(\xi/2)^2}\sum_{k\in\mathbb{Z}}\hat{\mu}(k)e^{-i(k-1)\xi}$$

$$= \sum_{k\in\mathbb{Z}} \hat{\mu}(k)\widehat{\Delta}(\xi)e^{-i(k-1)\xi} = \sum_{k\in\mathbb{Z}} \hat{\mu}(k)(\tau_{k-1}\Delta)^{\wedge}(\xi),$$

where $\tau_k\Delta(x) = \Delta(x-k)$. By taking the inverse Fourier transform we deduce

$$f(x) = \sum_{k\in\mathbb{Z}} \hat{\mu}(k)\Delta(x-k+1).$$

This shows that $f \in V_0$, since $\{\Delta(\cdot - k) : k \in \mathbb{Z}\}$ is a basis of V_0. ∎

Our aim is to prove that the sequence of subspaces $\{V_j : j \in \mathbb{Z}\}$ forms an MRA. The scaling function φ must belong to the space V_0 which we have characterized in Theorems 1.3 and 1.4. The wavelet ψ must belong to V_1. Two characterizations of this space can be easily obtained from the above theorems.

THEOREM 1.5 *Suppose* $g \in L^2(\mathbb{R})$.

(a) $g \in V_1$ *if and only if*

$$\hat{g}(\xi) = \frac{\sin^2(\xi/4)}{(\xi/4)^2}\, m_g(\xi),$$

where m_g *is a* 4π*-periodic function in* $L^2([0,4\pi])$. *Moreover,*

$$\|g\|_{L^2(\mathbb{R})} \approx \|m_g\|_{L^2([0,4\pi])}.$$

(b) $g \in V_1$ *if and only if* $\xi^2\hat{g}(\xi)$ *is a* 4π*-periodic function on* $\mathbb{R}$.

PROOF: For $g \in L^2(\mathbb{R})$ define $f(x) = g(\frac{1}{2}x)$. Then g belongs to V_1 if and only if f belongs to V_0. Moreover, $\hat{f}(\xi) = 2\hat{g}(2\xi)$. The result follows by using the characterizations of V_0 given in Theorem 1.3 and in Theorem 1.4. ∎

The system $\{\Delta(\cdot - k) : k \in \mathbb{Z}\}$ is a basis for V_0, but it is not an orthogonal system. As we stated above, we need to find a scaling function φ for which $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal basis of V_0. By Proposition 1.11 of Chapter 2, the orthonormality of the system $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ is equivalent to

$$\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + 2k\pi)|^2 = 1 \qquad \text{for } a.e.\ \xi \in \mathbb{R}. \tag{1.6}$$

Theorem 1.3 gives us

$$\hat{\varphi}(\xi) = \frac{\sin^2(\xi/2)}{(\xi/2)^2}\, m_\varphi(\xi), \tag{1.7}$$

where m_φ is a 2π-periodic function belonging to $L^2(\mathbb{T})$.

From (1.6) we obtain

$$\begin{aligned} 1 &= \sum_{k\in\mathbb{Z}} \Big(\frac{2\sin(\frac{1}{2}\xi + k\pi)}{\xi + 2k\pi}\Big)^4 |m_\varphi(\xi)|^2 \\ &= 16\sin^4(\tfrac{1}{2}\xi)|m_\varphi(\xi)|^2 \sum_{k\in\mathbb{Z}} \frac{1}{(\xi + 2k\pi)^4}\,. \end{aligned} \tag{1.8}$$

The following lemma gives us the value of the infinite sum on the right-hand side of the above equality.

LEMMA 1.9 *For every $\xi \in \mathbb{R}$ we have*

$$\sum_{k\in\mathbb{Z}} \frac{1}{(\xi + 2k\pi)^4} = \frac{1}{16\sin^4(\frac{1}{2}\xi)}\big\{1 - \tfrac{2}{3}\sin^2(\tfrac{1}{2}\xi)\big\}.$$

PROOF : The lemma is proved by differentiating twice both sides of the equality

$$\sum_{k\in\mathbb{Z}} \frac{1}{(\xi + 2k\pi)^2} = \frac{1}{4\sin^2(\frac{1}{2}\xi)}\,.$$

To prove this formula we consider the function $\chi = \chi_{[0,1]}$. Since

$$\hat{\chi}(\xi) = e^{-i\frac{\xi}{2}}\frac{\sin(\xi/2)}{\xi/2}$$

and $\{\chi(\cdot - k) : k \in \mathbb{Z}\}$ is clearly an orthonormal system in $L^2(\mathbb{R})$, we use Proposition 1.11 of Chapter 2 to obtain the result:

$$1 = \sum_{k\in\mathbb{Z}} |\hat{\chi}(\xi + 2k\pi)|^2 = \sum_{k\in\mathbb{Z}} \frac{4\sin^2(\frac{1}{2}\xi + k\pi)}{(\xi + 2k\pi)^2} = 4\sin^2(\tfrac{1}{2}\xi)\sum_{k\in\mathbb{Z}} \frac{1}{(\xi + 2k\pi)^2}\,.$$

∎

We now proceed with our search of the scaling function φ. Equality (1.8) and Lemma 1.9 give us

$$|m_\varphi(\xi)| = \bigl(1 - \tfrac{2}{3}\sin^2(\tfrac{1}{2}\xi)\bigr)^{-\frac{1}{2}}. \tag{1.10}$$

This shows that the orthonormality of the system $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ completely determines the absolute values of m_φ and $\hat{\varphi}$ (see (1.7)). We choose

$$\hat{\varphi}(\xi) = \frac{\sin^2(\xi/2)}{(\xi/2)^2}\bigl(1 - \tfrac{2}{3}\sin^2(\tfrac{1}{2}\xi)\bigr)^{-\frac{1}{2}}. \tag{1.11}$$

Reversing the above steps we also see that if φ is given by (1.11), $\hat{\varphi}$ satisfies (1.6), showing, therefore, that $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal system in V_0. We shall show that this system is also complete in V_0. To see this consider the bijection from V_0 to $L^2(\mathbb{T})$ given by $f \mapsto m_f$ (see Theorem 1.3). The completeness of the translates of φ is seen to be equivalent to the completeness of the system $\{m_\varphi(\xi)e^{-ik\xi} : k \in \mathbb{Z}\}$ in $L^2(\mathbb{T})$. Since

$$\bigl(1 - \tfrac{2}{3}\sin^2(\tfrac{1}{2}\xi)\bigr)^{-\frac{1}{2}} = \bigl(\tfrac{2}{3} + \tfrac{1}{3}\cos\xi\bigr)^{-\frac{1}{2}}$$

is bounded above and below on $\mathbb{T}$, (1.10) shows that the completeness of the above system is equivalent to the well known completeness of the system $\{e^{-ik\xi} : k \in \mathbb{Z}\}$ in $L^2(\mathbb{T})$. This proves the desired result.

We have produced an MRA, $\{V_j : j \in \mathbb{Z}\}$, where V_j is the set of continuous functions belonging to $L^2(\mathbb{R})$ which are linear on each interval of the form $[2^{-j}k, 2^{-j}(k+1)]$, $k \in \mathbb{Z}$, whose scaling function φ can be chosen so that it satisfies (1.11). The low-pass filter m_0 of this MRA, associated with this scaling function φ, can be obtained from the equation $\hat{\varphi}(2\xi) = m_0(\xi)\hat{\varphi}(\xi)$ (see (2.3) of Chapter 2) and (1.11):

$$\begin{aligned} m_0(\xi) &= \frac{(\sin\xi)^2\bigl(1 - \frac{2}{3}\sin^2(\frac{1}{2}\xi)\bigr)^{\frac{1}{2}}}{\bigl(2\sin(\frac{1}{2}\xi)\bigr)^2\bigl(1 - \frac{2}{3}\sin^2\xi\bigr)^{\frac{1}{2}}} \\ &= \bigl(\cos(\tfrac{1}{2}\xi)\bigr)^2\Bigl(\frac{1 - \frac{2}{3}\sin^2(\frac{1}{2}\xi)}{1 - \frac{2}{3}\sin^2\xi}\Bigr)^{\frac{1}{2}}. \end{aligned} \tag{1.12}$$

An orthonormal wavelet ψ associated with the MRA we have just constructed can be given by (2.12) of Chapter 2; ψ satisfies

$$\hat{\psi}(2\xi) = e^{i\xi}\,\overline{m_0(\xi + \pi)}\,\hat{\varphi}(\xi)$$

$$= e^{i\xi}\sin^2(\tfrac{1}{2}\xi)\Big(\frac{1-\frac{2}{3}\cos^2(\frac{1}{2}\xi)}{1-\frac{2}{3}\sin^2\xi}\Big)^{\frac{1}{2}}\Big(\frac{\sin(\xi/2)}{\xi/2}\Big)^2\Big(1-\tfrac{2}{3}\sin^2(\tfrac{1}{2}\xi)\Big)^{-\frac{1}{2}}$$

$$= e^{i\xi}\frac{\sin^4(\frac{1}{2}\xi)}{(\frac{1}{2}\xi)^2}\Big(\frac{1-\frac{2}{3}\cos^2(\frac{1}{2}\xi)}{(1-\frac{2}{3}\sin^2\xi)(1-\frac{2}{3}\sin^2(\frac{1}{2}\xi))}\Big)^{\frac{1}{2}}.$$

For future reference the results we have obtained are collected in the following theorem:

THEOREM 1.13 *For each $j \in \mathbb{Z}$, let V_j be the subspace of $L^2(\mathbb{R})$ of all continuous functions on $\mathbb{R}$ which are linear on each interval of the form $[2^{-j}k, 2^{-j}(k+1)]$, $k \in \mathbb{Z}$. Then the sequence $\{V_j : j \in \mathbb{Z}\}$ forms an MRA for $L^2(\mathbb{R})$. An associated scaling function φ can be defined by* (1.11). *The orthonormal wavelet ψ associated with this scaling function via* (2.12) *of* Chapter 2 *satisfies*

$$\hat{\psi}(\xi) = e^{i\frac{\xi}{2}}\frac{\sin^4(\frac{1}{4}\xi)}{(\frac{1}{4}\xi)^2}\Big(\frac{1-\frac{2}{3}\cos^2(\frac{1}{4}\xi)}{(1-\frac{2}{3}\sin^2(\frac{1}{2}\xi))(1-\frac{2}{3}\sin^2(\frac{1}{4}\xi))}\Big)^{\frac{1}{2}}. \tag{1.14}$$

The wavelet we have constructed is often called the **Franklin wavelet**. (Though this name is also applied to "equivalent" wavelets in the sense of Proposition 2.13 of Chapter 2.) We shall obtain several properties of the Franklin wavelet ψ and the scaling function φ. The ones contained in the next result are a consequence of formula (1.14) that describes the Fourier transform of the Franklin wavelet ψ.

PROPOSITION 1.15 *The Franklin wavelet ψ, whose Fourier transform is given by* (1.14), *satisfies:*

i) $\displaystyle\int_{\mathbb{R}}\psi(x)\,dx = 0 = \int_{\mathbb{R}} x\psi(x)\,dx;$

ii) ψ *is symmetric with respect to* $x = -\frac{1}{2}$.

PROOF: Part i) follows from $\hat{\psi}(0) = 0$ and $\frac{d\hat{\psi}}{d\xi}(0) = 0$, which is an easy consequence of (1.14). Part ii) follows if we show $\psi(-1-x) = \psi(x)$. Write $\hat{\psi}(\xi) = e^{i\frac{\xi}{2}}\gamma(\xi)$; then

$$\psi(-1-x) = \frac{1}{2\pi}\int_{\mathbb{R}}\hat{\psi}(\xi)e^{i\xi(-1-x)}\,d\xi = \frac{1}{2\pi}\int_{\mathbb{R}}\hat{\psi}(\xi)e^{-i\xi}e^{-ix\xi}\,d\xi$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}}\gamma(\xi)e^{-i\frac{\xi}{2}}e^{-ix\xi}\,d\xi.$$

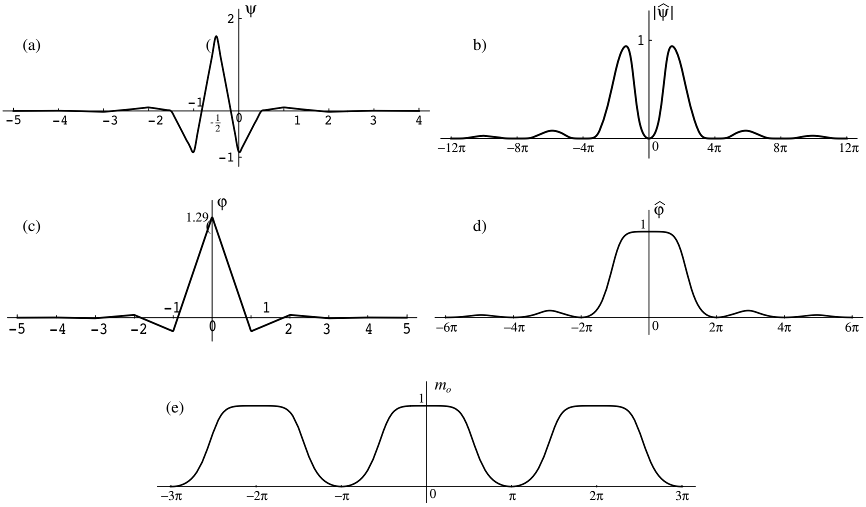

Figure 4.2: Graphs of several functions related to the Franklin wavelet:

(a) *the Franklin wavelet* ψ,
(b) *the modulus of the Fourier transform of the Franklin wavelet,*
(c) *the scaling function* φ *given in* (1.11),
(d) *the Fourier transform of the scaling function,*
(e) *the corresponding low-pass filter* m_0.

By (1.14) $\gamma(\xi)$ is even and we obtain

$$\psi(-1-x) = \frac{1}{2\pi}\int_{\mathbb{R}} \gamma(\xi) e^{i\frac{\xi}{2}} e^{ix\xi}\, d\xi = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\psi}(\xi) e^{ix\xi}\, dx = \psi(x).$$

∎

The next property refers to the decay at infinity of the Franklin wavelet and the associated scaling function. We say that a function g has **exponential decay** if there exist constants $C > 0$ and $\alpha > 0$ such that

$$|g(x)| \leq Ce^{-\alpha|x|} \qquad \text{for all } \; x \in \mathbb{R}.$$

PROPOSITION 1.16 *The Franklin wavelet* ψ *defined by* (1.14) *and its associated scaling function* φ *defined by* (1.11) *have exponential decay.*

PROOF : We start by proving the result for the wavelet ψ. Write

$$\hat{\psi}(\xi) = e^{i\frac{\xi}{2}}\gamma(\xi)$$

as in the proof of Proposition 1.15. Consider the region $\Gamma(\xi_o, \eta_o)$ described in Figure 4.3 with η_o small enough so that

$$\gamma(z) = \frac{\sin^4(\frac{1}{4}z)}{(\frac{1}{4}z)^2}\Big(\frac{1-\frac{2}{3}\cos^2(\frac{1}{4}z)}{(1-\frac{2}{3}\sin^2(\frac{1}{2}z))(1-\frac{2}{3}\sin^2(\frac{1}{4}z))}\Big)^{\frac{1}{2}}.$$

has no poles in $0 < |\Im m(z)| < \eta_o$. We then have

$$\begin{aligned} 0 = \int_{\partial\Gamma(\xi_o,\eta_o)} \gamma(z)e^{ixz}\,dz &= \int_{-\xi_o}^{\xi_o} \gamma(\xi)e^{ix\xi}\,d\xi + \int_0^{\eta_o} \gamma(\xi_o + i\eta)e^{ix(\xi_o+i\eta)}\,d\eta \\ &- \int_{-\xi_o}^{\xi_o} \gamma(\xi + i\eta_o)e^{ix(\xi+i\eta_o)}\,d\xi - \int_0^{\eta_o} \gamma(-\xi_o + i\eta)e^{ix(-\xi_o+i\eta)}\,d\eta. \end{aligned}$$

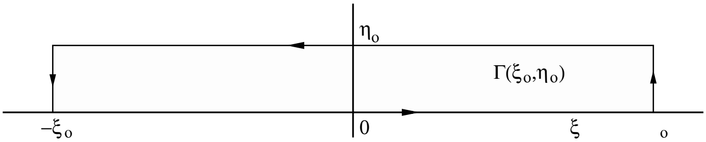

Figure 4.3: The region $\Gamma(\xi_o, \eta_o)$.

It is clear that there exists a constant C_o such that

$$\gamma(\xi + i\eta) \leq \frac{C_o}{\xi^2 + \eta^2} \qquad \text{for all} \quad \xi, \eta \in \mathbb{R},$$

which shows that the second and the fourth terms in the above equality tend to 0 as $|\xi_o| \to \infty$. Therefore,

$$\begin{aligned} \big|\psi(x - \tfrac{1}{2})\big| &= \frac{1}{2\pi}\Big|\int_{\mathbb{R}} \gamma(\xi)e^{ix\xi}\,d\xi\Big| \\ &= \frac{1}{2\pi}\Big|\lim_{|\xi_o|\to\infty} \int_{-\xi_o}^{\xi_o} \gamma(\xi + i\eta_o)e^{ix(\xi+i\eta_o)}\,d\xi\Big| \leq Ce^{-x\eta_o}. \end{aligned}$$

Since ψ is symmetric with respect to $-\frac{1}{2}$ we must have $|\psi(x - \frac{1}{2})| \leq Ce^{-|x|\eta_o}$, which shows the desired result.

The proof of the exponential decay of φ is similar since ξ^2 appears in the denominator of the formula for $\hat{\varphi}$ given in (1.11).

∎

We conclude this section by proving that the scaling function associated with the Franklin wavelet minimizes a certain functional.

THEOREM 1.17 *Among the functions $g \in V_0$ such that $\{g(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal system, the scaling function φ given by* (1.11) *is the unique one, up to a constant factor of absolute value* 1*, that minimizes the functional*

$$\int_{\mathbb{R}} x^2 |g(x)|^2 \, dx.$$

PROOF : From the boundedness and the exponential decay of φ we deduce that

$$\int_{\mathbb{R}} x^2 |\varphi(x)|^2 \, dx < \infty.$$

Hence, it suffices to consider only those $g \in V_0$ such that $\{g(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal system and $\int_{-\infty}^{\infty} x^2 |g(x)|^2 \, dx$ is finite. Thus, $\frac{d\hat{g}}{d\xi}$ exists almost everywhere and belongs to $L^2(\mathbb{R})$. As in the construction of the scaling function φ, the orthonormality of the system $\{g(\cdot - k) : k \in \mathbb{Z}\}$ completely determines $|\hat{g}(\xi)|$ (see (1.10) and (1.7)):

$$|\hat{g}(\xi)| = \Big(\frac{\sin(\xi/2)}{\xi/2} \Big)^2 \big(1 - \tfrac{2}{3} \sin^2(\tfrac{1}{2}\xi)\big)^{-\frac{1}{2}}.$$

Hence, by (1.11) we must have $\hat{g}(\xi) = e^{i\lambda(\xi)} \hat{\varphi}(\xi)$. Since both g and φ belong to V_0 we must have

$$e^{i\lambda(\xi)} = \frac{m_g(\xi)}{m_\varphi(\xi)},$$

where m_g and m_φ are the 2π-periodic functions associated with g and φ, respectively, given in Theorem 1.3. Moreover, $\lambda(\xi)$ can be chosen to be differentiable since $e^{i\lambda(\xi)}$ is the quotient of $\hat{g}(\xi)$ and $\hat{\varphi}(\xi)$ and this function never vanishes. Taking derivatives we obtain

$$\frac{d\hat{g}}{d\xi}(\xi) = i \frac{d\lambda}{d\xi}(\xi) e^{i\lambda(\xi)} \hat{\varphi}(\xi) + e^{i\lambda(\xi)} \frac{d\hat{\varphi}}{d\xi}(\xi).$$

Since $\frac{d\lambda}{d\xi}$, $\hat{\varphi}$, and $\frac{d\hat{\varphi}}{d\xi}$ are real valued we have

$$\begin{aligned} \int_{\mathbb{R}} |xg(x)|^2 \, dx &= \int_{\mathbb{R}} \Big| \frac{d\hat{g}}{d\xi}(\xi) \Big|^2 d\xi = \int_{\mathbb{R}} \Big| \frac{d\hat{\varphi}}{d\xi}(\xi) + i \frac{d\lambda}{d\xi}(\xi) \hat{\varphi}(\xi) \Big|^2 d\xi \\ &\geq \int_{\mathbb{R}} \Big| \frac{d\hat{\varphi}}{d\xi}(\xi) \Big|^2 d\xi = \int_{\mathbb{R}} x^2 |\varphi(x)|^2 \, dx. \end{aligned}$$

This proves the desired minimality property. To see uniqueness, observe that, if we have only equalities in the above estimations, we must have

$$\int_{\mathbb{R}} \left| \frac{d\lambda}{d\xi}(\xi)\hat{\varphi}(\xi) \right|^2 d\xi = 0.$$

Since $\hat{\varphi}$ is never zero (see the definition of $\hat{\varphi}$ given in (1.11)), it follows that $\frac{d\lambda}{d\xi} = 0$ almost everywhere. Thus, $\lambda(\xi)$ is equal to a constant and, thus, $\hat{g}(\xi) = \tilde{k}\hat{\varphi}(\xi)$ with $|\tilde{k}| = 1$. This is, as promised, the uniqueness up to a unimodular multiplicative constant.

▮

4.2 Spline wavelets on the real line

In section 4.1 we have constructed an orthonormal basis of piecewise linear continuous functions on $\mathbb{R}$. The main purpose of this section is to show that a similar construction produces wavelets with greater degrees of smoothness.

For $n = 0, 1, 2, \cdots$, let $\mathcal{P}_n$ be the collection of all polynomials of degree at most n and $C^n \equiv C^n(\mathbb{R})$ be the set of all functions f defined on $\mathbb{R}$ such that all the derivatives of f up to order n exist and $f^{(n)}$ is continuous on $\mathbb{R}$. For $n \in \mathbb{N}$, the space $\mathtt{S}^n \equiv \mathtt{S}^n(\mathbb{R})$ of **splines of order** n is the set of all $f \in C^{n-1}$ such that the restrictions of f to any interval of the form $[k, k+1]$, $k \in \mathbb{Z}$, are in $\mathcal{P}_n$.

A basis for $\mathtt{S}^1$ has been exhibited at the beginning of section 4.1. In fact, we observe that the representation (1.2) shows that $\{\Delta(\cdot - k) : k \in \mathbb{Z}\}$ is a basis of this space, with $\Delta = \chi * \chi$ and $\chi = \chi_{[0,1]}$. The linear independence follows immediately from the fact that $\Delta(\ell - k + 1) = \delta_{\ell,k}$ for all $\ell \in \mathbb{Z}$.

Let $\Delta^2(x) = (\Delta * \chi)(x)$; that is,

$$\Delta^2(x) = (\Delta * \chi)(x) = \int_{\mathbb{R}} \Delta(y)\chi(x-y)\,dy = \int_{x-1}^{x} \Delta(y)\,dy.$$

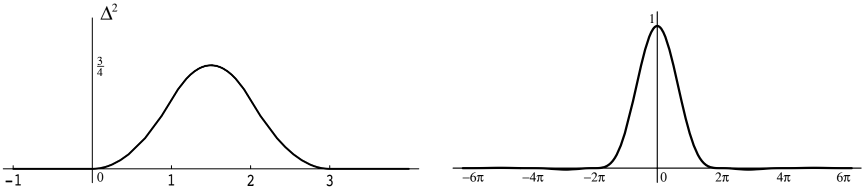

Figure 4.4: Graphs of Δ^2 (left) *and* $e^{i\frac{3}{2}\xi}\widehat{\Delta^2}(\xi)$ (right).

From this, it is easy to deduce

$$\Delta^2(x) = \begin{cases} \frac{1}{2}x^2 & \text{if } \ 0 \le x \le 1, \\ -(x-\frac{3}{2})^2 + \frac{3}{4} & \text{if } \ 1 \le x \le 2, \\ \frac{1}{2}(x-3)^2 & \text{if } \ 2 \le x \le 3, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\widehat{\Delta^2}(\xi) = e^{-i\frac{3}{2}\xi}\left(\frac{\sin(\xi/2)}{\xi/2}\right)^3.$$

It follows that $\Delta^2(\cdot - k) \in \mathtt{S}^2$ for all $k \in \mathbb{Z}$. The functions $\Delta^2(\cdot - k)$ are called the **basic splines of order** 2. (See the graph of Δ^2 in Figure 4.4.)

The procedure just described can be extended to obtain splines of order n. For $n = 2, 3, 4, \cdots$, define

$$\Delta^n = \Delta^{n-1} * \chi, \tag{2.1}$$

where $\Delta^1 \equiv \Delta$. By the definition of convolution we have

$$\Delta^n(x) = \int_{\mathbb{R}} \Delta^{n-1}(y)\chi(x-y)\,dy = \int_{x-1}^{x} \Delta^{n-1}(y)\,dy. \tag{2.2}$$

We can now prove by induction that $\Delta^n \in \mathtt{S}^n$, $n = 1, 2, 3, \cdots$. This is true for $n = 1$ (see Figure 4.1) and $n = 2$ (see Figure 4.4). If $x \in [k, k+1]$, (2.2) shows that

$$\Delta^n(x) = \int_{x-1}^{k} \Delta^{n-1}(y)\,dy + \int_{k}^{x} \Delta^{n-1}(y)\,dy,$$

where $\Delta^{n-1}\big|_{[x-1,k)}$ and $\Delta^{n-1}\big|_{[k,x)}$ are polynomials of degree at most $n-1$; thus, $\Delta^n(x)$ is a polynomial of degree at most n when restricted to $[k, k+1)$. Since $\Delta^{n-1} \in C^{n-2}$, it is clear from (2.2) that $\Delta^n \in C^{n-1}$. This shows the desired result. It follows that $\Delta^n(\cdot - k) \in \mathtt{S}^n$ for all $k \in \mathbb{Z}$. The functions $\Delta^n(\cdot - k)$, $k \in \mathbb{Z}$, are called the **basic splines of order** n. In the next lemma we collect some results about the basic splines.

LEMMA 2.3 *The basic spline of order n, Δ^n, satisfies the following properties:*

$$\text{(i)} \qquad \widehat{\Delta^n}(\xi) = e^{-i\frac{n+1}{2}\xi}\Big(\frac{\sin(\xi/2)}{\xi/2}\Big)^{n+1};$$

$$\text{(ii)} \qquad \textit{supp}\,(\Delta^n) = [0, n+1] \quad \textit{and} \quad \Delta^n(x) > 0 \quad \textit{for all} \quad x \in (0, n+1);$$

$$\text{(iii)} \qquad \sum_{k=-\infty}^{\infty} \Delta^n(x-k) = 1 \quad \textit{for all} \quad x \in \mathbb{R};$$

$$\text{(iv)} \qquad \Delta^n \textit{ is symmetric with respect to } \tfrac{1}{2}(n+1), \quad \textit{that is,}$$

$$\Delta^n\Big(\frac{n+1}{2} + x\Big) = \Delta^n\Big(\frac{n+1}{2} - x\Big) \qquad \textit{for all} \quad x \in \mathbb{R}.$$

PROOF: (i) follows from $\hat{\chi}(\xi) = e^{-i\frac{\xi}{2}}\frac{\sin(\xi/2)}{\xi/2}$ and (2.1). Property (ii) is easily deduced by induction. Induction is again the main tool to prove (iii). The case $n = 1$ is easy, and left to the reader; by assuming the inductive hypothesis, we use (2.2) to obtain

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \Delta^n(x-k) &= \sum_{k\in\mathbb{Z}} \int_{x-k-1}^{x-k} \Delta^{n-1}(y)\,dy = \sum_{k\in\mathbb{Z}} \int_{x-1}^{x} \Delta^{n-1}(z-k)\,dz \\ &= \int_{x-1}^{x} \Big\{\sum_{k\in\mathbb{Z}} \Delta^{n-1}(z-k)\Big\}\,dz = 1.\end{aligned}$$

Property (iv) follows, again, by induction and (2.2). Details are left to the reader.

$\blacksquare$

We shall now try to construct wavelets that belong to the class $\mathtt{S}^n$, $n > 1$. The case $n = 1$ was developed in section 4.1 where we began the construction by introducing the space V_0 which was used to introduce an associated multiresolution analysis that produced the desired wavelet by the method developed in Chapter 2. We shall see that the cases $n > 1$ are somewhat more complicated and there are some differences between the cases when n is odd and when n is even. In particular, it is not possible to proceed in a way that is very close to that employed in the last section; however, this treatment does provide some good motivation for the following construction.

We shall try to find a function $\varphi \in \mathrm{S}^n \cap L^2(\mathbb{R})$ that has the appropriate properties of a scaling function. Motivated by Theorems 1.3 and 1.4 we impose the condition that $\xi^{n+1}\hat{\varphi}(\xi) \equiv c_n(\xi)$ defines a 2π-periodic function in $L^2(\mathbb{T})$. It is also clear that we must impose the condition

$$\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + 2k\pi)|^2 = 1 \qquad \text{for } \; a.e. \; \xi \in \mathbb{R}, \tag{2.4}$$

which, by Proposition **??** of Chapter 2, guarantees the orthonormality of the system $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ in $L^2(\mathbb{R})$.

Remark: The condition $\varphi \in L^2(\mathbb{R}) \cap \mathrm{S}^n$ requires $\xi^{n+1}\hat{\varphi}(\xi)$ to be a 2π-periodic function in $L^2(\mathbb{T})$. To see this, observe that on the interval $(k, k+1)$ the function φ agrees with a polynomial of order at most n; thus, $(\frac{d^{n+1}}{dx^{n+1}}\varphi)(x) = 0$ on $(k, k+1)$ and, therefore, $(\frac{d^n}{dx^n}\varphi)(x)$ equals a constant on this interval. In the sense of distributions this means that near k, the function $\frac{d^{n+1}}{dx^{n+1}}\varphi$ should coincide with a multiple of the Dirac "δ-function" at k; that is,

$$(\frac{d^{n+1}}{dx^{n+1}}\varphi)(x) = \sum_{k\in\mathbb{Z}} a_k \delta(x - k),$$

where a_k is the "jump" at k exhibited by the piecewise constant function $\frac{d^n}{dx^n}\varphi$ at the point k. Since the Fourier transform of $\delta(\cdot - k)$ is the function $e^{-ik\xi}$ we have, formally,

$$\xi^{n+1}\hat{\varphi}(\xi) = (-i)^{n+1} \sum_{k\in\mathbb{Z}} a_k e^{-ik\xi}.$$

We may consider the right-hand side of this equality to be the Fourier series of the function we denoted by $c_n(\xi)$.

If we replace $\hat{\varphi}(\xi)$ in (2.4) by $\xi^{-n-1}c_n(\xi)$ $(= \hat{\varphi}(\xi))$ and use the 2π-periodicity of $c_n(\xi)$ we obtain

$$1 = |c_n(\xi)|^2 \sum_{k\in\mathbb{Z}} \frac{1}{(\xi + 2k\pi)^{2n+2}}\,. \tag{2.5}$$

Recall that in the proof of Lemma 1.9 we established a precise formula for the infinite sum in (2.5) when $n = 0$:

$$\sum_{k\in\mathbb{Z}} \frac{1}{(\xi + 2k\pi)^2} = \bigl(2\sin(\tfrac{1}{2}\xi)\bigr)^{-2}. \tag{2.6}$$

In fact, Lemma 1.9 gives us the value of the series in (2.5) when $n = 1$. For all of the cases $N \geq 1$ we have

PROPOSITION 2.7 *Suppose $N = 1, 2, 3, \cdots$, then*

$$\sum_{k\in\mathbb{Z}} \frac{1}{(\xi + 2k\pi)^{N+1}} = \bigl(2\sin(\tfrac{1}{2}\xi)\bigr)^{-N-1} P_N(\tfrac{1}{2}\xi), \tag{2.8}$$

where P_N is a trigonometric polynomial satisfying:

(i) *P_N is an even function, and $P_N(k\pi) = (-1)^{k(N-1)}$ for all $k \in \mathbb{Z}$;*

(ii) *when N is odd, P_N is π-periodic and $P_N(\xi) > 0$ for all $\xi \in \mathbb{R}$;*

(iii) *when N is even, $P_N(\xi + \pi) = -P_N(\xi)$ for all $\xi \in \mathbb{R}$.*

PROOF : We prove the various assertions in this proposition by induction. Equality (2.6) gives us the case $N = 1$, for which $P_1(\xi) \equiv 1$. In fact, Lemma 1.9 shows that $P_3(\xi) = 1 - \frac{2}{3}\sin^2\xi$. An even simpler calculation leads us to $P_2(\xi) = \cos\xi$. Let us assume the validity of equality (2.8) for the case $N - 1$:

$$\sum_{k\in\mathbb{Z}} \frac{1}{(\xi + 2k\pi)^N} = \frac{P_{N-1}(\frac{1}{2}\xi)}{(2\sin(\frac{1}{2}\xi))^N},$$

where $P_{N-1}(\xi)$ is a trigonometric polynomial. Differentiating both sides of this equality we obtain (2.8) with

$$P_N(\xi) = (\cos\xi)P_{N-1}(\xi) - \frac{1}{N}(\sin\xi)P'_{N-1}(\xi), \tag{2.9}$$

which is clearly a trigonometric polynomial (that is, a polynomial in $\sin\xi$ and $\cos\xi$). From this and $P_1 \equiv 1$ we see that

$$P_N(k\pi) = (-1)^k P_{N-1}(k\pi) = (-1)^{k(N-1)} P_1(k\pi) = (-1)^{k(N-1)}$$

for all $k \in \mathbb{Z}$. If we assume the validity of (i), (ii) and (iii) for $N-1$ we also obtain from (2.9) that

$$\begin{aligned} P_N(-\xi) &= (\cos\xi)P_{N-1}(\xi) + \frac{1}{N}(\sin\xi)P'_{N-1}(-\xi) \\ &= (\cos\xi)P_{N-1}(\xi) - \frac{1}{N}(\sin\xi)P'_{N-1}(\xi) = P_N(\xi), \end{aligned}$$

since the evenness of P_{N-1} implies that P'_{N-1} is odd. This establishes (i).

We now prove (ii) and (iii) simultaneously. In particular, let us assume that

$$P_{N-1}(\xi+\pi) = (-1)^{N-1}P_{N-1}(\xi).$$

Then a direct computation involving (2.9) shows that this relation holds for $P_N(\xi)$ as follows:

$$\begin{aligned} &P_N(\xi+\pi) = -(\cos\xi)P_{N-1}(\xi+\pi) + \frac{1}{N}(\sin\xi)P'_{N-1}(\xi+\pi) \\ &= (-1)^N(\cos\xi)P_{N-1}(\xi) - (-1)^N\frac{1}{N}(\sin\xi)P'_{N-1}(\xi) = (-1)^N P_N(\xi). \end{aligned}$$

The only thing remaining to be shown is that $P_N(\xi) > 0$ for all $\xi \in \mathbb{R}$ when N is odd. But from (2.8) we have

$$P_N(\xi) = \sum_{k\in\mathbb{Z}} \frac{(2\sin\xi)^{N+1}}{(2\xi+2k\pi)^{N+1}} = \sum_{k\in\mathbb{Z}}(-1)^{k(N+1)}\left(\frac{\sin(\xi+k\pi)}{\xi+k\pi}\right)^{N+1}.$$

When N is odd the last sum is clearly positive.

∎

We have already observed that $P_1(\xi) = 1$, $P_2(\xi) = \cos\xi$ and

$$P_3(\xi) = 1 - \tfrac{2}{3}\sin^2\xi = \tfrac{2}{3} + \tfrac{1}{3}\cos(2\xi).$$

Simple calculations show that

$$\left.\begin{aligned} P_4(\xi) &= \tfrac{1}{3}\cos^3\xi + \tfrac{2}{3}\cos\xi, \\ P_5(\xi) &= \tfrac{1}{30}\cos^2(2\xi) + \tfrac{13}{30}\cos(2\xi) + \tfrac{8}{15}. \end{aligned}\right\} \tag{2.10}$$

The graphs of P_2, P_3, P_4 and P_5 are given in Figure 4.5. We see, in particular, that there is an important difference between the polynomials

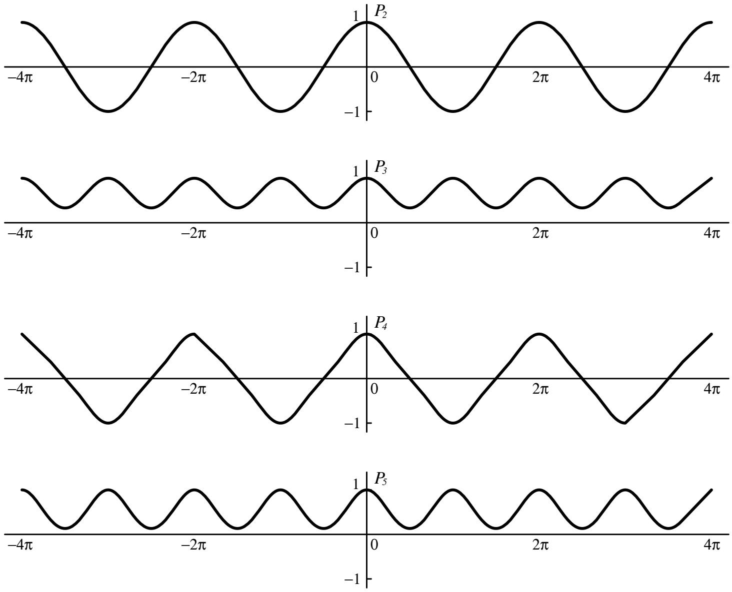

Figure 4.5: Graphs of P_2, P_3, P_4 *and* P_5*.*

P_N when N is even and N is odd: in the latter case $P_N(\xi)$ is always positive (see part (ii) of Proposition 2.7), while in the former case $P_N(\xi)$ assumes negative values (see part (iii) of Proposition 2.7). This phenomenon is consistent with equality (2.5), that must be satisfied by the scaling function φ we are seeking, since the sum on the right side has the form of the left side of (2.8) with $N = 2n + 1$. Since $\hat{\varphi}(\xi) = \xi^{-n-1}c_n(\xi)$, (2.5) and (2.8) completely determine $|\hat{\varphi}(\xi)|$. Indeed, we must have

$$|\hat{\varphi}(\xi)| = \left|\frac{\sin(\xi/2)}{\xi/2}\right|^{n+1} \frac{1}{\sqrt{P_{2n+1}(\xi/2)}}\,. \tag{2.11}$$

Thus, the scaling function we are seeking, φ^n, must have a Fourier transform $\widehat{\varphi^n}$ having absolute value given by the expression on the right in (2.11). For reasons that will become clear shortly, once again we find that the cases when n is odd or even must be treated differently. In fact, we choose φ^n so that

$$\widehat{\varphi^n}(\xi) = \begin{cases} \Big(\dfrac{\sin(\xi/2)}{\xi/2}\Big)^{n+1} \dfrac{1}{\sqrt{P_{2n+1}(\xi/2)}} & \text{if } n \text{ is odd,} \\ e^{-i\frac{\xi}{2}} \Big(\dfrac{\sin(\xi/2)}{\xi/2}\Big)^{n+1} \dfrac{1}{\sqrt{P_{2n+1}(\xi/2)}} & \text{if } n \text{ is even.} \end{cases} \tag{2.12}$$

It then follows from Proposition 2.7 that, in either case, $\xi^{n+1}\widehat{\varphi^n}(\xi)$ is a 2π-periodic function in $L^\infty(\mathbb{T}) \subset L^2(\mathbb{T})$. Moreover, $\frac{d^j\varphi^n}{dx^j}$ exists and belongs to $L^2(\mathbb{R})$ for $j = 0, 1, \cdots, n$. In fact, $\frac{d^{n+1}\varphi^n}{dx^{n+1}}$ also exists as a tempered distribution. Let $\vartheta(\xi) = \xi^n\widehat{\varphi^n}(\xi)$; thus, since $\xi\vartheta(\xi)$ is 2π-periodic we can represent it in $L^2(\mathbb{T})$ by its Fourier series:

$$\xi\vartheta(\xi) = \xi^{n+1}\widehat{\varphi^n}(\xi) = \sum_{\ell\in\mathbb{Z}} a_\ell e^{i\ell\xi}. \tag{2.13}$$

Then, if $t(x)$ is a tempered test function supported in the open interval $(\ell_o, \ell_o + 1)$,

$$\begin{aligned} &\int_{\ell_o}^{\ell_o+1} (\frac{d^n\varphi^n}{dx^n})(x)\,\overline{t'(x)}\,dx = \int_{-\infty}^{\infty} (\frac{d^n\varphi^n}{dx^n})(x)\,\overline{t'(x)}\,dx \\ &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \vartheta(\xi)\big(\xi\,\overline{\hat{t}(\xi)}\,\big)\,d\xi = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \overline{\hat{t}(\xi)}\sum_{\ell\in\mathbb{Z}} a_\ell e^{i\ell\xi}\,d\xi \\ &= -i\sum_{\ell\in\mathbb{Z}} a_\ell\,\overline{t(\ell)} = 0, \end{aligned}$$

since ℓ lies outside the support of t for all $\ell \in \mathbb{Z}$. This shows that, for each $\ell_o \in \mathbb{Z}$,

$$\int_{\ell_o}^{\ell_o+1} (\frac{d^n\varphi^n}{dx^n})(x)\,s(x)\,dx = 0$$

for any test function $s \in \mathcal{S}(\mathbb{R})$ supported in $(\ell_o, \ell_o + 1)$ with $\int_{\mathbb{R}} s(x)\,dx = 0$. But this means that $\frac{d^n\varphi^n}{dx^n}$ is constant in each interval of the form $(\ell_o, \ell_o + 1)$ and, consequently, $\varphi^n(x)$ restricted to $(\ell_o, \ell_o + 1)$ is a polynomial of order at most n. It is also evident that φ^n belongs to the class $C^{n-1}(\mathbb{R})$. (Observe that

$$(\frac{d^{n-1}\varphi^n}{dx^{n-1}})^\wedge(\xi) = \frac{b(\xi)}{1+\xi^2}\,,$$

where $b \in L^\infty(\mathbb{R})$; thus, the Fourier transform of $\frac{d^{n-1}\varphi^n}{dx^{n-1}}$ is in $L^1(\mathbb{R})$, and this implies $\frac{d^{n-1}\varphi^n}{dx^{n-1}}$ is continuous on $\mathbb{R}$.) We collect these results in the following:

PROPOSITION 2.14 *The functions φ^n defined by the equality* (2.12) *belong to* $\mathrm{S}^n(\mathbb{R}) \cap L^2(\mathbb{R})$ *and* $\{\varphi^n(\cdot - k) : k \in \mathbb{Z}\}$ *is an orthonormal system in* $L^2(\mathbb{R})$.

We now define $V_0 \subset L^2(\mathbb{R})$ to be the closure in the L^2-norm of the span of $\{\varphi^n(\cdot - k) : k \in \mathbb{Z}\}$, and for each $j \in \mathbb{Z}$ we let V_j be the closure of the span of $\{(\varphi^n)_{j,k} : k \in \mathbb{Z}\}$. Then conditions (1.2) and (1.5) (in Chapter 2) in the definition of an MRA are satisfied by the sequence $\{V_j\}_{j\in\mathbb{Z}}$. Suppose that condition (1.1) (also in Chapter 2) is satisfied; then the fact that $\widehat{\varphi^n}$ is continuous at 0 and $\widehat{\varphi^n}(0) = 1$, together with Theorems 1.6 and 1.7 in Chapter 2, imply the remaining two conditions in the definition of an MRA. To show (1.1) it suffices to show $V_{-1} \subset V_0$ and, by applying the inverse Fourier transform, this inclusion is true if there exists a 2π-periodic function $m_0^n \in L^2(\mathbb{T})$ such that $\widehat{\varphi^n}(2\xi) = \widehat{\varphi^n}(\xi) m_0^n(\xi)$ (see (2.3) of Chapter 2). But this is easily seen to be the case: we let

$$m_0^n(\xi) = \frac{\widehat{\varphi^n}(2\xi)}{\widehat{\varphi^n}(\xi)} = e^{i\alpha_n(\xi)} \big(\cos(\tfrac{1}{2}\xi)\big)^{n+1} \sqrt{\frac{P_{2n+1}(\frac{1}{2}\xi)}{P_{2n+1}(\xi)}}\,, \tag{2.15}$$

where

$$\alpha_n(\xi) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ -\frac{1}{2}\xi & \text{if } n \text{ is even.} \end{cases}$$

From Proposition 2.7 we see that the square root factor on the right is 2π-periodic and, because of our choice of $\alpha_n(\xi)$, the same is true for

$$e^{i\alpha_n(\xi)} \big(\cos(\tfrac{1}{2}\xi)\big)^{n+1}.$$

It is also clear that $m_0^n \in L^2(\mathbb{T})$. We have constructed, therefore, an MRA with an associated scaling function that is a spline of order n. The corresponding low-pass filter is given by equality (2.15).

We can now use formula (2.12) in Chapter 2 to produce an orthonormal wavelet ψ^n from the scaling function φ^n and the low-pass filter m_0^n; that is, the Fourier transform of ψ^n is given by the formula

$$\widehat{\psi^n}(2\xi) = \begin{cases} e^{i\xi}\dfrac{(\sin(\frac{1}{2}\xi))^{2n+2}}{(\frac{1}{2}\xi)^{n+1}}\sqrt{\dfrac{P_{2n+1}(\frac{1}{2}\xi+\frac{1}{2}\pi)}{P_{2n+1}(\xi)P_{2n+1}(\frac{1}{2}\xi)}} & \text{if } n \text{ is odd,} \\ \\ -ie^{i\xi}\dfrac{(\sin(\frac{1}{2}\xi))^{2n+2}}{(\frac{1}{2}\xi)^{n+1}}\sqrt{\dfrac{P_{2n+1}(\frac{1}{2}\xi+\frac{1}{2}\pi)}{P_{2n+1}(\xi)P_{2n+1}(\frac{1}{2}\xi)}} & \text{if } n \text{ is even.} \end{cases} \tag{2.16}$$

Let us examine these wavelets and their properties in more detail. In either case (when n is odd or n is even) formula (2.16) shows that

$$\widehat{\psi^n}(\xi) = e^{i\frac{\xi}{2}}b_n(\xi)\xi^{-n-1},$$

where b_n is an even function in $L^\infty(\mathbb{R})$; moreover, $\xi^{n+1}\widehat{\psi^n}(\xi) = e^{i\frac{\xi}{2}}b_n(\xi)$ is a 4π-periodic function. These properties are direct consequences of Proposition 2.7. For example, the fact that P_{2n+1} is even and π-periodic (part (i) and (ii) of Proposition 2.7) shows that

$$\begin{aligned} P_{2n+1}\Big(\frac{\xi+2\pi}{4}\Big) &= P_{2n+1}\Big(\frac{-\xi-2\pi}{4}\Big) \\ &= P_{2n+1}\Big(\frac{-\xi-2\pi}{4}+\pi\Big) = P_{2n+1}\Big(\frac{-\xi+2\pi}{4}\Big); \end{aligned}$$

thus, $\xi \mapsto P_{2n+1}\big(\frac{\xi+2\pi}{4}\big)$ is an even function. Even simpler calculations show that the other factors making up b_n are even. The 4π-periodicity is precisely what we need to modify the argument we presented before Proposition 2.14 to show that, for the wavelet ψ^n, $\psi^n(\frac{1}{2}x)$ belongs to the class $\mathtt{S}^n$ of splines of order n. We collect these facts in the following:

THEOREM 2.17 *The wavelets ψ^n defined by equality* (2.16) *belong to the space V_1, and $\psi^n(\frac{1}{2}x)$ are splines of order n. When n is odd, ψ^n is even about $x = -\frac{1}{2}$, while, when n is even, ψ^n is odd about $x = -\frac{1}{2}$; that is,*

$$\text{(i)} \qquad \psi^n(x) = \begin{cases} \psi^n(-1-x) & \textit{if } n \textit{ is odd,} \\ -\psi^n(-1-x) & \textit{if } n \textit{ is even.} \end{cases}$$

Moreover,

$$\text{(ii)} \qquad \int_{\mathbb{R}} x^k\psi^n(x)\,dx = 0 \qquad \textit{for } k = 0, 1, 2, \cdots, n.$$

PROOF : Since $\widehat{\psi^n}(\xi) = e^{i\frac{\xi}{2}} b_n(\xi)\xi^{-n-1}$ with $b_n(\xi)$ an even function, $b_n(\xi)\xi^{-n-1}$ is even when n is odd and, clearly, an odd function when n is even. The factor $e^{i\frac{\xi}{2}}$ represents a translation of ψ^n that shifts this parity property from 0 to $-\frac{1}{2}$.

The vanishing moment property (ii) follows from the fact that

$$\Big(\frac{d^k\widehat{\psi^n}}{d\xi^k}\Big)(0) = 0 \qquad \text{for } \; k = 0, 1, 2, \cdots, n,$$

which is a consequence of the identity (2.16). Observe that the power $2n+2$ of $\sin(\frac{1}{2}\xi)$ compensates for the (possible) singularity produced by ξ^{-n-1}. In the next proof we obtain an estimate that assures us of the integrability of the integrand in (ii).

∎

REMARK: The wavelet ψ^n discussed in Theorem 2.17 is often called a **spline wavelet of order n on the real line**. Notice that, although ψ^n is not in $\mathtt{S}^n(\mathbb{R})$, its dilation $\psi^n(2^{-1}\cdot)$ does belong to $\mathtt{S}^n(\mathbb{R})$.

In Proposition 1.16 we saw that the Franklin wavelet and its associated scaling function have exponential decays at ∞. This property, and more, is shared with the higher order splines:

THEOREM 2.18 *The spline wavelets ψ^n, $n = 1, 2, \cdots$, and their associated scaling functions φ^n, as well as their derivatives*

$$\frac{d^j\psi^n}{dx^j}, \; \frac{d^j\varphi^n}{dx^j}, \qquad j = 1, 2, \cdots, n-1,$$

have exponential decay at ∞.

PROOF : The proof of this theorem is done by using the same argument we employed for the proof of Proposition 1.16. It is easily seen that the same observation about the poles that was made about the analytic extensions of $\widehat{\psi^n}$ and $\widehat{\varphi^n}$ in the proof of Proposition 1.16 applies to the Fourier transforms of the derivatives of ψ^n and φ^n we are now considering.

∎

We end this section by establishing an identity that will prove to be most helpful for transferring the construction of spline wavelets from the real line to their periodic analogs on the torus $\mathbb{T}$.

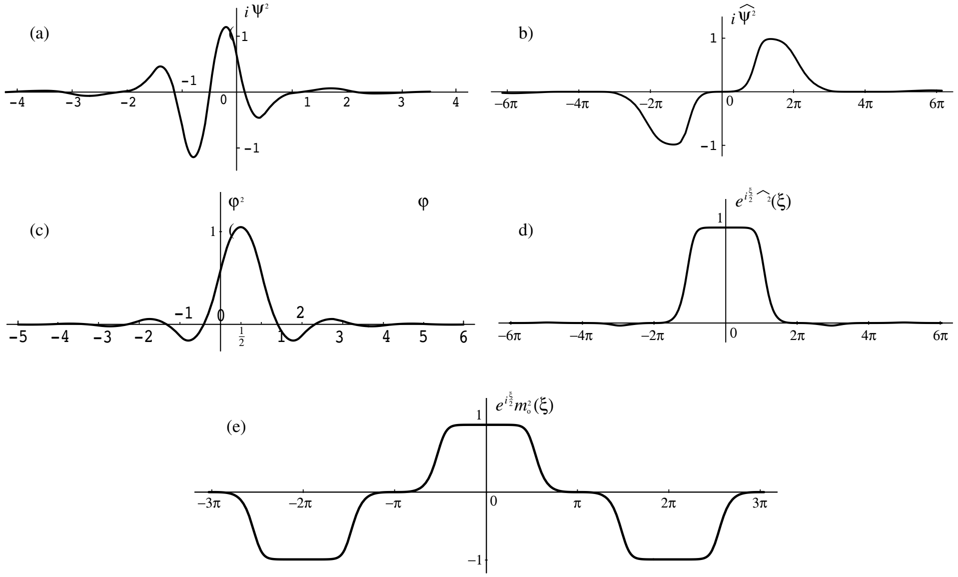

Figure 4.6: Graphs of several functions related to the spline wavelet ψ^2:

(a) $i\psi^2$,
(b) $ie^{-i\frac{\xi}{2}}\widehat{\psi^2}$,
(c) the scaling function φ^2 given in (2.12),
(d) $e^{i\frac{\xi}{2}}\widehat{\varphi^2}$,
(e) $e^{i\frac{\xi}{2}}m_0^2(\xi)$, where m_0^2 is the corresponding low-pass filter.

THEOREM 2.19 *For any $n = 1, 2, 3, \cdots$, the spline wavelet ψ^n satisfies*

$$\sum_{k\in\mathbb{Z}} \frac{\widehat{\psi^n}(\xi + 2k\pi)}{(\xi + 2k\pi)^{n+1}} = 0 \qquad \textit{for a.e. } \xi \in \mathbb{R}.$$

PROOF : Since $\widehat{\psi^n} \in L^\infty(\mathbb{R})$ it is clear that the above series converges for a.e. $\xi \in \mathbb{R}$. In order to obtain this identity we use an idea we exploited before in this book: we break up the sum over $\mathbb{Z}$ into the two sums obtained by considering the even and odd integers separately. Thus,

$$\sum_{k\in\mathbb{Z}} \frac{\widehat{\psi^n}(\xi + 2k\pi)}{(\xi + 2k\pi)^{n+1}} = \Sigma_e(\xi) + \Sigma_o(\xi),$$

where

$$\Sigma_e(\xi) = \sum_{\ell\in\mathbb{Z}} \frac{\widehat{\psi^n}(\xi + 4\ell\pi)}{(\xi + 4\ell\pi)^{n+1}}$$

and

$$\Sigma_o(\xi) = \sum_{\ell\in\mathbb{Z}} \frac{\widehat{\psi^n}(\xi + 2(2\ell+1)\pi)}{(\xi + 2(2\ell+1)\pi)^{n+1}}\,.$$

Let us first consider the case when n is odd. Using (2.16) and Proposition 2.7 we have

$$\widehat{\psi^n}(\xi + 4\ell\pi) = 4^{n+1} e^{i\frac{\xi}{2}} \frac{(\sin(\frac{1}{4}\xi))^{2n+2}}{(\xi+4\ell\pi)^{n+1}} \Big(\frac{P_{2n+1}(\frac{1}{4}(\xi+2\pi))}{P_{2n+1}(\frac{1}{2}\xi)P_{2n+1}(\frac{1}{4}\xi)}\Big)^{\frac{1}{2}}.$$

Substituting this value for $\widehat{\psi^n}(\xi+4\ell\pi)$ in the expression for $\Sigma_e(\xi)$ and using (2.8) with $N = 2n+1$ we then obtain

$$\begin{aligned}\Sigma_e(\xi) &= 4^{n+1} e^{i\frac{\xi}{2}} \Big(\frac{P_{2n+1}(\frac{1}{4}\xi+\frac{1}{2}\pi)}{P_{2n+1}(\frac{1}{2}\xi)P_{2n+1}(\frac{1}{4}\xi)}\Big)^{\frac{1}{2}} \sum_{\ell\in\mathbb{Z}} \frac{(\sin(\frac{1}{4}\xi))^{2n+2}}{(\xi+4\ell\pi)^{2n+2}} \\ &= 4^{-n-1} e^{i\frac{\xi}{2}} \Big(\frac{P_{2n+1}(\frac{1}{4}\xi+\frac{1}{2}\pi)P_{2n+1}(\frac{1}{4}\xi)}{P_{2n+1}(\frac{1}{2}\xi)}\Big)^{\frac{1}{2}}.\end{aligned}$$

In order to evaluate $\widehat{\psi^n}(\xi + 2(2\ell+1)\pi)$ we use (2.16) and Proposition 2.7 again, then substitute this value in the expression for $\Sigma_o(\xi)$, use (2.8) and obtain

$$\Sigma_o(\xi) = -4^{-n-1} e^{i\frac{\xi}{2}} \Big(\frac{P_{2n+1}(\frac{1}{4}\xi)P_{2n+1}(\frac{1}{4}\xi+\frac{1}{2}\pi)}{P_{2n+1}(\frac{1}{2}\xi)}\Big)^{\frac{1}{2}}.$$

Hence, $\Sigma_e(\xi) + \Sigma_o(\xi) = 0$, and we obtain the desired equality.

When n is even we can repeat this argument by using the other case in (2.16). The only difference is the factor $-i$ that does not affect the conclusion $\Sigma_e(\xi) + \Sigma_o(\xi) = 0$.

∎

4.3 Orthonormal bases of piecewise linear continuous functions for $L^2(\mathbb{T})$

The boundary of the unit disk $\mathbb{T} = \{e^{2\pi i x} : 0 \le x < 1\}$ can be identified with the interval $[0,1)$ via the correspondence

$$e^{2\pi i x} \longleftrightarrow x.$$

With this identification, the space $L^2(\mathbb{T})$ of square integrable functions on $\mathbb{T}$ can be viewed as the space of 1-periodic functions defined on $\mathbb{R}$ such that

$$\|f\|^2_{L^2([0,1))} = \int_0^1 |f(x)|^2\, dx < \infty.$$

The Haar function restricted to $[0,1)$

$$h(x) = \begin{cases} 1 & \text{if } 0 \le x < \frac{1}{2}, \\ -1 & \text{if } \frac{1}{2} \le x < 1, \end{cases}$$

can be used to find an orthonormal basis for $L^2(\mathbb{T})$. Let us define, for $j = 0, 1, \cdots$, and $k = 0, 1, \cdots, 2^j - 1$,

$$h_{j,k}(x) = 2^{\frac{j}{2}} h(2^j x - k), \qquad x \in [0,1). \tag{3.1}$$

Extend each $h_{j,k}$ to a 1-periodic function on $\mathbb{R}$. With the identification described above, the system of 1-periodic functions

$$\{h_{j,k} : j = 0, 1, \cdots, \text{ and } k = 0, 1, \cdots, 2^j - 1\}$$

is easily seen to be an orthonormal basis for $L^2(\mathbb{T})$. As was the case for the Haar wavelet on $\mathbb{R}$, this periodization of it furnishes us with a simple example of the periodic wavelets we shall construct in this section.

In fact, we shall construct orthonormal bases for $L^2(\mathbb{T})$ whose elements are piecewise linear continuous functions. One of these bases will be obtained by using the Franklin wavelet constructed in section 4.1 (see Theorem 1.13 in that section).

Let $\mathcal{B}$ be the space of continuous 1-periodic functions on $\mathbb{R}$. For $j \ge 0$, let $\mathcal{B}_j$ be the subspace of $\mathcal{B}$ consisting of all piecewise linear functions with nodes at $2^{-j}k$, $k = 0, 1, \cdots, 2^j - 1$. In particular, the functions in $\mathcal{B}_j$ are continuous on $[0,1)$ and linear on each interval of the form $[2^{-j}k, 2^{-j}(k+1)]$, $k = 0, 1, \cdots, 2^j - 1$. We shall write $N = N_j = 2^j - 1$ for convenience. The space $\mathcal{B}_0$ is the space of constant functions. Clearly,

$$\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}_2 \subset \cdots \subset \mathcal{B}_j \subset \cdots \subset \mathcal{B} \subset L^2(\mathbb{T}) = L^2([0,1)),$$

and the dimension of $\mathcal{B}_j$ is $2^j = N + 1$.

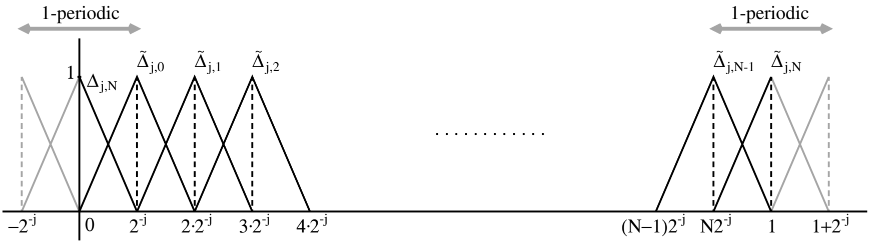

Figure 4.7: Graphs of $\tilde{\Delta}_{j,k}$, $k = 0, 1, \cdots, N$.

A basis for $\mathcal{B}_j$ can be obtained as follows. We first introduce the 1-periodic "triangle" functions defined by

$$\tilde{\Delta}_j(x) = \begin{cases} 2^j x & \text{if } \ 0 \le x < 2^{-j}, \\ 2 - 2^j x & \text{if } \ 2^{-j} \le x < 2^{-j+1}, \\ 0 & \text{if } \ 2^{-j+1} \le x < 1, \end{cases}$$

on $[0, 1)$, for $j = 1, 2, \cdots$. We then introduce the translates of these functions

$$\tilde{\Delta}_{j,k}(x) = \tilde{\Delta}_j(x - 2^{-j}k), \qquad k = 0, 1, 2, \cdots, N;$$

(see Figure 4.7). It is clear that

$$\{\tilde{\Delta}_{j,0}, \tilde{\Delta}_{j,1}, \cdots, \tilde{\Delta}_{j,N}\}$$

is a basis for $\mathcal{B}_j$. This is not, however, an orthonormal basis.

An orthonormal basis for $\mathcal{B}_j$ can be constructed with the aid of the group of $(2^j)^{\text{th}}$ roots of unity

$$\Gamma_j = \left\{1, e^{2\pi i 2^{-j}}, e^{2\pi i 2 \cdot 2^{-j}}, \cdots, e^{2\pi i N 2^{-j}}\right\}.$$

Let $U = U_j$ be the unitary operator in $L^2(\mathbb{T})$ defined by

$$(U_j f)(x) = f(x - 2^{-j}) \qquad \text{for all } \ f \in L^2(\mathbb{T}).$$

The powers of U give us an action of Γ_j on $\mathcal{B}_j$. Observe that

$$U^k(\tilde{\Delta}_j) = \tilde{\Delta}_{j,k} \qquad \text{for } \ k = 0, 1, \cdots, N,$$

and that $\tilde{\Delta}_j$ is a cyclic vector under this action: $U^{N+1}\tilde{\Delta}_j = \tilde{\Delta}_j$. The minimal polynomial of U is $x^{N+1} - 1$ (it coincides with its characteristic polynomial) and the (distinct) proper values of U are

$$\lambda_j = \lambda_j^1 = e^{2\pi i 2^{-j}},\ \lambda_j^2,\ \cdots,\ \lambda_j^N,\ \lambda_j^{N+1} = 1.$$

Since U is unitary and the proper values are distinct, there exists an orthonormal set of associated proper vectors. This is, then, an orthonormal basis for $\mathcal{B}_j$.

The family of spaces $\mathcal{B}_j$, $j = 0, 1, \cdots$, can be used to obtain an orthonormal basis for $L^2(\mathbb{T})$. Let $\mathcal{C}_j$ be the orthogonal complement of $\mathcal{B}_j$ in $\mathcal{B}_{j+1}$. Then $\mathcal{C}_j \oplus \mathcal{B}_j = \mathcal{B}_{j+1}$ and $\dim(\mathcal{C}_j) = 2^j$. Since

$$\mathcal{B}_0 \bigoplus \Big(\bigoplus_{j=0}^{\infty} \mathcal{C}_j \Big) = \overline{\mathcal{B}},$$

and $\mathcal{B}$ is dense in $L^2(\mathbb{T})$, a basis is obtained if we find an orthonormal basis for each $\mathcal{C}_j$, $j = 0, 1, \cdots$.

To do this, observe that U is unitary on $\mathcal{C}_j$ and on this space it has minimal polynomial $x^{N+1} - 1$. Therefore, there exists an orthonormal basis of proper vectors $\{z_0, z_1, \cdots, z_N\} \subset \mathcal{C}_j$ such that $U_j(z_\ell) = \lambda_j^\ell z_\ell$. Let

$$g_j = \frac{1}{\sqrt{N+1}} \sum_{\ell=0}^{N} z_\ell.$$

Clearly,

$$U_j^{N+1} g_j = \frac{1}{\sqrt{N+1}} \sum_{\ell=0}^{N} \lambda_j^{\ell(N+1)} z_\ell = g_j,$$

and, therefore, g_j is cyclic for U_j. Moreover,

$$\big(U_j^m g_j, U_j^n g_j\big) = \frac{1}{N+1} \sum_{\ell=0}^{N} \lambda_j^{\ell(m-n)} = \delta_{m,n}$$

since, for $n \neq m$, $\lambda_j^{(m-n)}$ is a root of the polynomial

$$x^N + x^{N-1} + \cdots + x + 1 = \frac{x^{N+1} - 1}{x - 1}.$$

We have proved the following result that presents us with an orthonormal basis for $L^2(\mathbb{T})$:

THEOREM 3.2 *If* $\mathcal{C}_j$ *is the orthogonal complement of* $\mathcal{B}_j$ *in* $\mathcal{B}_{j+1}$*, then there exists a* $g_j \in \mathcal{C}_j$ *such that* $\{g_j, Ug_j, \cdots, U^N g_j\}$ *is an orthonormal basis for* $\mathcal{C}_j$*,* $j = 0, 1, 2, \cdots$.

The orthonormal basis provided by Theorem 3.2 consists of piecewise linear continuous functions in $L^2(\mathbb{T})$. This was accomplished by finding a $g_j \in \mathcal{C}_j$ that generates a basis for $\mathcal{C}_j$, $j = 0, 1, 2, \cdots$. The functions g_j are chosen in each $\mathcal{C}_j$ without any concern for a relation connecting any two of them. What we want to do is to find a single function on $\mathbb{R}$ that will generate a basis for each $\mathcal{C}_j$ by appropriate dilations, periodizations and translations much in the spirit of (3.1) and the wavelet bases of previous chapters.

To do this we need to characterize the spaces $\mathcal{C}_j$ and $\mathcal{B}_j$. We begin by computing the Fourier coefficients of $\tilde{\Delta}_j$. Since we have identified $\mathbb{T}$ with the interval $[0, 1)$, the Fourier coefficients of a 1-periodic function f with $\|f\|_{L^2([0,1))} < \infty$ are given by

$$\mathcal{F}[f](k) \equiv \int_0^1 f(x) e^{-2\pi ikx}\, dx.$$

Recall that throughout this book we use the traditional symbol $\hat{f}(\xi)$ to denote the Fourier transform of an $L^2(\mathbb{R})$ function f, that is,

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi}\, dx.$$

LEMMA 3.3 *The Fourier coefficients of* $\tilde{\Delta}_j$ *are*

$$\mathcal{F}[\tilde{\Delta}_j](k) = \begin{cases} 2^{-j} & \text{if } k = 0, \\ -\dfrac{2^j}{4\pi^2}\Big(\dfrac{e^{-2\pi i 2^{-j} k} - 1}{k}\Big)^2 & \text{if } k \neq 0. \end{cases}$$

PROOF: The proof is a straightforward computation. It can also be proved by observing that $\tilde{\Delta}_j = \tilde{\chi}_j * \tilde{\chi}_j$, where $\tilde{\chi}_j$ is the 1-periodic function whose values on $[0, 1)$ are $2^{\frac{j}{2}} \chi_{[0,2^{-j})}$. Since

$$\mathcal{F}[\tilde{\chi}_j](k) = -\frac{2^{\frac{j}{2}}}{2\pi ik}(e^{-2\pi i 2^{-j} k} - 1) \qquad \text{for } k \neq 0,$$

the result follows from the equalities

$$\mathcal{F}[\tilde{\Delta}_j](k) = \mathcal{F}[\tilde{\chi}_j * \tilde{\chi}_j](k) = \mathcal{F}[\tilde{\chi}_j](k)\,\mathcal{F}[\tilde{\chi}_j](k).$$

∎

If we define

$$b_j(y) = \begin{cases} 1 & \text{if } \ y = 0, \\ \left(\dfrac{e^{-2\pi i 2^{-j} y} - 1}{2\pi i 2^{-j} y}\right)^2 & \text{if } \ y \neq 0, \end{cases} \tag{3.4}$$

we obtain a continuous function on $\mathbb{R}$. Moreover, we have

$$\mathcal{F}[\tilde{\Delta}_j](k) = 2^{-j} b_j(k) = \frac{1}{N+1} b_j(k) \qquad \text{for } \ k \in \mathbb{Z}. \tag{3.5}$$

In order to characterize those f that belong to $\mathcal{C}_j$ we shall use the fact that we must have

$$\sum_{k\in\mathbb{Z}} \mathcal{F}[g](k)\,\overline{\mathcal{F}[f](k)} = 0 \qquad \text{for all } \ g \in \mathcal{B}_j.$$

Since each $g \in \mathcal{B}_j$ can be expressed as

$$g = \sum_{m=0}^{N} a_m \tilde{\Delta}_{j,m} = \sum_{m=0}^{N} a_m \tilde{\Delta}_j(\cdot - 2^{-j} m),$$

we have

$$\sum_{m=0}^{N} \sum_{k\in\mathbb{Z}} \overline{\mathcal{F}[f](k)}\, a_m \mathcal{F}[\tilde{\Delta}_j] e^{-2\pi i 2^{-j} km} = 0.$$

For each $0 \le m \le N$, applying this to $g = \tilde{\Delta}_{j,m} \in \mathcal{B}_j$ and using (3.5), we obtain

$$0 = \frac{1}{N+1} \overline{\mathcal{F}[f](0)}\, b_j(0) + \frac{1}{N+1} \sum_{k\neq 0} \overline{\mathcal{F}[f](k)}\, b_j(k) e^{-2\pi i 2^{-j} km}.$$

Write $\lambda = e^{-2\pi i 2^{-j}}$, the non-zero k as $n + 2^j \ell$ with $n = 1, \cdots, N$ and $\ell \in \mathbb{Z}$, and observe that $\lambda^{mk} = \lambda^{mn}$. Then, the last equality is equivalent to

$$0 = \overline{\mathcal{F}[f](0)}\, b_j(0) + \sum_{n=1}^{N} (\lambda^m)^n \sum_{\ell\in\mathbb{Z}} \overline{\mathcal{F}[f](n + 2^j \ell)}\, b_j(n + 2^j \ell)$$

for $m = 0, 1, 2, \cdots, N$. Let

$$A_n = \sum_{\ell \in \mathbb{Z}} \overline{\mathcal{F}[f](n + 2^j \ell)}\, b_j(n + 2^j \ell) \qquad \text{for} \;\; n = 0, 1, 2, \cdots, N.$$

Since $b_j(2^j \ell) = \delta_{\ell,0}$, we have $A_0 = \overline{\mathcal{F}[f](0)}$. Thus, we have shown

$$0 = \sum_{n=0}^{N} (\lambda^m)^n A_n \qquad \text{for} \;\; m = 0, 1, 2, \cdots, N.$$

If we let

$$C_N = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \lambda & \lambda^2 & \cdots & \lambda^N \\ 1 & \lambda^2 & (\lambda^2)^2 & \cdots & (\lambda^2)^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda^N & (\lambda^N)^2 & \cdots & (\lambda^N)^N \end{pmatrix} \tag{3.6}$$

then we have

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = C_N \begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_N \end{pmatrix}$$

But $\frac{1}{\sqrt{N+1}} C_N$ is unitary (use the properties of the roots of unity). Thus,

$$A_m = 0 \qquad \text{for all} \;\; m = 0, 1, 2, \cdots, N.$$

Hence, since $0 = A_0 = \overline{\mathcal{F}[f](0)}$, we have $\mathcal{F}[f](0) = 0$. Moreover, for $n = 1, 2, \cdots, N$,

$$\begin{aligned} A_n &= \sum_{\ell \in \mathbb{Z}} \overline{\mathcal{F}[f](n + 2^j \ell)}\, b_j(n + 2^j \ell) \\ &= \sum_{\ell \in \mathbb{Z}} \overline{\mathcal{F}[f](n + 2^j \ell)}\, \frac{(\lambda^n - 1)^2}{(-4\pi^2) 2^{-2j} (n + 2^j \ell)^2} \end{aligned}$$

Therefore,

$$\sum_{\ell \in \mathbb{Z}} \frac{\mathcal{F}[f](n + 2^j \ell)}{(n + 2^j \ell)^2} = 0 \qquad \text{for} \;\; n = 1, 2, \cdots, N.$$

THEOREM 3.7 *$f \in \mathcal{C}_j$ if and only if $f \in \mathcal{B}_{j+1}$, $\mathcal{F}[f](0) = 0$ and*

$$\sum_{\ell \in \mathbb{Z}} \frac{\mathcal{F}[f](n + 2^j \ell)}{(n + 2^j \ell)^2} = 0 \qquad \text{for } \; n = 1, 2, \cdots, N (= 2^j - 1).$$

We have just shown the "only if" part; the "if" part follows by reversing the arguments given above.

Theorem 3.7 gives us a characterization of the classes $\mathcal{C}_j$, $j = 0, 1, 2, \cdots$. There is a simple characterization of the classes $\mathcal{B}_j$ as well. Since $\{\tilde{\Delta}_{j,m} : m = 0, 1, 2, \cdots, 2^j - 1\}$ is a basis for $\mathcal{B}_j$, if $g \in \mathcal{B}_j$, then

$$g = \sum_{m=0}^{N} a_m \tilde{\Delta}_{j,m},$$

and we have

$$\begin{aligned} \mathcal{F}[g](k) &= \sum_{m=0}^{N} a_m \mathcal{F}[\tilde{\Delta}_{j,m}](k) = \sum_{m=0}^{N} a_m \mathcal{F}[\tilde{\Delta}_j](k) e^{-2\pi i 2^{-j} km} \\ &= \sum_{m=0}^{N} a_m \frac{1}{N+1} b_j(k) e^{-2\pi i 2^{-j} km}, \end{aligned}$$

where $b_j(k)$ is defined in (3.4). This shows that $k^2 \mathcal{F}[g](k)$ is 2^j-periodic over $\mathbb{Z}$. It turns out that this is a characterization of the space $\mathcal{B}_j$. More precisely, we have

THEOREM 3.8 *Suppose $f \in \mathcal{B}$, the space of all 1-periodic continuous functions on $\mathbb{R}$. Then $f \in \mathcal{B}_j$, $j \geq 0$, if and only if $k^2 \mathcal{F}[f](k)$ is a 2^j-periodic function on $\mathbb{Z}$. In particular, $\mathcal{F}[f](2^j n) = 0$ if $n \neq 0$ and $\mathcal{F}[f](0)$ is arbitrary.*

PROOF: If $f \in \mathcal{B}_j$ we have seen that $k^2 \mathcal{F}[f](k)$ is 2^j-periodic. Suppose that $k^2 \mathcal{F}[f](k)$ is 2^j-periodic on $\mathbb{Z}$. It is enough to show that f is orthogonal to $\mathcal{C}_p$ for all $p \geq j$. Suppose $g \in \mathcal{C}_p$, $p \geq j$. By Theorem 3.7 we have $\mathcal{F}[g](0) = 0$ and

$$\sum_{\ell \in \mathbb{Z}} \frac{\mathcal{F}[g](n + 2^p \ell)}{(n + 2^p \ell)^2} = 0 \qquad \text{for } \; n = 1, 2, \cdots, 2^p - 1.$$

Thus, since $\mathcal{F}[g](0)$ and $\mathcal{F}[f](2^p\ell) = 0$ for $\ell \neq 0$, we have

$$\begin{aligned}
\big(g, f\big)_{L^2(\mathbb{T})} &= \overline{\mathcal{F}[f](0)}\,\mathcal{F}[g](0) + \sum_{k\neq 0} k^2\,\overline{\mathcal{F}[f](k)}\,\frac{\mathcal{F}[g](k)}{k^2} \\
&= 0 + \sum_{n=1}^{2^p-1}\sum_{\ell\in\mathbb{Z}} (n+2^p\ell)^2\,\overline{\mathcal{F}[f](n+2^p\ell)}\,\frac{\mathcal{F}[g](n+2^p\ell)}{(n+2^p\ell)^2} \\
&= \sum_{n=1}^{2^p-1}\sum_{\ell\in\mathbb{Z}} n^2\,\overline{\mathcal{F}[f](n)}\,\frac{\mathcal{F}[g](n+2^p\ell)}{(n+2^p\ell)^2} \qquad \text{(since } p \geq j\text{)} \\
&= \sum_{n=1}^{2^p-1} n^2\,\overline{\mathcal{F}[f](n)} \sum_{\ell\in\mathbb{Z}} \frac{\mathcal{F}[g](n+2^p\ell)}{(n+2^p\ell)^2} = 0.
\end{aligned}$$

∎

We now give a necessary and sufficient condition for the orthonormality of the dyadic translates of a 1-periodic function. This result is similar to Proposition 1.11 of Chapter 2.

THEOREM 3.9 *If $f \in L^2(\mathbb{T})$, then $\{f, Uf, \cdots, U^N f\}$, where $U \equiv U_j$, is an orthonormal system if and only if*

$$\sum_{\ell\in\mathbb{Z}} \big|\mathcal{F}[f](n+2^j\ell)\big|^2 = 2^{-j} \qquad \textit{for}\ \ n = 0, 1, \cdots, N = 2^j - 1. \tag{3.10}$$

PROOF : Let

$$a_n = \sum_{\ell\in\mathbb{Z}} \big|\mathcal{F}[f](n+2^j\ell)\big|^2 \qquad \text{for}\ \ n = 0, 1, \cdots, N = 2^j - 1,$$

and suppose that $\{f, Uf, \cdots, U^N f\}$ is an orthonormal system. Then, by the Plancherel theorem, the fact that $\big\|f\big\|_{L^2(\mathbb{T})} = 1$ is equivalent to

$$\sum_{n=0}^{N} a_n = 1; \tag{3.11}$$

the orthogonality $(f, U^m f)_{L^2(\mathbb{T})} = 0$ for $m = 1, 2, \cdots, N$, in terms of the Fourier coefficients of f, is equivalent to

$$0 = \sum_{k\in\mathbb{Z}} \mathcal{F}[f](k)\,\overline{\mathcal{F}[f](k)}\,e^{2\pi i 2^{-j}km}$$

$$= \sum_{n=0}^{N} \Big\{ \sum_{\ell \in \mathbb{Z}} \big| \mathcal{F}[f](n + 2^j \ell) \big|^2 \Big\} e^{2\pi i 2^{-j} mn} = \sum_{n=0}^{N} a_n \big(\overline{\lambda}^{\,m} \big)^n \qquad (3.12)$$

for $m = 1, 2, \cdots, N$, where $\lambda = e^{-2\pi i 2^{-j}}$. Thus, if we let A be the column vector

$$\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{pmatrix},$$

equalities (3.11) and (3.12) can be expressed matricially by the equality

$$C_N^* A = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where C_N is the matrix given by (3.6) and used in the proof of Theorem 3.7. Since $\frac{1}{\sqrt{N+1}} C_N^*$ is unitary, this means that

$$A = \frac{1}{N+1} C_N \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 2^{-j} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

But this is precisely equality (3.10).

Conversely, suppose condition (3.10) is satisfied by an $f \in L^2(\mathbb{T})$. We can "reverse" the equalities we have just derived to obtain (3.11) and (3.12). But these are equivalent to $\|f\|^2_{L^2(\mathbb{T})} = (f, f)_{L^2(\mathbb{T})} = 1$ and $(f, U^m f)_{L^2(\mathbb{T})} = 0$ for $m = 1, 2, \cdots, N$. Since U is unitary we have

$$\big(U^m f, U^{m+n} f \big)_{L^2(\mathbb{T})} = \big(f, U^n f \big)_{L^2(\mathbb{T})} \quad \text{whenever} \ \ 0 \le m \le m + n \le N.$$

This shows that $\{f, Uf, \cdots, U^N f\}$ is an orthonormal system in $L^2(\mathbb{T})$. ∎

We can now describe the promised "wavelet-like" basis for $L^2(\mathbb{T})$. Let ψ be the Franklin wavelet on the real line (see (1.14) in Theorem 1.13). For

$j = 0, 1, 2, \cdots$, $k = 0, 1, \cdots, 2^j - 1$, let

$$\tilde{\psi}_{j,k}(x) = 2^{\frac{j}{2}} \sum_{\ell \in \mathbb{Z}} \psi(2^j(x + \ell) - k). \tag{3.13}$$

Observe that

$$\tilde{\psi}_{j,k}(x) = \tilde{\psi}_{j,0}(x - 2^{-j}k) = (U_j^k \tilde{\psi}_{j,0})(x) = \sum_{\ell \in \mathbb{Z}} \psi_{j,k}(x + \ell), \tag{3.14}$$

where $\psi_{j,k}(x) = 2^{\frac{j}{2}} \psi(2^j x - k)$. Thus, each $\tilde{\psi}_{j,k}$ is a "periodization" of $\psi_{j,k}$. The exponential decay of ψ (see Proposition 1.16) shows that the series in (3.13) is uniformly convergent on $[0, 1]$. Thus, the functions $\tilde{\psi}_{j,k}$ are all continuous.

THEOREM 3.15 *The system*

$$\{1, \tilde{\psi}_{j,k} : j = 0, 1, 2, \cdots, \; k = 0, 1, \cdots, 2^j - 1\}$$

is an orthonormal basis for $L^2(\mathbb{T}) \approx L^2([0, 1))$.

REMARK: This system is called the **Franklin periodic wavelet** basis.

PROOF : Since

$$\overline{\mathcal{B}} = \mathcal{B}_0 \bigoplus \Big(\bigoplus_{j=0}^{\infty} \mathcal{C}_j \Big),$$

where $\mathcal{B}$ is the space of 1-periodic continuous functions on $\mathbb{R}$, and $\mathcal{B}$ is dense in $L^2(\mathbb{T})$, it is enough to show that the system

$$\{\tilde{\psi}_{j,k} : k = 0, 1, \cdots, N = 2^j - 1\}$$

is an orthonormal basis of $\mathcal{C}_j$, for each $j = 0, 1, 2, \cdots$.

To prove the orthonormality we shall use Theorem 3.9, and, hence, all we need to establish is that

$$\{\tilde{\psi}_{j,0}, \tilde{\psi}_{j,1}, \cdots, \tilde{\psi}_{j,N}\} = \{\tilde{\psi}_{j,0}, U\tilde{\psi}_{j,0}, \cdots, U^N \tilde{\psi}_{j,0}\}$$

satisfies (3.10). We claim that

$$\mathcal{F}[\tilde{\psi}_{j,0}](n) = 2^{-\frac{j}{2}} \hat{\psi}(2^{-j} n 2\pi). \tag{3.16}$$

This is shown by a simple periodization argument:

$$\begin{aligned}\mathcal{F}[\tilde{\psi}_{j,0}](n) &= \int_0^1 \Big\{2^{\frac{j}{2}} \sum_{\ell\in\mathbb{Z}} \psi(2^j(x+\ell))\Big\} e^{-2\pi i n x}\, dx \\ &= 2^{-\frac{j}{2}} \sum_{\ell\in\mathbb{Z}} \int_{2^j\ell}^{2^j(\ell+1)} \psi(t) e^{-2\pi i n 2^{-j} t}\, dt \\ &= 2^{-\frac{j}{2}} \int_{-\infty}^{\infty} \psi(t) e^{-2\pi i n 2^{-j} t}\, dt = 2^{-\frac{j}{2}} \hat{\psi}(2^{-j} n 2\pi).\end{aligned}$$

Since ψ is an orthonormal wavelet, by Proposition **??** of Chapter 2 we have

$$\sum_{k\in\mathbb{Z}} |\hat{\psi}(\xi + 2k\pi)|^2 = 1.$$

Putting $\xi = 2^{-j} n 2\pi$ in this equality and using (3.16) we obtain

$$1 = \sum_{k\in\mathbb{Z}} |\hat{\psi}(2^{-j} n 2\pi + 2k\pi)|^2 = 2^j \sum_{k\in\mathbb{Z}} \left|\mathcal{F}[\tilde{\psi}_{j,0}](n + 2^j k)\right|^2,$$

which is equality (3.10) for $\tilde{\psi}_{j,0}$.

The proof of the theorem is not finished yet. We still have to show that the functions $\tilde{\psi}_{j,k}$ are elements of $\mathcal{C}_j$, $j = 0, 1, \cdots$. This can be accomplished by using Theorem 3.7. Because of (3.14) it is enough to show that $\tilde{\psi}_{j,0} \in \mathcal{C}_j$. By (3.16) and (1.14) we have $\mathcal{F}[\tilde{\psi}_{j,0}](0) = 2^{-\frac{j}{2}}\hat{\psi}(0) = 0$, and

$$k^2 \mathcal{F}[\tilde{\psi}_{j,0}](k) = 2^{-\frac{j}{2}} k^2 \hat{\psi}(2^{-j} k 2\pi).$$

A simple computation, together with the fact that $\xi^2\hat{\psi}(\xi)$ is 4π-periodic (see part (b) of Theorem 1.5), shows that $k^2\mathcal{F}[\tilde{\psi}_{j,0}](k)$ is 2^{j+1}-periodic. Hence, by Theorem 3.8, $\tilde{\psi}_{j,0} \in \mathcal{B}_{j+1}$.

Finally, it remains to show the equality that appears in Theorem 3.7. Putting $\xi = 2^{-j} m 2\pi$, $m = 1, 2, \cdots, 2^j - 1$, in Theorem 2.19 (with $n = 1$) and using (3.16) again, we obtain

$$0 = \sum_{k\in\mathbb{Z}} \frac{\hat{\psi}(2^{-j} m 2\pi + 2k\pi)}{(2^{-j} m 2\pi + 2k\pi)^2} = \frac{2^{2j} 2^{\frac{j}{2}}}{4\pi^2} \sum_{k\in\mathbb{Z}} \frac{\mathcal{F}[\tilde{\psi}_{j,0}](m + 2^j k)}{(m + 2^j k)^2}.$$

Thus, the last condition in Theorem 3.7 is established. This completes the proof of our theorem.

▮

4.4 Orthonormal bases of periodic splines

The finite dimensional function spaces $\mathcal{B}_j \subset L^2(\mathbb{T})$ of section 4.3 were defined using linear interpolation between the values at the nodes $2^{-j}k$. All the functions in $\mathcal{B}_j$ were contained in the space $\mathcal{B}$ of 1-periodic continuous functions. New scales of spaces are obtained if we use polynomials of higher degree to do this interpolation and assume, globally, a certain degree of smoothness.

More precisely, for $m \geq 1$ and $j \geq 0$, we consider the space $\mathcal{B}_j^{(m)}$ of continuous 1-periodic functions on $\mathbb{R}$ which are in the class C^{m-1} and agree with a polynomial of degree at most m on each interval of the form $[2^{-j}(k-1), 2^{-j}k]$, $k = 1, 2, \cdots, 2^j$. Thus, we have $\mathcal{B}_j^{(1)} = \mathcal{B}_j$ and

$$\mathcal{B}_0^{(m)} \subset \mathcal{B}_1^{(m)} \subset \mathcal{B}_2^{(m)} \subset \cdots \subset \mathcal{B} \subset L^2(\mathbb{T}).$$

Observe that $\mathcal{B}_0^{(m)}$ consists of the constant functions. To see this we let $f \in \mathcal{B}_0^{(m)}$ and write

$$f(x) = a_0 + a_1 x + \cdots + a_m x^m \qquad \text{on} \quad [0,1];$$

since $f^{(m-1)}(x) = (m-1)!\,a_{m-1} + m!\,a_m x$ we must have

$$(m-1)!\,a_{m-1} + m!\,a_m = f^{(m-1)}(1) = f^{(m-1)}(0) = (m-1)!\,a_{m-1}.$$

Thus, $a_m = 0$. The argument can be repeated to prove that $\mathcal{B}_0^{(m)}$ contains only the constant functions.

We can now prove a result that is a generalization of Theorem 3.8. It gives us a characterization of the functions that belong to $\mathcal{B}_j^{(m)}$, and allows us to find the dimension of these spaces.

THEOREM 4.1 *Let $m \geq 1$, $j \geq 0$ and $f \in \mathcal{B}$. Then $f \in \mathcal{B}_j^{(m)}$ if and only if $k^{m+1}\mathcal{F}[f](k)$ is a 2^j-periodic function on $\mathbb{Z}$. In particular, $\mathcal{F}[f](2^j n) = 0$ for $n \neq 0$, $\mathcal{F}[f](0)$ is arbitrary and the dimension of $\mathcal{B}_j^{(m)}$ is 2^j.*

PROOF : The case $m = 1$ is Theorem 3.8. For a general $m > 1$ we proceed by induction on m. Let us consider the case $m = 2$. If $f \in \mathcal{B}_j^{(2)}$ we can write

$$f(x) = \sum_{k\in\mathbb{Z}} \mathcal{F}[f](k)e^{2\pi ikx}.$$

Thus, $f' \in \mathcal{B}_j^{(1)}$ and

$$f'(x) = \sum_{k\in\mathbb{Z}} k\mathcal{F}[f](k)(2\pi i)e^{2\pi ikx}.$$

By Theorem 3.8, $k^2(k\mathcal{F}[f](k))$ is 2^j-periodic.

Suppose now that $f \in \mathcal{B}$ and $k^3\mathcal{F}[f](k)$ is 2^j-periodic. This implies $\mathcal{F}[f](k) = O(k^{-3})$ as $|k| \to \infty$, and allows us to define

$$h(x) = (2\pi i)\sum_{k\in\mathbb{Z}} k\mathcal{F}[f](k)e^{2\pi ikx};$$

thus, $\mathcal{F}[h](k) = (2\pi i)k\mathcal{F}[f](k)$. Therefore, $k^2\mathcal{F}[h](k) = (2\pi i)k^3\mathcal{F}[f](k)$ is 2^j-periodic. By Theorem 3.8, $h \in \mathcal{B}_j^{(1)}$. But $h = f'$ so that f has one more degree of smoothness and $f \in \mathcal{B}_j^{(2)}$. The induction proof for $m > 2$ is similar.

∎

With the aid of this theorem we can give examples of elements in $\mathcal{B}_j^{(m)}$. Let $\tilde{\chi}_j$ be the 1-periodic function whose restriction to $[0,1)$ is $2^{\frac{j}{2}}\chi_{[0,2^{-j})}$ (see the proof of Lemma 3.3). If $\tilde{\Delta}_j$ is the 1-periodic "triangle" function of section 4.3 we then have $\tilde{\Delta}_j = \tilde{\chi}_j * \tilde{\chi}_j$. For $m \in \mathbb{N}$ we define, by induction,

$$\tilde{\Delta}_j^{(m)} = \tilde{\Delta}_j^{(m-1)} * \tilde{\chi}_j\,, \qquad m \geq 1,$$

with $\tilde{\Delta}_j^{(0)} = \tilde{\chi}_j$. Since

$$\mathcal{F}[\tilde{\Delta}_j^{(m)}](k) = \begin{cases} 2^{-(m+1)\frac{j}{2}} & \text{if } k = 0, \\ 2^{-(m+1)\frac{j}{2}}\Big(\dfrac{e^{-2\pi i2^{-j}k} - 1}{-2\pi i2^{-j}k}\Big)^{m+1} & \text{if } k \neq 0. \end{cases} \tag{4.2}$$

(see the proof of Lemma 3.3), the function $k^{m+1}\mathcal{F}[\tilde{\Delta}_j^{(m)}](k)$ is 2^j-periodic on $\mathbb{Z}$. Thus, $\tilde{\Delta}_j^{(m)} \in \mathcal{B}_j^{(m)}$, $j = 1, 2, \cdots$. It follows that the translations of this function, namely,

$$\tilde{\Delta}_{j,k}^{(m)}(x) = \tilde{\Delta}_j^{(m)}(x - 2^{-j}k), \qquad k = 0, 1, \cdots, 2^j - 1, \tag{4.3}$$

are also elements of $\mathcal{B}_j^{(m)}$. The collection of elements in (4.3) is a basis for $\mathcal{B}_j^{(m)}$. (We prove this at the end of section 4.6.) There is, however, a simpler way for finding a basis for this space. By Theorem 4.1 the functions

$$g_{j,k}^{(m)}(x) = \begin{cases} 1 & \text{if } \ k = 0, \\ \displaystyle\sum_{\ell \in \mathbb{Z}} \frac{k^{m+1}}{(k + 2^j \ell)^{m+1}} \, e^{2\pi i (k + 2^j \ell) x} & \text{for } \ k = 1, \cdots, 2^j - 1, \end{cases} \tag{4.4}$$

belong to $\mathcal{B}_j^{(m)}$. They are linearly independent since they involve a disjoint collection of characters. Since there are 2^j of them, they form a basis for $\mathcal{B}_j^{(m)}$.

The functions in the spaces $\mathcal{B}_j^{(m)}$ will be called **periodic splines of order** m. We want to find an orthonormal basis for $L^2(\mathbb{T})$ whose elements are periodic splines of order m. The case $m = 1$ was done in section 4.3. For the case $m > 1$ we proceed in a similar manner.

Let $\mathcal{C}_j^{(m)}$ be the orthogonal complement of $\mathcal{B}_j^{(m)}$ in $B_{j+1}^{(m)}$. Then

$$\mathcal{C}_j^{(m)} \oplus \mathcal{B}_j^{(m)} = \mathcal{B}_{j+1}^{(m)} \qquad \text{and} \qquad \dim(\mathcal{C}_j^{(m)}) = 2^j.$$

Since

$$\mathcal{B}_0^{(m)} \bigoplus \Big(\bigoplus_{j=0}^{\infty} \mathcal{C}_j^{(m)} \Big) = \overline{\mathcal{B}},$$

and $\mathcal{B}$ is dense in $L^2(\mathbb{T})$, a basis for $L^2(\mathbb{T})$ can be obtained by finding an orthonormal basis for each $\mathcal{C}_j^{(m)}$, $j = 0, 1, \cdots$. This will be done by an appropriate periodization of the spline wavelet basis of Theorem 2.17.

We begin with the characterization of the space $\mathcal{C}_j^{(m)}$. The result is similar to Theorem 3.7.

THEOREM 4.5 *$f \in \mathcal{C}_j^{(m)}$ if and only if $f \in \mathcal{B}_{j+1}^{(m)}$, $\mathcal{F}[f](0) = 0$ and*

$$\sum_{\ell \in \mathbb{Z}} \frac{\mathcal{F}[f](n + 2^j \ell)}{(n + 2^j \ell)^{m+1}} = 0 \qquad \textit{for} \ \ n = 1, 2, \cdots, N (= 2^j - 1).$$

PROOF: A function $f \in \mathcal{C}_j^{(m)}$ if and only if $f \in \mathcal{B}_{j+1}^{(m)}$ and f is orthogonal to all the elements in $\mathcal{B}_j^{(m)}$. The last assertion is equivalent to showing that

$$\sum_{k \in \mathbb{Z}} \mathcal{F}[g_{j,n}^{(m)}](k) \, \overline{\mathcal{F}[f](k)} = 0 \qquad \text{for} \ \ n = 0, 1, \cdots, N,$$

since (4.4) provides us with a basis $\{g_{j,n}^{(m)} : n = 0, 1, \cdots, N\}$ for $\mathcal{B}_j^{(m)}$.

Suppose $f \in \mathcal{C}_j^{(m)}$, then the above condition is satisfied. For $n = 0$ we have $\mathcal{F}[g_{j,0}^{(m)}](k) = \delta_{k,0}$; hence,

$$0 = \sum_{k \in \mathbb{Z}} \mathcal{F}[g_{j,0}^{(m)}](k)\, \overline{\mathcal{F}[f](k)} = \mathcal{F}[f](0).$$

When $1 \leq n \leq N$,

$$\mathcal{F}[g_{j,n}^{(m)}](k) = \begin{cases} \dfrac{n^{m+1}}{(n + 2^j \ell)^{m+1}} & \text{if } \ k = n + 2^j \ell \ \text{ for some } \ \ell \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$0 = \sum_{k \in \mathbb{Z}} \mathcal{F}[g_{j,n}^{(m)}](k)\, \overline{\mathcal{F}[f](k)} = \sum_{\ell \in \mathbb{Z}} \frac{n^{m+1}}{(n + 2^j \ell)^{m+1}}\, \overline{\mathcal{F}[f](n + 2^j \ell)}\,.$$

This establishes the required result for $n = 1, 2, \cdots, N$. The converse is obtained by reversing the above argument.

∎

We are now ready to describe the promised "wavelet-like" basis for $L^2(\mathbb{T})$ involving splines of order m. Let ψ^m be the spline wavelet of order m on the real line (see formula (2.16) and Theorem 2.17). For $j = 0, 1, 2, \cdots$ and $k = 0, 1, \cdots, 2^j - 1$, let

$$\tilde{\psi}_{j,k}^m(x) = \sum_{\ell \in \mathbb{Z}} \psi_{j,k}^m(x + \ell) \tag{4.6}$$

be the periodization of the elements $\psi_{j,k}^m(x) = 2^{\frac{j}{2}} \psi^m(2^j x - k)$ with j, k ranging as described above. Observe that

$$\tilde{\psi}_{j,k}^m(x) = 2^{\frac{j}{2}} \sum_{\ell \in \mathbb{Z}} \psi^m(2^j(x + \ell) - k)$$

and

$$\tilde{\psi}_{j,k}^m(x) = \tilde{\psi}_{j,0}^m(x - 2^{-j} k) = (U_j^k \tilde{\psi}_{j,0}^m)(x). \tag{4.7}$$

The exponential decay of ψ^m (see Theorem 2.18) shows that the series in (4.6) is uniformly convergent on $[0, 1]$. Thus, each of the functions $\tilde{\psi}_{j,k}^m$ is continuous.

We claim that these functions belong to $\mathcal{C}_j^{(m)}$. By (4.7) it is enough to show that $\tilde{\psi}_{j,0}^m \in \mathcal{C}_j^{(m)}$. To do this we use Theorem 4.5.

As in (3.16) the relation between the Fourier coefficients of the "periodized" function (as in (3.13) or (4.6)) and the Fourier transform of the original function is given by

$$\mathcal{F}[\tilde{\psi}_{j,0}^m](n) = 2^{-\frac{j}{2}}\widehat{\psi^m}(2^{-j}n2\pi). \tag{4.8}$$

It follows from the definition of $\widehat{\psi^m}$ (in (2.16)) that $\xi^{m+1}\widehat{\psi^m}(\xi)$ is 4π-periodic. Thus, $k^{m+1}\mathcal{F}[\tilde{\psi}_{j,0}^m](k)$ is 2^{j+1}-periodic on $\mathbb{Z}$:

$$\begin{aligned}
&(k+2^{j+1})^{m+1}\mathcal{F}[\tilde{\psi}_{j,0}^m](k+2^{j+1}) \\
&= (k+2^{j+1})^{m+1}2^{-\frac{j}{2}}\widehat{\psi^m}(2^{-j}(k+2^{j+1})2\pi) \\
&= (k+2^{j+1})^{m+1}2^{-\frac{j}{2}}\widehat{\psi^m}(2^{-j}k2\pi+4\pi) \\
&= (k+2^{j+1})^{m+1}2^{-\frac{j}{2}}\widehat{\psi^m}(2^{-j}k2\pi)\frac{(2^{-j}k2\pi)^{m+1}}{(2^{-j}k2\pi+4\pi)^{m+1}} \\
&= k^{m+1}\mathcal{F}[\tilde{\psi}_{j,0}^m](k)\,(k+2^{j+1})^{m+1}\frac{1}{(k+2^{j+1})^{m+1}} = k^{m+1}\mathcal{F}[\tilde{\psi}_{j,0}^m](k).
\end{aligned}$$

Theorem 4.1 allows us to conclude that $\tilde{\psi}_{j,0}^m \in \mathcal{B}_{j+1}^{(m)}$.

From (4.8) and the definition of $\widehat{\psi^m}$ (2.16) it follows that $\mathcal{F}[\tilde{\psi}_{j,0}^m](0) = 0$. It remains to show the equality that appears in Theorem 4.5. Putting $\xi = 2^{-j}\ell 2\pi$, $\ell = 1, 2, \cdots, 2^j - 1$, in Theorem 2.19 and using (4.8) again, we obtain

$$0 = \sum_{k\in\mathbb{Z}} \frac{\widehat{\psi^m}(2^{-j}\ell 2\pi + 2k\pi)}{(2^{-j}\ell 2\pi + 2k\pi)^{m+1}} = \frac{2^{\frac{j}{2}}}{(2^{-j}2\pi)^{m+1}} \sum_{k\in\mathbb{Z}} \frac{\mathcal{F}[\tilde{\psi}_{j,0}^m](\ell + 2^j k)}{(\ell + 2^j k)^2}\,.$$

This finishes the proof of our claim: $\tilde{\psi}_{j,k}^m \in \mathcal{C}_j^{(m)}$, $k = 0, 1, \cdots, 2^j - 1$.

Theorem 4.9 *For each $m \in \mathbb{N}$ the system*

$$\left\{1, \tilde{\psi}_{j,k}^m : j = 0, 1, 2, \cdots,\ k = 0, 1, \cdots, 2^j - 1\right\}$$

is an orthonormal basis for $L^2(\mathbb{T}) \approx L^2([0,1))$.

REMARK: Such a basis will be called a basis of **periodic spline wavelets**.

PROOF : Since

$$\overline{\mathcal{B}} = \mathcal{B}_0^{(m)} \bigoplus \Big(\bigoplus_{j=0}^{\infty} \mathcal{C}_j^{(m)} \Big),$$

where $\mathcal{B}$ is the space of all 1-periodic continuous functions on $\mathbb{R}$, and $\mathcal{B}$ is dense in $L^2(\mathbb{T})$, it is enough to show that the system

$$\{\tilde{\psi}_{j,k}^m : k = 0, 1, \cdots, N = 2^j - 1\} \tag{4.10}$$

is an orthonormal basis of $\mathcal{C}_j^{(m)}$, for each $j = 0, 1, 2, \cdots$ (recall that $\mathcal{B}_0^{(m)}$ consists of constant functions). The argument preceeding the statement of Theorem 4.9 shows that the elements of the system (4.10) belong to $\mathcal{C}_j^{(m)}$. It suffices to show that (4.10) is an orthonormal system. Since each element in the system (4.10) is a translate of the 1-periodic function $\tilde{\psi}_{j,0}^m$, all we need to do is to establish equality (3.10) for this $\tilde{\psi}_{j,0}^m$.

Since ψ^m is an orthonormal wavelet, Proposition **??** of Chapter 2 gives us

$$\sum_{k \in \mathbb{Z}} |\widehat{\psi^m}(\xi + 2k\pi)|^2 = 1.$$

Putting $\xi = 2^{-j} n 2\pi$, $n = 0, 1, \cdots, 2^j - 1$, in this equality and using (4.8) we obtain

$$1 = \sum_{k \in \mathbb{Z}} |\widehat{\psi^m}(2^{-j} n 2\pi + 2k\pi)|^2 = 2^j \sum_{k \in \mathbb{Z}} \big| \mathcal{F}[\tilde{\psi}_{j,0}^m](n + 2^j k) \big|^2.$$

This finishes the proof of our theorem.

∎

Splines are used to approximate functions by interpolation. It is clear that if $f \in \mathcal{B}_j^{(1)} = \mathcal{B}_j$ the values $y_k = f(2^{-j} k)$, $k = 0, 1, \cdots, 2^j - 1$, completely determine f, since this function coincides with the linear interpolation obtained by using y_k, $k = 0, 1, \cdots, 2^j - 1$, as nodes. We shall show that this interpolation property is shared by elements of $\mathcal{B}_j^{(m)}$ **only** when m is odd.

THEOREM 4.11 *Suppose that $y_0, y_1, \cdots, y_N$ are any $2^j = N + 1$ complex numbers and m is an odd positive integer. Then there exists a unique $f \in \mathcal{B}_j^{(m)}$ such that $f(2^{-j} k) = y_k$ for $k = 0, 1, \cdots, N$.*

Proof: It suffices to show the existence of a unique $\vartheta_j \equiv \vartheta_j^{(m)}$ in $\mathcal{B}_j^{(m)}$ satisfying

$$\vartheta_j(0) = 0,\ \vartheta_j(2^{-j}) = 1, \quad \text{and} \quad \vartheta_j(2^{-j}k) = 0 \ \text{ for } \ k = 2, \cdots, N. \tag{4.12}$$

This is clear since, then, $\vartheta_{j,k} = \vartheta_j(x - 2^{-j}k)$, $k = 0, 1, 2, \cdots, N$, is a linearly independent system of 2^j functions in $\mathcal{B}_j^{(m)}$ and, thus, a basis. The desired unique function is

$$f = \sum_{k=0}^{N} y_k \vartheta_{j,k}.$$

We construct ϑ_j first by showing that its Fourier coefficients are uniquely determined and, then, finding them. Let $\gamma(n) = \mathcal{F}[\vartheta_j](n)$, $n \in \mathbb{Z}$, be the n^{th} Fourier coefficient of ϑ_j. Then, since we want $\vartheta_j \in \mathcal{B}_j^{(m)}$, $n^{m+1}\gamma(n)$ has to be 2^j-periodic on $\mathbb{Z}$ (by Theorem 4.1). Hence, we can write

$$\begin{aligned} \vartheta_j(x) &= \gamma(0) + \sum_{n=1}^{N} \sum_{\ell \in \mathbb{Z}} (n + 2^j\ell)^{m+1} \gamma(n + 2^j\ell) \, \frac{e^{2\pi i (n+2^j\ell)x}}{(n + 2^j\ell)^{m+1}} \\ &= \gamma(0) + \sum_{n=1}^{N} n^{m+1} \gamma(n) e^{2\pi i n x} \sum_{\ell \in \mathbb{Z}} \frac{e^{2\pi i 2^j \ell x}}{(n + 2^j\ell)^{m+1}} \, . \end{aligned}$$

Putting $x = 2^{-j}k$, $k = 0, 1, \cdots, N$, in the above equality we obtain the system of equations

$$\begin{aligned} 0 &= \gamma(0) + \sum_{n=1}^{N} n^{m+1} \gamma(n) A_n^{(m)} \\ 1 &= \gamma(0) + \sum_{n=1}^{N} n^{m+1} \gamma(n) \, (\,\overline{\lambda}\,)^n A_n^{(m)} \\ 0 &= \gamma(0) + \sum_{n=1}^{N} n^{m+1} \gamma(n) \, (\,\overline{\lambda}\,)^{nk} A_n^{(m)}, \qquad k = 2, 3, \cdots, N, \end{aligned}$$

where $\lambda = e^{-2\pi i 2^{-j}}$ as before, and

$$A_n^{(m)} = \sum_{\ell \in \mathbb{Z}} \frac{1}{(n + 2^j\ell)^{m+1}} \, , \qquad n = 1, 2, \cdots, N.$$

Letting $b_n = n^{m+1}\gamma(n)A_n^{(m)}$, $n = 1, 2, \cdots, N$, and $b_0 = \gamma(0)$, the above system of $N + 1$ equations can be written matricially as

$$C_N^* \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where C_N is the same matrix as the one used in the proof of Theorem 3.7 (see (3.6)). As we have seen, $\frac{1}{\sqrt{N+1}}C_N$ is unitary. Therefore, multiplying both sides by $\frac{1}{N+1}C_N = 2^{-j}C_N$ we obtain

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix} = 2^{-j}C_N \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 2^{-j} \begin{pmatrix} 1 \\ \lambda \\ \lambda^2 \\ \vdots \\ \lambda^N \end{pmatrix}.$$

Thus,

$$b_n = 2^{-j}\lambda^n \qquad \text{for all } \ n = 0, 1, \cdots, N.$$

Since m is odd, $m + 1$ is even, and hence $A_n^{(m)} > 0$ for all $n = 1, 2, \cdots, N$. This allows us to write

$$\gamma(n) = \begin{cases} 2^{-j} & \text{if } \ n = 0, \\ \dfrac{2^{-j}\lambda^n}{n^{m+1}A_n^{(m)}} & \text{for } \ n = 1, \cdots, N. \end{cases}$$

The 2^j-periodicity of $n^{m+1}\gamma(n)$ determines all the Fourier coefficients of ϑ_j, and hence, ϑ_j is completely determined. This gives us the function ϑ_j satisfying (4.12).

∎

The proof of the above theorem allows us to do some explicit calculations. The formula we used to prove Lemma 1.9 gives us, when $m = 1$,

$$A_n^{(1)} = \sum_{\ell \in \mathbb{Z}} \frac{1}{(n + 2^j\ell)^2} = \frac{(2^{-j}\pi)^2}{\sin^2(2^{-j}n\pi)} \qquad \text{for } \ n = 1, 2, \cdots, N.$$

Hence, $\gamma(0) = 2^{-j}$ and

$$\gamma(n) = 2^{-j} e^{-2\pi i 2^{-j} n} \frac{\sin^2(2^{-j} n\pi)}{(2^{-j} n\pi)^2} = -\frac{2^j}{4\pi^2} \Big(\frac{e^{-2\pi i 2^{-j} n} - 1}{n} \Big)^2$$

for $n = 1, 2, \cdots, N$. This shows that the function $\vartheta \equiv \vartheta_j^{(1)}$ constructed in the proof of Theorem 4.11 coincides with the function $\tilde{\Delta}_j$ whose graph is given in Figure 4.7 (see Lemma 3.3 to obtain this result).

Lemma 1.9 gives us, for $n = 1, 2, \cdots, N$,

$$A_n^{(3)} = \sum_{\ell \in \mathbb{Z}} \frac{1}{(n + 2^j \ell)^4} = \frac{(2^{-j}\pi)^4}{\sin^4(2^{-j} n\pi)} \big(1 - \tfrac{2}{3} \sin^2(2^{-j} n\pi)\big).$$

Using this formula we can find the Fourier coefficients of the function $\vartheta_j^{(3)}$ in $\mathcal{B}_j^{(3)}$.

The property stated in Theorem 4.11 is not shared by the even order splines in $\mathcal{B}_j^{(2)}$. We can see this by computing $A_n^{(2)}$ for some $1 \le n \le N$. Differentiating once the formula

$$\sum_{n \in \mathbb{Z}} \frac{1}{(x + n)^2} = \frac{\pi^2}{\sin^2(\pi x)}$$

(see the proof of Lemma 1.9) we obtain

$$A_n^{(2)} = (2^{-j}\pi)^3 \frac{\cos(2^{-j} n\pi)}{\sin^3(2^{-j} n\pi)} \qquad n = 1, 2, \cdots, N.$$

For $n = 2^{j-1}$ we obtain $A_{2^{j-1}}^{(2)} = 0$. This forces $b_{2^{j-1}}$ to be 0, where b_n is as in the proof of Theorem 4.11. But we have shown in this theorem that $b_n = 2^{-j}\lambda^n$ with $\lambda = e^{-2\pi i 2^{-j}}$ and, thus, they cannot be zero. This shows that the interpolation property is not shared by the elements in $\mathcal{B}_j^{(2)}$. This is a general fact: **the interpolation property fails when m is even.** To see this observe that

$$A_{2^{j-1}}^{(m)} = \sum_{\ell \in \mathbb{Z}} \frac{1}{(2^{j-1} + 2^j \ell)^{m+1}}$$

$$= 2^{-(j-1)(m+1)} \Big\{ \sum_{\ell=0}^{\infty} \frac{1}{(1 + 2\ell)^{m+1}} + \sum_{\ell=1}^{\infty} \frac{1}{(1 - 2\ell)^{m+1}} \Big\}$$

$$= 2^{-(j-1)(m+1)}\Big\{\sum_{\ell=0}^{\infty}\frac{1}{(1+2\ell)^{m+1}} + (-1)^{m+1}\sum_{k=0}^{\infty}\frac{1}{(1+2k)^{m+1}}\Big\} = 0$$

when m is even. We notice that for $m = 2$ it is an easy exercise of algebra to show that there is no function $\vartheta_j^{(2)}$ in $\mathcal{B}_j^{(2)}$ that would satisfy condition (4.12) in the proof of Theorem 4.11.

We now consider the case $m = 3$. We shall show that, in this case, the unique function f whose existence is found in Theorem 4.11 is also the one that minimizes a certain functional.

Let $\mathcal{H}^2$ be the Sobolev space of 1-periodic functions with two derivatives in $L^2([0,1))$. Observe that $\mathcal{B}_j^{(3)} \subset \mathcal{H}^2$. On $\mathcal{H}^2$ we consider the functional

$$\mathcal{T}(f) = \int_0^1 \big|f''(x)\big|^2\, dx. \tag{4.13}$$

THEOREM 4.14 *Suppose that* $y_0, y_1, \cdots, y_N$ *are any* $2^j = N+1$ *complex numbers. Among all* $f \in \mathcal{H}^2$ *satisfying the specific interpolation property*

$$f(2^{-j}k) = y_k, \qquad k = 0, 1, \cdots, N,$$

the unique element of $\mathcal{B}_j^{(3)}$ *satisfying this property minimizes the functional* $\mathcal{T}$ *given by* (4.13).

PROOF : Let $\tilde{\Delta}_{j,k}$, $k = 0, 1, \cdots, N$ be the "triangle" functions given at the beginning of section 4.3 (see Figure 4.7), which form a basis of $\mathcal{B}_j^{(1)}$. Suppose that a function $g \in \mathcal{H}^2$ satisfies $g(2^{-j}k) = y_k$ for $k = 0, 1, \cdots, N$. Then

$$\begin{aligned}\langle g'', \tilde{\Delta}_{j,k}\rangle &= \int_0^1 g''(x)\,\overline{\tilde{\Delta}_{j,k}(x)}\, dx = \int_{2^{-j}k}^{2^{-j}(k+2)} g''(x)\,\overline{\tilde{\Delta}_{j,k}(x)}\, dx \\ &= 2^j(y_{k+2} - 2y_{k+1} + y_k)\end{aligned}$$

for $k = 0, 1, \cdots, 2^j - 1$ (we put $y_{2^j} = y_0$ and $y_{2^j+1} = y_1$). The last formula can be proved using integration by parts. Since $\{\tilde{\Delta}_{j,k} : k = 0, 1, \cdots, N\}$ is a basis for $\mathcal{B}_j^{(1)}$, this shows that $f'' - g''$ is orthogonal to $\mathcal{B}_j^{(1)}$ for all $f \in \mathcal{H}^2$ satisfying the specific interpolation property. We can restate this as

$$P(f'') = P(g''),$$

where P is the orthogonal projection of L^2 onto $\mathcal{B}_j^{(1)}$. If f is the unique function in $\mathcal{B}_j^{(3)}$ satisfying the specific interpolation property we then have $f'' \in \mathcal{B}_j^{(1)}$ and, hence, $P(f'') = f''$. It follows that

$$\begin{aligned}\int_0^1 \left|f''(x)\right|^2 dx &= \left\|f''\right\|^2_{L^2(\mathbb{T})} = \left\|P(f'')\right\|^2_{L^2(\mathbb{T})} \\ &= \left\|P(g'')\right\|^2_{L^2(\mathbb{T})} \le \left\|g''\right\|^2_{L^2(\mathbb{T})} = \int_0^1 \left|g''(x)\right|^2 dx\end{aligned}$$

so that the functional $\mathcal{T}$ of (4.13) is minimized by $f \in \mathcal{B}_j^{(3)}$.

$\blacksquare$

4.5 Periodization of wavelets defined on the real line

In sections 4.3 and 4.4 we have constructed periodic Franklin wavelets, as well as more general spline wavelets, by appropriate periodizations of the corresponding wavelets on the real line. We now generalize this procedure to other wavelets.

Throughout this section we shall assume that ψ is an orthonormal wavelet that arises from an MRA with scaling function φ. We shall also assume that both ψ and φ are in $L^1(\mathbb{R})$. This allows us to define, for each $j, k \in \mathbb{Z}$,

$$\widetilde{\varphi_{j,k}}(x) \equiv \sum_{\ell\in\mathbb{Z}} \varphi_{j,k}(x+\ell) = 2^{\frac{j}{2}} \sum_{\ell\in\mathbb{Z}} \varphi(2^j(x+\ell)-k) \tag{5.1}$$

and

$$\widetilde{\psi_{j,k}}(x) \equiv \sum_{\ell\in\mathbb{Z}} \psi_{j,k}(x+\ell) = 2^{\frac{j}{2}} \sum_{\ell\in\mathbb{Z}} \psi(2^j(x+\ell)-k), \tag{5.2}$$

which are 1-periodic functions belonging to $L^1([0,1])$.

It is easy to see that if $j \le 0$, $\widetilde{\varphi_{j,k}} = \widetilde{\varphi_{j,0}}$ and $\widetilde{\psi_{j,k}} = \widetilde{\psi_{j,0}}$ for all $k \in \mathbb{Z}$. We prove this only for the scaling function, since the proof is the same for the wavelet. The main ingredient is the fact that $2^{-j}k \in \mathbb{Z}$ when $j, k \in \mathbb{Z}$ and $j \le 0$:

$$\widetilde{\varphi_{j,k}}(x) = 2^{\frac{j}{2}} \sum_{\ell\in\mathbb{Z}} \varphi(2^j(x+\ell)-k) = 2^{\frac{j}{2}} \sum_{\ell\in\mathbb{Z}} \varphi(2^j(x+\ell-2^{-j}k))$$

$$= 2^{\frac{j}{2}} \sum_{m\in\mathbb{Z}} \varphi(2^j(x+m)) = \widetilde{\varphi_{j,0}}(x).$$

It turns out that when $j \le 0$ all these functions are constants. This is proved by computing the Fourier coefficients of $\widetilde{\varphi_{j,0}}$ for $j \le 0$. Recall that

$$\mathcal{F}[f](k) = \int_0^1 f(x)e^{-2\pi ikx}\,dx.$$

For $j, k \in \mathbb{Z}$, a similar calculation as the one that proves (3.16) gives us

$$\mathcal{F}[\widetilde{\varphi_{j,0}}](k) = 2^{-\frac{j}{2}}\hat{\varphi}(2^{-j}2k\pi), \qquad j,k \in \mathbb{Z}. \tag{5.3}$$

Observe that $\hat{\varphi}$ is continuous since $\varphi \in L^1(\mathbb{R})$, and, hence, it makes sense to write the value of $\hat{\varphi}$ at a particular point $2^{-j}2k\pi$. By Proposition 2.17 of Chapter 2, $\hat{\varphi}(2\ell\pi) = 0$ for all $\ell \neq 0$, $\ell \in \mathbb{Z}$. From (5.3) we deduce that, when $j \le 0$, all the Fourier coefficients of $\widetilde{\varphi_{j,0}}$ are zero except the one corresponding to $k = 0$. This shows that the functions $\widetilde{\varphi_{j,0}}$ are constants for all $j \le 0$.

We claim that when $j \le -1$, all the functions $\widetilde{\psi_{j,0}}$ are zero. A calculation similar to the one that shows (3.16) gives us

$$\mathcal{F}[\widetilde{\psi_{j,0}}](k) = 2^{-\frac{j}{2}}\hat{\psi}(2^{-j}2k\pi), \qquad j,k \in \mathbb{Z}. \tag{5.4}$$

By Proposition 2.17 of Chapter 2, $\hat{\psi}(4\ell\pi) = 0$ for all $\ell \in \mathbb{Z}$. Hence, all the Fourier coefficients of $\widetilde{\psi_{j,0}}$ are zero when $j \le -1$. This proves the claim.

For $j > 0$, many of the functions in (5.1) coincide; the same is true for the functions in (5.2). We claim that if $j > 0$, $m \in \mathbb{Z}$ and $0 \le k \le 2^j - 1$

$$(\varphi_{j,2^jm+k})^{\sim} = \widetilde{\varphi_{j,k}} \qquad \text{and} \qquad (\psi_{j,2^jm+k})^{\sim} = \widetilde{\psi_{j,k}}\,.$$

We write the proof for the wavelet, since the case of the scaling function is proved similarly. Let $k' = 2^jm + k$; then

$$\begin{aligned}\widetilde{\psi_{j,k'}}(x) &= 2^{\frac{j}{2}} \sum_{\ell\in\mathbb{Z}} \psi(2^j(x+\ell-m)-k)\\ &= 2^{\frac{j}{2}} \sum_{n\in\mathbb{Z}} \psi(2^j(x+n)-k) = \widetilde{\psi_{j,k}}(x).\end{aligned}$$

We summarize these results as follows:

LEMMA 5.5 *Let ψ be an orthonormal wavelet that arises from an MRA with a scaling function φ, and suppose that both ψ and φ are elements of $L^1(\mathbb{R})$. If $\widetilde{\varphi_{j,k}}$ and $\widetilde{\psi_{j,k}}$ are defined by* (5.1) *and* (5.2)*, respectively, we have*

(a) $\widetilde{\varphi_{j,k}} = \widetilde{\varphi_{j,0}}$ *for all $j \le 0$, $k \in \mathbb{Z}$; moreover $\widetilde{\varphi_{j,0}}$ is constant for $j \le 0$.*

(b) $\widetilde{\psi_{j,k}} \equiv 0$ *for all $j \le -1$ and $k \in \mathbb{Z}$.*

(c) *For $j \ge 0$, $0 \le k \le 2^j - 1$ and $m \in \mathbb{Z}$,*

$$(\varphi_{j,2^j m+k})^\sim = \widetilde{\varphi_{j,k}} \qquad \textit{and} \qquad (\psi_{j,2^j m+k})^\sim = \widetilde{\psi_{j,k}}\,.$$

We have shown that each of the collections of functions given by (5.1) and (5.2) for fixed non-negative j consists of, at most, 2^j distinct functions $\widetilde{\varphi_{j,k}}$ and $\widetilde{\psi_{j,k}}$. Some properties of these functions are inherited from the corresponding properties for the collections $\{\varphi_{j,k}\}$ and $\{\psi_{j,k}\}$, $j,k \in \mathbb{Z}$.

THEOREM 5.6 *For $j \ge 0$,*

(a) *the set $\{\widetilde{\varphi_{j,k}} : 0 \le k \le 2^j - 1\}$ is an orthonormal system in $L^2(\mathbb{T})$;*

(b) *the set $\{\widetilde{\psi_{j,k}} : 0 \le k \le 2^j - 1\}$ is an orthonormal system in $L^2(\mathbb{T})$;*

(c) $\langle \widetilde{\psi_{j,k}}, \widetilde{\varphi_{j,\ell}} \rangle = 0$ *for all $k, \ell \in \mathbb{Z}$ such that $0 \le k, \ell \le 2^j - 1$.*

PROOF: Let us prove (a). Let $k, m \in \mathbb{Z}$ be such that $0 \le k \le 2^j - 1$ and $0 \le m \le 2^j - 1$. Then

$$\begin{aligned}\langle \widetilde{\varphi_{j,k}}, \widetilde{\varphi_{j,m}} \rangle &= \sum_{\ell\in\mathbb{Z}} \int_0^1 \widetilde{\varphi_{j,k}}(x)\, \overline{\varphi_{j,m}(x+\ell)}\, dx \\ &= \sum_{\ell\in\mathbb{Z}} \int_\ell^{\ell+1} \widetilde{\varphi_{j,k}}(y-\ell)\, \overline{\varphi_{j,m}(y)}\, dy \\ &= \int_{\mathbb{R}} \widetilde{\varphi_{j,k}}(y)\, \overline{\varphi_{j,m}(y)}\, dy,\end{aligned}$$

where the last equality is due to the fact that $\widetilde{\varphi_{j,k}}$ is 1-periodic. Since $j \ge 0$, we can write

$$\widetilde{\varphi_{j,k}}(y) = \sum_{\ell\in\mathbb{Z}} \varphi_{j,k-2^j\ell}(y).$$

This gives

$$\langle \widetilde{\varphi_{j,k}}, \widetilde{\varphi_{j,m}} \rangle = \sum_{\ell \in \mathbb{Z}} \langle \varphi_{j,k-2^j\ell}, \varphi_{j,m} \rangle = \sum_{\ell \in \mathbb{Z}} \delta_{k-2^j\ell,m} \, .$$

Since $0 \le k, m \le 2^j - 1$, $k - 2^j\ell$ coincides with m only when $\ell = 0$ and, then, $k = m$. Hence $\langle \widetilde{\varphi_{j,k}}, \widetilde{\varphi_{j,m}} \rangle = \delta_{k,m}$, as we wanted to show.

The proofs for (b) and (c) are similar.

▮

REMARK: Parts (a) and (b) of Theorem 5.6 can also be proved by using Theorem 3.9. Observe that for $j \ge 0$ fixed, $\widetilde{\varphi_{j,k}}$ is a translation of $\widetilde{\varphi_{j,0}}$ by $2^{-j}k$, so that in the notation of Theorem 3.9, $\widetilde{\varphi_{j,k}} = U\widetilde{\varphi_{j,0}}$, $0 \le k \le 2^j - 1$.

We want to construct an orthonormal basis for $L^2(\mathbb{T})$ associated with an MRA orthonormal wavelet. For the wavelet ψ and the scaling function φ we shall assume $|\psi(x)| \le R_1(|x|)$ and $|\varphi(x)| \le R_2(|x|)$, where R_1 and R_2 are bounded decreasing functions belonging to $L^1([0,\infty))$. We shall call these kind of functions **radial decreasing L^1-majorants** of ψ and φ, respectively (see also (3.1) of Chapter 5).

For $j = 0, 1, 2, \cdots$, we define $\widetilde{V}_j$ as the subspace of $L^2(\mathbb{T})$ generated by $\{\widetilde{\varphi_{j,k}} : k = 0, 1, \cdots, 2^j - 1\}$; similarly, we define $\widetilde{W}_j$ as the subspace of $L^2(\mathbb{T})$ generated by $\{\widetilde{\psi_{j,k}} : k = 0, 1, \cdots, 2^j - 1\}$. Obviously, $\widetilde{V}_j$ and $\widetilde{W}_j$ have dimension 2^j. Some properties of these subspaces are inherited from the corresponding properties of the subspaces V_j and W_j associated with the MRA we are considering.

THEOREM 5.7 *For $j = 0, 1, 2, \cdots$, we have the following inclusions:*

$$\widetilde{V}_j \subset \widetilde{V}_{j+1} \qquad \text{and} \qquad \widetilde{W}_j \subset \widetilde{V}_{j+1}.$$

Moreover,

$$\widetilde{V}_j \oplus \widetilde{W}_j = \widetilde{V}_{j+1}.$$

PROOF : We only prove the inclusion $\widetilde{W}_j \subset \widetilde{V}_{j+1}$. The proof of the other inclusion is similar. Once these results are proved, the equality stated in the theorem follows from part (c) of Theorem 5.6 and the fact that $\dim \widetilde{V}_j = 2^j = \dim \widetilde{W}_j$.

Let

$$\widetilde{\psi_{j,k}}(x) = \sum_{\ell \in \mathbb{Z}} \psi_{j,k}(x+\ell)$$

be one of the functions that generate $\widetilde{W}_j$. Since $\psi_{j,k} \in V_{j+1}$ we can write

$$\psi_{j,k}(x) = \sum_{s \in \mathbb{Z}} \alpha_s^{(j,k)} \varphi_{j+1,s}(x).$$

The conditions imposed on ψ and φ imply that the sequence of coefficients $\{\alpha_s^{(j,k)} : s \in \mathbb{Z}\}$ belongs to $\ell^1(\mathbb{Z})$:

$$\begin{aligned} \sum_{s \in \mathbb{Z}} |\alpha_s^{(j,k)}| &= \sum_{s \in \mathbb{Z}} \Big| \int_{\mathbb{R}} \psi_{j,k}(x)\, \overline{\varphi_{j+1,s}(x)}\, dx \Big| \\ &\le \big\| \psi_{j,k} \big\|_{L^1} \sup_{x \in \mathbb{R}} \sum_{s \in \mathbb{Z}} |\varphi_{j+1,s}(x)| \le c \big\| \psi_{j,k} \big\|_{L^1} \big\| R_2 \big\|_{L^1([0,\infty))}. \end{aligned}$$

Hence,

$$\widetilde{\psi_{j,k}}(x) = \sum_{\ell \in \mathbb{Z}} \sum_{s \in \mathbb{Z}} \alpha_s^{(j,k)} \varphi_{j+1,s}(x+\ell) = \sum_{s \in \mathbb{Z}} \alpha_s^{(j,k)} (\varphi_{j+1,s})^{\sim}(x).$$

Writing $s = n + 2^{j+1}m$, $m \in \mathbb{Z}$, $s = 0, 1, \cdots, 2^{j+1}-1$ and observing that $(\varphi_{j+1,n+2^j m})^{\sim}(x) = (\varphi_{j+1,n})^{\sim}(x)$ (part (c) of Lemma 5.5), we obtain

$$\widetilde{\psi_{j,k}}(x) = \sum_{n=0}^{2^{j+1}-1} \Big\{ \sum_{s \in \mathbb{Z}} \alpha_{n+2^{j+1}m}^{(j,k)} \Big\} (\varphi_{j+1,n})^{\sim}(x) \equiv \sum_{n=0}^{2^{j+1}-1} c_n^{(j,k)} (\varphi_{j+1,n})^{\sim}(x).$$

This proves that $\widetilde{\psi_{j,k}} \in \widetilde{V}_{j+1}$ and establishes the desired inclusion. ∎

We have constructed an increasing sequence

$$\widetilde{V}_0 \subset \widetilde{V}_1 \subset \widetilde{V}_2 \subset \cdots$$

of subspaces of $L^2(\mathbb{T})$. To find an orthonormal basis we want to show that

$$\bigcup_{j=0}^{\infty} \widetilde{V}_j$$

is dense in $L^2(\mathbb{T})$. Towards this end we need a result that will be presented in Corollary 3.18 of Chapter 5. This corollary deals with the norm convergence of the projection operators P_j from $L^2(\mathbb{R})$ to V_j (see (3.2) of Chapter 5 for the definition) and requires the scaling function φ to have a radial decreasing L^1-majorant (see the definition in (3.1) of Chapter 5). The reader might find it helpful to look at section 5.3 before continuing with this one.

THEOREM 5.8 *Let ψ be an MRA orthonormal wavelet with scaling function φ such that both ψ and φ have bounded decreasing L^1-majorants. Then, the linear space*

$$\bigcup_{j=0}^{\infty} \widetilde{V}_j$$

is dense in the space $L^2(\mathbb{T})$ of all 2π-periodic functions that are square integrable on $\mathbb{T}$.

PROOF : By Corollary 3.18 of Chapter 5,

$$\lim_{j\to\infty} \left\| g - P_j(g) \right\|_{L^\infty} = 0$$

for all bounded uniformly continuous g, where P_j is the orthogonal projection from $L^2(\mathbb{R})$ onto V_j (see (3.2) of Chapter 5). P_j has a natural extension to the bounded functions on $\mathbb{R}$. Thus, if we take $f \in \widetilde{V}_j$ we only need to show that $P_j(f)$ belongs to $\widetilde{V}_j$ (observe that $\widetilde{V}_j \subset L^\infty(\mathbb{R})$ for $j \geq 0$). Since f is 1-periodic,

$$\begin{aligned} \langle f, \varphi_{j,k+2^j\ell} \rangle &= \int_{\mathbb{R}} f(x)\, \overline{\varphi_{j,k+2^j\ell}(x)}\, dx = \int_{\mathbb{R}} f(y+\ell)\, \overline{\varphi_{j,k}(y)}\, dy \\ &= \int_{\mathbb{R}} f(y)\, \overline{\varphi_{j,k}(y)}\, dy = \langle f, \varphi_{j,k} \rangle. \end{aligned}$$

Thus,

$$\begin{aligned} P_j f(x) &= \sum_{k\in\mathbb{Z}} \langle f, \varphi_{j,k} \rangle\, \varphi_{j,k}(x) = \sum_{n=0}^{2^j-1} \sum_{\ell\in\mathbb{Z}} \langle f, \varphi_{j,n+2^j\ell} \rangle\, \varphi_{j,n+2^j\ell}(x) \\ &= \sum_{n=0}^{2^j-1} \langle f, \varphi_{j,n} \rangle \sum_{\ell\in\mathbb{Z}} \varphi_{j,n+2^j\ell}(x) = \sum_{n=0}^{2^j-1} \langle f, \varphi_{j,n} \rangle\, \widetilde{\varphi_{j,n}}(x). \end{aligned}$$

∎

We have all the ingredients needed to construct orthonormal "wavelet" basis for $L^2(\mathbb{T})$.

THEOREM 5.9 *Let ψ be an MRA orthonormal wavelet with scaling function φ such that both ψ and φ have bounded decreasing L^1-majorants. Then the system*

$$\left\{1, \widetilde{\psi_{j,k}} : j = 0, 1, \cdots,\ k = 0, 1, \cdots, 2^j - 1\right\}$$

is an orthonormal basis of $L^2(\mathbb{T})$, where

$$\widetilde{\psi_{j,k}}(x) = \sum_{\ell \in \mathbb{Z}} \psi_{j,k}(x + \ell).$$

This basis is called the **basis of periodic wavelets** *for $L^2(\mathbb{T})$ associated with the wavelet ψ.*

PROOF : Theorems 5.8 and 5.7 show that

$$L^2(\mathbb{T}) = \widetilde{V}_0 \bigoplus \Big(\bigoplus_{j=0}^{\infty} \widetilde{W}_j \Big).$$

The result follows from Theorem 5.6.

∎

We can apply the procedure described above to the Franklin and spline wavelets of sections 4.1 and 4.2. We obtain the periodized Franklin and spline wavelets of sections 4.3 and 4.4.

Let us apply this procedure to a Lemarié-Meyer wavelet of the form $\hat{\psi}(\xi) = e^{i\frac{\xi}{2}} b(\xi)$ (see Corollary 4.7 of Chapter 1) where $b \in C^\infty$ is an even bell function (as in Theorem 4.5 of Chapter 1). We obtain that the $\widetilde{\psi_{j,k}}$ are trigonometric polynomials. More precisely, for $j = 0$ and $k = 0$, (5.4) gives us

$$\mathcal{F}[\widetilde{\psi_{0,0}}](\ell) = \hat{\psi}(2\ell\pi) = \begin{cases} -\dfrac{1}{\sqrt{2}} & \text{if } \ell = 1 \text{ or } -1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$\widetilde{\psi_{0,0}}(x) = -\frac{e^{-2\pi i x} + e^{2\pi i x}}{\sqrt{2}} = -\sqrt{2}\cos(2\pi x).$$

In general, if $j \geq 1$, (5.4) gives us

$$\mathcal{F}[\widetilde{\psi_{j,0}}](\ell) = 2^{-\frac{j}{2}}\hat{\psi}(2\pi 2^{-j}\ell) = 2^{-\frac{j}{2}} e^{\pi i 2^{-j}\ell} b(2\pi 2^{-j}\ell).$$

Hence, $\mathcal{F}[\widetilde{\psi_{j,0}}](\ell) = 0$ when $2\pi 2^{-j}\ell \notin \operatorname{supp}(b)$; that is, $|2\pi 2^{-j}\ell| \leq \pi - \varepsilon$ or $|2\pi 2^{-j}\ell| \geq 2\pi + 2\varepsilon$. This shows that $\widetilde{\psi_{j,k}}(x)$ is a finite trigonometric polynomial for these Lemarié-Meyer wavelets.

4.6 Notes and references

1. The Haar wavelet appeared in 1910 ([Haa]). In 1928, Ph. Franklin ([Fra]) introduced a set of continuous orthogonal functions, generalizing the Haar functions, and proved that they form a basis for $L^2([0,1))$. Around 1980, P. Wojtazczyk ([Wo1]) was able to show that this basis is an unconditional basis for $H^1([0,1))$ and J.O. Strömberg ([Str]) introduced modified Franklin systems with higher degree of smoothness, proving that they are unconditional bases for Hardy spaces over $\mathbb{R}^n$. The subject of unconditionality of wavelet basis will be treated in Chapter 5.

2. Splines have been used for several years in the theory of approximation of functions. The reader is referred to the books of I.J. Schoenberg ([Sch]), C. de Boor ([Boo]), and G. Nürnberger ([Nür]) for the theory and applications of this subject. The connection between wavelets and splines was first made explicit by Y. Meyer ([Me3]); most of the material presented in sections 4.3 and 4.4 has, in fact, appeared in [Me4]. The construction of spline wavelets in $\mathbb{Z}$ presented in sections 4.1 and 4.2 is essentially due to G. Battle ([Bat1]) (see also note 3 in this section). There is another way of constructing these wavelets presented by P.G. Lemarié in [Le4]. A comprehensive treatment of splines, spline wavelets and applications can be found in [Chu]. A more detailed study of the graph of the Franklin wavelet is presented in [Ber].

3. The way we constructed the Franklin and spline wavelets in sections 4.1 and 4.2 is related to the notion of an MRA with a Riesz basis, as was introduced after the definition of an MRA in section 2.1. Let

$$V_0 = \overline{\operatorname{span}\{\Delta^n(\cdot - k) : k \in \mathbb{Z}\}},$$

where $\Delta^1 = \chi_{[0,1]} * \chi_{[0,1]}$ and Δ^n is defined inductively in (2.1). For $j \in \mathbb{Z}$, $j \neq 0$, define

$$V_j = \{f \in L^2(\mathbb{R}) : f_{-j,0} \in V_0\},$$

where $f_{-j,0}(x) = 2^{-\frac{j}{2}} f(2^{-j}x)$. Clearly, (1.2) of the definition of an MRA (section 2.1) is satisfied. With the notation of Lemma 1.8 of Chapter 2 and using Proposition 2.7 we obtain

$$\begin{aligned}\sigma_{\Delta^n}(f) &= \Big(\sum_{k\in\mathbb{Z}} \big|\widehat{\Delta^n}(\xi + 2k\pi)\big|^2\Big)^{\frac{1}{2}} \\ &= \big|2\sin(\tfrac{1}{2}\xi)\big|^{n+1}\Big(\sum_{k\in\mathbb{Z}} \frac{1}{(\xi+2k\pi)^{2(n+1)}}\Big)^{\frac{1}{2}} = \sqrt{P_{2n+1}(\tfrac{1}{2}\xi)}\,,\end{aligned}$$

since $P_{2n+1}(\xi) > 0$ for all $\xi \in \mathbb{R}$. This last fact and the π-periodicity of P_{2n+1} allows us to conclude $\sqrt{A_n} \leq \sigma_{\Delta^n}(\xi) \leq \sqrt{B_n}$, $0 < A_n \leq B_n < \infty$, for all $\xi \in \mathbb{R}$. Proceeding now as in the argument that follows Lemma 1.8 of Chapter 2 we conclude that $\{\gamma^n(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal basis for V_0, where γ^n is defined by

$$\widehat{\gamma^n} = \widehat{\Delta^n}/\sigma_{\Delta^n}.$$

This establishes (1.5) of section 2.1. To verify (1.1) in the definition of an MRA (see Chapter 2), it is enough to show $V_0 \subset V_1$ (sometimes referred to as the **two scale equation**). This will be done by exhibiting a remarkable relation for the basic splines, namely

$$\Delta^n(x) = \sum_{k=0}^{n+1} 2^{-n}\binom{n+1}{k}\Delta^n(2x-k). \tag{6.1}$$

To see this, we use Lemma 2.3 to obtain

$$\frac{\widehat{\Delta^n}(\xi)}{\widehat{\Delta^n}(\frac{1}{2}\xi)} = e^{-i\frac{1}{4}(n+1)\xi}\big(\cos(\tfrac{1}{4}\xi)\big)^{n+1} = \Big(\frac{1+e^{-i\frac{\xi}{2}}}{2}\Big)^{n+1},$$

so that

$$\widehat{\Delta^n}(\xi) = \frac{1}{2^{n+1}}\sum_{k=0}^{n+1}\binom{n+1}{k}e^{-ik\frac{\xi}{2}}\widehat{\Delta^n}(\tfrac{1}{2}\xi).$$

The proof of (6.1) is finished by taking the inverse Fourier transform. The results proved in section 2.1 show that $\{V_j : j \in \mathbb{Z}\}$ is an MRA and the scaling function is given by γ^n, where

$$\widehat{\gamma^n}(\xi) = \frac{\widehat{\Delta^n}(\xi)}{\sigma_{\Delta^n}(\xi)},$$

which coincides (except for the factor $e^{-i\frac{1}{2}(n+1)\xi}$ if n is odd and $e^{-i\frac{n}{2}\xi}$ if n is even) with $\widehat{\varphi^n}$ given by (2.12).

4. Let us make some observations about the optimal values of A_n and B_n for which $\sqrt{A_n} \le \sigma_{\Delta^n}(\xi) \le \sqrt{B_n}$ (see the above paragraph). The optimal B_n is always 1. The best value of A_n is related to the roots of the so-called "Euler-Frobenius polynomials" (see [Chu] for details). From the formula for the polynomial P_3 given in the equality preceding (2.10) and the fact that $[\sigma_{\Delta^n}(\xi)]^2 = P_{2n+1}(\xi/2)$, proved in note 3, we obtain $A_1 = \frac{1}{3}$.

5. Another approach for finding spline wavelets is given in [Le4]. A spline wavelet ψ of order n must belong to $L^2(\mathbb{R}) \cap C^{n-1}(\mathbb{R})$ and is a polynomial of degree at most n on each interval of the form $(\frac{1}{2}k, \frac{1}{2}(k+1))$. An argument similar to the one given in the remark that follows (2.4) leads us to consider

$$\hat{\psi}(\xi) = \xi^{-(n+1)} \mathcal{A}_n(\xi), \tag{6.2}$$

where $\mathcal{A}_n(\xi)$ is a 4π-periodic function. To determine $|\mathcal{A}_n(\xi)|$ we can use equations (1.1) and (1.2) of Chapter 3, that characterize the orthonormality of the system $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$. Writing

$$M_n(\xi) = \sum_{k \in \mathbb{Z}} \frac{1}{|\xi + 4k\pi|^{2(n+1)}}$$

(see Proposition 2.7), equations (1.1) and (1.2) of Chapter 3 here are equivalent to

$$M_n(\xi)\big|\mathcal{A}_n(\xi)\big|^2 + M_n(\xi + 2\pi)\big|\mathcal{A}_n(\xi + 2\pi)\big|^2 = 1 \qquad \text{for } \ a.e.\ \xi \in \mathbb{R} \tag{6.3}$$

and

$$M_n(\xi)\mathcal{A}_n(\xi) + M_n(\xi + 2\pi)\mathcal{A}_n(\xi + 2\pi) = 0 \qquad \text{for } \ a.e.\ \xi \in \mathbb{R}, \tag{6.4}$$

respectively. Solving (6.4) for $\mathcal{A}_n(\xi + 2\pi)$ and replacing this value in (6.3) one obtains

$$\big|\mathcal{A}_n(\xi)\big| = \frac{\sqrt{M_n(\xi + 2\pi)}}{\sqrt{M_n(\xi)}} \cdot \frac{1}{\sqrt{M_n(\xi) + M_n(\xi + 2\pi)}}\,.$$

Guided by the general form of a wavelet arising from an MRA (see Proposition 2.13 of Chapter 2), one is led to choose

$$\mathcal{A}_n(\xi) = \nu(\xi) e^{i\frac{\xi}{2}} \frac{\sqrt{M_n(\xi + 2\pi)}}{\sqrt{M_n(\xi)}} \cdot \frac{1}{\sqrt{M_n(\xi) + M_n(\xi + 2\pi)}},$$

where $\nu(\xi)$ is a 2π-periodic function such that $|\nu(\xi)| = 1$ for a.e. $\xi \in \mathbb{T}$. Reversing the above argument we easily see that the system

$$\{\psi_{j,k} : j, k \in \mathbb{Z}\},$$

with this choice of ψ, is orthonormal. In particular $\|\psi\|_2 = 1$.

To prove that ψ is an orthonormal wavelet, Theorem 1.1 of Chapter 7 can be used and, hence, all that we need is to show that (1.2) and (1.3) of Chapter 7 are satisfied. In this setting, these two equations are equivalent to

$$\sum_{j\in\mathbb{Z}} 4^{-j(n+1)} \left|\mathcal{A}_n(2^j\xi)\right|^2 = |\xi|^{2(n+1)} \qquad \text{for } \ a.e.\ \xi \in \mathbb{R} \tag{6.5}$$

and

$$\sum_{j=1}^{\infty} 4^{-j(n+1)} \left|\mathcal{A}_n(2^j\xi)\right|^2 = -\mathcal{A}_n(\xi)\, \overline{\mathcal{A}_n(\xi + 2\pi)} \qquad \text{for } \ a.e.\ \xi \in \mathbb{R}, \tag{6.6}$$

respectively. Equality (6.6) follows by a remarkable equation satisfied by M_n, namely,

$$M_n(\xi) + M_n(\xi + 2\pi) = 4^{n+1} M_n(2\xi). \tag{6.7}$$

This equation can be proved by starting with the expression for $M_n(2\xi)$ in a series form and summing over the even and odd integers separately. Using (6.7) it can be shown that both the left-hand and the right-hand sides of (6.6) coincide with

$$\frac{1}{M_n(\xi) + M_n(\xi + 2\pi)}.$$

In computing the right-hand side of (6.6) the factor $e^{i\frac{\xi}{2}}$ that appears in the definition of $\mathcal{A}_n(\xi)$ is essential. Equation (6.5) now follows from (6.6) and the fact that

$$4^{(n+1)\pi} M_n(2\xi) \sim \xi^{-2(n+1)}$$

in a neighborhood of the origin (to prove this fact use Proposition 2.7 of this chapter).

It can be seen that the wavelet ψ just constructed coincides with the wavelet ψ^n given by (2.16), except for the factor $\nu(\xi)$ when n is odd and $-i\nu(\xi)$ when n is even. All these wavelets arise from an MRA. This is proved directly in [Le4] by exhibiting the corresponding scaling functions. It also follows from Corollary 3.16 of Chapter 7 (observe that ψ has exponential decay at infinity). Another way to prove this result is to show that

$$D_\psi(\xi) = 1 \qquad \text{for } \ a.e. \ \xi \in \mathbb{R}$$

(see Theorem 3.2 of Chapter 7). The equality $D_\psi(\xi) = 1$ for a.e. $\xi \in \mathbb{R}$ follows, in this setting, from (6.6) and (6.7).

We have shown that the spline wavelets of order n, ψ^n, have exponential decay at infinity. That is, $|\psi^n(x)| \le ce^{-\alpha(n)|x|}$. The exponent of decay $\alpha(n)$ depends on n. It is shown in [Le4] that

$$\alpha(n) \sim \frac{\pi^2}{4n} \qquad \text{as } \ n \to \infty.$$

Another amusing property (also proved in [Le4]) is that for a.e. $\xi \in \mathbb{R}$

$$\lim_{n\to\infty} \beta(n)\widehat{\psi^n}(\xi) = e^{i\frac{\xi}{2}}\chi_{[-2\pi,\pi]\cup[\pi,2\pi]}(\xi),$$

where $\beta(n) = 1$ if n is odd and $\beta(n) = i$ if n is even (see (2.16)). Hence, the Shannon wavelet is recovered as a limit of spline wavelets.

6. There is no C^∞ orthonormal wavelet with exponential decay. To see this suppose that ψ is such a wavelet. By Theorem 3.4 in Chapter 2, all the moments of ψ are zero. On the other hand, the exponential decay of ψ, say, $|\psi(x)| \le ce^{-\alpha|x|}$ for all $x \in \mathbb{R}$, implies that the function $\hat{\psi}(z)$, defined by

$$\hat{\psi}(z) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-izx}\,\psi(x)\,dx,$$

is an analytic function on $|\Im m z| < \alpha$. Thus,

$$\frac{d^n}{d\xi^n}\hat{\psi}(0) = 0 \qquad \text{for all } \ n = 0, 1, 2, \cdots,$$

since all the moments of ψ are zero. The expansion of $\hat{\psi}(z)$ in powers of z around the origin shows that $\hat{\psi} \equiv 0$ in a neighborhood of $z = 0$. Since $\{z \in \mathbb{C} : |\Im m z| < \alpha\}$ contains the real line in its interior, ψ must be the zero function on $\mathbb{R}$. This contradiction establishes our claim.

Also, **it is impossible to find a band-limited orthonormal wavelet ψ with exponential decay**. In fact, under these conditions, $\psi \in L^1(\mathbb{R})$ and, thus, $\hat{\psi}$ is continuous. Hence, by Theorem 2.7 of Chapter 3, all the moments of ψ are zero, and the same argument as above shows $\psi \equiv 0$.

On the other hand, if we relax the exponential decay condition, it is possible to construct such wavelets. A function g defined on $\mathbb{R}$ is said to have **subexponential decay** if for every $\varepsilon > 0$, there exist $C_\varepsilon > 0$ and $\alpha_\varepsilon > 0$ such that

$$|g(x)| \le C_\varepsilon e^{-\alpha_\varepsilon |x|^{1-\varepsilon}}, \qquad x \in \mathbb{R}.$$

The precise result is: there exists a band-limited orthonormal wavelet $\psi \in C^\infty$ such that ψ has subexponential decay. The proof of this result is constructive and the construction is accomplished by using the Gevrey class of functions (see [DH] for details).

7. The set of elements given by (4.3) is a basis for the space $\mathcal{B}_j^{(m)}$, $j = 1, 2, \cdots$, $m = 1, 2, \cdots$. Since the space $\mathcal{B}_j^{(m)}$ has dimension 2^j it is enough to show that the collection

$$\tilde{\Delta}_{j,k}^{(m)}(x) = \tilde{\Delta}_j^{(m)}(x - 2^{-j}k), \qquad k = 0, 1, \cdots, 2^j - 1 \equiv N,$$

is linearly independent. This is proved by induction on m. The case $m = 1$ is clear: see the graphs of the triangle functions $\tilde{\Delta}_{j,k}^{(1)}$ given in Figure 4.7. Let us assume that the result in true for $m - 1$. Our purpose is to show that if

$$a_0\tilde{\Delta}_j^{(m)}(x) + \cdots + a_N\tilde{\Delta}_j^{(m)}(x - 1 + 2^{-j}) = 0 \tag{6.8}$$

is valid for all $x \in \mathbb{R}$, we must have $a_0 = a_1 = \cdots = a_N = 0$. Since $\tilde{\Delta}_j^{(m)} = \tilde{\Delta}_j^{(m-1)} * \tilde{\chi}_j$ we deduce

$$\tilde{\Delta}_j^{(m)}(x) = 2^{\frac{j}{2}} \int_{x-2^{-j}}^{x} \tilde{\Delta}_j^{(m-1)}(t)\, dt.$$

Differentiating (6.8), rearranging terms and using the 1-periodicity of $\tilde{\Delta}_j^{(m-1)}$ we obtain

$$2^{\frac{j}{2}}(a_1 - a_0)\tilde{\Delta}_j^{(m-1)}(x - 2^{-j}) + \cdots + 2^{\frac{j}{2}}(a_0 - a_N)\tilde{\Delta}_j^{(m-1)}(x) = 0$$

for all $x \in \mathbb{R}$. By the induction hypothesis, $a_0 = a_1 = \cdots = a_N$. Hence, (6.8) can be written as

$$\begin{aligned} 0 &= a_0\bigl[\tilde{\Delta}_j^{(m)}(x) + \cdots + \tilde{\Delta}_j^{(m)}(x - 1 + 2^{-j})\bigr] \\ &= a_0 2^{\frac{j}{2}} \int_{x-1}^{x} \tilde{\Delta}_j^{(m-1)}(t)\,dt = a_0 2^{\frac{j}{2}} \int_0^1 \tilde{\Delta}_j^{(m-1)}(t)\,dt, \end{aligned}$$

due to the 1-periodicity of $\tilde{\Delta}_j^{(m-1)}$. Since $\tilde{\Delta}_j^{(m-1)}$ is a positive, not identically zero, function, we conclude that $a_0 = 0$, and, consequently,

$$a_1 = \cdots = a_N = a_0 = 0.$$

We are grateful to Pascal Auscher for relating this proof to us.