CHAPTER 2

Continuous-wave Modulation

Problem 2.1

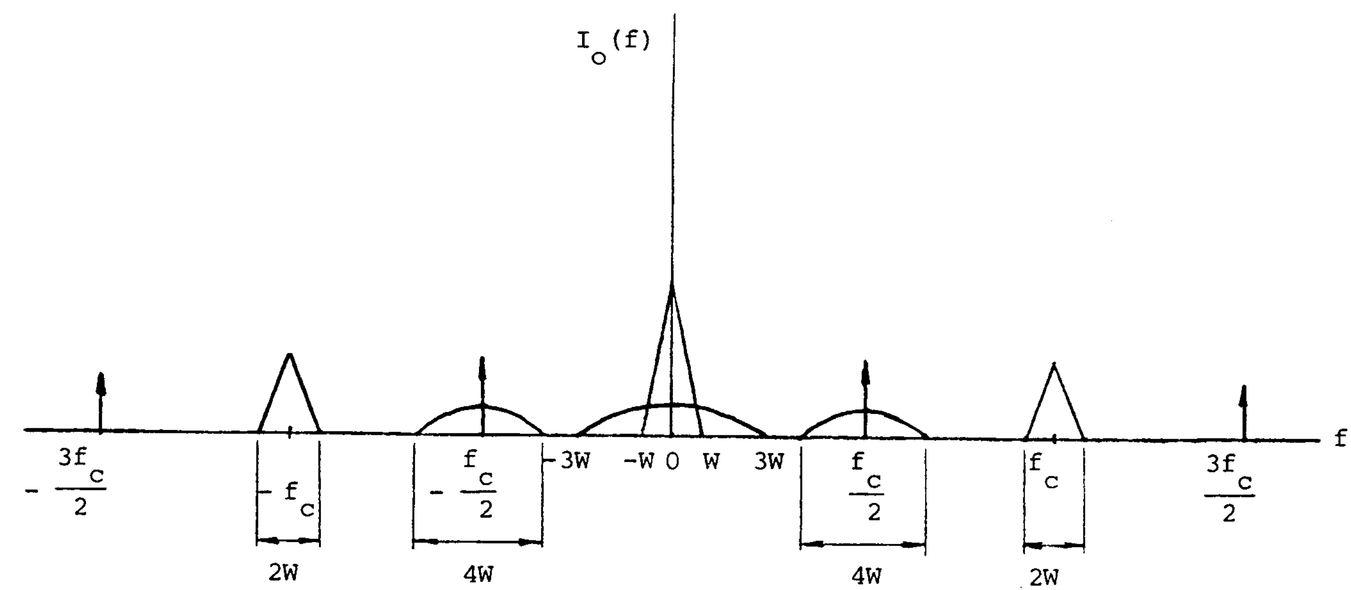
(a) Let the input voltage v_i consist of a sinusoidal wave of frequency $\frac{1}{2} f_c$ (i.e., half the desired carrier frequency) and the message signal m(t):

$$v_i = A_c \cos(\pi f_c t)+m(t)$$

Then, the output current i_o is

$$\begin{aligned} i_o &= a_1 v_i + a_3 v_i^3 \\ &= a_1[A_c\cos(\pi f_c t)+m(t)]+a_3[A_c\cos(\pi f_c t)+m(t)]^3 \\ &= a_1[A_c\cos(\pi f_c t)+m(t)] + \frac{1}{4}a_3A_c^3 [\cos(3\pi f_c t)+3\cos(\pi f_c t)] \\ &\quad + \frac{3}{2}a_3A_c^2 m(t)[1+\cos(2\pi f_c t)] + 3a_3A_c\cos(\pi f_c t)m^2(t) + a_3m^3(t) \end{aligned}$$

Assume that m(t) occupies the frequency interval $-W \leq f \leq W$. Then, the amplitude spectrum of the output current i_o is as shown below.

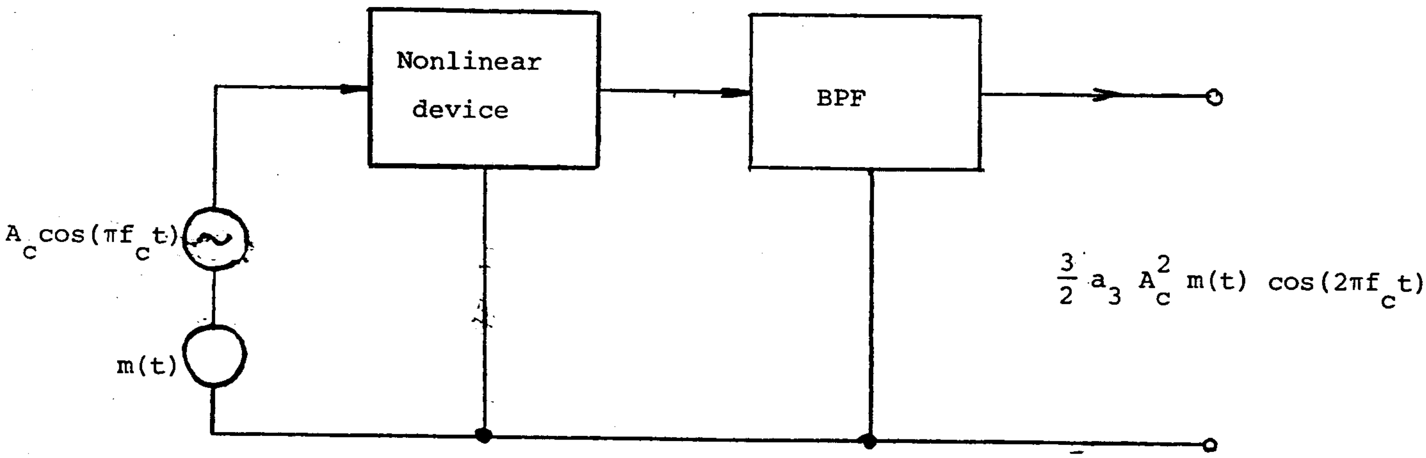


From this diagram we see that in order to extract a DSBSC wave, with carrier frequency f_c from i_o, we need a band-pass filter with mid-band frequency f_c and bandwidth 2W, which satisfy the requirement:

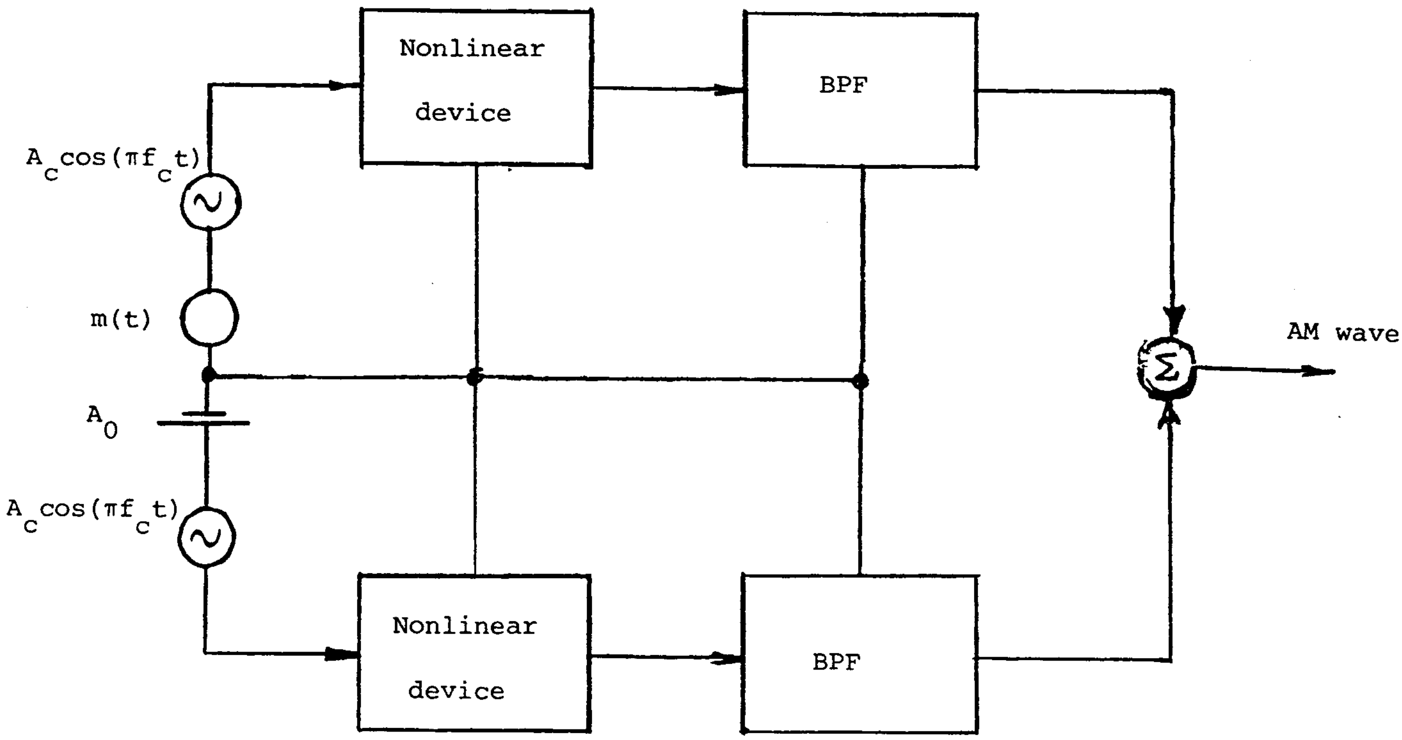
$$f_c - W > \frac{f_c}{2} + 2W$$

that is, $f_c > 6W$

Therefore, to use the given nonlinear device as a product mmodulator, we may use the following configuration:



(b) To generate an AM wave with carrier frequency f_c we require a sinusoidal component of frequency f_c to be added to the DSBSC generated in the manner described above. To achieve this requirement, we may use the following configuration involving a pair of the nonlinear devices and a pair of identical band-pass filters.



The resulting AM wave is therefore $\frac{3}{2} a_3 A_c^2[A_0+m(t)]\cos(2\pi f_c t)$. Thus, the choice of the dc level A_0 at the input of the lower branch controls the percentage modulation of the AM wave.

Problem 2.2

Consider the square-law characteristic:

$$v_2(t) = a_1 v_1(t) + a_2 v_1^2(t) \tag{1}$$

where a_1 and a_2 are constants. Let

$$v_1(t) = A_c \cos(2\pi f_c t) + m(t) \tag{2}$$

Therefore substitutingEq. (2) into (1), and expanding terms:

$$v_2(t) = a_1 A_c\left[1 + \frac{2a_2}{A_1} m(t)\right]\cos(2\pi f_c t) \tag{3}$$

$$+ a_1 m(t) + a_2 m^2(t) + a_2 A_c^2 \cos^2(2\pi f_c t)$$

The first term in Eq. (3) is the desired AM signal with $k_a = 2a_2/a_1$. The remaining three terms are unwanted terms that are removed by filtering.

Let the modulating wave m(t) be limited to the band $-W \le f \le W$, as in Fig. 1(a). Then, from Eq. (3) we find that the amplitude spectrum $|V_2(f)|$ is as shown in Fig. 1(b). It follows therefore that the unwanted terms may be removed from $v_2(t)$ by designing the tuned filter at the modulator output of Fig. P2.2 to have a mid-band frequency f_c and bandwidth 2W, which satisfy the requirement that $f_c > 3W$.

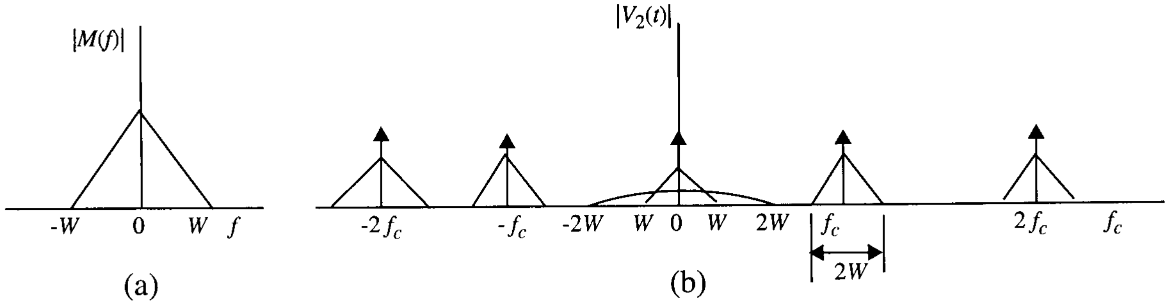


Figure 1

Problem 2.3

The generation of an AM wave may be accomplished using various devices; here we describe one such device called a *switching modulator.* Details of this modulator are shown in Fig. P2.3a, where it is assumed that the carrier wave $c(t)$ applied to the diode is large in amplitude, so that it swings right across the characteristic curve of the diode. We assume that the diode acts as an *ideal switch*, that is, it presents zero impedance when it is forward-biased [corresponding to $c(t) > 0$]. We may thus approximate the transfer characteristic of the diode-load resistor combination by a *piecewise-linear characteristic*, as shown in Fig. P2.3b. Accordingly, for an input voltage $v_1(t)$ consisting of the sum of the carrier and the message signal:

$$v_1(t) = A_c\cos(2\pi f_c t) + m(t) \tag{1}$$

where $|m(t)| << A_c$, the resulting load voltage $v_2(t)$ is

$$v_2(t) \approx \begin{cases} v_1(t), & c(t) > 0 \\ 0, & c(t) < 0 \end{cases} \tag{2}$$

That is, the load voltage $v_2(t)$ varies periodically between the values $v_1(t)$ and zero at a rate equal to the carrier frequency f_c. In this way, by assuming a modulating wave that is weak compared with the carrier wave, we have effectively replaced the nonlinear behavior of the diode by an approximately equivalent piecewise-linear time-varying operation.

We may express Eq. (2) mathematically as

$$v_2(t) \approx A_c \cos(2\pi f_c t) + m(t) g_{T_0}(t) \tag{3}$$

where $g_{T_0}(t)$ is a periodic pulse train of duty cycle equal to one-half, and period $T_0 = 1/f_c$, as in Fig. 1. Representing this $g_{T_0}(t)$ by its Fourier series, we have

$$g_{T_0}(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\cos[2\pi f_c t(2n-1)] \tag{4}$$

Therefore, substituting Eq. (4) in (3), we find that the load voltage $v_2(t)$ consists of the sum of two components:

1. The component

$$\frac{A_c}{2}\left[1 - \frac{4}{\pi A_c}m(t)\right]\cos(2\pi f_c t)$$

which is the desired AM wave with amplitude sensitivity $k_a = 4\pi A_c$. The switching modulator is therefore made more sensitive by reducing the carrier amplitude A_c; however, it must be maintained large enough to make the diode act like an ideal switch.

2. Unwanted components, the spectrum of which contains delta functions at 0, $\pm 2f_c$, $\pm 4f_c$, and so on, and which occupy frequency intervals of width $2W$ centered at 0, $\pm 3f_c$. $\pm 5f_c$, and so on, where W is the message bandwidth.

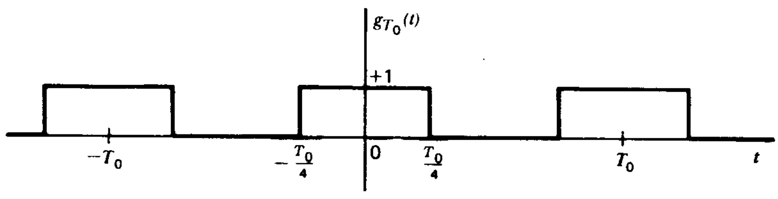


Fig. 1: Periodic pulse train

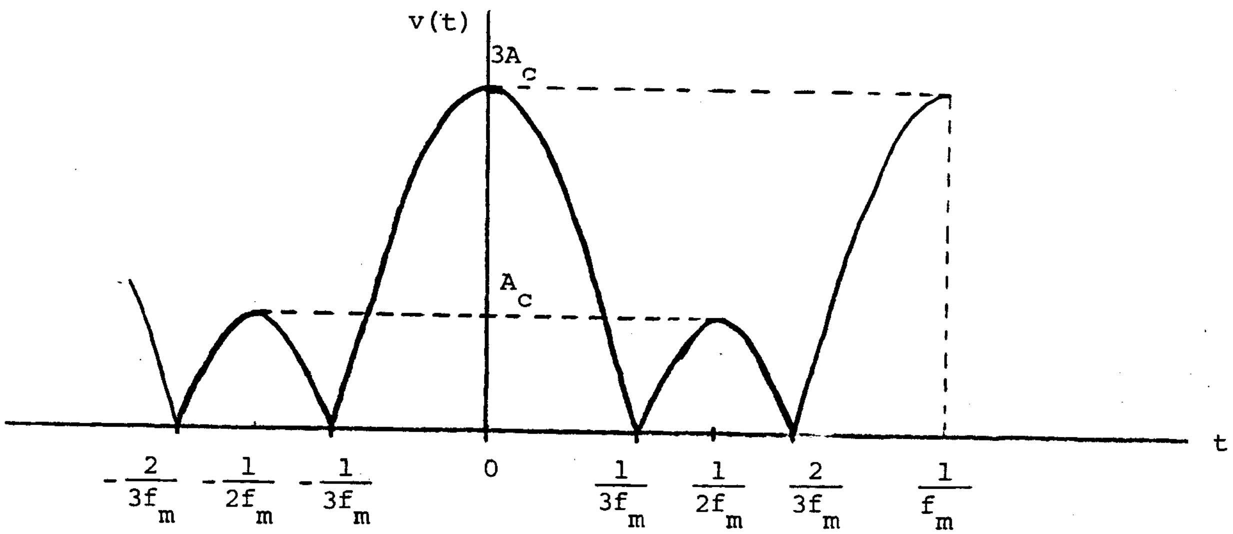
The unwanted terms are removed from the load voltage $v_2(t)$ by means of a band-pass filter with mid-band frequency f_c and bandwidth $2W$, provided that $f_c > 2W$. This latter condition ensures that the frequency separations between the desired AM wave the unwanted components are large enough for the band-pass filter to suppress the unwanted components.

Problem 2.4

(a) The envelope detector output is

$$v(t) = A_c|1+ \mu\cos(2\pi f_m t)|$$

which is illustrated below for the case when $\mu=2$.



We see that v(t) is periodic with a period equal to f_m, and an even function of t, and so we may express v(t) in the form:

$$v(t) = a_0 + 2 \sum_{n=1}^{\infty} a_n \cos(2n\pi f_m t)$$

where $$a_0 = 2f_m \int_0^{1/2f_m} v(t)dt$$

$$= 2A_c f_m \int_0^{1/3f_m} [1+2\cos(2\pi f_m t)]dt + 2A_c f_m \int_{1/3f_m}^{1/2f_m} [-1-2\cos(2\pi f_m t)]dt$$

$$= \frac{A_c}{3} + \frac{4A_c}{\pi}\sin(\frac{2\pi}{3}) \tag{1}$$

$$a_n = 2f_m \int_0^{1/2f_m} v(t)\cos(2n\pi f_m t)\,dt$$

$$= 2A_c f_m \int_0^{1/3f_m} [1+2\cos(2\pi f_m t)]\cos(2n\pi f_m t)\,dt$$

$$+ 2A_c f_m \int_{1/3f_m}^{1/2f_m} [-1-2\cos(2\pi f_m t)]\cos(2n\pi f_m t)\,dt$$

$$= \frac{A_c}{n\pi}\left[2\sin\left(\frac{2n\pi}{3}\right) - \sin(n\pi)\right] + \frac{A_c}{(n+1)\pi}\left\{2\sin\left[\frac{2\pi}{3}(n+1)\right] - \sin[\pi(n+1)]\right\}$$

$$+ \frac{A_c}{(n-1)\pi}\left\{2\sin\left[\frac{2\pi}{3}(n-1)\right] - \sin[\pi(n-1)]\right\} \qquad (2)$$

For n=0, Eq. (2) reduces to that shown in Eq. (1).

(b) For n=1, Eq. (2) yields

$$a_1 = A_c\left(\frac{\sqrt{3}}{2\pi} + \frac{1}{3}\right)$$

For n=2, it yields

$$a_2 = \frac{A_c\sqrt{3}}{2\pi}$$

Therefore, the ratio of second-harmonic amplitude to fundamental amplitude in v(t) is

$$\frac{a_2}{a_1} = \frac{3\sqrt{3}}{2\pi+3\sqrt{3}} = 0.452$$

Problem 2.5

(a) The demodulation of an AM wave can be accomplished using various devices; here, we describe a simple and yet highly effective device known as the *envelope detector.* Some version of this demodulator is used in almost all commercial AM radio receivers. For it to function properly, however, the AM wave has to be narrow-band, which requires that the carrier frequency be large compared to the message bandwidth. Moreover, the percentage modulation must be less than 100 percent.

An envelope detector of the series type is shown in Fig. P2.5, which consists of a diode and a resistor-capacitor (RC) filter. The operation of this envelope detector is as follows. On a positive half-cycle of the input signal, the diode is forward-biased and the capacitor C charges up rapidly to the peak value of the input signal. When the input signal falls below this value, the diode becomes reverse-biased and the capacitor C discharges slowly through the load resistor R_l. The discharging process continues until the next positive half-cycle. When the input signal becomes greater than the voltage across the capacitor, the diode conducts again and the process is repeated. We assume that the diode is ideal, presenting resistance r_f to current flow in the forward-biased region and infinite resistance in the reverse-biased region. We further assume that the AM wave applied to the envelope detector is supplied by a voltage source of internal impedance R_s. The charging time constant $(r_f + R_s)\, C$ must be short compared with the carrier period $1/f_c$, that is

$$(r_f + R_s)C \ll \frac{1}{f_c} \tag{1}$$

so that the capacitor C charges rapidly and thereby follows the applied voltage up to the positive peak when the diode is conducting.

(b) The discharging time constant R_lC must be long enough to ensure that the capacitor discharges slowly through the load resistor R_l between positive peaks of the carrier wave, but not so long that the capacitor voltage will not discharge at the maximum rate of change of the modulating wave, that is

$$\frac{1}{f_c} \ll R_lC \ll \frac{1}{W} \tag{2}$$

where W is the message bandwidth. The result is that the capacitor voltage or detector output is nearly the same as the envelope of the AM wave.

Problem 2.6

Let

$$v_1(t) = A_c[1 + k_a m(t)]\cos(2\pi f_c t)$$

(a) Then the output of the square-law device is

$$v_2(t) = a_1 v_1(t) + a_2 v_1^2(t)$$

$$= a_1 A_c[1 + k_a m(t)]\cos(2\pi f_c t)$$

$$+ \frac{1}{2} a_2 A_c^2[1 + 2k_a m(t) + k_a^2 m^2(t)]\,[1 + \cos(4\pi f_c t)]$$

(b) The desired signal, namely $a_2 A_c^2 k_a m(t)$, is due to the $a_2 v_1^2(t)$ - hence, the name "square-law detection". This component can be extracted by means of a low-pass filter. This is not the only contribution within the baseband spectrum, because the term $1/2\ a_2 A_c^2 k_a^2 m^2(t)$ will give rise to a plurality of similar frequency components. The ratio of wanted signal to distortion is $2/k_a m(t)$. To make this ratio large, the percentage modulation, that is, $|k_a m(t)|$ should be kept small compared with unity.

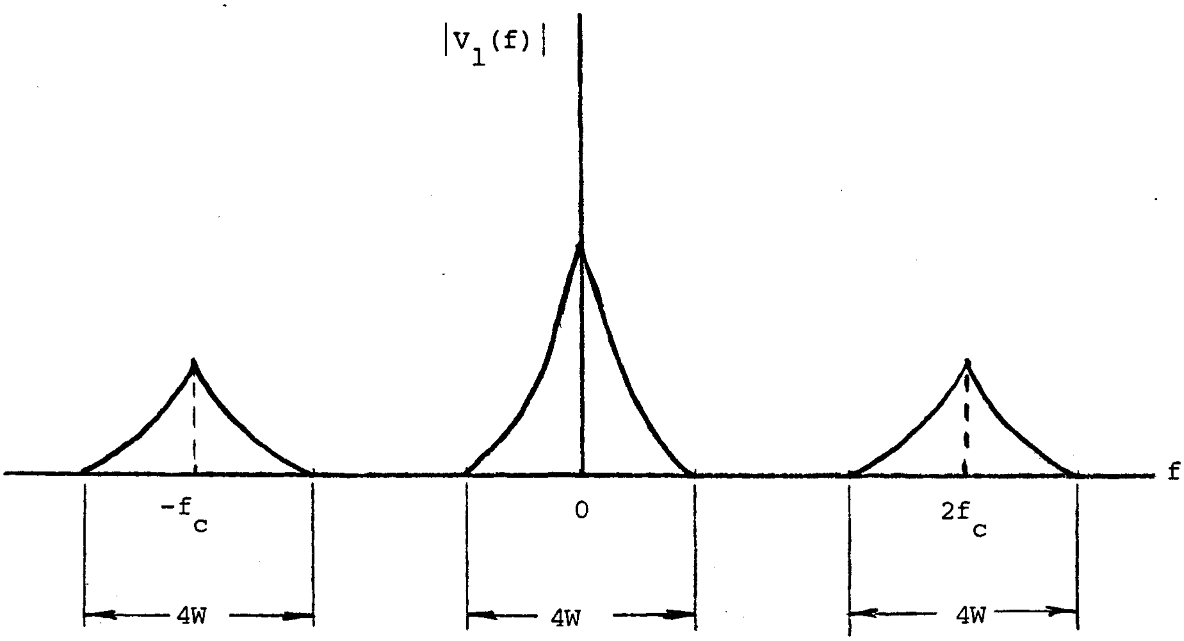
Problem 2.7

The squarer output is

$$v_1(t) = A_c^2\,[1+k_a m(t)]^2 \cos^2(2\pi f_c t)$$

$$= \frac{A_c^2}{2}\,[1+2k_a m(t)+m^2(t)][1+\cos(4\pi f_c t)]$$

The amplitude spectrum of $v_1(t)$ is therefore as follows, assuming that m(t) is limited to the interval $-W \le f \le W$:



Since $f_c > 2W$, we find that $2f_c-2W > 2W$. Therefore, by choosing the cutoff frequency of the low-pass filter greater than 2W, but less than $2f_c-2W$, we obtain the output

$$v_2(t) = \frac{A_c^2}{2}\,[1+k_a m(t)]^2$$

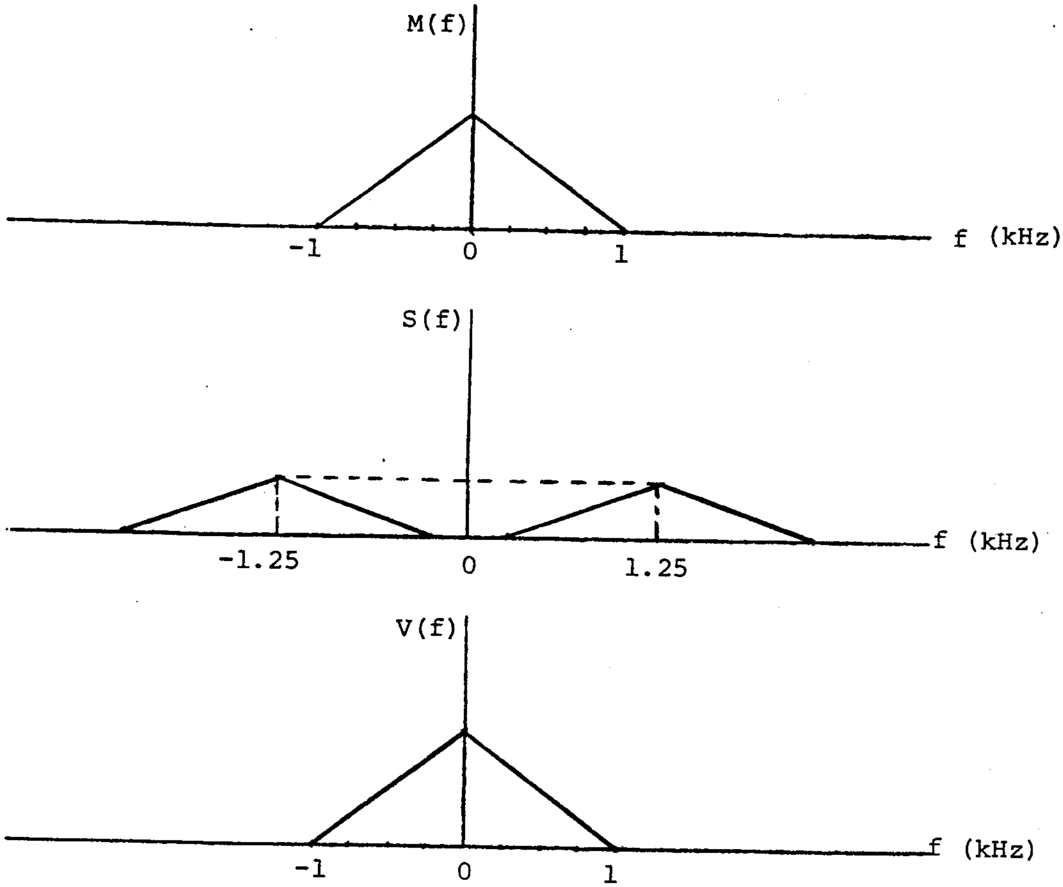
Hence, the square-rooter output is

$$v_3(t) = \frac{A_c}{\sqrt{2}}\,[1+k_a m(t)]$$

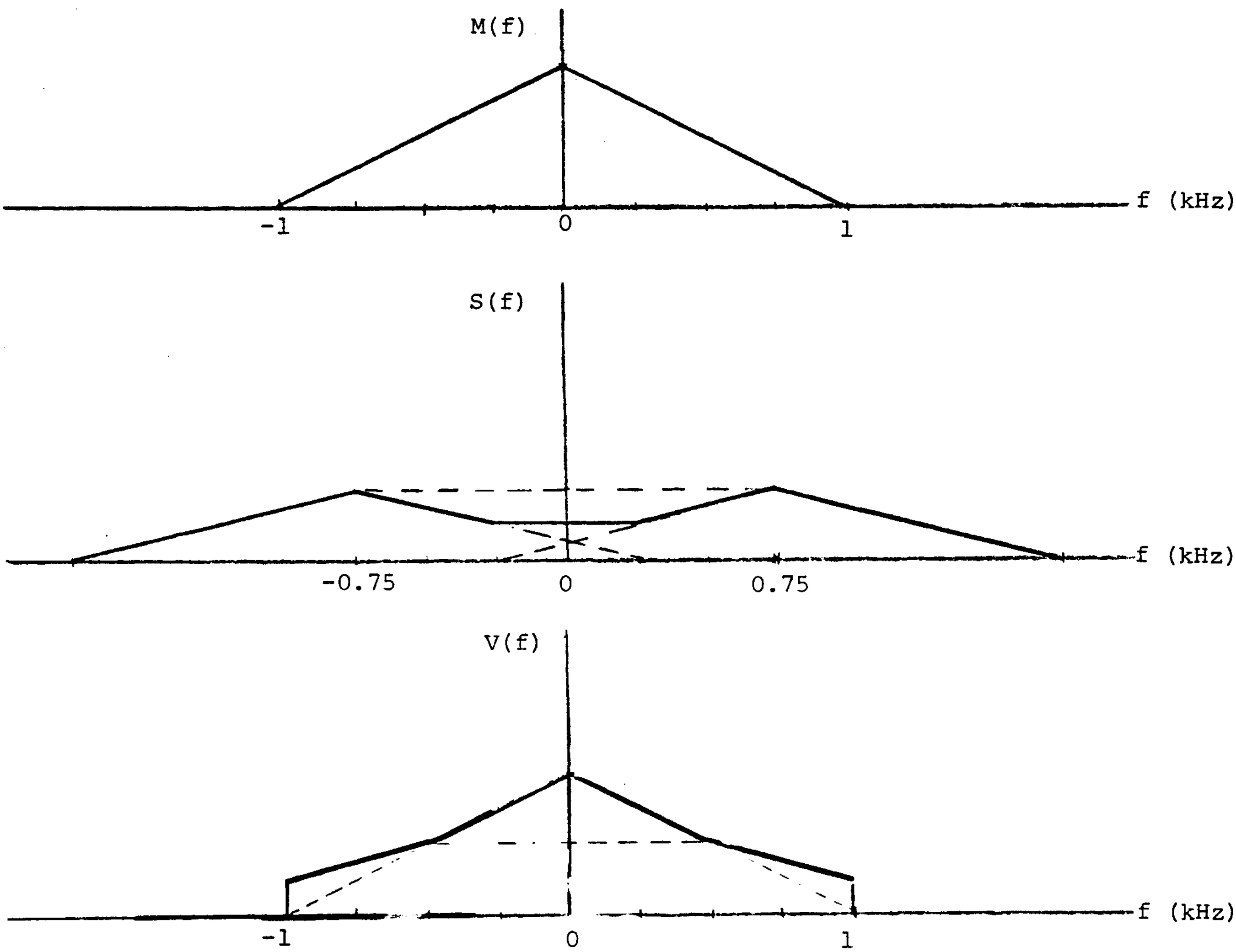
which, except for the dc component $\frac{A_c}{\sqrt{2}}$, is proportional to the message signal m(t).

Problem 2.8

(a) For f_c = 1.25 kHz, the spectra of the message signal m(t), the product modulator output s(t), and the coherent detector output v(t) are as follows, respectively:



(b) For the case when $f_c = 0.75$, the respective spectra are as follows:



To avoid sideband-overlap, the carrier frequency f_c must be greater than or equal to 1 kHz. The lowest carrier frequency is therefore 1 kHz for each sideband of the modulated wave s(t) to be uniquely determined by m(t).

Problem 2.9

The two AM modulator outputs are

$$s_1(t) = A_c[1 + k_a m(t)]\cos(2\pi f_c t)$$

$$s_2(t) = A_c[1 - k_a m(t)]\cos(2\pi f_c t)$$

Subtracting $s_2(t)$ from $s_1(t)$:

$$\begin{aligned} s(t) &= s_2(t) - s_1(t) \\ &= 2k_a A_c m(t)\cos(2\pi f_c t) \end{aligned}$$

which represents a DSB-SC modulated wave.

Problem 2.10

(a) Multiplying the signal by the local oscillator gives:

$$s_1(t) = A_c m(t)\cos(2\pi f_c t)\cos[2\pi(f_c+\Delta f)t]$$

$$= \frac{A_c}{2} m(t)\{\cos(2\pi\Delta f t) + \cos[2\pi(2f_c+\Delta f)t]\}$$

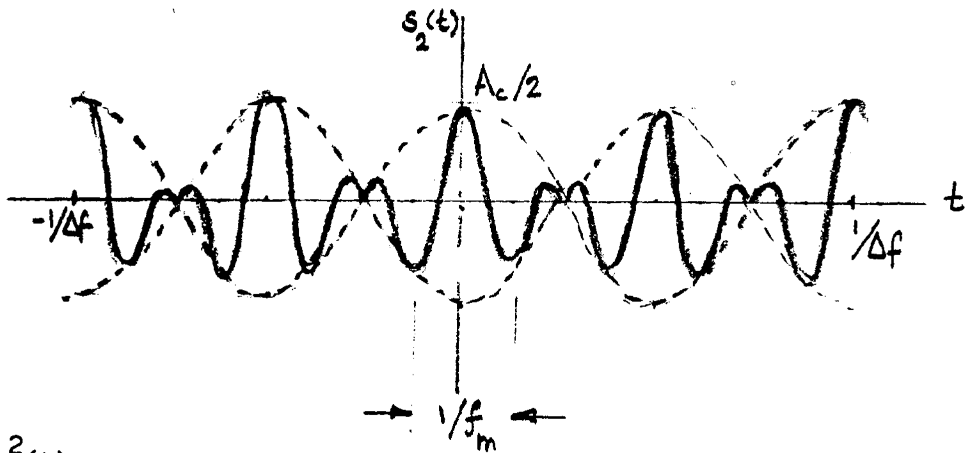
Low pass filtering leaves:

$$s_2(t) = \frac{A_c}{2} m(t)\cos(2\pi\Delta f t)$$

Thus the output signal is the message signal modulated by a sinusoid of frequency Δf.

(b) If $m(t) = \cos(2\pi f_m t)$,

then $s_2(t) = \frac{A_c}{2}\cos(2\pi f_m t)\cos(2\pi\Delta f t)$



Problem 2.11

(a) $y(t) = s^2(t)$

$$= A_c^2\cos^2(2\pi f_c t)m^2(t)$$

$$= \frac{A_c^2}{2}[1+\cos(4\pi f_c t)]m^2(t)$$

Therefore, the spectrum of the multiplier output is

$$Y(f) = \frac{A_c^2}{2}\int_{-\infty}^{\infty} M(\lambda)M(f-\lambda)d\lambda + \frac{A_c^2}{4}\left[\int_{-\infty}^{\infty} M(\lambda)M(f-2f_c-\lambda)d\lambda + \int_{-\infty}^{\infty} M(\lambda)M(f+2f_c-\lambda)d\lambda\right]$$

where $M(f) = F[m(t)]$.

(b) At $f=2f_c$, we have

$$Y(2f_c) = \frac{A_c^2}{2} \int_{-\infty}^{\infty} M(\lambda)M(2f_c-\lambda)d\lambda$$

$$+ \frac{A_c^2}{4} [\int_{-\infty}^{\infty} M(\lambda)M(-\lambda)d\lambda + \int_{-\infty}^{\infty} M(\lambda)M(4f_c-\lambda)d\lambda]$$

Since $M(-\lambda) = M^*(\lambda)$, we may write

$$Y(2f_c) = \frac{A_c^2}{2} \int_{-\infty}^{\infty} M(\lambda)M(2f_c-\lambda)d\lambda$$

$$+ \frac{A_c^2}{4} [\int_{-\infty}^{\infty} |M(\lambda)|^2 d\lambda + \int_{-\infty}^{\infty} M(\lambda)M(4f_c-\lambda)d\lambda] \qquad (1)$$

With $m(t)$ limited to $-W \leq f \leq W$ and $f_c > W$, we find that the first and third integrals reduce to zero, and so we may simplify Eq. (1) as follows

$$Y(2f_c) = \frac{A_c^2}{4} \int_{-\infty}^{\infty} |M(\lambda)|^2 d\lambda$$

$$= \frac{A_c^2 E}{4}$$

where E is the signal energy (by Rayleigh's energy theorem). Similarly, we find that

$$Y(-2f_c) = \frac{A_c^2}{4} E$$

The band-pass filter output, in the frequency domain, is therefore defined by

$$V(f) \simeq \frac{A_c^2}{4} E \ \Delta f[\delta(f-2f_c) + \delta(f+2f_c)]$$

Hence,

$$v(t) \simeq \frac{A_c^2}{2} E \ \Delta f \cos(4\pi f_c t)$$

Problem 2.12

The multiplexed signal is

$$s(t) = A_c\, m_1(t) \cos(2\pi f_c t) + A_c\, m_2(t) \sin(2\pi f_c t)$$

Therefore,

$$S(f) = \frac{A_c}{2} [M_1(f-f_c)+M_1(f+f_c)] + \frac{A_c}{2j} [M_2(f-f_c)-M_2(f+f_c)]$$

where $M_1(f) = F[m_1(t)]$ and $M_2(f) = F[m_2(t)]$. The spectrum of the received signal is therefore

$$R(f) = H(f)S(f)$$

$$= \frac{A_c}{2} H(f) [M_1(f-f_c)+M_1(f+f_c)+ \frac{1}{j}\, M_2(f-f_c)- \frac{1}{j} M_2(f+f_c)]$$

To recover $m_1(t)$, we multiply $r(t)$, the inverse Fourier transform of $R(f)$, by $\cos(2\pi f_c t)$ and then pass the resulting output through a low-pass filter, producing a signal with the following spectrum

$$F[r(t)\cos(2\pi f_c t)] = \frac{1}{2} [R(f-f_c)+R(f+f_c)]$$

$$= \frac{A_c}{4} H(f-f_c)[M_1(f-2f_c) + M_1(f) + \frac{1}{j} M_2(f-2f_c) - \frac{1}{j} M_2(f)]$$

$$+ \frac{A_c}{4} H(f+f_c)[M_1(f) + M_1(f+2f_c) + \frac{1}{j} M_2(f) - \frac{1}{j} M_2(f+2f_c)] \qquad (1)$$

The condition $H(f_c+f) = H^*(f_c-f)$ is equivalent to $H(f+f_c)=H(f-f_c)$; this follows from the fact that for a real-valued impulse response $h(t)$, we have $H(-f)=H^*(f)$. Hence, substituting this condition in Eq. (1), we get

$$F[r(t)\cos(2\pi f_c t)] = \frac{A_c}{2} H(f-f_c)M_1(f)$$

$$+ \frac{A_c}{4} H(f-f_c)[M_1(f-2f_c) + \frac{1}{j} M_2(f-2f_c)+M_1(f+2f_c) - \frac{1}{j} M_2(f+2f_c)]$$

The low-pass filter output, therefore, has a spectrum equal to $(A_c/2)\, H(f-f_c)M_1(f)$.

Similarly, to recover $m_2(t)$, we multiply $r(t)$ by $\sin(2\pi f_c t)$, and then pass the resulting signal through a low-pass filter. In this case, we get an output with a spectrum equal to $(A_c/2)\, H(f-f_c)M_2(f)$.

Problem 2.13

When the local carriers have a phase error ϕ, we may write

$$\cos(2\pi f_c t + \phi) = \cos(2\pi f_c t)\cos\phi - \sin(2\pi f_c t)\sin\phi$$

In this case, we find that by multiplying the received signal $r(t)$ by $\cos(2\pi f_c t+\phi)$, and passing the resulting output through a low-pass filter, the corresponding low-pass filter output in the receiver has a spectrum equal to $(A_c/2)\, H(f-f_c)\, [\cos\phi\, M_1(f) - \sin\phi\, M_2(f)]$. This indicates that there is cross-talk at the demodulator outputs.

Problem 2.14

The transmitted signal is given by

$$s(t) = A_c m_1(t)\cos(2\pi f_c t) + A_c m_2(t)\sin(2\pi f_c t)$$

$$= A_c[V_0 + m_l(t) + m_r(t)]\cos(2\pi f_c t) + A_c[m_l(t) - m_r(t)]\sin(2\pi f_c t)$$

(a) The envelope detection of $s(t)$ yields

$$y_1(t) = A_c\sqrt{(V_0 + m_l(t) + m_r(t))^2 + (m_l(t) - m_r(t))^2}$$

$$= A_c(V_0 + m_l(t) + m_r(t))\sqrt{1 + \left(\frac{m_l(t) - m_r(t)}{V_0 + m_l(t) + m_r(t)}\right)^2}$$

To minimize the distortion in the envelope detector output due to the quadrature component, we choose the DC offset V_0 to be large. We may then approximate $y_1(t)$ as

$$y_1(t) \approx A(V_0 + m_l(t) + m_r(t))$$

which, except for the DC component $A_c V_0$, is proportional to the sum $m_l(t) + m_r(t)$.

(b) For coherent detection at the receiver, we need a replica of the carrier $A_c\cos(2\pi f_c t)$. This requirement can be satisfied by passing the received signal $s(t)$ through a narrow-band filter of mid-band frequency f_c. However, to extract the difference $m_l(t)$ - $m_r(t)$, we need $\sin(2\pi f_c t)$, which is obtained by passing the narrow-band filter output through a 90°-phase shifter. Then, multiplying $s(t)$ by $\sin(2\pi f_c t)$ andt low-pass filtering, we obtain a signal proportional to $m_l(t)$ - $m_r(t)$.

(c) To recover the original loudspeaker signals $m_l(t)$ and $m_r(t)$, we proceed as follows:

- Equalize the outputs of the envelope detector and coherent detector.
- Pass the equalized outputs through an audio demixer to produce $m_l(t)$ and $m_r(t)$.

Problem 2.15

(a) $s(t) = A_c(1 + k_a m(t))\cos(2\pi f_c t)$

$$= A_c\left(1 + \frac{k_a}{1 + t^2}\right)\cos(2\pi f_c t)$$

To ensure 50 percent modulation, $k_a = 1$, in which case we get

$$s(t) = A_c\left(1 + \frac{1}{1 + t^2}\right)\cos(2\pi f_c t)$$

(b) $s(t) = A_c m(t)\cos(2\pi f_c t)$

$$= \frac{A_c}{1 + t^2}\cos(2\pi f_c t)$$

(c) $s(t) = \frac{A_c}{2}[m(t)\cos(2\pi f_c t) - \hat{m}(t)\sin(2\pi f_c t)]$

$$= \frac{A_c}{2}\left[\frac{1}{1 + t^2}\cos(2\pi f_c t) - \frac{t}{1 + t^2}\sin(2\pi f_c t)\right]$$

(d) $s(t) = \frac{A_c}{2}\left[\frac{1}{1 + t^2}\cos(2\pi f_c t) + \frac{t}{1 + t^2}\sin(2\pi f_c t)\right]$

As an aid to the sketching of the modulated signals in (c) and (d), the envelope of either SSB wave is

$$a(t) = \frac{1}{2}\sqrt{\frac{t^2 + 1}{(1 + t^2)^2}} = \frac{1}{2}\sqrt{\frac{1}{1 + t^2}}$$

Plots of the modulated signals in (a) to (d) are presented in Fig. 1 on the next page.

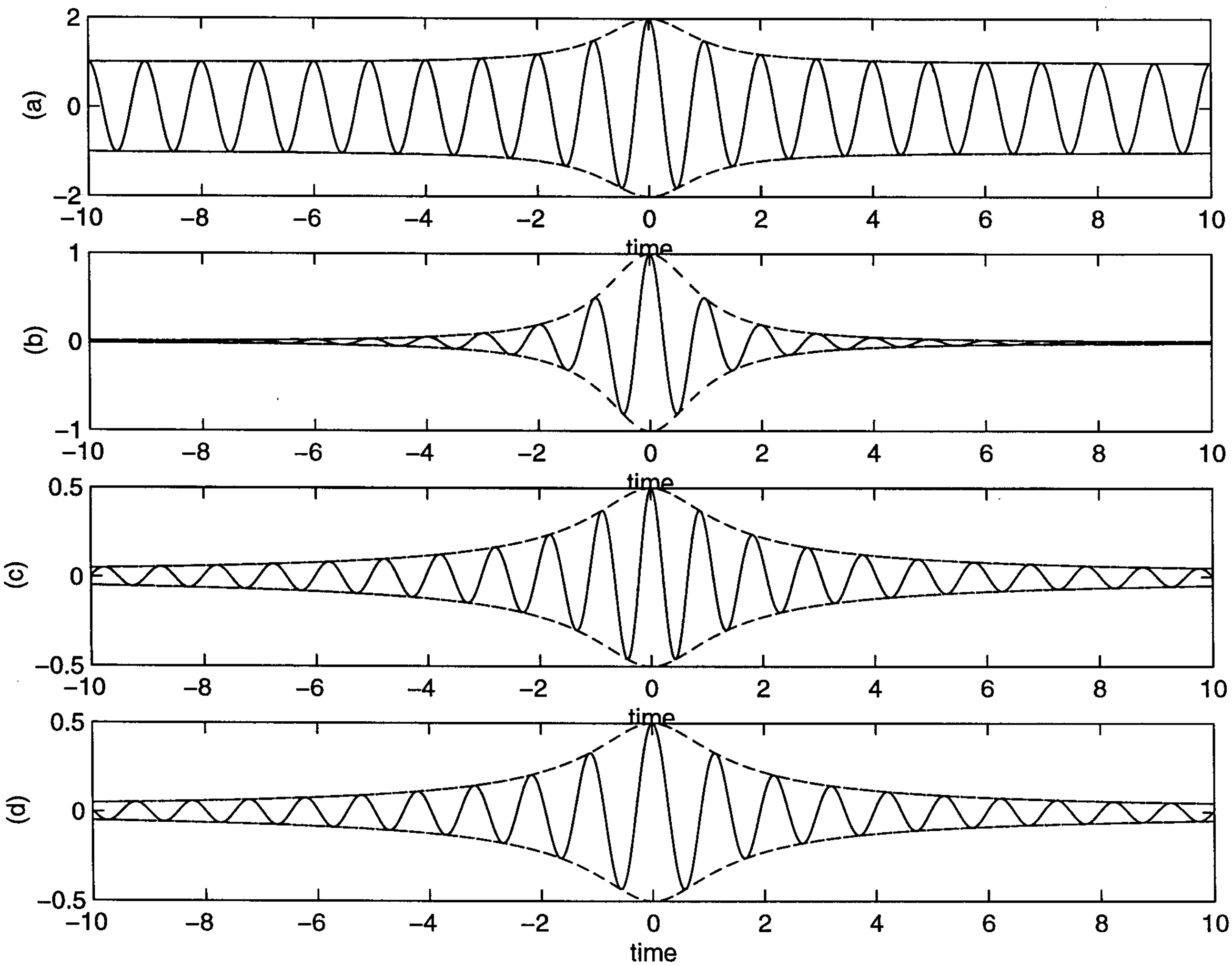


Figure 1

Problem 2.16

Consider first the modulated signal

$$s(t) = \frac{1}{2}m(t)\cos(2\pi f_c t) - \frac{1}{2}\hat{m}(t)\sin(2\pi f_c t) \tag{1}$$

Let $S(f) = F[s(t)]$, $M(f) = F[m(t)]$, and $\hat{M}(f) = f[\hat{m}(t)]$ where $\hat{m}(t)$ is the Hilbert transform of the message signal $m(t)$. Then applying the Fourier transform to Eq. (1), we obtain

$$S(f) = \frac{1}{4}[M(f-f_c)+M(f+f_c)] - \frac{1}{4j}[\hat{M}(f-f_c)-\hat{M}(f+f_c)] \tag{2}$$

From the definition of the Hilbert transform, we have

$$\hat{M}(f) = -j\,\mathrm{sgn}(f)M(f)$$

where sgn(f) is the signum function. Equivalently, we may write

$$-\frac{1}{j}\hat{M}(f-f_c) = \mathrm{sgn}(f-f_c)M(f-f_c)$$

$$-\frac{1}{j}\hat{M}(f+f_c) = \mathrm{sgn}(f+f_c)M(f+f_c)$$

(i) From the definition of the signum function, we note the following for $f > 0 =$ and $f > f_c$:

$$\mathrm{sgn}(f-f_c) = \mathrm{sgn}(f+f_c) = +1$$

Correspondingly, Eq. (2) reduces to

$$S(f) = \frac{1}{4}[M(f-f_c)+M(f+f_c)] + \frac{1}{4}[M(f-f_c)-M(f+f_c)]$$

$$= \frac{1}{2}M(f-f_c)$$

In words, we may thus state that, except for a scaling factor, the spectrum of the modulated signal $s(t)$ defined in Eq. (1) is the same as that of the DSB-SC modulated signal for $f > f_c$.

(ii) For $f > 0$ and $f < f_c$, we have

$$\operatorname{sgn}(f - f_c) = -1$$
$$\operatorname{sgn}(f + f_c) = +1$$

Correspondingly, Eq. (2) reduces to

$$S(f) = \frac{1}{4}[M(f - f_c) + M(f + f_c)] + \frac{1}{4}[-M(f - f_c) - M(f - f_c)]$$
$$= 0$$

In words, we may now state that for $f < f_c$, the modulated signal $s(t)$ defined in Eq. (1) is zero.

Combining the results for parts (i) and (ii), the modulated signal $s(t)$ of Eq. (1) represents a single sideband modulated signal containing only the upper sideband. This result was derived for $f > 0$. This result also holds for $f < 0$, the proof for which is left as an exercise for the reader.

Following a procedure similar to that described above, we may show that the modulated signal

$$s(t) = \frac{1}{2}m(t)\cos(2\pi f_c t) + \frac{1}{2}\hat{m}(t)\sin(2\pi f_c t) \tag{3}$$

represents a single sideband modulated signal containing only the lower sideband.

Problem 2.17

An error Δf in the frequency of the local oscillator in the demodulation of an SSB signal, measured with respect to the carrier frequency f_c, gives rise to distortion in the demodulated signal. Let the local oscillator output be denoted by $A_c' \cos(2\pi(f_c + \Delta f)t)$. The resulting demodulated signal is given by (for the case when the upper sideband only is transmitted)

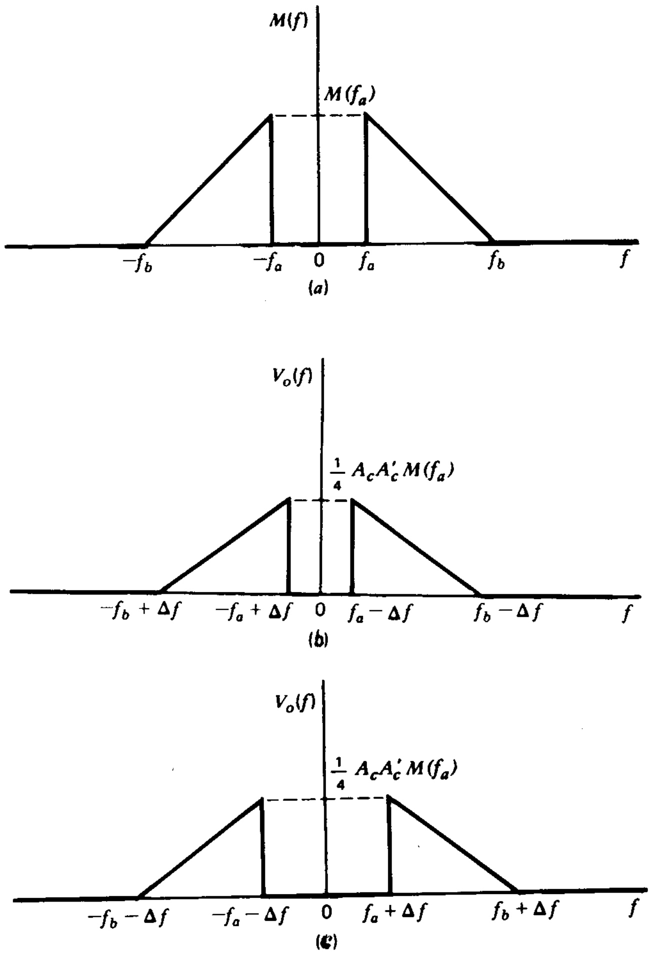
$$v_0(t) = \frac{1}{4} A_c A_c' \, [m(t)\cos(2\pi\Delta f t) + m(t)\sin(2\pi\Delta f t)]$$

This demodulated signal represents an SSB wave corresponding to a carrier frequency Δf.

The effect of frequency error Δf in the local oscillator may be interpreted as follows:

(a) If the SSB wave s(t) contains the upper sideband and the frequency error Δf is positive, or equivalently if s(t) contains the lower sideband and Δf is negative, then the frequency components of the demodulated signal $v_0(t)$ are shifted inward by the amount Δf compared with the baseband signal m(t), as illustrated in Fig. 1(b).

(b) If the incoming SSB wave s(t) contains the lower sideband and the frequency error Δf is positive, or equivalently if s(t) contains the upper sideband and Δf is negative, then the frequency components of the demodulated signal $v_0(t)$ are shifted outward by the amount Δf, compared with the baseband signal m(t). This is illustrated in Fig. 1c for the case of a baseband signal (e.g., voice signal) with an energy gap occupying the interval $-f_a \le f \le f_a$, in part (a) of the figure.

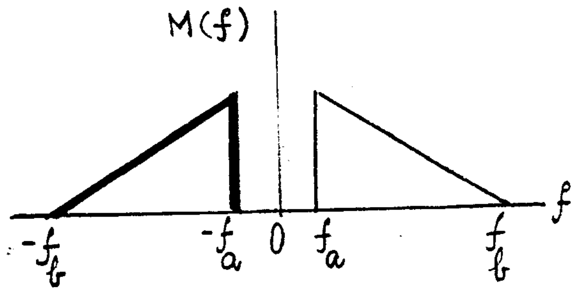


M(f)
M(f_a)
−f_b
−f_a
0
f_a
f_b
f
(a)
V_o(f)
1/4 A_c A'_c M(f_a)
−f_b + Δf
−f_a + Δf
0
f_a − Δf
f_b − Δf
f
(b)
V_o(f)
1/4 A_c A'_c M(f_a)
−f_b − Δf
−f_a − Δf
0
f_a + Δf
f_b + Δf
f
(c)

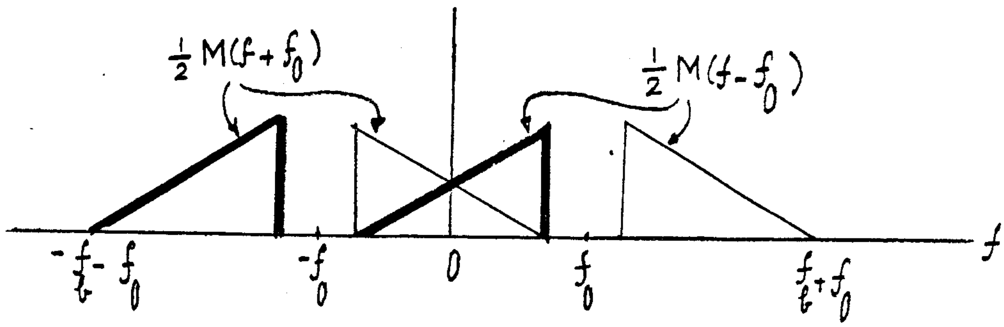
Fig. 1

Problem 2.18

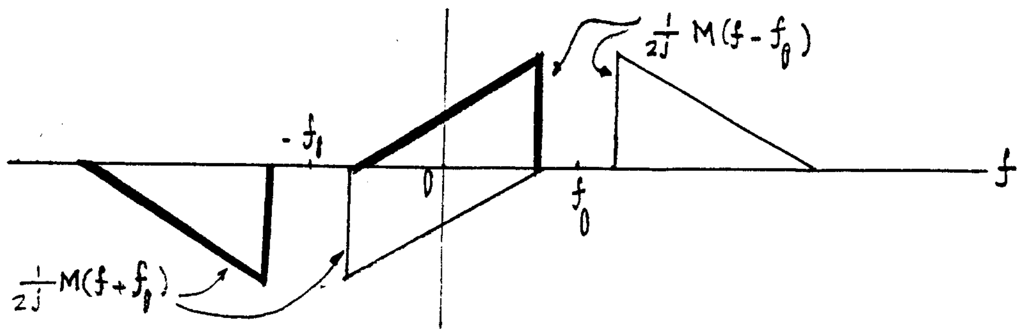
(a,b) The spectrum of the message signal is illustrated below:



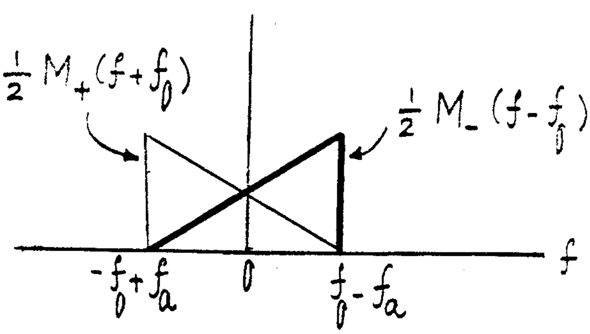
Correspondingly, the output of the upper first product modulator has the following spectrum:



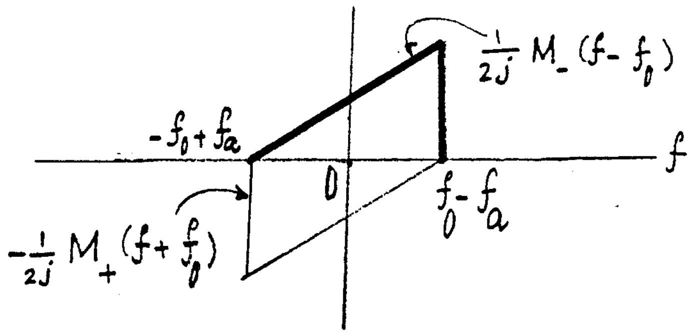
The output of the lower first product modulator has the spectrum:



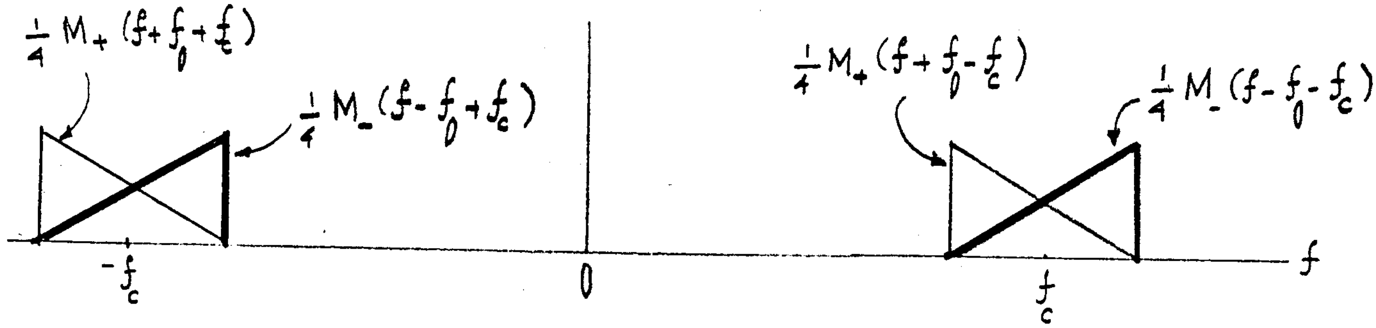
The output of the upper low pass filter has the spectrum:



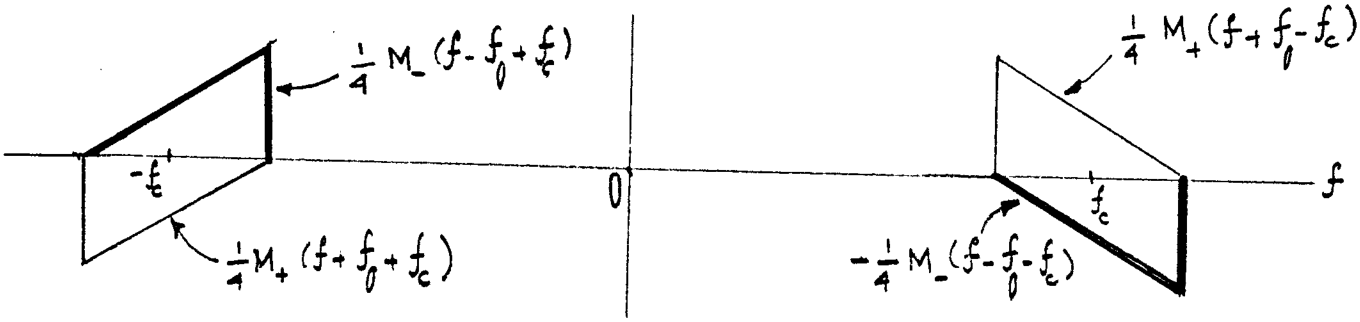
The output of the lower low pass filter has the spectrum:



The output of the upper second product modulator has the spectrum:



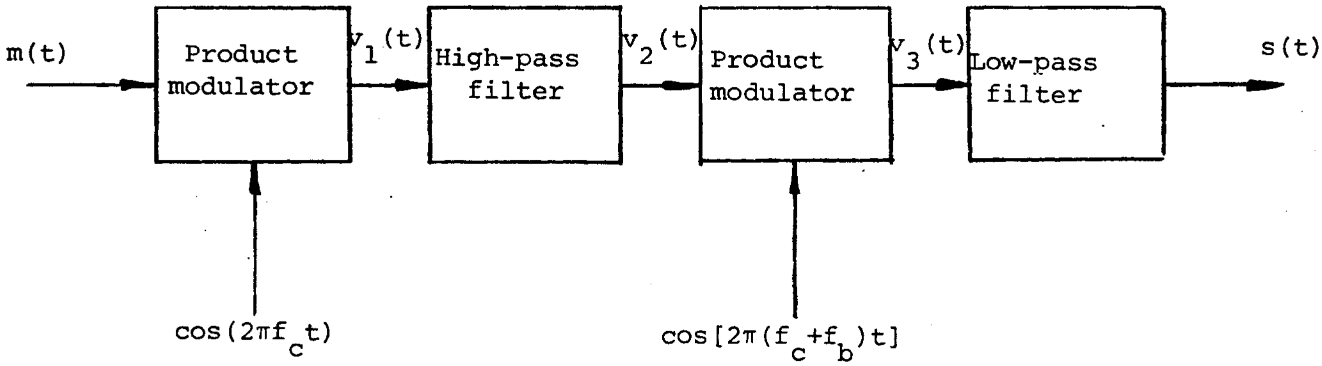
The output of the lower second product modulator has the spectrum:



Adding the two second product modulator outputs, their upper sidebands add constructively while their lower sidebands cancel each other.

(c) To modify the modulator to transmit only the lower sideband, a single sign change is required in one of the channels. For example, the lower first product modulator could multiply the message signal by $-\sin(2\pi f_o t)$. Then, the upper sideband would be cancelled and the lower one transmitted.

Problem 2.19



(a) The first product modulator output is

$$v_1(t) = m(t)\cos(2\pi f_c t)$$

The second product modulator output is

$$v_3(t) = v_2(t)\cos[2\pi(f_c+f_b)t]$$

The amplitude spectra of $m(t)$, $v_1(t)$, $v_2(t)$, $v_3(t)$ and $s(t)$ are illustrated on the next page:

We may express the voice signal $m(t)$ as

$$m(t) = \frac{1}{2}[m_+(t) + m_-(t)]$$

where $m_+(t)$ is the pre-envelope of $m(t)$, and $m_-(t) = m_+^*(t)$ is its complex conjugate. The Fourier transforms of $m_+(t)$ and $m_-(t)$ are defined by (see Appendix 2)

$$M_+(f) = \begin{cases} 2M(f), & f > 0 \\ 0, & f < 0 \end{cases}$$

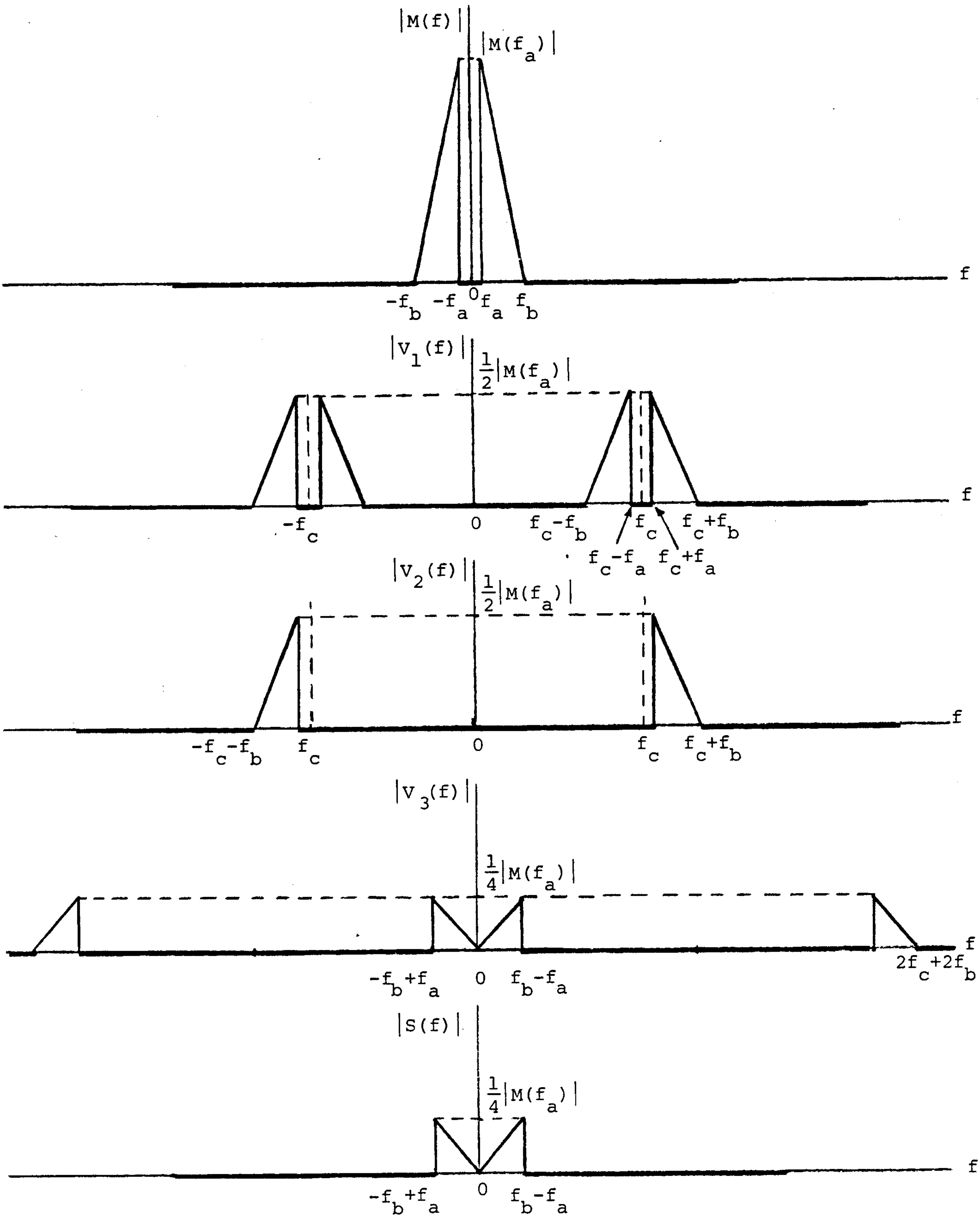
$$M_-(f) = \begin{cases} 0, & f > 0 \\ 2M(f), & f < 0 \end{cases}$$

Comparing the spectrum of $s(t)$ with that of $m(t)$, we see that $s(t)$ may be expressed in terms of $m_+(t)$ and $m_-(t)$ as follows:

$$\begin{aligned} s(t) &= \frac{1}{8} m_+(t)\exp(-j2\pi f_b t) + \frac{1}{8} m_-(t)\exp(j2\pi f_b t) \\ &= \frac{1}{8}[m(t)+j\hat{m}(t)]\exp(-j2\pi f_b t) + \frac{1}{8}[m(t)-j\hat{m}(t)]\exp(j2\pi f_b t) \\ &= \frac{1}{4} m(t)\cos(2\pi f_b t) + \frac{1}{4}\hat{m}(t)\sin(2\pi f_b t) \end{aligned}$$

(b) With $s(t)$ as input, the first product modulator output is

$$v_1(t) = s(t)\cos(2\pi f_c t)$$


|M(f)|
|M(f_a)|
f
-f_b -f_a 0 f_a f_b
|V_1(f)|
1/2|M(f_a)|
f
-f_c
0
f_c-f_b
f_c
f_c+f_b
f_c-f_a
f_c+f_a
|V_2(f)|
1/2|M(f_a)|
f
-f_c-f_b
f_c
0
f_c
f_c+f_b
|V_3(f)|
1/4|M(f_a)|
f
-f_b+f_a
0
f_b-f_a
2f_c+2f_b
|S(f)|
1/4|M(f_a)|
f
-f_b+f_a
0
f_b-f_a

C19

Problem 2.20

(a) Consider the system described in Fig. 1a, where $u(t)$ denotes the product modulator output, as shown by

$$u(t) = A_c m(t)\cos(2\pi f_c t)$$

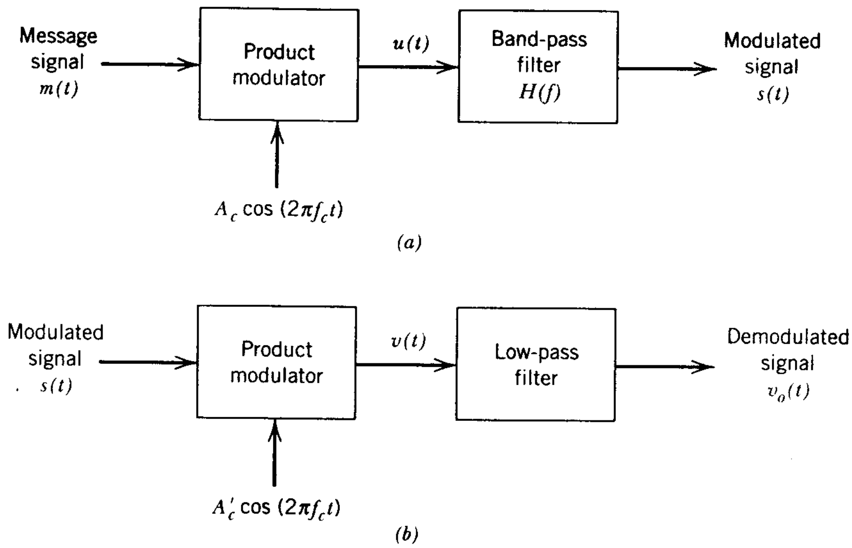


Figure 1: (a) Filtering scheme for processing sidebands. (b) Coherent detector for recovering the message signal.

Let $H(f)$ denote the transfer function of the filter following the product modulator. The spectrum of the modulated signal $s(t)$ produced by passing $u(t)$ through the filter is given by

$$S(f) = U(f)H(f)$$

$$= \frac{A_c}{2}[M(f-f_c) + M(f+f_c)]H(f) \qquad (1)$$

where $M(f)$ is the Fourier transform of the message signal $m(t)$. The problem we wish to address is to determine the particular $H(f)$ required to produce a modulated signal $s(t)$ with desired spectral characteristics, such that the original message signal $m(t)$ may be recovered from $s(t)$ by coherent detection.

The first step in the coherent detection process involves multiplying the modulated signal $s(t)$ by a locally generated sinusoidal wave $A'_c\cos(2\pi f_c t)$, which is synchronous with the carrier wave $A_c\cos(2\pi f_c t)$, in both frequency and phase as in Fig. 1b. We may thus write

$$v(t) = A'_c \cos(2\pi f_c t) s(t)$$

Transforming this relation into the frequency domain gives the Fourier transform of $v(t)$ as

$$V(f) = \frac{A'_c}{2}[S(f - f_c) + S(f + f_c)] \tag{2}$$

Therefore, substitution of Eq. (1) in (2) yields

$$V(f) = \frac{A_c A'_c}{4} M(f)[H(f - f_c) + H(f + f_c)]$$

$$+ \frac{A_c A'_c}{4}[M(f - 2f_c)H(f - f_c) + M(f + 2f_c)H(f + f_c)] \tag{3}$$

(b) The high-frequency components of $v(t)$ represented by the second term in Eq. (3) are removed by the low-pass filter in Fig. 1b to produce an output $v_o(t)$, the spectrum of which is given by the remaining components:

$$V_o(f) = \frac{A_c A'_c}{2} M(f)[H(f - f_c) + H(f + f_c)] \tag{4}$$

For a distortionless reproduction of the original baseband signal $m(t)$ at the coherent detector output, we require $V_o(f)$ to be a scaled version of $M(f)$. This means, therefore, that the transfer function $H(f)$ must satisfy the condition

$$H(f - f_c) + H(f + f_c) = 2H(f_c) \tag{5}$$

where $H(f_c)$, the value of $H(f)$ at $f = f_c$, is a constant. When the message (baseband) spectrum $M(f)$ is zero outside the frequency range $-W \leq f \leq W$, we need only satisfy Eq. (5) for values of f in this interval. Also, to simplify the exposition, we set $H(f_c) = 1/2$. We thus require that $H(f)$ satisfies the condition:

$$H(f - f_c) + H(f + f_c) = 1, \qquad -W \leq f \leq W \tag{6}$$

Under the condition described in Eq. (6), we find from Eq. (4) that the coherent detector output in Fig. 1b is given by

$$v_o(t) = \frac{A_c A'_c}{2} m(t) \tag{7}$$

Equation (1) defines the spectrum of the modulated signal $s(t)$. Recognizing that $s(t)$ is a band-pass signal, we may formulate its time-domain description in terms of in-phase and quadrature components. In particular, $s(t)$ may be expressed in the canonical form

$$s(t) = s_I(t)\cos(2\pi f_c t) - s_Q(t)\sin(2\pi f_c t) \tag{8}$$

where $s_I(t)$ is the in-phase component of $s(t)$, and $s_Q(t)$ is its quadrature component. To determine $s_I(t)$, we note that its Fourier transform is related to the Fourier transform of $s(t)$ as follows:

$$S_I(f) = \begin{cases} S(f-f_c) + S(f+f_c), & -W \le f \le W \\ 0, & \text{elsewhere} \end{cases} \tag{9}$$

Hence, substituting Eq. (1) in (9), we find that the Fourier transform of $s_I(t)$ is given by

$$\begin{aligned} S_I(f) &= \frac{1}{2}A_c M(f)[H(f-f_c) + H(f+f_c)] \\ &= \frac{1}{2}A_c M(f); \qquad -W \le f \le W \end{aligned} \tag{10}$$

where, in the second line, we have made use of the condition in Eq. (6) imposed on $H(f)$. From Eq. (10) we readily see that the in-phase component of the modulated signal $s(t)$ is defined by

$$s_I(t) = \frac{1}{2}A_c m(t) \tag{11}$$

which, except for a scaling factor, is the same as the original message signal m(t).

To determine the quadrature component $s_Q(t)$ of the modulated signal $s(t)$, we recognize that its Fourier transform is defined in terms of the Fourier transform of $s(t)$ as follows:

$$S_Q(f) = \begin{cases} j[S(f-f_c) - S(f+f_c)] & -W \le f \le W \\ 0, & \text{elsewhere} \end{cases} \tag{12}$$

Therefore, substituting Eq. (11) in (12), we get

$$S_Q(f) = \frac{j}{2}A_c M(f)[H(f-f_c) - H(f+f_c)] \tag{13}$$

This equation suggests that we may generate $s_Q(t)$, except for a scaling factor, by passing the message signal $m(t)$ through a new filter whose transfer function is related to that of the filter in Fig. 1a as follows:

$$H_Q(f) = j[H(f-f_c)-H(f+f_c)], \qquad -W \le f \le W \tag{14}$$

Let $m'(t)$ denote the output of this filter produced in response to the input $m(t)$. Hence, we may express the quadrature component of the modulated signal $s(t)$ as

$$s_Q(t) = \frac{1}{2}A_c m'(t) \tag{15}$$

Accordingly, substituting Eqs. (11) and (15) in (8), we find that $s(t)$ may be written in the canonical form

$$m(t) = \frac{1}{2}A_c m(t)\cos(2\pi f_c t) - \frac{1}{2}A_c m'(t)\sin(2\pi f_c t) \tag{16}$$

There are two important points to note here:

1. The in-phase component $s_I(t)$ is completely independent of the transfer function $H(f)$ of the band-pass filter involved in the generation of the modulated wave $s(t)$ in Fig. 1a, so long as it satisfies the condition of Eq. (6).

2. The spectral modification attributed to the transfer function $H(f)$ is confined solely to the quadrature component $s_Q(t)$.

The role of the quadrature component is merely to interfere with the in-phase component, so as to reduce or eliminate power in one of the sidebands of the modulated signal $s(t)$, depending on the application of interest.

Problem 2.21

(a) Expanding s(t), we get

$$s(t) = \frac{1}{2} a\ A_m A_c \cos(2\pi f_c t) \cos(2\pi f_m t)$$
$$- \frac{1}{2} a\ A_m A_c \sin(2\pi f_c t) \sin(2\pi f_m t) + \frac{1}{2}(1-a)\ A_c A_m \cos(2\pi f_c t) \cos(2\pi f_m t)$$
$$+ \frac{1}{2}(1-a)\ A_m A_c \sin(2\pi f_c t) \sin(2\pi f_m t)$$
$$= \frac{1}{2} A_m A_c \cos(2\pi f_c t) \cos(2\pi f_m t)$$
$$+ \frac{1}{2} A_m A_c\ (1-2a) \sin(2\pi f_c t) \sin(2\pi f_m t)$$

Therefore, the quadrature component is:

$$- \frac{1}{2} A_c A_m (1-2a) \sin(2\pi f_m t)$$

(b) After adding the carrier, the signal will be:

$$s(t) = A_c[1 + \frac{1}{2} A_m \cos(2\pi f_m t)] \cos(2\pi f_c t)$$
$$+ \frac{1}{2} A_c A_m (1-2a) \sin(2\pi f_m t) \sin(2\pi f_c t)$$

The envelope equals

$$a(t) = A_c \sqrt{[1 + \frac{1}{2} A_m \cos(2\pi f_m t)]^2 + [\frac{1}{2} A_m (1-2a) \sin(2\pi f_m t)]^2}$$

$$= A_c\ [1 + \frac{1}{2} A_m \cos(2\pi f_m t)] \sqrt{1 + \left[\frac{\frac{1}{2} A_m (1-2a) \sin(2\pi f_m t)}{1 + \frac{1}{2} A_m \cos(2\pi f_m t)}\right]^2}$$

$$= A_c\ [1 + \frac{1}{2} A_m \cos(2\pi f_m t)]\ d(t)$$

where d(t) is the distortion, defined by

$$d(t) = \sqrt{1 + \left[\frac{\frac{1}{2} A_m (1-2a) \sin(2\pi f_m t)}{1 + \frac{1}{2} A_m \cos(2\pi f_m t)}\right]^2}$$

(c) d(t) is greatest when a = 0.

Problem 2.22

Consider an incoming narrow-band signal of bandwidth 10 kHz, and mid-band frequency which may lie in the range 0.535–1.605 MHz. It is required to translate this signal to a fixed frequency band centered at 0.455 MHz. The problem is to determine the range of tuning that must be provided in the local oscillator.

Let f_c denote the mid-band frequency of the incoming signal, and f_l denote the local oscillator frequency. Then we may write

$$0.535 < f_c < 1.605$$

and

$$f_c - f_l = 0.455$$

where both f_c and f_l are expressed in MHz. That is,

$$f_l = f_c - 0.455$$

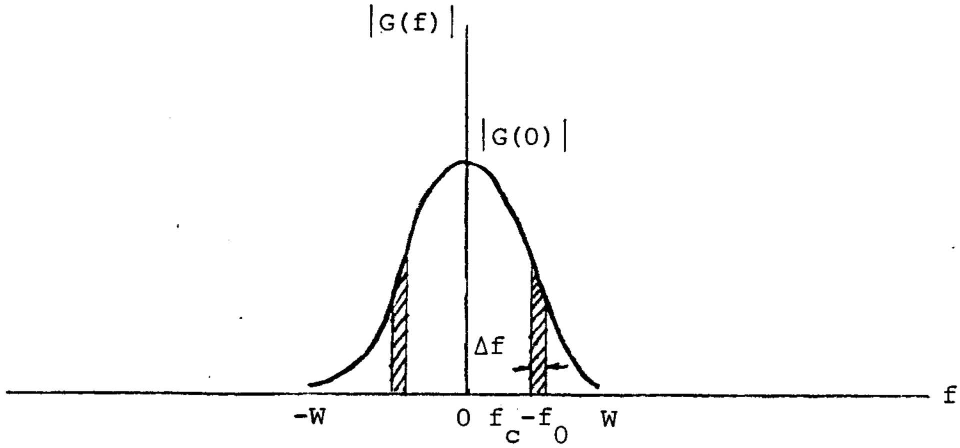
When $f_c = 0.535$ MHz, we get $f_l = 0.08$ MHz; and when $f_c = 1.605$ MHz, we get $f_l = 1.15$ MHz. Thus the required range of tuning of the local oscillator is 0.08–1.15 MHz.

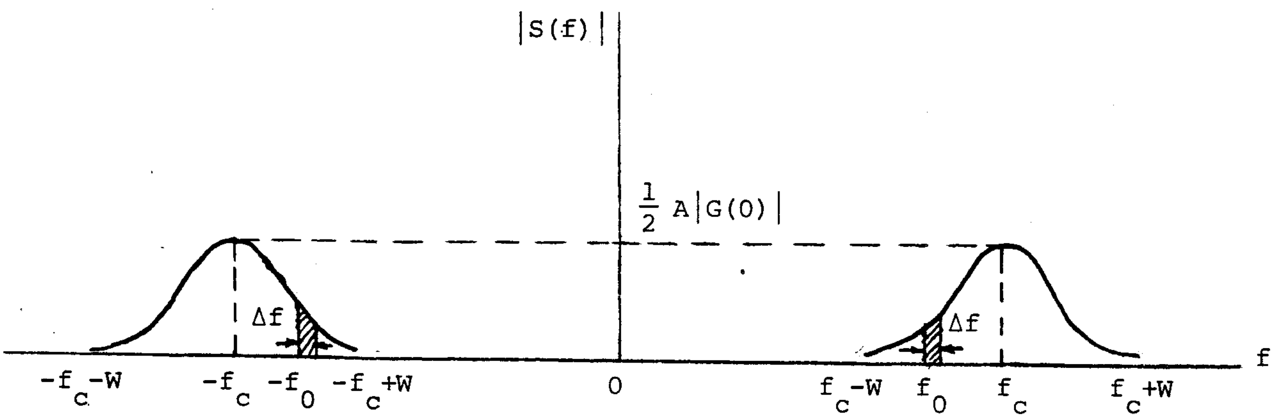
Problem 2.23

Let s(t) denote the multiplier output, as shown by

$s(t) = A\ g(t) \cos(2\pi f_c t)$

where f_c lies in the range f_0 to f_0+W. The amplitude spectra of s(t) and g(t) are related as follows:





With v(t) denoting the band-pass filter output, we thus find that the Fourier transform of v(t) is approximately given by

$$V(f) \simeq \frac{1}{2} A\ G(f_c - f_0) \quad , \quad f_0 - \frac{\Delta f}{2} \le |f| \le f_0 + \frac{\Delta f}{2}$$

The rms meter output is therefore (by using Rayleigh's energy theorem)

$$V_{rms} = [\int_{-\infty}^{\infty} v^2(t)dt]^{1/2}$$

$$= [\int_{-\infty}^{\infty} |V(f)|^2\ df]^{1/2} = [2(\frac{1}{4} A^2\ |G(f_c - f_o|^2)\ \Delta f]^{1/2}$$

$$= \frac{A}{\sqrt{2}} |G(f_c - f_o)|\ \sqrt{\Delta f}$$

Problem 2.24

For the PM case,

$$s(t) = A_c \cos[2\pi f_c t + k_p\ m(t)].$$

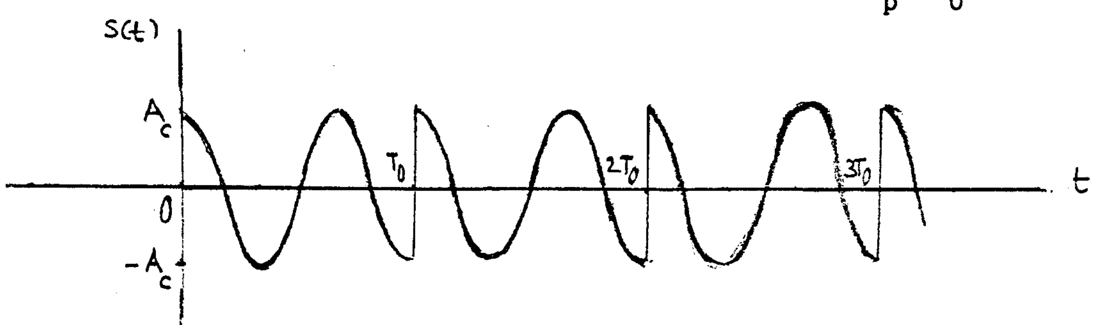
The angle equals

$$\theta_i(t) = 2\pi f_c t + k_p\ m(t).$$

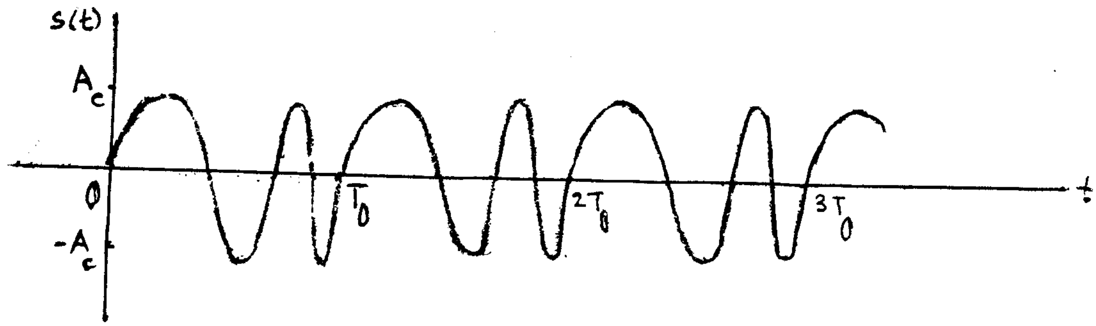
The instantaneous frequency,

$$f_i(t) = f_c + \frac{Ak_p}{2\pi T_0} - \sum_n \frac{Ak_p}{2\pi} \delta(t - nT_0),$$

is equal to $f_c + Ak_p/2\pi T_0$ except for the instants that the message signal has discontinuities. At these instants, the phase shifts by $-k_p A/T_0$ radians.

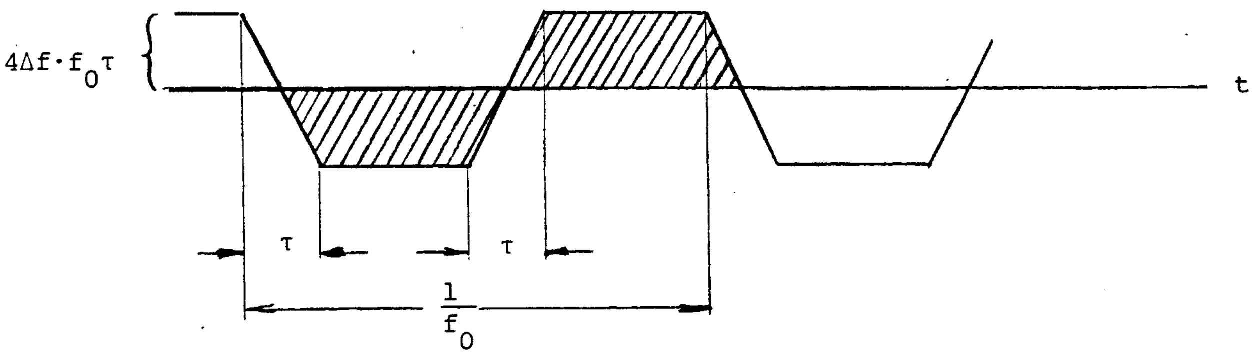


For the FM case, $f_i(t) = f_c + k_f\, m(t)$



Problem 2.25

The instantaneous frequency of the mixer output is as shown below:



The presence of negative frequency merely indicates that the phasor representing the difference frequency at the mixer output has reversed its direction of rotation.

Let N denote the number of beat cycles in one period. Then, noting that N is equal to the shaded area shown above, we deduce that

$$N = 2[4\Delta f \cdot f_0 \tau (\frac{1}{2f_0} - \tau) + 2\Delta f \cdot f_0 \tau^2]$$

$$= 4\Delta f \cdot \tau (1 - f_0 \tau)$$

Since $f_0 \tau << 1$, we have

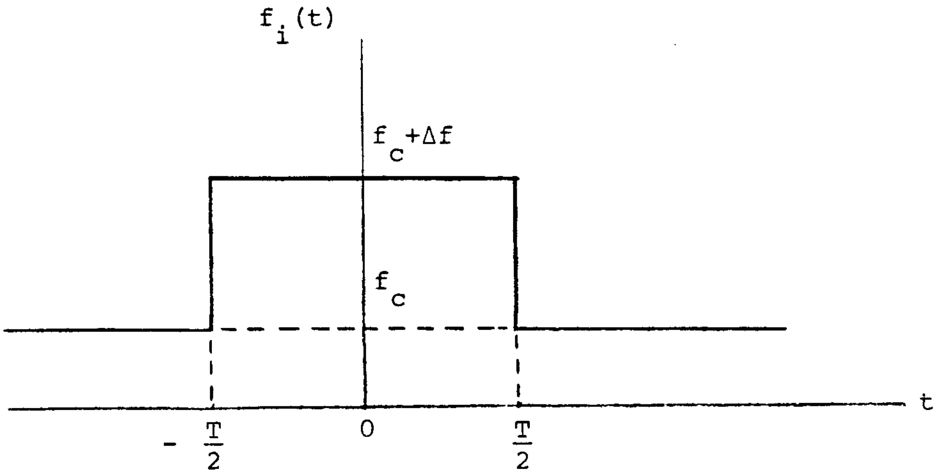
$$N \simeq 4\Delta f \cdot \tau$$

Therefore, the number of beat cycles counted over one second is equal to

$$\frac{N}{1/f_0} = 4\Delta f \cdot f_0 \tau.$$

Problem 2.26

The instantaneous frequency of the modulated wave s(t) is as shown below:



We may thus express s(t) as follows

$$s(t) = \begin{cases} \cos(2\pi f_c t), & t < -\frac{T}{2} \\ \cos[2\pi(f_c+\Delta f)t], & -\frac{T}{2} \le t \le \frac{T}{2} \\ \cos[2\pi f_c t), & \frac{T}{2} < t \end{cases}$$

The Fourier transform of s(t) is therefore

$$S(f) = \int_{-\infty}^{-T/2} \cos(2\pi f_c t) \exp(-j2\pi ft)\, dt$$

$$+ \int_{-T/2}^{T/2} \cos[2\pi(f_c+\Delta f)t] \exp(-j2\pi ft)\, dt$$

$$+ \int_{T/2}^{\infty} \cos(2\pi f_c t) \exp(-j2\pi ft)\, dt$$

$$= \int_{-\infty}^{\infty} \cos(2\pi f_c t) \exp(-j2\pi ft)\, dt$$

$$+ \int_{-T/2}^{T/2} \{\cos[2\pi(f_c+\Delta f)t - \cos(2\pi f_c t)\} \exp(-j2\pi ft)\, dt \qquad (1)$$

The second term of Eq. (1) is recognized as the difference between the Fourier transforms of two RF pulses of unit amplitude, one having a frequency equal to $f_c+\Delta f$ and the other having a frequency equal to f_c. Hence, assuming that $f_cT \gg 1$, we may express S(f) as follows:

$$S(f) \approx \begin{cases} \frac{1}{2}\delta(f-f_c) + \frac{T}{2}\,\mathrm{sinc}[T(f-f_c-\Delta f)] - \frac{T}{2}\,\mathrm{sinc}[T(f-f_c)], & f > 0 \\ \\ \frac{1}{2}\delta(f+f_c) + \frac{T}{2}\,\mathrm{sinc}[T(f+f_c+\Delta f)] - \frac{T}{2}\,\mathrm{sinc}[T(f+f_c)], & f < 0 \end{cases}$$

Problem 2.27

For SSB modulation, the modulated wave is

$$s(t) = \frac{A_c}{2}\,[m(t)\,\cos(2\pi f_c t) \pm \hat{m}(t)\,\sin(2\pi f_c t)],$$

the minus sign applying when transmitting the upper sideband and the plus sign applying when transmitting the lower one.

Regardless of the sign, the envelope is

$$a(t) = \frac{A_c}{2}\sqrt{m^2(t) + \hat{m}^2(t)}\ .$$

(a) For upper sideband transmission, the angle,

$$\theta_i(t) = 2\pi f_c t + \tan^{-1}\left(\frac{\hat{m}(t)}{m(t)}\right)\ .$$

The instantaneous frequency is,

$$f_i(t) = \frac{1}{2\pi}\frac{d\theta_i(t)}{dt}$$

$$= f_c + \frac{m(t)\,\hat{m}'(t) - \hat{m}(t)\,m'(t)}{2\pi(m^2(t) + \hat{m}^2(t))}\ ,$$

where ' denotes time derivative.

(b) For lower sideband transmission, we have

$$\theta_i(t) = 2\pi f_c t + \tan^{-1}\left(-\frac{\hat{m}(t)}{m(t)}\right)\ ,$$

and

$$f_i(t) = f_c + \frac{\hat{m}(t)\,m'(t) - m(t)\,\hat{m}'(t)}{2\pi(m^2(t) + \hat{m}^2(t))}\ .$$

Problem 2.28

(a) The envelope of the FM wave s(t) is

$$a(t) = A_c \sqrt{1+\beta^2 \sin^2(2\pi f_m t)}$$

The maximum value of the envelope is

$$a_{max} = A_c \sqrt{1+\beta^2}$$

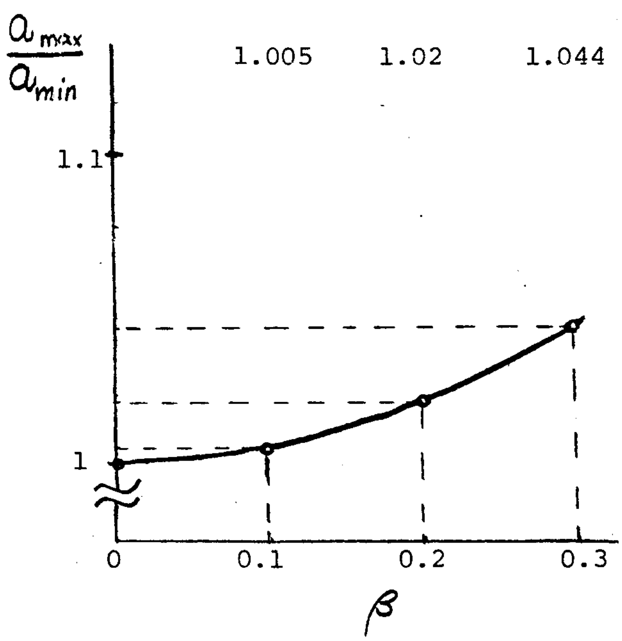
and its minimum value is

$$a_{min} = A_c$$

Therefore,

$$\frac{a_{max}}{a_{min}} = \sqrt{1+\beta^2}$$

This ratio is shown plotted below for $0 < \beta < 0.3$:



(b) Expressing s(t) in terms of its frequency components:

$$s(t) = A_c \cos(2\pi f_c t) + \frac{1}{2}\beta A_c \cos[2\pi(f_c+f_m)t] - \frac{1}{2}\beta A_c \cos[2\pi(f_c-f_m)t]$$

The mean power of s(t) is therefore

$$P_1 = \frac{A_c^2}{2} + \frac{\beta^2 A_c^2}{8} + \frac{\beta^2 A_c^2}{8}$$

$$= \frac{A_c^2}{2}\left(1 + \frac{\beta^2}{2}\right)$$

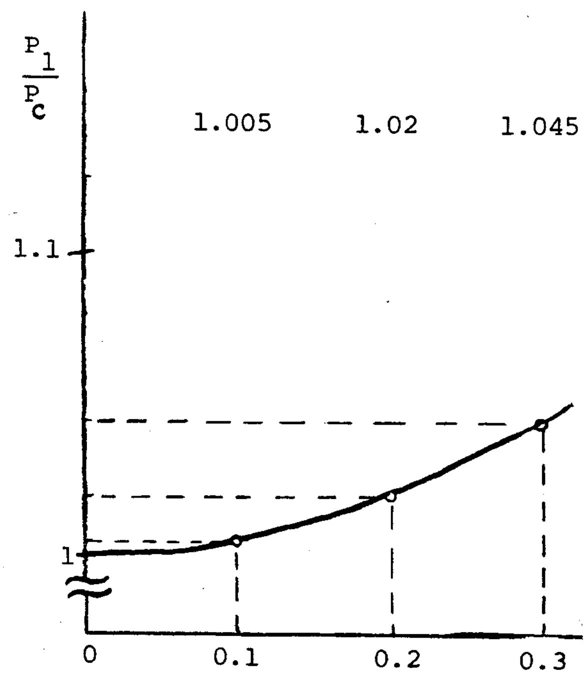
The mean power of the unmodulated carrier is

$$P_c = \frac{A_c^2}{2}$$

Therefore,

$$\frac{P_1}{P_c} = 1 + \frac{\beta^2}{2}$$

which is shown plotted below for $0 \leq \beta \leq 0.3$:



(c) The angle $\theta_i(t)$, expressed in terms of the in-phase component, $s_I(t)$, and the quadrature component, $s_Q(t)$, is:

$$\theta_i(t) = 2\pi f_c t + \tan^{-1}\left[\frac{s_I(t)}{s_Q(t)}\right]$$

$$= 2\pi f_c t + \tan^{-1}\left[\beta \sin(2\pi f_m t)\right]$$

Since $\tan^{-1}(x) \approx x - x^3/3 + \ldots$,

$$\theta_i(t) \approx 2\pi f_c t + \beta \sin(2\pi f_m t) - \frac{\beta^3}{3} \sin^3(2\pi f_m t)$$

The harmonic distortion is the power ratio of the third and first harmonics:

$$D_h = \left(\frac{\frac{1}{3}\beta^3}{\beta}\right)^2 = \frac{\beta^4}{9}$$

For $\beta = 0.3$, $D_h = 0.09\%$

Problem 2.29

(a) The phase-modulated wave is

$$s(t) = A_c \cos[2\pi f_c t + k_p A_m \cos(2\pi f_m t)]$$

$$= A_c \cos[2\pi f_c t + \beta_p \cos(2\pi f_m t)]$$

$$= A_c \cos(2\pi f_c t) \cos[\beta_p \cos(2\pi f_m t)] - A_c \sin(2\pi f_c t) \sin[\beta_p \cos(2\pi f_m t)] \quad (1)$$

If $\beta_p \leq 0.5$, then

$$\cos[\beta_p \cos(2\pi f_m t)] \simeq 1$$

$$\sin[\beta_p \cos(2\pi f_m t)] \simeq \beta_p \cos(2\pi f_m t)$$

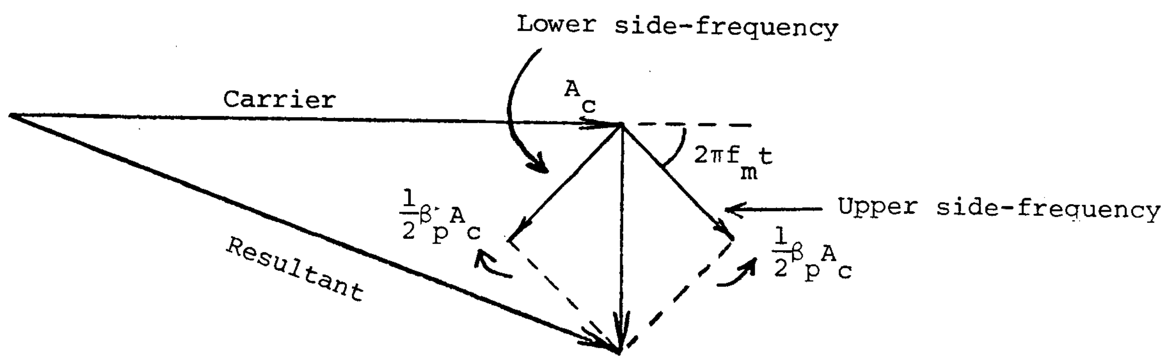
Hence, we may rewrite Eq. (1) as

$$\begin{aligned} s(t) &\simeq A_c \cos(2\pi f_c t) - \beta_p A_c \sin(2\pi f_c t) \cos(2\pi f_m t) \\ &= A_c \cos(2\pi f_c t) - \frac{1}{2} \beta_p A_c \sin[2\pi(f_c+f_m)t] \\ &\qquad - \frac{1}{2} \beta_p A_c \sin[2\pi(f_c-f_m)t] \end{aligned} \tag{2}$$

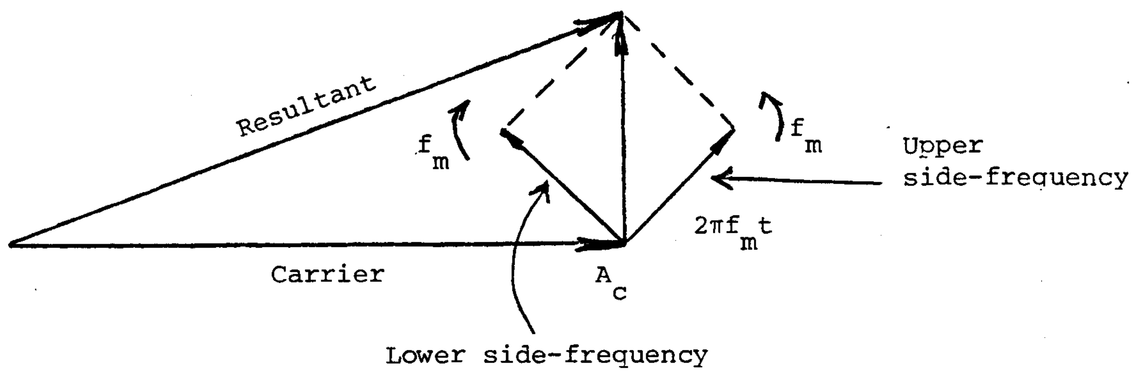
The spectrum of s(t) is therefore

$$\begin{aligned} S(f) &\simeq \frac{1}{2} A_c[\delta(f-f_c) + \delta(f+f_c)] \\ &- \frac{1}{4j} \beta_p A_c[\delta(f-f_c-f_m) - \delta(f+f_c+f_m)] \\ &- \frac{1}{4j} \beta_p A_c[\delta(f-f_c+f_m) - \delta(f+f_c-f_m)] \end{aligned}$$

(b) The phasor diagram for s(t) is deduced from Eq. (2) to be as follows:



The corresponding phasor diagram for the narrow-band FM wave is as follows:



Comparing these two phasor diagrams, we see that, except for a phase difference, the narrow-band PM and FM waves are of exactly the same form.

Problem 2.30

The phase-modulated wave is

$$s(t) = A_c \cos[2\pi f_c t + \beta_p \cos(2\pi f_m t)]$$

The complex envelope of s(t) is

$$\tilde{s}(t) = A_c \exp[j\beta_p \cos(2\pi f_m t)]$$

Expressing $\tilde{s}(t)$ in the form of a complex Fourier series, we have

$$\tilde{s}(t) = \sum_{n=-\infty}^{\infty} c_n \exp(j2\pi n f_m t)$$

where

$$c_n = f_m \int_{-1/2f_m}^{1/2f_m} \tilde{s}(t) \exp(-j2\pi n f_m t)\, dt$$

$$= A_c f_m \int_{-1/2f_m}^{1/2f_m} \exp[j\beta_p \cos(2\pi f_m t) - j2\pi n f_m t]\, dt \qquad (1)$$

Let $2\pi f_m t = \pi/2 - \phi$.

Then, we may rewrite Eq. (1) as

$$c_n = -\frac{A_c}{2\pi} \exp(-\frac{jn\pi}{2}) \int_{3\pi/2}^{-\pi/2} \exp[j\beta_p \sin(\phi) + jn\phi]\, d\phi$$

The integrand is periodic with respect to ϕ with a period of 2π. Hence, we may rewrite this expression as

$$c_n = \frac{A_c}{2\pi} \exp(-\frac{jn\pi}{2}) \int_{-\pi}^{\pi} \exp[j\beta_p \sin(\phi) + jn\phi] \, d\phi$$

However, from the definition of the Bessel function of the first kind of order n, we have

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(j\, x \sin\phi - nj\phi) \, d\phi$$

Therefore,

$$c_n = A_c \exp(-\frac{jn\pi}{2}) J_{-n}(\beta_p)$$

We may thus express the PM wave s(t) as

$$s(t) = \mathrm{Re}[\tilde{s}(t) \exp(j2\pi f_c t)]$$

$$= A_c \mathrm{Re}[\sum_{n=-\infty}^{\infty} J_{-n}(\beta_p) \exp(-\frac{jn\pi}{2}) \exp(j2\pi n f_m t) \exp(j2\pi f_c t)]$$

$$= A_c \sum_{n=-\infty}^{\infty} J_{-n}(\beta_p) \cos[2\pi(f_c+nf_m)t - \frac{n\pi}{2}]$$

The band-pass filter only passes the carrier, the first upper side-frequency, and the first lower side-frequency, so that the resulting output is

$$s_o(t) = A_c J_0(\beta_p) \cos(2\pi f_c t) + A_c J_{-1}(\beta_p) \cos[2\pi(f_c+f_m)t - \frac{\pi}{2}]$$
$$+ A_c J_1(\beta_p) \cos[2\pi(f_c-f_m)t + \frac{\pi}{2}]$$
$$= A_c J_0(\beta_p) \cos(2\pi f_c t) + A_c J_{-1}(\beta_p) \sin[2\pi(f_c+f_m)t]$$
$$- A_c J_1(\beta_p) \sin[2\pi(f_c-f_m)t]$$

But

$$J_{-1}(\beta_p) = -J_1(\beta_p)$$

Therefore,

$$s_o(t) = A_c J_0(\beta_p) \cos(2\pi f_c t)$$
$$- A_c J_1(\beta_p) \{\sin[2\pi(f_c+f_m)t] + \sin[2\pi(f_c-f_m)t]\}$$
$$= A_c J_0(\beta_p) \cos(2\pi f_c t) - 2 A_c J_1(\beta_p) \cos(2\pi f_m t) \sin(2\pi f_c t)$$

The envelope of $s_o(t)$ equals

$$a(t) = A_c \sqrt{J_0^2(\beta_p) + 4J_1^2(\beta_p) \cos^2(2\pi f_m t)}$$

The phase of $s_o(t)$ is

$$\phi(t) = -\tan^{-1} \left[\frac{2\, J_1(\beta_p)}{J_0(\beta_p)} \cos(2\pi f_m t)\right]$$

The instantaneous frequency of $s_o(t)$ is

$$f_i(t) = f_c + \frac{1}{2\pi} \frac{d\phi(t)}{dt}$$

$$= f_c + \frac{2\, J_0(\beta_p)\, J_1(\beta_p) \sin(2\pi f_m t)}{J_0^2(\beta_p) + 4J_1^2(\beta_p) \cos^2(2\pi f_m t)}$$

Problem 2.31

(a) From Table A4.1, we find (by interpolation) that $J_0(\beta)$ is zero for

$\beta = 2.44$,

$\beta = 5.52$,

$\beta = 8.65$,

$\beta = 11.8$,

and so on.

(b) The modulation index is

$$\beta = \frac{\Delta f}{f_m} = \frac{k_f\, A_m}{f_m}$$

Therefore,

$$k_f = \frac{\beta f_m}{A_m}$$

Since $J_0(\beta) = 0$ for the first time when $\beta = 2.44$, we deduce that

$$k_f = \frac{2.44 \times 10^3}{2}$$

$$= 1.22 \times 10^3 \text{ hertz/volt}$$

Next, we note that $J_0(\beta) = 0$ for the second time when $\beta = 5.52$. Hence, the corresponding value of A_m for which the carrier component is reduced to zero is

$$A_m = \frac{\beta f_m}{k_f}$$

$$= \frac{5.52 \times 10^3}{1.22 \times 10^3}$$

$$= 4.52 \text{ volts}$$

Problem 2.32

For $\beta = 1$, we have

$J_0(1) = 0.765$

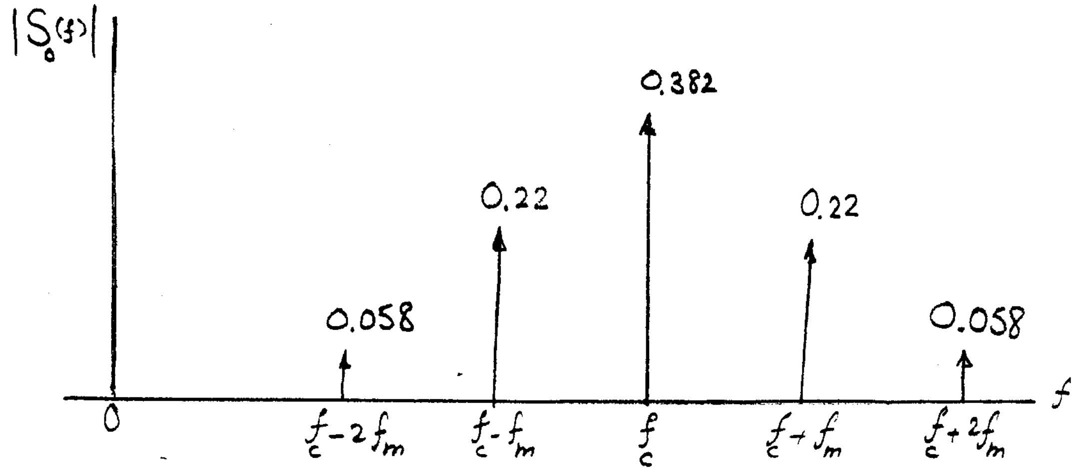
$J_1(1) = 0.44$

$J_2(1) = 0.115$

Therefore, the band-pass filter output is (assuming a carrier amplitude of 1 volt)

$$s_o(t) = 0.765 \cos(2\pi f_c t)$$
$$+ 0.44 \{\cos[2\pi(f_c+f_m)t] - \cos[2\pi(f_c-f_m)t]\}$$
$$+ 0.115 \{\cos[2\pi(f_c+2f_m)t] + \cos[2\pi(f_c-2f_m)t]\} ,$$

and the amplitude spectrum (for positive frequencies) is



Problem 2.33

(a) The frequency deviation is

$$\Delta f = k_f A_m = 25 \times 10^3 \times 20 = 5 \times 10^5 \text{ Hz}$$

The corresponding value of the modulation index is

$$\beta = \frac{\Delta f}{f_m} = \frac{5 \times 10^5}{10^5} = 5$$

The transmission bandwidth of the FM wave, using Carson's rule, is therefore

$$B_T = 2f_m(1+\beta) = 2\text{x}100\ (1+5) = 1200 \text{ kHz} = 1.2 \text{ MHz}.$$

(b) Using the universal curve of Fig. 3-36 we find that for β=5:

$$\frac{B_T}{\Delta f} = 3$$

Therefore,

$$B_T = 3\text{x}500 = 1500 \text{ kHz} = 1.5 \text{ MHz}$$

(c) If the amplitude of the modulating wave is doubled, we find that

$$\Delta f = 1 \text{ MHz and } \beta = 10$$

Thus, using Carson's rule we obtain

$$B_T = 2\text{x}100\ (1+10) = 2200 \text{ kHz} = 2.2 \text{ MHz}$$

Using the universal curve of Fig. 3-36, we get

$$\frac{B_T}{\Delta f} = 2.75$$

and B_T = 2.75 MHz.

(d) If f_m is doubled, β = 2.5. Then, using Carson's rule, B_T = 1.4 MHz. Using the universal curve, $B_T/\Delta f$ = 4, and

$$B_T = 4\Delta f = 2 \text{ MHz}.$$

Problem 2.34

(a) The angle of the PM wave is

$$\theta_i(t) = 2\pi f_c t + k_p\, m(t)$$
$$= 2\pi f_c t + k_p\, A_m \cos(2\pi f_m t)$$
$$= 2\pi f_c t + \beta_p \cos(2\pi f_m t)$$

where $\beta_p = k_p\, A_m$. The instantaneous frequency of the PM wave is therefore

$$f_i(t) = \frac{1}{2\pi}\frac{d\theta_i(t)}{dt}$$
$$= f_c - \beta_p\, f_m \sin(2\pi f_m t)$$

We see that the maximum frequency deviation in a PM wave varies linearly with the modulation frequency f_m.

Using Carson's rule, we find that the transmission bandwidth of the PM wave is approximately (for the case when $\beta_p \gg 1$)

$$B_T \simeq 2(f_m + \beta_p\, f_m) = 2f_m(1 + \beta_p) \simeq 2f_m\, \beta_p$$

This shows that B_T varies linearly with f_m.

(b) In an FM wave, the transmission bandwidth B_T is approximately equal to $2\Delta f$, if the modulation index $\beta \gg 1$. Therefore, for an FM wave, B_T is effectively independent of the modulation frequency f_m.

Problem 2.35

The filter input is

$$v_1(t) = g(t)\, s(t)$$
$$= g(t) \cos(2\pi f_c t - \pi k t^2)$$

The complex envelope of $v_1(t)$ is

$$\tilde{v}_1(t) = g(t) \exp(-j\pi k t^2)$$

The impulse response h(t) of the filter is defined in terms of the complex impulse response $\tilde{h}(t)$ as follows

$$h(t) = \mathrm{Re}[\tilde{h}(t) \exp(j2\pi f_c t)]$$

With

$$h(t) = \cos(2\pi f_c t + \pi k t^2),$$

we have

$$\tilde{h}(t) = \exp(j\pi k t^2)$$

The complex envelope of the filter output is therefore (See Appendix 2)

$$\tilde{v}_0(t) = \frac{1}{2}\tilde{h}(t) \star \tilde{v}_i(t)$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} g(\tau)\ \exp(-j\pi k\tau^2)\ \exp[j\pi k(t-\tau)]^2 d\tau$$

$$= \frac{1}{2}\exp(j\pi kt^2)\int_{-\infty}^{\infty} g(\tau)\ \exp(-j2\pi kt\tau)\ d\tau$$

$$= \frac{1}{2}\exp(j\pi kt^2)\ G(kt)$$

Hence,

$$|\tilde{v}_0(t)| = \frac{1}{2}\,|G(kt)|$$

This shows that the envelope of the filter output is, except for the scale factor of 1/2, equal to the magnitude of the Fourier transform of the input signal $g(t)$, with kt playing the role of frequency f.

Problem 2.36

The overall frequency multiplication ratio is

$$n = 2\text{x}3 = 6$$

Assume that the instantaneous frequency of the FM wave at the input of the first frequency multiplier is

$$f_{i1}(t) = f_c + \Delta f\ \cos(2\pi f_m t)$$

The instantaneous frequency of the resulting FM wave at the output of the second frequency multiplier is therefore

$$f_{i2}(t) = nf_c + n\Delta f\ \cos(2\pi f_m t)$$

Thus, the frequency deviation of this FM wave is equal to

$$n\Delta f = 6\text{x}10 = 60\ \text{kHz}$$

and its modulation index is equal to

$$\frac{n\Delta f}{f_m} = \frac{60}{5} = 12$$

The frequency separation of the adjacent side-frequencies of this FM wave is unchanged at $f_m = 5$ kHz.

Problem 2.37

(a) Figure 1 shows the simplified block diagram of a typical FM transmitter (based on the indirect method) used to transmit audio signals containing frequencies in the range 100 Hz to 15 kHz. The narrow-band phase modulator is supplied with a carrier signal of frequency f_1 = 0.2 MHz by a crystal-controlled oscillator. The desired FM signal at the transmitter output is to have a carrier frequency f_c = 100 MHz and a minimum frequency deviation Δf = 75 kHz.

In order to limit the harmonic distortion produced by the narrow-band phase modulator, we restrict the modulation index β_1 of this modulator to a maximum value of 0.3 radians. Consider then the value β_1 = 0.2 radians, which certainly satisfies this requirement. The lowest modulation frequencies of 100 Hz produce a frequency deviation of Δf_1 = 20 Hz at the narrow-band phase modulator output, whereas the highest modulation frequencies of 15 kHz produce a frequency deviation of Δf_1 = 3 kHz. The lowest modulation frequencies are therefore of immediate concern, as they produce a much lower frequency deviation than the highest modulation frequencies. The requirement is therefore to ensure that the frequency deviation produced by the lowest modulation frequencies of 100 Hz is raised to 75 kHz.

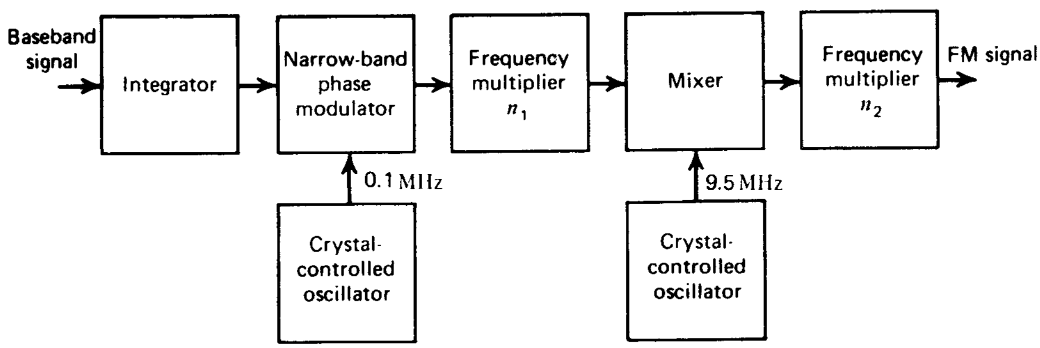


Figure 1

To produce a frequency deviation of Δf = 75 kHz at the FM transmitter output, the use of frequency multiplication is obviously required. Specifically, with Δf_1 = 20 Hz and Δf = 75 kHz, we require a total frequency multiplication ratio of 3750. However, using a straight frequency multiplication equal to this value would produce a much higher carrier frequency at the transmitter output than the desired value of 100 MHz. To generate an FM signal having both the desired frequency deviation and carrier frequency, we therefore need to use a *two-stage frequency multiplier* with an intermediate stage of frequency translation as illustrated in Fig. 1. Let n_1 and n_2 denote the respective frequency multiplication ratios, so that

$$n_1 n_2 = \frac{\Delta f}{\Delta f_1} = \frac{75000}{20} = 3750 \qquad (1)$$

The carrier frequency $n_1 f_1$ at the first frequency multiplier output is translated downward to $(f_2 - n_1 f_1)$ by mixing it with a sinusoidal wave of frequency f_2 = 95 MHz, which is supplied by a second crystal-controlled oscillator. However, the carrier frequency at the input of the second frequency multiplier is required to equal f_c/n_2. Equating these two frequencies, we thus get

$$f_2 - n_1 f_1 = \frac{f_c}{n_2}$$

Hence, with f_1 = 0.1 MHz, f_2 = 9.5 MHz, and f_c = 100 MHz, we have

$$9.5 - 0.1 n_1 = \frac{100}{n_2} \qquad (2)$$

Solving Eqs. (1) and (2) for n_1 and n_2, we obtain

$n_1 = 75$
$n_2 = 50$

(b) Using these frequency multiplication ratios, we get the set of values indicated in the table below:

Table -Values of Carrier Frequency and Frequency Deviation at the Various Points in the Wide-band Frequency Modulator of Fig. 1

	At the Phase Modulator Output	At the First Frequency Multiplier Output	At the Mixer Output	At the Second Frequency Multiplier Output
Carrier frequency	0.1 MHz	7.5 MHz	2.0 MHz	100 MHz
Frequency deviation	20 Hz	1.5 kHz	1.5 kHz	75 kHz

Problem 2.38

(a) Let L denote the inductive component, C the capacitive component, and C_0 the capacitance of each varactor diode due to the bias voltage V_b acting alone. Then, we have

$$C_0 = 100\ V_b^{-1/2}\ \text{pF}$$

and the corresponding frequency of oscillation is

$$f_0 = \frac{1}{2\pi\sqrt{L(C+C_0/2)}}$$

Therefore,

$$10^6 = \frac{1}{2\pi\sqrt{200 \times 10^{-6}\ (100 \times 10^{-12} + 50\ V_b^{-1/2} \times 10^{-12})}}$$

Solving for V_b, we get

$$V_b = 3.52\ \text{volts}$$

(b) The frequency multiplication ratio is 64. Therefore, the modulation index of the FM wave at the frequency multiplier input is

$$\beta = \frac{5}{64} = 0.078$$

This indicates that the FM wave produced by the combination of L, C and the varactor diodes is a narrow-band one, which in turn means that the amplitude A_m of the modulating wave is small compared to V_b. We may thus express the instantaneous frequency of this FM wave as follows:

$$f_i(t) = \frac{1}{2\pi}\ [200 \times 10^{-6}\ \{100 \times 10^{-12} + 50 \times 10^{-12}\ [3.52 + A_m \sin(2\pi f_m t)]^{-1/2}\}]^{-1/2}$$

$$= \frac{10^7}{2\sqrt{2}\pi}\ \{1 + 0.266\ [1 + \frac{A_m}{3.52} \sin(2\pi f_m t)]^{-1/2}\}^{-1/2}$$

$$\simeq \frac{10^7}{2\sqrt{2}\pi}\ \{1 + 0.266\ [1 - \frac{A_m}{7.04} \sin(2\pi f_m t)]\}^{-1/2}$$

$$= 10^6\ [1 - 0.03\ A_m \sin(2\pi f_m t)]^{-1/2}$$

$$\simeq 10^6\ [1 + 0.015\ A_m \sin(2\pi f_m t)]$$

With a modulation index of 0.078, the corresponding value of the frequency deviation is

$$\Delta f = \beta f_m$$

$$= 0.078 \times 10^4 \text{ Hz}$$

Therefore,

$$0.015 A_m \times 10^6 = 0.078 \times 10^4$$

where A_m is in volts. Solving for A_m, we get

$$A_m = 52 \times 10^{-3} \text{ volts.}$$

Problem 2.39

The transfer function of the RC filter is

$$H(f) = \frac{j2\pi fCR}{1+j2\pi fCR}$$

If $2\pi fCR << 1$ for all frequencies of interest, then we may approximate H(f) as

$$H(f) \simeq j2\pi fCR$$

However, multiplication by $j2\pi f$ in the frequency domain is equivalent to differentiation in the time domain. Therefore, denoting the RC filter output as $v_1(t)$, we may write

$$v_1(t) \simeq CR \frac{ds(t)}{dt}$$

$$= CR \frac{d}{dt} \{A_c \cos[2\pi f_c t + 2\pi k_f \int_0^t m(t)\, dt]\}$$

$$= -CR\, A_c[2\pi f_c + 2\pi k_f m(t)] \sin[2\pi f_c t + 2\pi k_f \int_0^t m(t)\, dt]$$

The corresponding envelope detector output is

$$v_2(t) \simeq 2\pi f_c CR\, A_c \left|1 + \frac{k_f}{f_c} m(t)\right|$$

Since $k_f|m(t)| < f_c$ for all t, then

$$v_2(t) \simeq 2\pi f_c CR\, A_c[1 + \frac{k_f}{f_c} m(t)]$$

which shows that, except for a dc bias, the output is proportional to the modulating signal m(t).

Problem 2.40

The envelope detector input is

$$v(t) = s(t) - s(t-T)$$

$$= A_c \cos[2\pi f_c t + \phi(t)] - A_c \cos[2\pi f_c (t-T) + \phi(t-T)]$$

$$= -2A_c \sin[\frac{2\pi f_c(2t-T) + \phi(t) + \phi(t-T)}{2}] \sin[\frac{2\pi f_c T + \phi(t) - \phi(t-T)}{2}] \quad (1)$$

where

$$\phi(t) = \beta \sin(2\pi f_m t)$$

The phase difference $\phi(t) - \phi(t-T)$ is

$$\phi(t) - \phi(t-T) = \beta \sin(2\pi f_m t) - \beta \sin[2\pi f_m (t-T)]$$

$$= \beta[\sin(2\pi f_m t) - \sin(2\pi f_m t) \cos(2\pi f_m T) + \cos(2\pi f_m t) \sin(2\pi f_m T)]$$

$$\simeq \beta[\sin(2\pi f_m t) - \sin(2\pi f_m t) + 2\pi f_m T \cos(2\pi f_m t)]$$

$$= 2\pi \Delta f T \cos(2\pi f_m t)$$

where

$$\Delta f = \beta f_m.$$

Therefore, noting that $2\pi f_c T = \pi/2$, we may write

$$\sin[\frac{2\pi f_c T + \phi(t) - \phi(t-T)}{2}] \simeq \sin[\pi f_c T + \pi \Delta f T \cos(2\pi f_m t)]$$

$$= \sin[\frac{\pi}{4} + \pi \Delta f T \cos(2\pi f_m t)]$$

$$= \sqrt{2} \cos[\pi \Delta f T \cos(2\pi f_m t)] + \sqrt{2} \sin[\pi \Delta f T \cos(2\pi f_m t)]$$

$$= \sqrt{2} + \sqrt{2}\, \pi \Delta f T \cos(2\pi f_m t)$$

where we have made use of the fact that $\pi \Delta f T << 1$. We may therefore rewrite Eq. (1) as

$$v(t) \simeq -2\sqrt{2}\, A_c[1 + \pi \Delta f T \cos(2\pi f_m t)] \sin[\pi f_c (2t-T) + \frac{\phi(t) + \phi(t-T)}{2}]$$

Accordingly, the envelope detector output is

$$a(t) \simeq 2\sqrt{2}\, A_c[1+\pi \Delta f T \cos(2\pi f_m t)]$$

which, except for a bias term, is proportional to the modulating wave.

Problem 2.41

(a) In the time interval $t-(T_1/2)$ to $t+(T_1/2)$, assume there are n zero crossings. The phase difference is $\theta_i(t+T_1/2) - \theta_i(t-T_1/2) = n\pi$. Also, the angle of an FM wave is

$$\theta_i(t) = 2\pi f_c t + 2\pi k_f \int_0^t m(t)\, dt.$$

Since m(t) is assumed constant, equal to m_1, $\theta_i(t) = 2\pi f_c t + 2\pi k_f m_1 t$. Therefore,

$$\theta_i(t+T_1/2) - \theta_i(t-T_1/2) = (2\pi f_c + 2\pi k_f m_1)\, [t+T_1/2 - (t-T_1/2)].$$

$$= (2\pi f_c + 2\pi k_f m_1)\, T_1.$$

But

$$f_i(t) = \frac{d\theta_i(t)}{dt} = 2\pi f_c + 2\pi k_f m_1 \;.$$

Thus,

$$\theta_i(t+T_1/2) - \theta_i(t-T_1/2) = f_i(t)\, T_1.$$

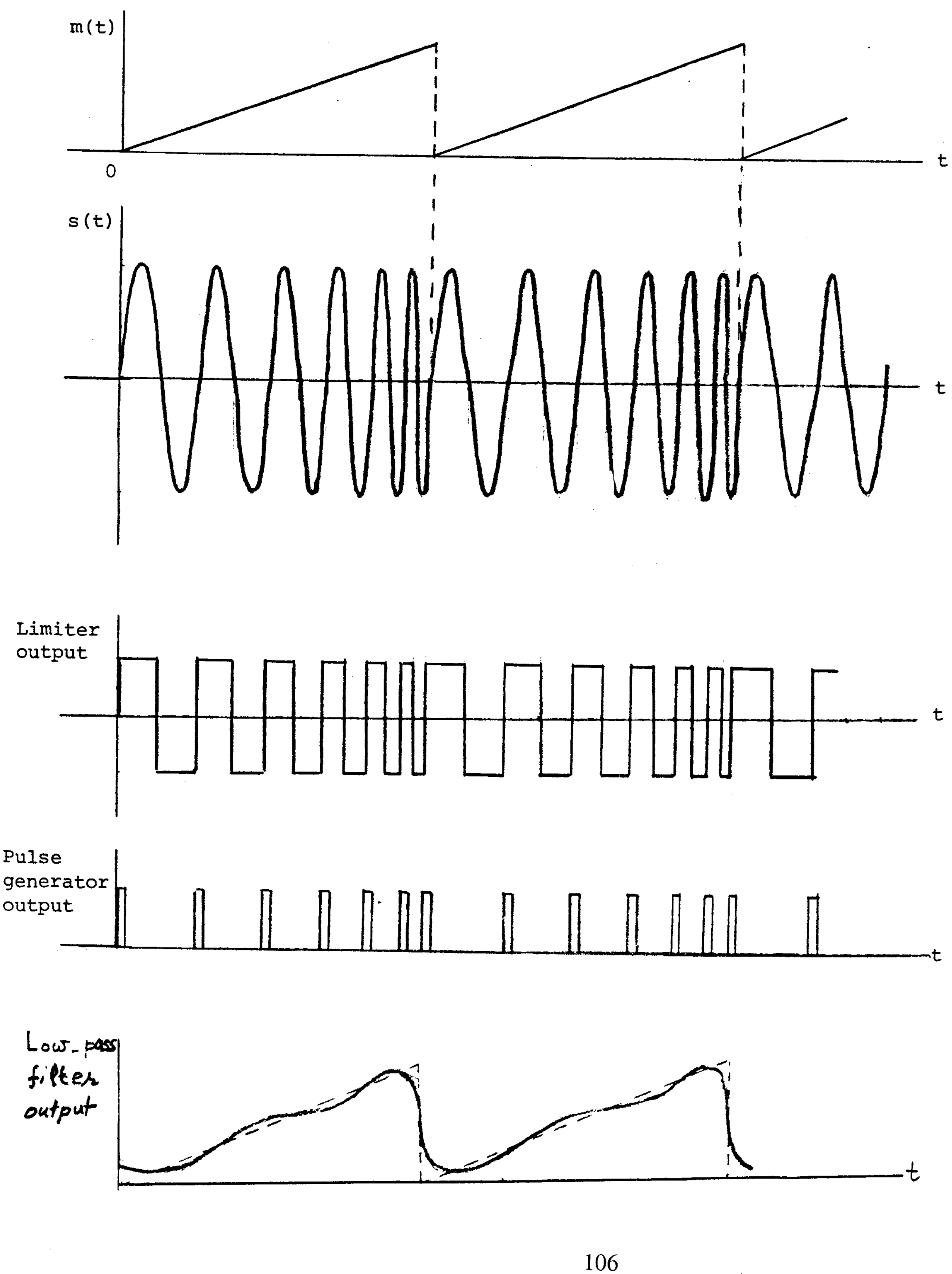
But this phase difference also equals $n\pi$. So,

$$f_i(t)\, T_1 = n\pi$$

and

$$f_i(t) = n\pi/T_1$$

(b) For a repetitive ramp as the modulating wave, we have the following set of waveforms


m(t)
0
t
s(t)
t
Limiter output
t
Pulse generator output
t
Low-pass filter output
t

Problem 2.42

The complex envelope of the modulated wave s(t) is

$$\tilde{s}(t) = a(t) \exp[j\phi(t)]$$

Since a(t) is slowly varying compared to $\exp[j\phi(t)]$, the complex envelope $\tilde{s}(t)$ is restricted effectively to the frequency band $-B_T/2 \leq f \leq B_T/2$. An ideal frequency discriminator consists of a differentiator followed by an envelope detector. The output of the differentiator, in response to $\tilde{s}(t)$, is

$$\tilde{v}_o(t) = \frac{d}{dt}\tilde{s}(t)$$

$$= \frac{d}{dt}\{a(t) \exp[j\phi(t)]\}$$

$$= \frac{da(t)}{dt}\exp[j\phi(t)] + j\frac{d\phi(t)}{dt}a(t)\exp[j\phi(t)]$$

$$= a(t)\exp[j\phi(t)]\left[\frac{1}{a(t)}\frac{da(t)}{dt} + j\frac{d\phi(t)}{dt}\right]$$

Since a(t) is slowly varying compared to $\phi(t)$, we have

$$\left|\frac{d\phi(t)}{dt}\right| \gg \left|\frac{1}{a(t)}\frac{da(t)}{dt}\right|$$

Accordingly, we may approximate $\tilde{v}_o(t)$ as

$$\tilde{v}_o(t) \simeq j\, a(t)\frac{d\phi(t)}{dt}\exp[j\phi(t)]$$

However, by definition

$$\phi(t) = 2\pi k_f \int_0^t m(t)\, dt$$

Therefore,

$$\tilde{v}_o(t) = j2\pi k_f\, a(t)\, m(t) \exp[j\phi(t)]$$

Hence, the envelope detector output is proportional to a(t) m(t) as shown by

$$|\tilde{v}_o(t)| \simeq 2\pi k_f\, a(t)\, m(t)$$

Problem 2.43

(a) The limiter output is

$$z(t) = \mathrm{sgn}\{a(t)\cos[2\pi f_c t + \phi(t)]\}$$

Since a(t) is of positive amplitude, we have

$$z(t) = \operatorname{sgn}\{\cos[2\pi f_c t + \phi(t)]\}$$

Let

$$\psi(t) = 2\pi f_c t + \phi(t)$$

Then, we may write

$$\operatorname{sgn}[\cos\psi] = \sum_{n=-\infty}^{\infty} c_n \exp(jn\psi)$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{sgn}[\cos\psi]\exp(-jn\psi)\,d\psi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-\pi/2}(-1)\exp(-jn\psi)d\psi + \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(+1)\exp(-jn\psi)\,d\psi \qquad (1)$$

$$+ \frac{1}{2\pi}\int_{\pi/2}^{\pi}(-1)\exp(-jn\psi)\,d\psi$$

If $n \neq 0$, then

$$c_n = \frac{1}{2\pi(-jn)}\left[-\exp\left(\frac{jn\pi}{2}\right)+\exp(jn\pi)+\exp\left(\frac{-jn\pi}{2}\right)-\exp\left(\frac{jn\pi}{2}\right)-\exp(-jn\pi)+\exp\left(\frac{-jn\pi}{2}\right)\right]$$

$$= \frac{1}{\pi n}\left[2\sin\left(\frac{n\pi}{2}\right)-\sin(n\pi)\right]$$

$$= \begin{cases} \frac{2}{\pi n}(-1)^{(n-1)/2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

If $n=0$, we find from Eq. (1) that $c_n=0$. Therefore,

$$\operatorname{sgn}[\cos\psi] = \frac{2}{\pi}\sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \frac{1}{n}(-1)^{(n-1)/2}\exp(jn\psi)$$

$$= \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cos[\psi(2k+1)]$$

We may thus express the limiter output as

$$z(t) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cos[2\pi f_c t(2k+1) + \phi(t)(2k+1)] \tag{2}$$

(b) Consider the term

$$\cos[2\pi f_c t(2k+1) + \phi(t)\ (2k+1)] = \mathrm{Re}\{\exp[j2\pi f_c t(2k+1)]\exp[j\phi(t)(2k+1)]\}$$
$$= \mathrm{Re}\{\exp[j2\pi f_c t(2k+1)][\exp(j\phi(t))]^{2k+1}\}$$

The function $\exp[j\phi(t)]$, representing the complex envelope of the FM wave with unit amplitude, is effectively low-pass in nature. Therefore, this term represents a band-pass signal centered about $\pm f_c(2k+1)$. Furthermore, the Fourier transform of $\{\exp[j\phi(t)]\}^{2k+1}$ is equal to that of $\exp[j\phi(t)]$ convolved with itself 2k times. Therefore, assuming that $\exp[j\phi(t)]$ is limited to the interval $-B_T/2 \leq f \leq B_T/2$, we find that $\left(\exp[j\phi(t)]\right)^{2k+1}$ is limited to the interval $-(B_T/2)(2k+1) \leq f \leq (B_T/2)(2k+1)$.

Assuming that $f_c > B_T$, as is usually the case, we find that none of the terms corresponding to values of k greater than zero will overlap the spectrum of the term corresponding to k=0. Thus, if the limiter output is applied to a band-pass filter of bandwidth B_T and mid-band frequency f_c, all terms, except the term corresponding to k=0 in Eq. (2), are removed by the filter. The resulting filter output is therefore

$$y(t) = \frac{4}{\pi} \cos[2\pi f_c t + \phi(t)]$$

We thus see that by using the amplitude limiter followed by a band-pass filter, the effect of amplitude variation, represented by a(t) in the modulated wave s(t), is completely removed.

Problem 2.44

(a) Let the FM wave be defined by

$$s(t) = A_c \cos[2\pi f_c t + 2\pi k_f \int_0^t m(t)\, dt]$$

Assuming that f_c is large compared to the bandwidth of s(t), we may express the complex envelope of s(t) as

$$\tilde{s}(t) = A_c \exp[j2\pi k_f \int_0^t m(t)\, dt]$$

But, by definition, the pre-envelope of s(t) is (see Appendix 2)

$$s_+(t) = \tilde{s}(t)\exp(j2\pi f_c t)$$
$$= s(t) + j\,\hat{s}(t)$$

where $\hat{s}(t)$ is the Hilbert transform of $s(t)$. Therefore,

$$s(t) + j\hat{s}(t) = A_c \exp[j2\pi k_f \int_0^t m(t)\,dt]\exp(j2\pi f_c t)$$

$$= A_c\{\cos[2\pi f_c t + 2\pi k_f \int_0^t m(t)\,dt] + j\sin[2\pi f_c t + 2\pi k_f \int_0^t m(t)\,dt]\}$$

Equating real and imaginary parts, we deduce that

$$\hat{s}(t) = A_c \sin[2\pi f_c t + 2\pi k_f \int_0^t m(t)\,dt] \qquad (1)$$

(b) For the case of sinusoidal modulation, we have

$$m(t) = A_m \cos(2\pi f_m t)$$

The corresponding FM wave is

$$s(t) = A_c \cos[2\pi f_c t + \beta \sin(2\pi f_m t)]$$

where

$$\beta = k_f A_m$$

Expanding $s(t)$ in the form of a Fourier series, we get

$$s(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta)\cos[2\pi(f_c+nf_m)t]$$

Noting that the Hilbert transform of $\cos[2\pi(f_c+nf_m)t]$ is equal to $\sin[2\pi(f_c+nf_m)t]$, and using the linearity property of the Hilbert transform, we find that the Hilbert transform of $s(t)$ is

$$\hat{s}(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta)\sin[2\pi(f_c+nf_m)\,t]$$

$$= A_c \sin[2\pi f_c t + \beta\sin(2\pi f_m t)]$$

This is exactly the same result as that obtained by using Eq. (1). In the case of sinusoidal modulation, therefore, there is no error involved in using Eq. (1) to evaluate the Hilbert transform of the corresponding FM wave.

Problem 2.45

(a) The modulated wave $s(t)$ is

$$s(t) = \exp[-\hat{\phi}(t)]\cos[2\pi f_c t + \phi(t)]$$

$$= \mathrm{Re}\{\exp[-\hat{\phi}(t)]\,\exp[j2\pi f_c t + j\phi(t)]\}$$

$$= \mathrm{Re}\{\exp[j2\pi f_c t + j(\phi(t) + j\hat{\phi}(t))]\}$$

$$= \mathrm{Re}\{\exp[j2\pi f_c t + j\phi_+(t)]\} \qquad (1)$$

where $\phi_+(t)$ is the pre-envelope of the phase function $\phi(t)$, that is,

$$\phi_+(t) = \phi(t) + j\hat{\phi}(t)$$

Expanding the exponential function $\exp[j\phi_+(t)]$ in the form of an infinite series:

$$\exp[j\phi_+(t)] = \sum_{n=0}^{\infty} \frac{j^n}{n!}\,\phi_+^n(t) \qquad (2)$$

Taking the Fourier transform of both sides of this relation, we may write

$$F\{\exp[j\phi_+(t)]\} = \sum_{n=0}^{\infty} \frac{j^n}{n!}\,F[\phi_+^n(t)]$$

For $n \geq 2$, we may express $\phi_+^n(t)$ as the product of $\phi_+(t)$ and $\phi_+^{n-1}(t)$. Hence,

$$F[\phi_+^n(t)] = \Phi_+(f) \star F[\phi_+^{n-1}(t)]$$

where $\phi_+(t) \rightleftharpoons \Phi_+(f)$, and $\star$ denotes convolution. Since $\Phi_+(f) = 0$ for $f < 0$, it follows that for all $n \geq 0$,

$$F[\Phi_+^n(t)] = 0, \qquad \text{for } f < 0$$

Hence,

$$F\{\exp[j\phi_+(t)]\} = 0 \qquad \text{for } f < 0$$

By using the frequency-shifting property of the Fourier transform, it follows that

$$F\{\exp[j\phi_+(t)]\,\exp(j2\pi f_c t)\} = 0 \quad \text{for } f < f_c \qquad (3)$$

From Eq. (1),

$$s(t) = \frac{1}{2}\,\{\exp[j2\pi f_c t + j\phi_+(t)] + \exp[-j2\pi f_c t - j\phi_+^*(t)]\}$$

where $\phi_+^*(t)$ is the complex conjugate of $\phi_+(t)$. Therefore,

$$F[s(t)] = \frac{1}{2}\,F\{\exp[j2\pi f_c t + j\phi_+(t)]\} + \frac{1}{2}\,F\{\exp[-j2\pi f_c t - j\phi_+^*(t)\}$$

Applying the conjugate-function property of the Fourier transform to Eq. (3), we find that

$$F\{\exp[-j2\pi f_c t - j\phi_+^*(t)]\} = 0, \quad \text{for } f > -f_c.$$

Hence, it follows that the spectrum of s(t) is zero for $-f_c < f < f_c$. However, this spectrum is of infinite extent, because the expansion of s(t) contains an infinite number of terms, as in eq. (2).

(b) With

$$\phi(t) = \beta \sin(2\pi f_m t),$$

we find that

$$\hat{\phi}(t) = -\beta \cos(2\pi f_m t)$$

Therefore,

$$\begin{aligned}\phi_+(t) &= \beta \sin(2\pi f_m t) - j\beta \cos(2\pi f_m t)\\ &= -\, j\beta[\cos(2\pi f_m t) + j \sin(2\pi f_m t)]\\ &= -\, j\beta \exp(j2\pi f_m t)\end{aligned}$$

Hence,

$$\begin{aligned}\exp[j\phi_+(t)] &= \exp[\beta \exp(j2\pi f_m t)]\\ &= \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \exp(j2\pi n f_m t)\end{aligned}$$

The modulated wave s(t) is therefore

$$\begin{aligned}s(t) &= \mathrm{Re}\{\exp(j2\pi f_c t) \exp[j\phi_+(t)]\}\\ &= \mathrm{Re}[\exp(j2\pi f_c t) \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \exp(j2\pi n f_m t)]\\ &= \mathrm{Re}\{ \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \exp[j2\pi(f_c+nf_m)t]\}\\ &= \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \cos[2\pi(f_c+nf_m)t]\end{aligned}$$

Problem 2.46

After passing the received signal through a narrow-band filter of bandwidth 8kHz centered on $f_c = 200$kHz, we get

$$x(t) = A_c m(t)\cos(2\pi f_c t) + n'(t)$$

$$= A_c m(t)\cos(2\pi f_c t) + n'_I(t)\cos(2\pi f_c t) - n'_I(t)\sin(2\pi f_c t)$$

$$= (A_c m(t) + n_I(t))\cos(2\pi f_c t) - n'_Q(t)\sin(2\pi f_c t)$$

where $n'(t)$ is the narrow-band noise produced at the filter output, and $n'_I(t)$ and $n'_Q(t)$ are its in-phase and quadrature components. Coherent detection of $x(t)$ yields the output

$$y(t) = A_c m(t) + n'_I(t)$$

The average power of the modulated wave is

$$\frac{A_c^2 P}{4} = 10W$$

where P is the average power of $m(t)$. To calculate the average power of the in-phase noise component $n'_I(t)$, we refer to the spectra shown in Fig. 1:

- Part (a) of Fig. 1 shows the power spectral density of the noise $n(t)$, and a superposition of the frequency response of the narrow-band filter.
- Part (b) shows the power spectral density of the noise $n'_I(t)$ produced at the filter output.
- Part (c) shows the power spectral density of the in-phase component $n'_I(t)$ of $n'(t)$.

Note that since the bandwidth of the filter is small compared to the carrier frequency f_c, we have approximated the spectral characteristic of $n'(t)$ to be flat at the level of 0.5 x 10^{-6} watts/Hz. Hence, the average power of $n'_I(t)$ is (from Fig. 1c):

$$(10^{-6} \text{ watts/Hz})\ (8 \times 10^3) = 0.008 \text{ watts}$$

The output signal-to-noise ratio (SNR) is therefore

$$\frac{10}{0.008} = 1,250$$

Expressing this result in decibels, we have an output SNR of 31 dB.

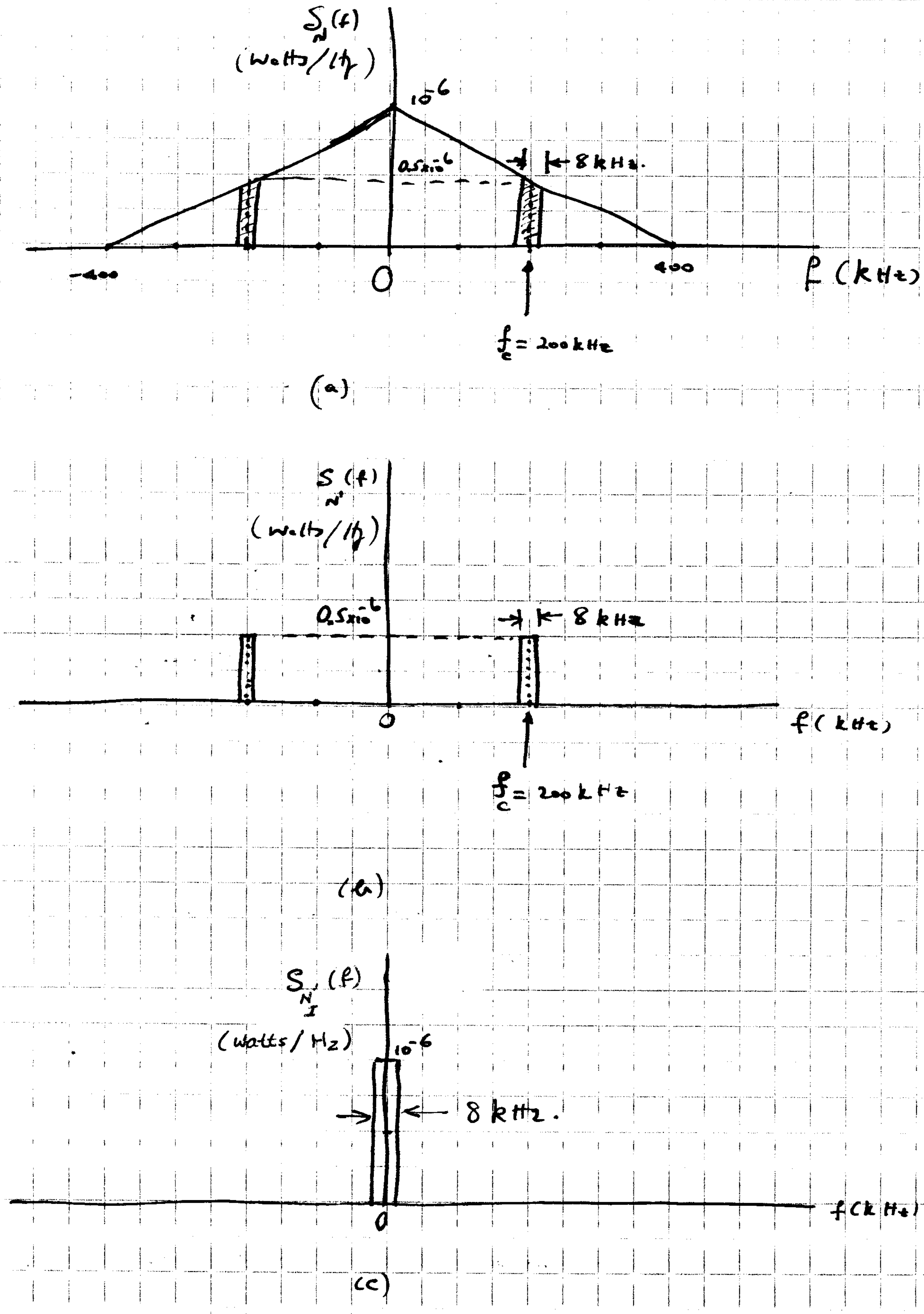

$S_N(f)$
(Watts/Hz)
10^{-6}
0.5×10^{-6}
8 kHz.
-400
0
400
f (kHz)
f_c = 200 kHz
(a)
$S_{N'}(f)$
(Watts/Hz)
0.5×10^{-6}
8 kHz
0
f (kHz)
f_c = 200 kHz
(b)
$S_{N'_I}(f)$
(Watts/Hz)
10^{-6}
8 kHz.
0
f (kHz)
(c)

Figure 1

Problem 2.47

From Problem 5.38, we note that the quadrature components of a narrow-band noise have autocorrelations:

$$R_{N_I}(\tau) = R_{N_Q}(\tau) = R_N(\tau)\cos(2\pi f_c\tau) + \hat{R}_N(\tau)\sin(2\pi f_c\tau)$$

where $R_N(\tau)$ is the autocorrelation of the narrow-band noise, $\hat{R}_N(\tau)$ is the Hilbert transform of $R_N(\tau)$, and f_c is the band center. The cross-correlations of the quadrature components are

$$R_{N_I N_Q}(\tau) = -R_{N_I N_Q}(\tau) = R_N(\tau)\sin(2\pi f_c\tau) - \hat{R}_N(\tau)\cos(2\pi f_c\tau)$$

(a) For a DSBSC system,

$$R_{N_I}(\tau) = R_{N_Q}(\tau) = R_N(\tau)\cos(2\pi f_c\tau) + \hat{R}_N(\tau)\sin(2\pi f_c\tau)$$

$$R_{N_I N_Q}(\tau) = -R_{N_Q N_I}(\tau) = R_N(\tau)\sin(2\pi f_c t) - \hat{R}_N(\tau)\cos(2\pi f_c\tau)$$

where f_c is the carrier frequency, and $R_N(\tau)$ is the autocorrelation function of the narrow-band noise on the interval $f_c - W \leq f \leq f_c + W$.

(b) For an SSB system using the lower sideband,

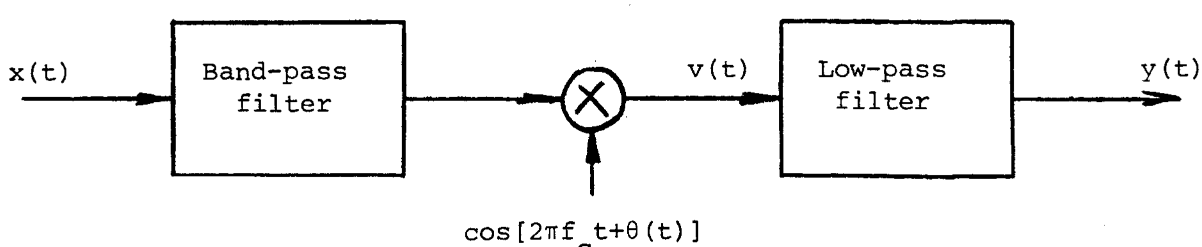
$$R_{N_I}(\tau) = R_{N_Q}(\tau) = R_N(\tau)\cos(2\pi(f_c - \frac{W}{2})\tau) + \hat{R}_N(\tau)\sin(2\pi(f_c - \frac{W}{2})\tau)$$

$$R_{N_I N_Q}(\tau) = -R_{N_Q N_I}(\tau) = R_N(\tau)\sin(2\pi(f_c - \frac{W}{2})\tau) - \hat{R}_N(\tau)\cos(2\pi(f_c - \frac{W}{2})\tau)$$

where in this case, $R_N(\tau)$ is the autocorrelation of the narrow-band noise on the interval $f_c - W \leq f \leq f_c$.

(c) For an SSB system with only the upper sideband transmitted, the correlations are similar to (b) above, except that $(f_c - \frac{W}{2})$ is replaced by $(f_c + \frac{W}{2})$, and the narrow-band noise is on the interval $f_c \leq f \leq f_c + W$.

Problem 2.48



The signal at the mixer input is equal to $s(t) + n(t)$, where $s(t)$ is the modulated wave, and $n(t)$ is defined by

$$n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)$$

with

$$E[n_I^2(t)] = E[n_Q^2(t)] = N_0 B_T$$

The $s(t)$ is defined by for DSB-SC modulation

$$s(t) = A_c m(t)\cos(2\pi f_c t)$$

The mixer output is

$$v(t) = [s(t) + n(t)]\cos[2\pi f_c t + \theta(t)]$$

$$= \{[A_c m(t) + n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)\}\cos 2\pi f_c t + \theta(t)]$$

$$= \frac{1}{2}[A_c m(t) + n_I(t)\{\cos[\theta(t)]\} + \cos[4\pi f_c t + \theta(t)]$$

$$+ \frac{1}{2}A_c n_Q(t)\{\sin[\theta(t)] - \sin[4\pi f_c t + \theta(t)]\}$$

The postdetection low-pass filter removes the high frequency components of $v(t)$, producing the output

$$y(t) = \frac{1}{2}[[A_c m(t) + n_I(t)]\cos[\theta(t)] + \frac{1}{2}A_c n_Q(t)\sin[\theta(t)] \qquad (1)$$

When the phase error $\theta(t)$ is zero, we find that the message signal component of the receiver output is $\frac{1}{2}A_c m(t)$. The error at the receiver output is therefore

$$e(t) = y(t) - \frac{A_c}{2}m(t)$$

The mean-square value of this error is

$$\varepsilon = E[e^2(t)]$$

$$= E\left[\left(y(t) - \frac{A_c}{2}m(t)\right)^2\right] \qquad (2)$$

Substituting Eq. (1) into (2), expanding the expectation, and noting that the processes $m(t)$, $\theta(t)$, $n_I(t)$ and $n_Q(t)$ are all independent of one another, we get

$$\varepsilon = \frac{A_c^2}{4}E[m^2(t)]E[(\cos^2\theta(t))] + \frac{1}{4}E[n_I^2(t)]E[\cos^2\theta(t)]$$

$$+ \frac{1}{4}E[n_Q^2(t)]E[\sin^2\theta(t)]$$

$$+ \frac{A_c^2}{4}E[m^2(t)] - \frac{A_c^2}{2}E[m^2(t)]E[\cos\theta(t)]$$

We now note that

$$E[n_I^2(t)] = E[n_Q^2(t)] = \sigma_N^2$$

$$E[n_I^2(t)]E[\cos^2\theta(t)] + E[n_Q^2(t)]E[\sin^2\theta(t)] = \sigma_N^2$$

Therefore,

$$\varepsilon = \frac{A_c^2}{2}E[m^2(t)]E\{[1 - \cos\theta(t)]^2\} + \frac{\sigma_N^2}{4}$$

$$= \frac{A_c^2 P}{4}E\{[1 + \cos\theta(t)]^2\} + \frac{\sigma_N^2}{4}$$

where $P = E[m^2(t)]$.

For small values of $\theta(t)$, we may use the approximation

$$1 - \cos\theta(t) \approx \frac{\sigma_N^2}{2}$$

Hence,

$$\varepsilon = \frac{A_c^2 P}{16} E[\theta^4(t)] + \frac{\sigma_N^2}{4}$$

Since $\theta(t)$ is Gaussian-distributed with zero mean and variance σ_θ^2, we have

$$E[\theta^4(t)] = 3\sigma_\theta^4$$

The mean-square error for the case of a DSBSC system is therefore

$$\varepsilon = \frac{3A_c^2 P \sigma_\theta^4}{16} + \frac{\sigma_N^2}{4}$$

Problem 2.49

Consider the case of a receiver using coherent detection, with an incoming single-sideband (SSB) modulated wave. We assume that only the lower sideband is transmitted, so that we can express the modulated wave as

$$s(t) = \frac{1}{2}CA_c\cos(2\pi f_c t)m(t) + \frac{1}{2}CA_c\sin(2\pi f_c t)\hat{m}(t) \tag{1}$$

where $\hat{m}(t)$ is the Hilbert transform of the message signal $m(t)$. The system-dependent scaling factor C is included to make the signal component $s(t)$ have the same units as the noise component $n(t)$. We may make the following observations concerning the in-phase and quadrature components of $s(t)$ in Eq. (1):

1. The two components $m(t)$ and $\hat{m}(t)$ are orthogonal to each other. Therefore, with the message signal $m(t)$ assumed to have zero mean, which is a reasonable assumption to make, it follows that $m(t)$ and $\hat{m}(t)$ are uncorrelated; hence, their power spectral densities are additive.
2. The Hilbert transform $\hat{m}(t)$ is obtained by passing $m(t)$ through a linear filter with a transfer function $-j\text{sgn}(f)$. The squared magnitude of this transfer function is equal to one for all f. Accordingly, we find that both $m(t)$ and $\hat{m}(t)$ have the same power spectral density.

Thus, using a procedure similar to that in Section 2.11, we find that the in-phase and quadrature components of the modulated signal $s(t)$ contribute an average power of $C^2A_c^2P/8$ each, where P is the average power of the message signal $m(t)$. The average power of $s(t)$ is therefore $C^2A_c^2P/4$. This result is half that in the DSB-SC receiver, which is intuitively satisfying.

The average noise power in the message bandwidth W is WN_0, as in the DSB-SC receiver. Thus the channel signal-to-noise ratio of a coherent receiver with SSB modulation is

$$(\text{SNR})_{C,\,\text{SSB}} = \frac{C^2A_c^2P}{4WN_0} \tag{2}$$

As illustrated in Fig. 1a, in an SSB system the transmission bandwidth B_T is W and the mid-band frequency of the power spectral density $S_N(f)$ of the narrow-band noise $n(t)$ is offset from the carrier frequency f_c by $W/2$. Therefore, we may express $n(t)$ as

$$n(t) = n_I(t)\cos\left[2\pi\left(f_c - \frac{W}{2}\right)t\right] - n_Q(t)\sin\left[2\pi\left(f_c - \frac{W}{2}\right)t\right] \tag{3}$$

The output of the coherent detector, due to the combined influence of the modulated signal $s(t)$ and noise $n(t)$, is thus given by

$$y(t) = \frac{1}{4}CA_c m(t) + \frac{1}{2}n_I(t)\cos(\pi W t) + \frac{1}{2}n_Q(t)\sin(\pi W t) \tag{4}$$

As expected, we see that the quadrature component $\hat{m}(t)$ of the modulated message signal $s(t)$ has been eliminated from the detector output, but unlike the case of DSB-SC modulation, the quadrature component of the narrow-band noise $n(t)$ now appears at the receiver output.

The message component in the receiver output is $CA_c m(t)/4$, and so we may express the average power of the recovered message signal as $C^2A_c^2P/16$. The noise component in the receiver output is $[n_I(t)\cos(\pi Wt) + n_Q(t)\sin(\pi Wt)]/2$. To determine the average power of the output noise, we note the following:

1. The power spectral density of both $n_I(t)$ and $n_Q(t)$ is as shown in Fig. 1b.
2. The sinusoidal wave $\cos(\pi Wt)$ is independent of both $n_I(t)$ and $n_Q(t)$. Hence, the power spectral density of $n'_I(t) = n_I(t)\cos(\pi W t)$ is obtained by shifting $S_{N_I}(f)$ to the left by $W/2$, shifting it to the right by $W/2$, adding the shifted spectra, and dividing the result by 4. The power spectral density of $n'_Q(t) = n_Q(t)\sin(\pi W t)$ is obtained in a similar way. The power spectral density of both $n'_I(t)$ and $n'_Q(t)$, obtained in this manner, is shown sketched in Fig. 1c.

From Fig. 1c we see that the average power of the noise component $n'_I(t)$ or $n'_Q(t)$ is $WN_0/2$. Therefore from Eq. (4), the average output noise power is $WN_0/4$. We thus find that the output signal-to-noise ratio of a system, using SSB modulation in the transmitter and coherent detection in the receiver, is given by

$$(\mathrm{SNR})_{O,\,\mathrm{SSB}} = \frac{C^2A_c^2P}{4WN_0} \tag{5}$$

Hence, from Eqs. (2) and (5), the figure of merit of such a system is

$$\left.\frac{(\mathrm{SNR})_O}{(\mathrm{SNR})_C}\right|_{\mathrm{SSB}} = 1 \tag{6}$$

where again we see that the factor C^2 cancels out.

Comparing Eqs. (5) and (6) with the corresponding results for DSB-SC modulation, we conclude that *for the same average transmitted (or modulated message) signal power and the same average noise power in the message bandwidth, an SSB receiver will have exactly the same output signal-to-noise ratio as a DSB-SC receiver, when both receivers use coherent detection for the recovery of the message signal.* Furthermore, in both cases, the noise performance of the receiver is the as that obtained by simply transmitting the message signal itself in the presence of the same noise. The only effect of the modulation process is to translate the message signal to a different frequency band to facilitate its transmission over a band-pass channel.

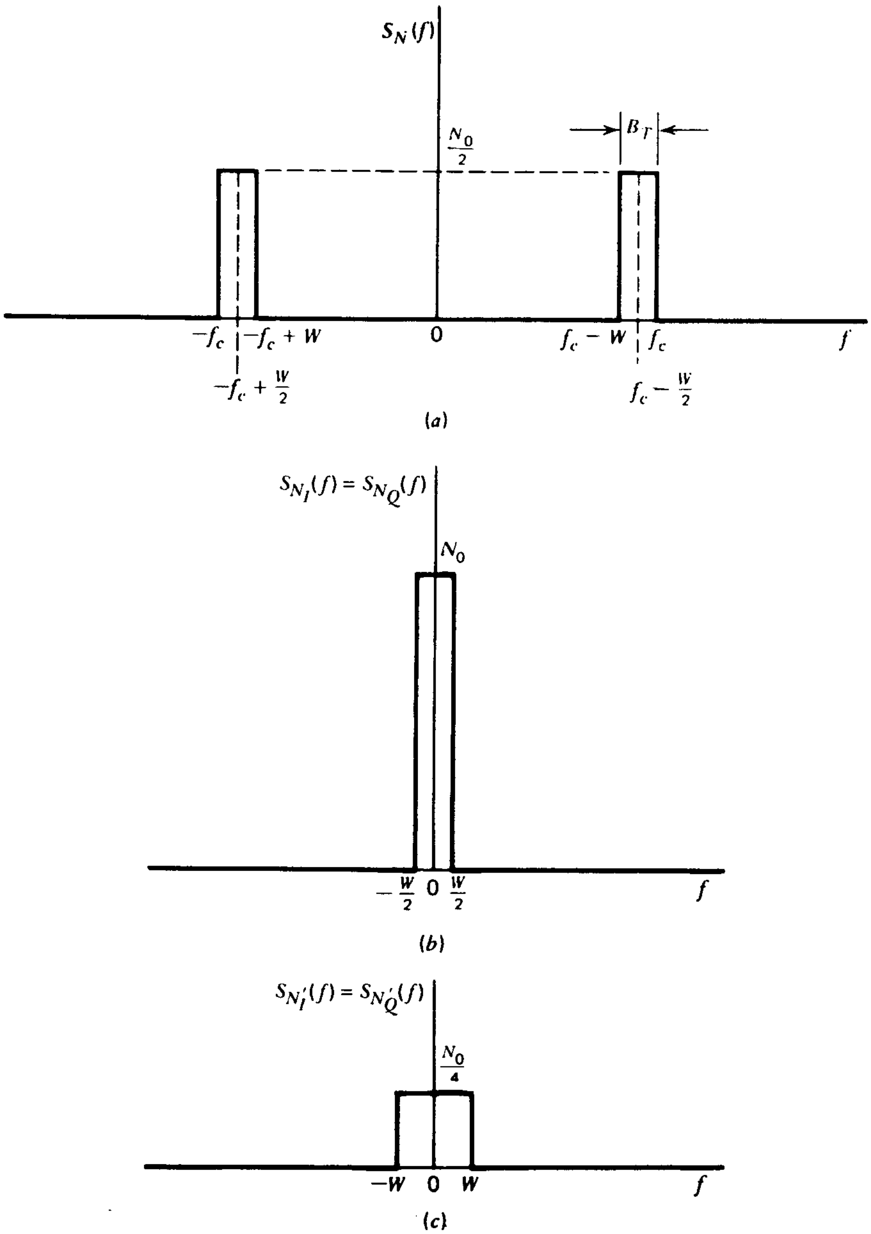
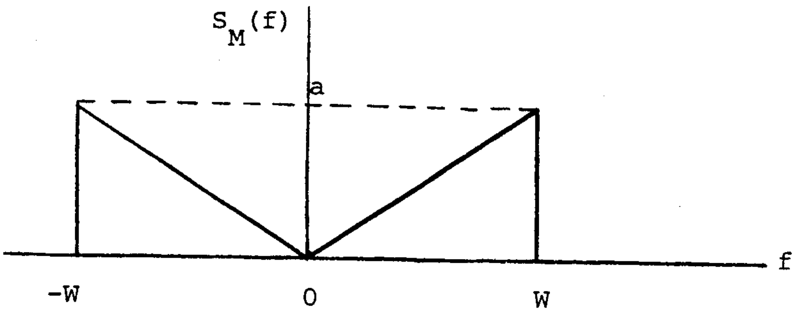


Figure 1

Problem 2.50

The power spectral density of the message signal m(t) is as follows



The average signal power is therefore

$$P = \int_{-\infty}^{\infty} S_M(f)\,df$$

$$= 2\int_0^W a\,\frac{f}{W}\,df$$

$$= aW$$

The corresponding value of the output signal-to-noise ratio of the SSB receiver is therefore, (using the solution to Problem 2.49)

$$(SNR)_O = \frac{A_c^2 P}{4WN_0}$$

$$= \frac{A_c^2\, aW}{4WN_0}$$

$$= \frac{a\,A_c^2}{4N_0}$$

Problem 2.51

(a) If the probability

$$P(|n_s(t)| > \varepsilon A_c|1 + k_a m(t)|) \le \delta_1 ,$$

then, with a probability greater than $1 - \delta_1$, we may say that

$$y(t) \approx \{[A_c + A_c\, k_a\, m(t) + n_c(t)]^2\}^{1/2}$$

That is, the probability that the quadrature component $n_s(t)$ is negligibly small is greater than $1 - \delta_1$.

(b) Next, we note that if $k_a\, m(t) < -1$, then we get overmodulation, so that even in the absence of noise, the envelope detector output is badly distorted. Therefore, in order to avoid overmodulation, we assume that k_a is adjusted relative to the message signal m(t) such that the probability

$$P(A_c + A_c\, k_a\, m(t) + n_c(t) < 0) = \delta_2$$

Then, the probability of the event

$$y(t) \approx A_c[1 + k_a\, m(t)] + n_c(t)$$

for any value of t, is greater than $(1 - \delta_1)(1 - \delta_2)$.

(c) When δ_1 and δ_2 are both small compared with unity, we find that the probability of the event

$$y(t) \approx A_c[1 + k_a\, m(t)] + n_c(t)$$

for any value of t, is very close to unity. Then, the output of the envelope detector is approximately the same as the corresponding output of a coherent detector.

Problem 2.52

The received signal is

$$x(t) = A_c \cos(2\pi f_c t) + n(t)$$

$$= A_c \cos(2\pi f_c t) + n_c(t) \cos(2\pi f_c t) - n_s(t) \sin(2\pi f_c t)$$

$$= [A_c + n_c(t)] \cos(2\pi f_c t) - n_s(t) \sin(2\pi f_c t)$$

The envelope detector output is therefore

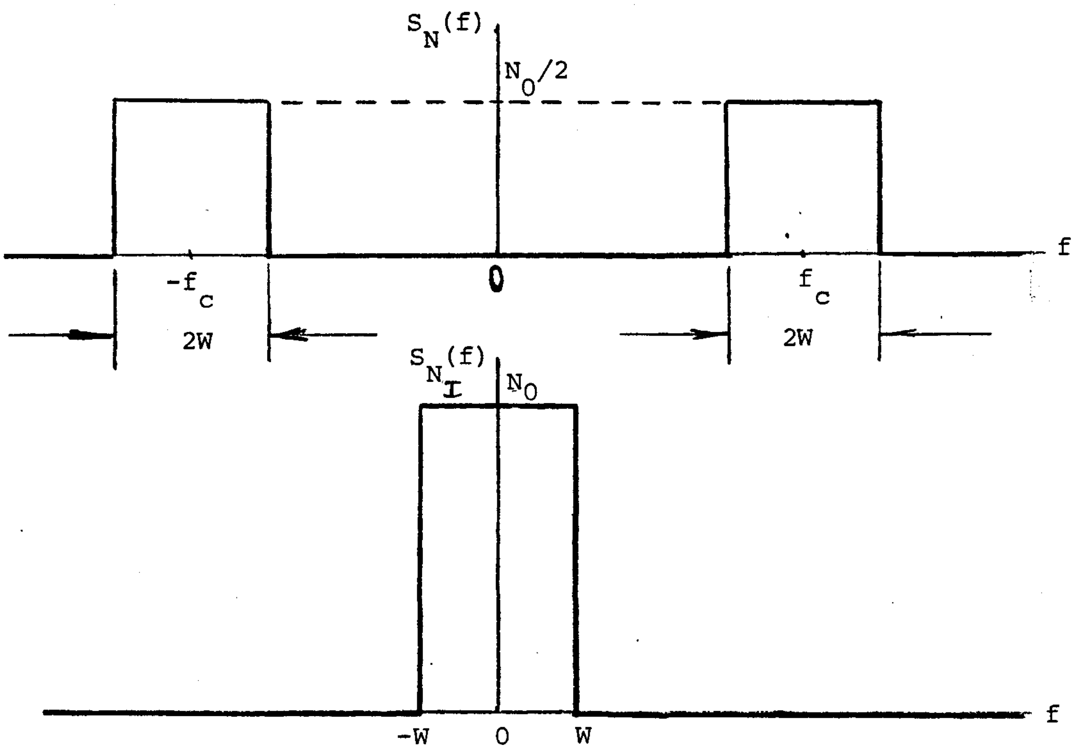
$$a(t) = \{[A_c + n_c(t)]^2 + n_s^2(t)\}^{1/2}$$

For the case when the carrier-to-noise ratio is high, we may approximate this result as

$$a(t) \simeq A_c + n_c(t)$$

The term A_c represents the useful signal component. The output signal power is thus A_c^2.

The power spectral densities of n(t) and $n_I(t)$ are as shown below:



The output noise power is $2N_0W$. The output signal-to-noise ratio is therefore

$$(SNR)_0 = \frac{A_c^2}{2N_0W}$$

Problem 2.53

(a) From Section 1.12 of the textbook we recall that the envelope $r(t)$ of the narrow-band noise $n(t)$ is Rayleigh distributed; that is

$$f_R(r) = \frac{r}{\sigma_N^2}\exp\left(-\frac{r^2}{2\sigma_N^2}\right)$$

where σ_N^2 is the variance of the noise $n(t)$. For an AM system, the variance σ_N^2 is $2WN_0$. Therefore, the probability of the event that the envelope R of the narrow-band noise $n(t)$ is large compared to the carrier amplitude A_c is defined by

$$P(R \geq A_c) = \int_{A_c}^{\infty} f_R(r)dr$$

$$= \int_{A_c}^{\infty} \frac{r}{2WN_0}\exp\left(-\frac{r^2}{4WN_0}\right)dr$$

$$= \exp\left(-\frac{A_c^2}{4WN_0}\right) \tag{1}$$

Define the *carrier to noise ratio* as

$$\rho = \frac{\text{average carrier power}}{\text{average noise power in bandwidth of the modulated message signal}} \tag{2}$$

Since the bandwidth of the AM signal is $2W$, the average noise power in this bandwidth is $2WN_0$. The average power of the carrier is $A_c^2/2$. The carrier-to-noise ratio is therefore

$$\rho = \frac{A_c^2}{4WN_0} \tag{3}$$

(b) We may now use this definition to rewrite Eq. (1) in the compact form

$$P(R \geq A_c) = \exp(-\rho) \tag{4}$$

Solving $P(R \geq A_c) = 0.5$ for ρ, we get

$\rho = \log 2 = 0.69$

Similarly, for $P(R \geq A_c) = 0.01$, we get

$\rho = \log 100 = 4.6$

Thus with a carrier-to-noise ratio $10\log_{10}0.69 = -1.6$ dB, the envelope detector is expected to be well into the threshold region, whereas with a carrier-to-noise ratio $10\log_{10}4.6 = 6.6$ dB, the detector is expected to be operating satisfactorily. We ordinarily need a signal-to-noise ratio considerably greater than 6.6 dB for satisfactory intelligibility, and therefore threshold effects are seldom of great importance in AM receivers using envelope detection.

Problem 2.54

(a) Following a procedure similar to that described for the case of an FM system, we find that the input of the phase detector is

$$v(t) = A_c \cos[2\pi f_c t + \theta(t)]$$

where

$$\theta(t) = k_p\, m(t) + \frac{n_Q(t)}{A_c}$$

with $n_Q(t)$ denoting the quadrature noise component. The output of the phase discriminator is therefore,

$$y(t) = k_p\, m(t) + \frac{n_Q(t)}{A_c}$$

The message signal component of y(t) is equal to $k_p\, m(t)$. Hence, the average output signal power is $k_p^2 P$, where P is the message signal power.

With the post detection low-pass filter following the phase detector restricted to the message bandwidth W, we find that the average output noise power is $2WN_0/A_c^2$.

Hence, the output signal-to-noise ratio of the PM system is

$$(SNR)_0 = \frac{k_p^2 \, P \, A_c^2}{2WN_0}$$

(b) The channel signal-to-noise ratio of the PM system is the same as that of the corresponding FM system. That is,

$$(SNR)_0 = \frac{A_c^2}{2WN_0}$$

The figure of merit of the PM system is therefore equal to k_p^2 P.

For the case of sinusoidal modulation, we have

$$m(t) = A_m \cos(2\pi f_m t)$$

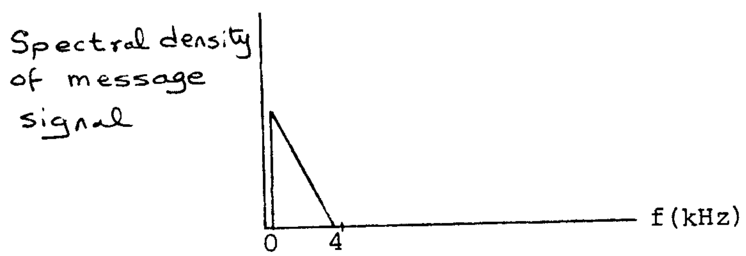
Hence,

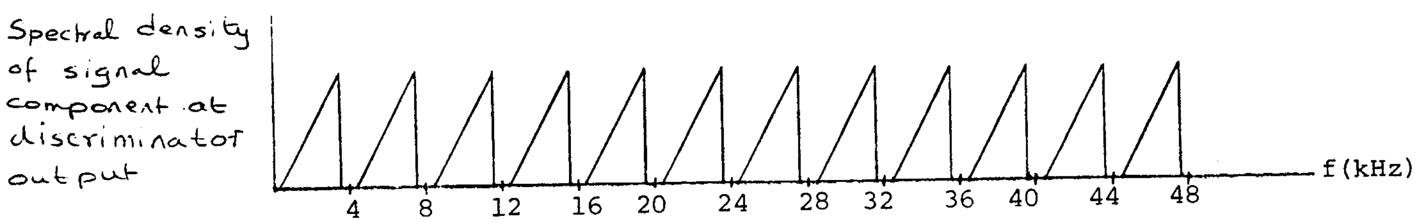
$$P = \frac{A_m^2}{2}$$

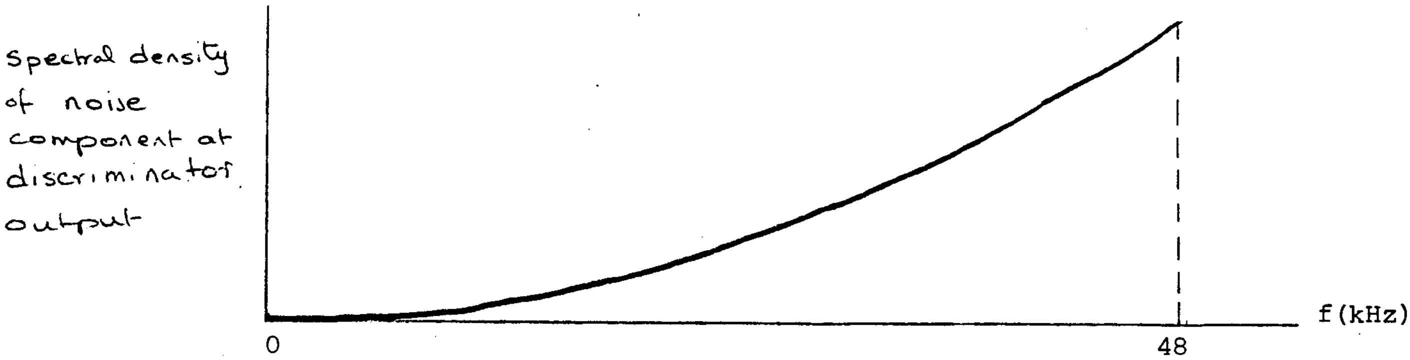
The corresponding value of the figure of merit for a PM system is thus equal to $\frac{1}{2} \beta_p^2$, where $\beta_p = k_p A_m$. On the other hand, the figure of merit for an FM system with sinusoidal modulation is equal to $\frac{3}{2} \beta^2$. We see therefore that for a specified phase deviation, the FM system is 3 times as good as the PM system.

Problem 2.55

(a) The power spectral densities of the original message signal, and the signal and noise components at the frequency discriminator output (for positive frequencies) are illustrated below:







(b) Each SSB modulated wave contains only the lower sideband. Let A_k and kf_0 denote the amplitude and frequency of the carrier used to generate the kth modulated wave, where f_0 = 4 kHz, and k = 1, 2, ..., 12. Then, we find that the kth modulated wave occupies the frequency interval $(k - 1)f_0 \leq |f| \leq kf_0$. We may define this modulated wave by

$$s_k(t) = \frac{A_k}{2} m(t) \cos(2\pi k f_0 t) + \frac{A_k}{2} \hat{m}(t) \sin(2\pi k f_0 t)$$

where m(t) is the original message signal, and $\hat{m}(t)$ is its Hilbert transform. Therefore, the average power of $s_k(t)$ is $A_k^2 P/4$, where P is the mean power of m(t).

We may express the output signal-to-noise ratio for the kth SSB modulated wave as follows:

$$(SNR)_0 = \frac{3A_c^2 k_f^2 (A_k^2 P/4)}{2N_0[k^3 f_0^3 - (k - 1)^3 f_0^3]}$$

$$= \frac{3A_c^2 A_k^2 k_f^2 P}{8N_0 f_0^3 (3k^2 - 3k + 1)}$$

where A_c is the carrier amplitude of the FM wave. For equal signal-to-noise ratios, we must therefore choose the A_k so as to satisfy the condition

$$\frac{A_k^2}{3k^2 - 3k + 1} = \text{constant for } k = 1, 2, \ldots, 12.$$

Problem 2.56

The envelope r(t) and phase ψ(t) of the narrow-band noise n(t) are defined by

$$r(t) = \sqrt{n_I^2(t) + n_Q^2(t)}$$

$$\psi(t) = \tan^{-1}\left(\frac{n_Q(t)}{n_I(t)}\right)$$

For a positive-going click to occur, we therefore require the following:

$n_I(t) \approx -A_c$

$n_Q(t)$ has a small positive value

$$\frac{d}{dt}\tan^{-1}\left(\frac{n_Q(t)}{n_I(t)}\right) > 0$$

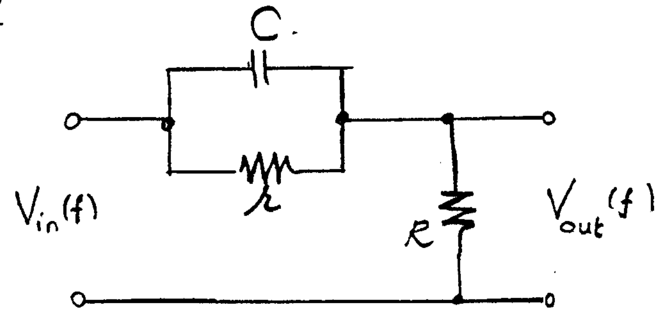
Correspondingly, for a negative-going click to occur, we require

$n_I(t) \approx A_c$

$n_Q(t)$ has a small negative value

$$\frac{d}{dt}\tan^{-1}\left(\frac{n_Q(t)}{n_I(t)}\right) < 0$$

Problem 2.57



Let H(f) be $V_{out}(f)/V_{in}(f)$, or the transfer function of the filter. At low frequencies, the capacitor behaves as an open circuit. Then,

$$H(f) \simeq \frac{R}{r + R} \simeq \frac{R}{r}$$

Thus, the low frequencies of the input are frequency-modulated. At high frequencies, the capacitor behaves as a short circuit in relation to the resistor. Then,

$$H(f) \simeq \frac{R}{R + \frac{1}{j2\pi fC}} \simeq j2\pi fCR \ ,$$

and

$$v_{out}(t) \simeq RC \frac{d}{dt} v_{in}(t)$$

Frequency modulating the derivative of a waveform is equivalent to phase modulating the waveform. Thus, the high frequencies of the input are phase modulated.

Problem 2.58

(a) For the average power of the emphasized signal to be the same as the average power of the original message signal, we must choose the transfer function $H_{pe}(f)$ of the pre-emphasis filter so as to satisfy the relation

$$\int_{-\infty}^{\infty} S_M(f)\,df = \int_{-\infty}^{\infty} |H_{pe}|^2 S_M(f)\,df$$

With

$$S_M(f) = \begin{cases} \dfrac{S_0}{1 + (f/f_0)2}, & -W \le f \le W \\ 0, & \text{elsewhere.} \end{cases}$$

$$H_{pe}(f) = k(1 + \frac{jf}{f_0})$$

we have

$$\int_{-W}^{W} \frac{df}{1 + (f/f_0)^2} = k^2 \int_{-W}^{W} df$$

Solving for k, we get

$$k = \left[\frac{f_0}{W} \tan^{-1}\left(\frac{W}{f_0}\right)\right]^{1/2} \tag{1}$$

(b) The improvement in output signal-to-noise ratio obtained by using pre-emphasis in the transmitter and de-emphasis in the receiver is defined by the ratio

$$D = \frac{2W^3}{3\int_{-W}^{W} f^2 |H_{de}(f)|^2 \, df}$$

$$= \frac{2W^3}{3\int_{-W}^{W} \frac{f^2}{k^2} \frac{df}{1 + (f/f_0)^2}}$$

$$= \frac{k^2 (W/f_0)^3}{3[(W/f_0) - \tan^{-1}(W/f_0)]} \tag{2}$$

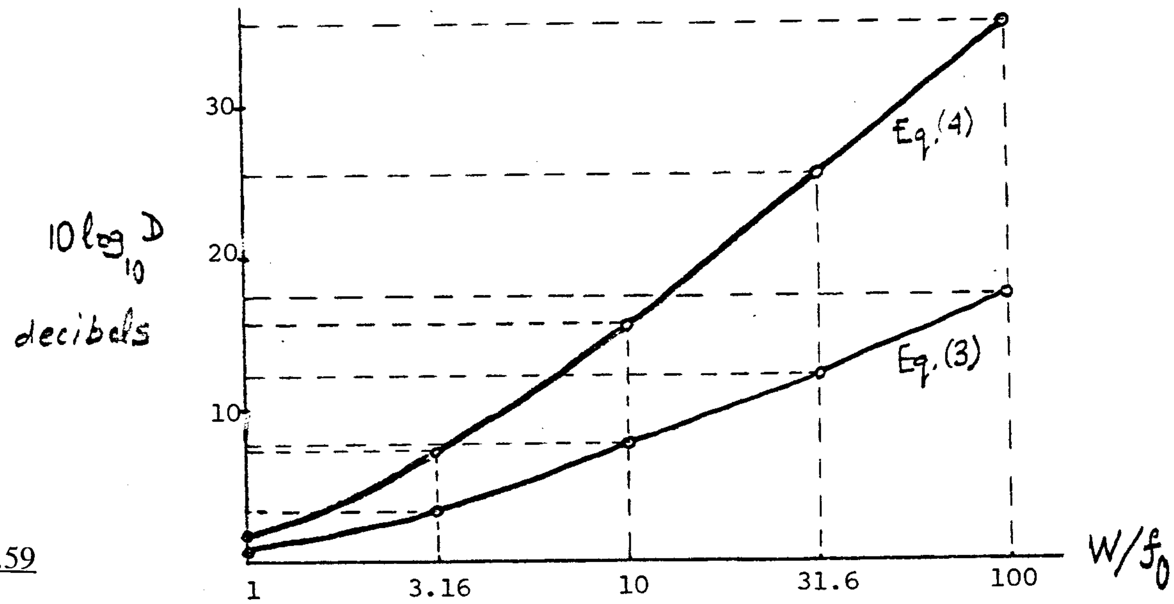
Substituting Eq. (1) in (2), we get

$$D = \frac{(W/f_0)^2 \tan^{-1}(W/f_0)}{3[(W/f_0) - \tan^{-1}(W/f_0)]} \tag{3}$$

This result applies to the case when the rms bandwidth of the FM system is maintained the same with or without pre-emphasis. When, however, there is no such constraint, we find from Example 4 of Chapter 6 that the corresponding value of D is

$$D = \frac{(W/f_0)^3}{3[(w/f_0) - \tan^{-1}(W/f_0)]} \tag{4}$$

In the diagram below, we have plotted the improvement D (expressed in decibels) versus the ratio W/f_0 for the two cases; when there is a transmission bandwidth constraint and when there is no such constraint:



Problem 2.59

In a PM system, the power spectral density of the noise at the phase discriminator output (in the absence of pre-emphasis and de-emphasis) is approximately constant. Therefore, the improvement in output signal-to-noise ratio obtained by using pre-emphasis in the transmitter and de-emphasis in the receiver of a PM system is given by

$$D = \frac{\int_0^W df}{\int_0^W |H_{de}(f)|^2 df}$$

With the transfer function $H_{de}(f)$ of the de-emphasis filter defined by

$$H_{de}(f) = \frac{1}{1 + (jf/f_0)},$$

we find that the corresponding value of D is

$$D = \frac{W}{\int_0^W \frac{df}{1 + (f/f_0)^2}}$$

$$= \frac{W/f_0}{\tan^{-1}(W/f_0)}$$

For the case when W = 15 kHz, f_0 = 2.1 kHz, we find that D = 5, or 7 dB. The corresponding value of the improvement ratio D for an FM system is equal to 13 dB (see Example 4 of Chapter 5). Therefore, the improvement obtained by using pre-emphasis and de-emphasis in a PM system is smaller by an amount equal to 6 dB.

Problem 2.60

Matlab codes

```
% Amplitude demodulation
%problem 2.60, CS: Haykin
% Mathini Sellathurai

clear all
Ac=1;
mue=0.5;
fc=20000;
fm=1000;
ts=1e-5;

% message signal
t=[0:250]*1e-5;
m=sin(2*pi*fm.*t);
plot(t, m)
xlabel('time (s)')
ylabel('Amplitude')
pause

% amplitude modulated signal
u=AM_mod(mue,m,ts,fc);
plot(t,u)
xlabel('time (s)')
ylabel('Amplitude')
pause

% demodulated signal
[t1, dem1]=AM_demod(mue,u,ts,fc);
plot(t1*ts, dem1)
xlabel('time (s)')
ylabel('Amplitude')


axis([0 2.5e-3 0 2])
```

```
function u=AM_mod(mue,m,ts,fc)
% Amplitude modulation
%used in problem 2.60, CS: Haykin
% Mathini Sellathurai
%

t=[0:length(m)-1]*ts;
c=cos(2*pi*fc.*t);
m_n=m/max(abs(m));
u=(1+mue*m_n).*c;
```

```
function [t, env]=AM_demod(mue,m,ts,fc)
% Amplitude demodulation
%used in problem 2.60, CS: Haykin
% Mathini Sellathurai
%

fs=1/ts;
fsofc=round(fs/fc);
n2=length(m);
v=zeros(1,round(n2/fsofc)); % initializing the envelope
R_L=1000; % load
C=0.01e-6; % capacitor


%demodulate the envelope
l=0; v(1)=m(1);
for k=1:fsofc:n2-fsofc
l=l+2;
v(l)=m(k)*exp(-ts/(R_L*C)/fsofc); % discharging
v(l+1)=m(k+fsofc); %charging
end

% envelope
t =0:fsofc/2:(length(v)-1)*fsofc/2;
env=v;
```

Answer to Problem 2.60

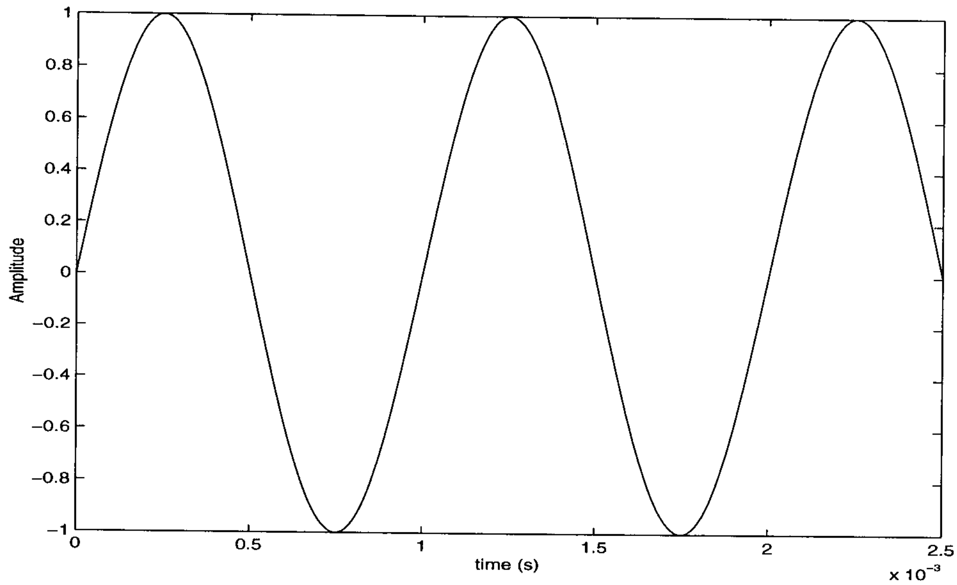


Figure 1: Message signal

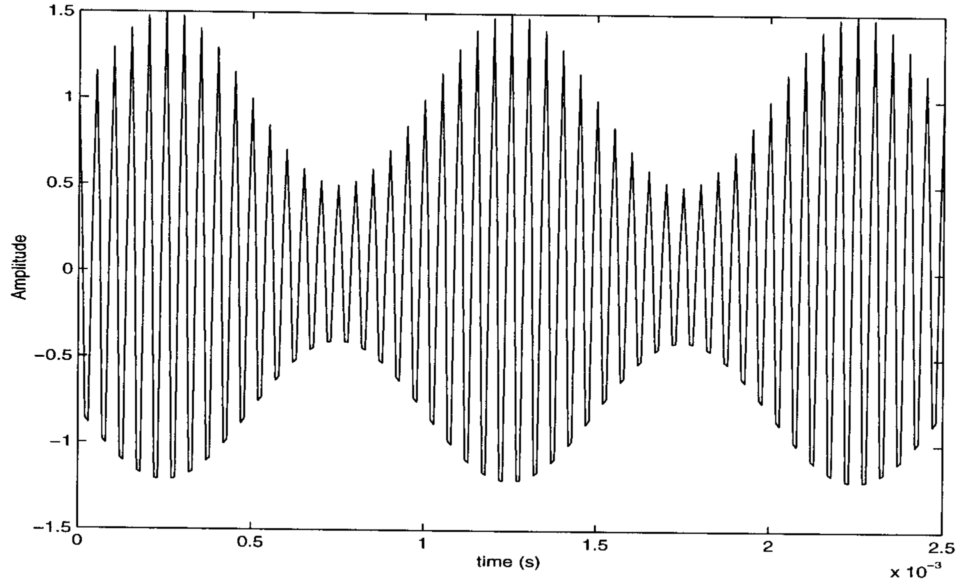


Figure 1: Amplitude modulated signal

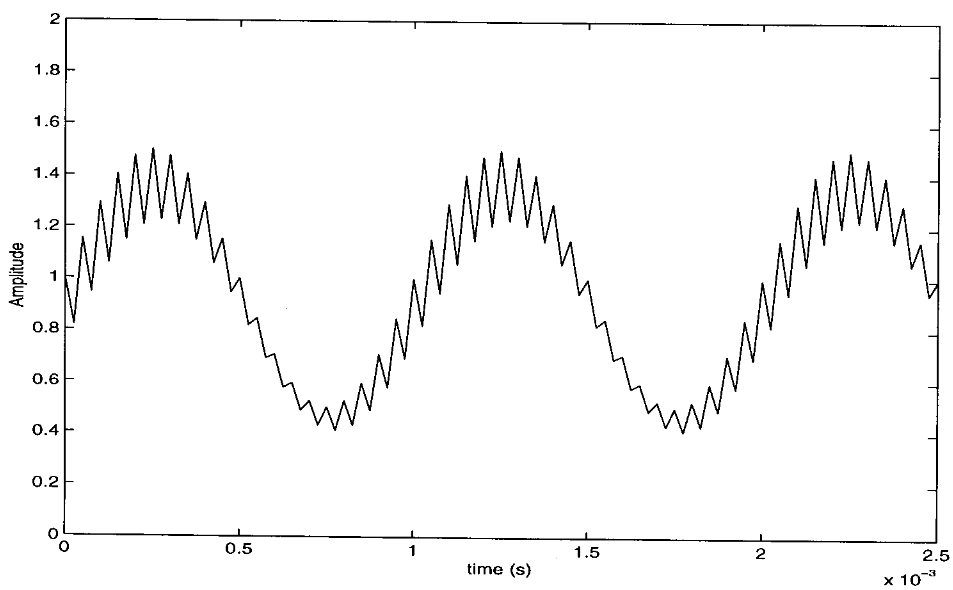


Figure 3: Demodulated signal

Problem 2.61

Matlab codes

```
% Problem 2.61 CS: Haykin
% phase lock loop and cycle slipping
% M. Sellathurai

% time interval
t0=0;tf=25;

% frequency step =0.125 Hz
delf=0.125;
u0=[0 -delf*2*pi];
[t,u]=ode23('lin',[t0 tf],u0); plot(t,u(:,2)/2/pi+delf);
xlabel('Time (s)')
ylabel('f_i (t), (Hz)')
pause

% frequency step =0.51 Hz
delf=0.5;
u0=[0 -delf*pi*2]';
[t,u]=ode23('lin',[t0 tf],u0); plot(t,u(:,2)/2/pi+delf);
xlabel('Time (s)');
ylabel('f_i (t), (Hz)');
pause;

% frequency step =7/12 Hz
delf=7/12;
u0=[0 -delf*pi*2]';
[t,u]=ode23('lin',[t0 tf],u0); plot(t,u(:,2)/2/pi+delf);
xlabel('Time (s)');
ylabel('f_i (t), (Hz)');
pause;

% frequency step =2/3 Hz
delf=2/3;
u0=[0 -delf*pi*2]';
[t,u]=ode23('lin',[t0 tf],u0); plot(t,u(:,2)/2/pi+delf);
xlabel('Time (s)');
ylabel('f_i (t), (Hz)');
```

```
function uprim =lin(t,u)
% used in Problem 2.61, CS: Haykin
% PLL
% Transfer function (1+as)/(1+bs),
% gain K=50/2/pi,
% natural frequency 1/2/pi
% damping 0.707
% Mathini Sellathurai

uprim(1)=u(2);
uprim(2)=-(1/50+1.2883*cos(u(1)))*u(2)-sin(u(1));
uprim=uprim';
```

Answer to Problem 2.61

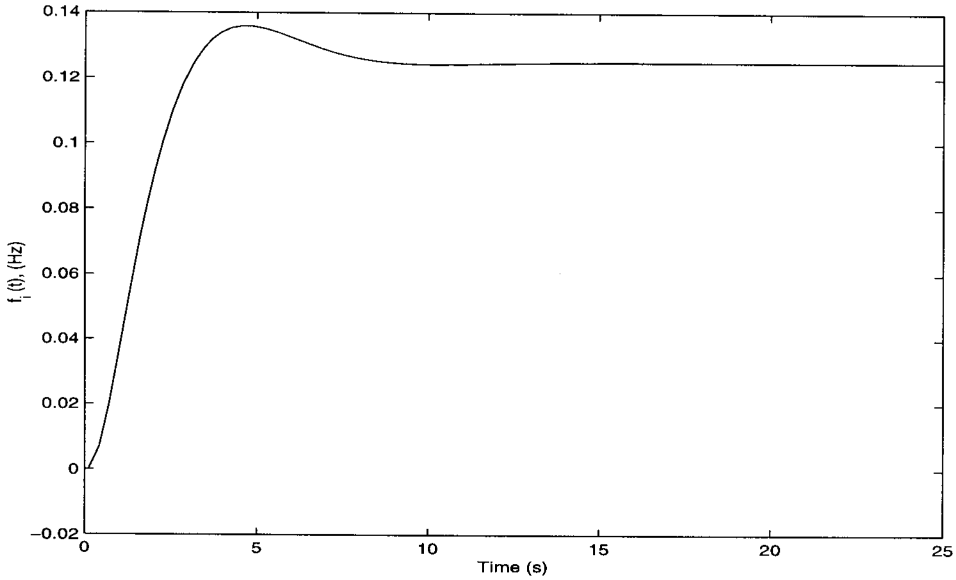


Figure 1: Variation in the instantaneous frequency of the PLL's voltage controlled oscillator for varying frequency step Δ f. (a) Δ f = 0.125 Hz

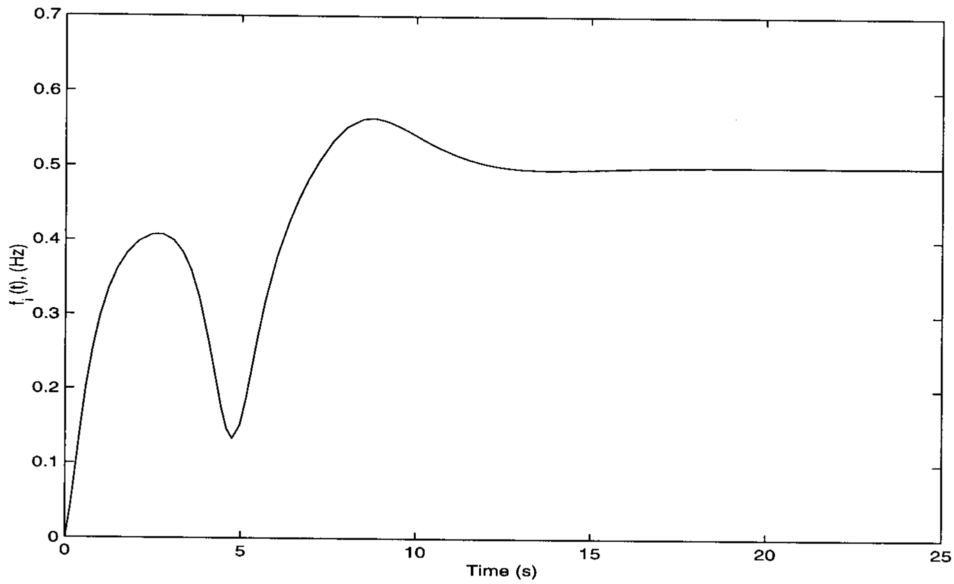


Figure 2: (b) Δ f = 0.5 Hz

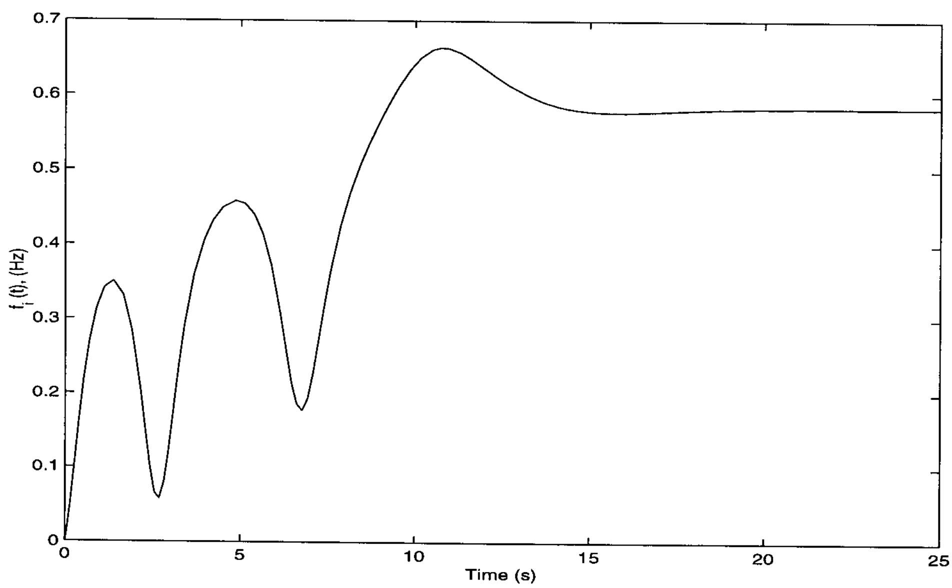


Figure 3: (b) Δ f = 7/12 Hz

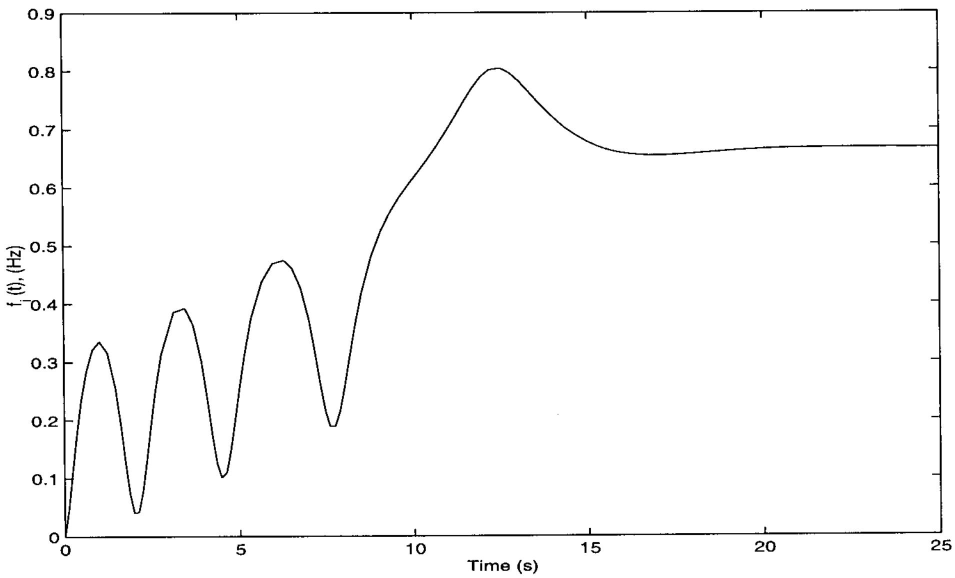


Figure 4: (b) Δ f = 2/3 Hz