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Sine and Cosine Transforms

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3.1 Introduction

Transforms with cosine and sine functions as the transform kernels represent an important area of analysis. It is based on the so-called half-range expansion of a function over a set of cosine or sine basis functions. Because the cosine and the sine kernels lack the nice properties of an exponential kernel, many of the transform properties are less elegant and more involved than the corresponding ones for the Fourier transform kernel. In particular, the convolution property, which is so important in many applications, will be much more complex.

Despite these basic mathematical limitations, sine and cosine transforms have their own areas of applications. In spectral analysis of real sequences, in solutions of some boundary value problems, and in transform domain processing of digital signals, both cosine and sine transforms have shown their special applicability. In particular, the discrete versions of these transforms have found favor among the digital signal-processing community. Many data compression techniques now employ, in one way or another, the discrete cosine transform (DCT), which has been found to be asymptotically equivalent to the optimal Karhunen-Loeve transform (KLT) for signal decorrelation.

In this chapter, the basic properties of cosine and sine transforms are presented, together with some selected transforms. To show the versatility of these transforms, several applications are discussed. Computational algorithms are also presented. The chapter ends with a table of sine and cosine transforms, which is not meant to be exhaustive. The reader is referred to the References for more details and for more exhaustive listings of the cosine and sine transforms.

3.2. The Fourier Cosine Transform (FCT)

3.2.1 Definitions and Relations to the Exponential Fourier Transforms

Given a real- or complex-valued function $f(t)$, which is defined over the positive real line $t \geq 0$, for $\omega \geq 0$, the Fourier cosine transform of $f(t)$ is defined as

$$F_c(\omega) = \int_0^\infty f(t) \cos \omega t \, dt, \quad \omega \geq 0, \tag{3.2.1}$$

subject to the existence of the integral. The definition is sometimes more compactly represented as an operator $\mathscr{F}_c$ applied to the function $f(t)$, so that

$$\mathscr{F}_c\left[f(t)\right] = F_c(\omega) = \int_0^\infty f(t) \cos \omega t \, dt. \tag{3.2.2}$$

The subscript c is used to denote the fact that the kernel of the transformation is a cosine function. The unit normalization constant used here provides for a definition for the inverse Fourier cosine transform, given by

$$\mathscr{F}_c^{-1}\left[F_c(\omega)\right] = \frac{2}{\pi} \int_0^\infty F_c(\omega) \cos \omega t \, d\omega, \quad t \geq 0, \tag{3.2.3}$$

again subject to the existence of the integral used in the definition. The functions $f(t)$ and $F_c(\omega)$, if they exist, are said to form a Fourier cosine transform pair.

Because the cosine function is the real part of an exponential function of purely imaginary argument, that is,

$$\cos(\omega t) = \mathrm{Re}\left[e^{j\omega t}\right] = \frac{1}{2}\left[e^{j\omega t} + e^{-j\omega t}\right], \tag{3.2.4}$$

it is easy to understand that there exists a very close relationship between the Fourier transform and the cosine transform. To see this relation, consider an even extension of the function $f(t)$ defined over the entire real line so that

$$f_e(t) = f(|t|), \quad t \in R. \tag{3.2.5}$$

Its Fourier transform is defined as

$$\mathscr{F}\left[f_e(t)\right] = \int_{-\infty}^\infty f_e(t) e^{-j\omega t} dt, \quad \omega \in R. \tag{3.2.6}$$

The integral in (3.2.6) can be evaluated in two parts over (–∞, 0] and [0, ∞). Then using (3.2.5) and changing the integrating variable in the (–∞, 0] integral from t to $-t$, we have

$$\mathcal{F}\left[f_e(t)\right]=\left[\int_0^{\infty} f(t)e^{-j\omega t}dt+\int_0^{\infty} f(t)e^{j\omega t}dt\right]=2\int_0^{\infty} f(t)\cos\omega t\,dt\,,$$

by (3.2.4), and thus

$$\mathcal{F}[f_e(t)] = 2\mathcal{F}_c[f(t)], \quad \text{if } f_e(t) = f(|t|)\,. \tag{3.2.7}$$

Many of the properties of the Fourier cosine transforms can be derived from the properties of Fourier transforms of symmetric, or even, functions. Some of the basic properties and operational rules are discussed in Section 3.2.2.

3.2.2 Basic Properties and Operational Rules

1. *Inverse Transformation*: As stated in (3.2.3), the inverse transformation is exactly the same as the forward transformation except for the normalization constant. This leads to the so-called Fourier cosine integral formula, which states that

$$\begin{aligned} f(t)&=\frac{2}{\pi}\int_0^{\infty} F_c(\omega)\cos\omega t\,d\omega \\ &=\frac{2}{\pi}\int_0^{\infty}\left[\int_0^{\infty} f(\tau)\cos\omega\tau\,d\tau\right]\cos\omega t\,d\omega\,. \end{aligned} \tag{3.2.8}$$

   The sufficient conditions for the inversion formula (3.2.3) are that $f(t)$ be absolutely integrable in [0, ∞) and that $f'(t)$ be piece-wise continuous in each bounded subinterval of [0, ∞). In the range where the function $f(t)$ is continuous, (3.2.8) represents f. At the point t_0 where $f(t)$ has a jump discontinuity, (3.2.8) converges to the mean of $f(t_0 + 0)$ and $f(t_0 - 0)$, that is,

$$\frac{2}{\pi}\int_0^{\infty}\left[\int_0^{\infty} f(\tau)\cos(\omega\tau)\,d\tau\right]\cos(\omega t_0)d\omega=\frac{1}{2}\left[f(t_0+0)+f(t_0-0)\right]. \tag{3.2.8$'$}$$

2. *Transforms of Derivatives*: It is easy to show, because of the Fourier cosine kernel, that the transforms of even-order derivatives are reduced to multiplication by even powers of the conjugate variable ω, much as in the case of the Laplace transforms. For the second-order derivative, using integration by parts, we can show that,

$$\begin{aligned} \mathcal{F}_c\left[f''(t)\right]&=\int_0^{\infty} f''(t)\cos(\omega t)\,dt \\ &=-f'(0)-\omega^2\int_0^{\infty} f(t)\cos\omega t\,dt \\ &=-\omega^2 F_c(\omega)-f'(0) \end{aligned} \tag{3.2.9}$$

   where we have assumed that $f(t)$ and $f'(t)$ vanish as $t \to \infty$. These form the sufficient conditions for (3.2.9) to be valid. As the transform is applied to higher order derivatives, corresponding conditions for higher derivatives of f are required for the operational rule to be valid. Here, we also assume that the function $f(t)$ and its derivative $f'(t)$ are continuous everywhere in $[0, \infty)$. If $f(t)$ and $f'(t)$ have a jump discontinuity at t_0 of magnitudes d and d' respectively, (3.2.9) is modified to

$$\mathcal{F}_c[f''(t)] = -\omega^2 F_c(\omega) - f'(0) - \omega d \sin \omega t_0 - d' \cos \omega t_0 \tag{3.2.10}$$

Higher even-order derivatives of functions with jump continuities have similar operational rules that can be easily generalized from (3.2.10). For example, the Fourier cosine transform of the fourth-order derivative is

$$\mathcal{F}_c[f^{(iv)}(t)] = \omega^4 F_c(\omega) + \omega^2 f'(0) - f'''(0) \tag{3.2.11}$$

if $f(t)$ is continuous to order three everywhere in $[0, \infty)$, and f, f', and f'' vanish as $t \to \infty$. If $f(t)$ has a jump discontinuity at t_0 to order three of magnitudes d, d', d'', and d''', then (3.2.11) is modified to

$$\begin{aligned} \mathcal{F}_c[f^{(iv)}(t)] = \omega^4 F_c(\omega) + \omega^2 f'(0) - f'''(0) + \omega^3 d \sin \omega t_0 \\ + \omega^2 d' \cos \omega t_0 - \omega d'' \sin \omega t_0 - d''' \cos \omega t_0 \end{aligned} \tag{3.2.12}$$

Here, and in (3.2.10), we have defined the magnitudes of the jump discontinuity at t_0 as

$$\begin{aligned} d = f(t_0 + 0) - f(t_0 - 0); \quad d' = f'(t_0 + 0) - f'(t_0 - 0); \\ d'' = f''(t_0 + 0) - f''(t_0 - 0); \quad d''' = f'''(t_0 + 0) - f'''(t_0 - 0). \end{aligned} \tag{3.2.13}$$

For derivatives of odd order, the operational rules require the definition for the Fourier sine transform, given in Section 3.3. For example, the Fourier cosine transform of the first order derivative is given by

$$\begin{aligned} \mathcal{F}_c\left[f'(t)\right] = \int_0^\infty f'(t) \cos \omega t \, dt = -f(0) + \omega \int_0^\infty f(t) \sin \omega t \, dt \\ = \omega \mathcal{F}_s\left[f(t)\right] - f(0) = \omega F_s(\omega) - f(0), \end{aligned} \tag{3.2.14}$$

if f vanishes as $t \to \infty$, and where the operator $\mathcal{F}_s$ and the function $F_s(\omega)$ are defined in (3.3.1). When $f(t)$ has a jump discontinuity of magnitude d at $t = t_0$, (3.2.14) is modified to

$$\mathcal{F}_c[f'(t)] = \omega F_s(\omega) - f(0) - d \cos(\omega t_0). \tag{3.2.15}$$

Generalization to higher odd-order derivatives with jump discontinuities is similar to that for even-order derivatives in (3.2.12).

3. *Scaling*: Scaling in the t domain translates directly to scaling in the ω domain. Expansion by a factor of a in t results in the contraction by the same factor in ω, together with a scaling down of the magnitude of the transform by the factor a. Thus, as we can show,

$$\begin{aligned} \mathcal{F}_c\left[f(at)\right] = \int_0^\infty f(at) \cos \omega t \, dt = \frac{1}{a} \int_0^\infty f(\tau) \cos \frac{\omega \tau}{a} d\tau, \quad \text{by letting } \tau = at \\ = \frac{1}{a} F_c\left(\frac{\omega}{a}\right), \quad a > 0. \end{aligned} \tag{3.2.16}$$

4. *Shifting*:
   (a) Shifting in the t-domain: The shift-in-t property for the cosine transform is somewhat less direct compared with the exponential Fourier transform for two reasons. First, a shift to the left will require extending the definition of the function $f(t)$ onto the negative real line. Secondly, a shift-in-t in the transform kernel does not result in a constant phase factor as in the case of the exponential kernel.

   If $f_e(t)$ is defined as the even extension of the function $f(t)$ such that $f_e(t) = f(|t|)$, and if $f(t)$ is piece-wise continuous and absolutely integrable over $[0, \infty)$, then

$$\mathcal{F}_c\left[f_e(t+a)+f_e(t-a)\right]=\int_0^\infty \left[f_e(t+a)+f_e(t-a)\right]\cos\omega t\,dt$$

$$=\int_a^\infty f_e(\tau)\cos\omega(\tau+a)\,d\tau$$

$$+\int_{-a}^\infty f_e(\tau)\cos\omega(\tau-a)\,d\tau\,.$$

   By expanding the compound cosine functions and using the fact that the function $f_e(\tau)$ is even, these combine to give:

$$\mathcal{F}_c[f_e(t+a)+f_e(t-a)] = 2F_c(\omega)\cos a\omega\,, \qquad a>0. \tag{3.2.17}$$

   This is sometimes called the kernel-product property of the cosine transform. In terms of the function $f(t)$, it can be written as:

$$\mathcal{F}_c[f(t+a)+f(|t-\mathrm{a}|)] = 2F_c(\omega)\cos a\omega\,. \tag{3.2.18}$$

   Similarly, the kernel-product $2F_c(\omega)\sin(a\omega)$ is related to the Fourier sine transform:

$$\mathcal{F}_s[f(|t-\mathrm{a}|)-f(t+a)] = 2F_c(\omega)\sin a\omega\,, \quad a>0. \tag{3.2.19}$$

   (b) Shifting in the ω-domain:
   To consider the effect of shifting in ω by the amount of $\beta(>0)$, we examine the following,

$$F_c(\omega+\beta)=\int_0^\infty f(t)\cos(\omega+\beta)t\,dt$$

$$=\int_0^\infty f(t)\cos\beta t\cos\omega t\,dt-\int_0^\infty f(t)\sin\beta t\sin\omega t\,dt \tag{3.2.20}$$

$$=\mathcal{F}_c\left[f(t)\cos\beta t\right]-\mathcal{F}_s\left[f(t)\sin\beta t\right].$$

   Similarly,

$$F_c(\omega-\beta) = \mathcal{F}_c[f(t)\cos\beta t] + \mathcal{F}_s[f(t)\sin\beta t]. \tag{3.2.20$'$}$$

   Combining (3.2.20) and (3.2.20′) produces a shift-in-ω operational rule involving only the Fourier cosine transform as

$$\mathcal{F}_c\left[f(t)\cos\beta t\right]=\frac{1}{2}\left[F_c(\omega+\beta)+F_c(\omega-\beta)\right]. \tag{3.2.21}$$

More generally, for $a, \beta > 0$, we have,

$$\mathcal{F}_c\left[f(at)\cos\beta t\right] = \frac{1}{2a}\left[F_c\left(\frac{\omega+\beta}{a}\right) + F_c\left(\frac{\omega-\beta}{a}\right)\right]. \tag{3.2.22}$$

Similarly, we can easily derive:

$$\mathcal{F}_c\left[f(at)\sin\beta t\right] = \frac{1}{2a}\left[F_s\left(\frac{\omega+\beta}{a}\right) - F_s\left(\frac{\omega-\beta}{a}\right)\right]. \tag{3.2.22$'$}$$

5. *Differentiation in the ω domain*: Similar to differentiation in the t domain, the transform operation reduces a differentiation operation into multiplication by an appropriate power of the conjugate variable. In particular, even-order derivatives in the ω domain are transformed as:

$$F_c^{(2n)}(\omega) = \mathcal{F}_c\left[(-1)^n t^{2n} f(t)\right]. \tag{3.2.23}$$

We show here briefly, the derivation for $n = 1$:

$$\begin{aligned} F_c^{(2)}(\omega) &= \frac{d^2}{d\omega^2}\int_0^\infty f(t)\cos\omega t\,dt \\ &= \int_0^\infty f(t)\frac{d^2}{d\omega^2}\cos\omega t\,dt \\ &= \int_0^\infty f(t)(-1)t^2\cos\omega t\,dt \\ &= \mathcal{F}_c\left[(-1)t^2 f(t)\right]. \end{aligned}$$

For odd orders, these are related to Fourier sine transforms

$$F_c^{(2n+1)}(\omega) = \mathcal{F}_s\left[(-1)^{n+1} t^{2n+1} f(t)\right]. \tag{3.2.24}$$

In both (3.2.23) and (3.2.24), the existence of the integrals in question is assumed. This means that $f(t)$ should be piece-wise continuous and that $t^{2n}f(t)$ and $t^{2n+1}f(t)$ should be absolutely integrable over $[0, \infty)$.

6. *Asymptotic behavior*: When the function $f(t)$ is piece-wise continuous and absolutely integrable over the region $[0, \infty)$, the Reimann-Lebesque theorem for Fourier series* can be invoked to provide the following asymptotic behavior of its cosine transform:

*The Reimann-Lebesque theorem states that if a function $f(t)$ is piece-wise continuous over an interval $a < t < b$, then

$$\lim_{\gamma\to\infty}\int_a^b f(t)\cos\gamma t\,dt = \lim_{\gamma\to\infty}\int_a^b f(t)\sin\gamma t\,dt = 0.$$

$$\lim_{\omega\to\infty} F_c(\omega) = 0 . \tag{3.2.25}$$

7. *Integration*:
   (a) Integration in the t domain:
   Integration in the t domain is transformed to division by the conjugate variable, very similar to the cases of Laplace transforms and Fourier transforms, except the resulting transform is a Fourier sine transform. Thus,

$$\mathcal{F}_c\left[\int_t^\infty f(\tau)\,d\tau\right] = \int_0^\infty \int_t^\infty f(\tau)\,d\tau \cos\omega t\,dt$$
$$= \int_0^\infty \left[\int_0^\tau \cos\omega t\,dt\right] f(\tau)\,d\tau$$

   by reversing the order of integration. The inner integral results in a sine function and is the kernel for the Fourier sine transform. Therefore,

$$\mathcal{F}_c\left[\int_t^\infty f(\tau)\,d\tau\right] = \frac{1}{\omega}\mathcal{F}_s\left[f(\tau)\right] = \frac{1}{\omega}F_s(\omega). \tag{3.2.26}$$

   Here, again, $f(t)$ is subject to the usual sufficient conditions of being piece-wise continuous and absolutely integrable in $[0, \infty)$.
   (b) Integration in the ω domain:
   A similar and symmetric relation exists for integration in the ω-domain.

$$\mathcal{F}_s^{-1}\left[\int_\omega^\infty F_c(\beta)\,d\beta\right] = -\frac{1}{t}f(t). \tag{3.2.27}$$

   Note that the integral transform inversion is of the Fourier sine type instead of the cosine type. Also the aysmptotic behavior of $F_c(\omega)$ has been invoked.
8. *The convolution property*: Let $f(t)$ and $g(t)$ be defined over $[0, \infty)$ and satisfy the sufficiency condition for the existence of F_c and G_c. If $f_e(t) = f(|t|)$ and $g_e(t) = g(|t|)$ are the even extensions of f and g, respectively, over the entire real line, then the convolution of f_e and g_e is given by:

$$f_e * g_e = \int_{-\infty}^\infty f_e(\tau)\,g_e(t-\tau)\,d\tau \tag{3.2.28}$$

   where $*$ has been used to denote the convolution operation. It is easy to see that in terms of f and g, we have:

$$f_e * g_e = \int_0^\infty f(\tau)\left[g(t+\tau) + g(|t-\tau|)\right]d\tau \tag{3.2.29}$$

   which is an even function. Applying the exponential Fourier transform on both sides and using (3.2.7) and the convolution property of the exponential Fourier transform, we obtain the convolution property for the cosine transform:

$$2F_c(\omega)G_c(\omega) = \mathscr{F}_c\left\{\int_0^\infty f(\tau)\left[g(t+\tau)+g(|t-\tau|)\right]d\tau\right\}. \tag{3.2.30}$$

In a similar way, the cosine transform of the convolution of odd extended functions is related to the sine transforms. Thus,

$$2F_s(\omega)G_s(\omega) = \mathscr{F}_c\left\{\int_0^\infty f(\tau)\left[g(t+\tau)+g_o(t-\tau)\right]d\tau\right\}. \tag{3.2.31}$$

where

$$\begin{aligned} g_o(t) &= g(t) && \text{for } t>0, \\ &= -g(-t) && \text{for } t<0, \end{aligned} \tag{3.2.32}$$

is defined as the odd extension of the function $g(t)$.

3.2.3 Selected Fourier Cosine Transforms

In this section, the Fourier cosine transforms of some typical functions are given. Most are selected for their simplicity and application. For a more complete listing of cosine transforms, see Section 3.7 where a more extensive table is provided.

3.2.3.1 FCT of Algebraic Functions

1. *The unit rectangular function*:

$$f(t) = U(t) - U(t-a), \quad \text{where } U(t) = 0 \quad \text{for } t<0 \\ \qquad\qquad\qquad\qquad\qquad\qquad = 1 \quad \text{for } t>0, \tag{3.2.33}$$

is the Heaviside unit step function.

$$\mathscr{F}_c\left[f(t)\right] = \int_0^a \cos\omega t\,dt = \frac{1}{\omega}\sin\omega a. \tag{3.2.34}$$

2. *The unit height tent function*:

$$\begin{aligned} f(t) &= t/a && 0<t<a, \\ &= (2a-t)/a && a<t<2a, \\ &= 0 && t>2a. \end{aligned}$$

$$\begin{aligned} \mathscr{F}_c\left[f(t)\right] &= \int_0^a \frac{t}{a}\cos\omega t\,dt + \int_a^{2a}\frac{2a-t}{a}\cos\omega t\,dt \\ &= \frac{1}{a\omega^2}\left[2\cos a\omega - \cos 2a\omega - 1\right]. \end{aligned} \tag{3.2.35}$$

3. *Delayed inverse*:

$$f(t) = U(t - a)/t.$$

$$\mathcal{F}_c\left[f(t)\right] = \int_a^\infty \frac{1}{t}\cos\omega t\, dt = \int_{a\omega}^\infty \frac{1}{\tau}\cos\tau\, d\tau = -\text{Ci}(a\omega), \tag{3.2.36}$$

where $\text{Ci}(y) = -\int_y^\infty \frac{1}{\tau}\cos\tau\, d\tau$ is defined as the cosine integral function.

4. *The inverse square root*:

$$f(t) = 1/\sqrt{t},$$

$$\mathcal{F}_c\left[f(t)\right] = \int_0^\infty \frac{1}{\sqrt{t}}\cos\omega t\, dt = \sqrt{\frac{\pi}{2\omega}}. \tag{3.2.37}$$

(3.2.27) is obtained by letting $t = z^2$, and considering the integral,

$$2\int \cos z^2 dz$$

in the complex plane (see Appendix 1). Using contour integration around a pie-shape region with angle $\pi/4$, the result is obtained directly from the identity:

$$\int_0^\infty e^{jt^2} dt = \frac{1}{2}\sqrt{\frac{\pi}{2}}(1+j).$$

5. *Inverse linear function*:

$$f(t) = (\alpha + t)^{-1} \qquad \left|\arg(\alpha)\right| < \pi.$$

$$\begin{aligned}\mathcal{F}_c\left[f(t)\right] &= \int_0^\infty (\alpha+t)^{-1}\cos\omega t\, dt \\ &= -\cos\alpha\omega\,\text{Ci}(\alpha\omega) - \sin\alpha\omega\,\text{si}(\alpha\omega).\end{aligned} \tag{3.2.38}$$

(3.2.38) is obtained by shifting the integrating variable to $\alpha + t$, and then expanding the compound cosine function. Here, si(y) is related to the sine integral function Si(y), and is defined as:

$$\begin{aligned}\text{si}(y) &= -\int_y^\infty \frac{\sin x}{x} dx \\ &= \int_0^y \frac{\sin x}{x} dx - \int_0^\infty \frac{\sin x}{x} dx = \text{Si}(y) - (\pi/2).\end{aligned} \tag{3.2.39}$$

6. *Inverse quadratic functions*:
(a) $f(t) = (\alpha^2 + t^2)^{-1} \qquad \text{Re}(\alpha) > 0.$

$$\mathcal{F}_c\left[f(t)\right] = \int_0^\infty \left(\alpha^2 + t^2\right)^{-1} \cos\omega t\,dt$$
$$= \frac{\pi}{2\alpha} e^{-\alpha\omega}, \tag{3.2.40}$$

which is obtained also by a properly chosen contour integration over the upper half-plane.

(b) $f(t) = (a^2 - t^2)^{-1}$ $\qquad a > 0,$

$$\mathcal{F}_c\left[f(t)\right] = \text{P.V.}\int_0^\infty \left(a^2 + t^2\right)^{-1} \cos\omega t\,dt$$
$$= \frac{\pi}{2a} \sin a\omega \tag{3.2.41}$$

where "P.V." stands for "principal value" and the integral can be obtained by a proper contour integration in the complex plane.

(c) $f(t) = \dfrac{\beta}{\beta^2 + (\alpha - t)^2} + \dfrac{\beta}{\beta^2 + (\alpha + t)^2}$ $\qquad \text{Im}|\alpha| < \text{Re}(\beta),$

$$\mathcal{F}_c\left[f(t)\right] = \int_0^\infty \left[\frac{\beta}{\beta^2 + (\alpha - t)^2} + \frac{\beta}{\beta^2 + (\alpha + t)^2}\right] \cos\omega t\,dt$$
$$= \pi \cos\alpha\omega\; e^{-\beta\omega} \tag{3.2.42}$$

where the integral can be obtained easily by considering a shift in t, applied to the result in (3.2.40).

(d) $f(t) = \dfrac{\alpha - t}{\beta^2 + (\alpha - t)^2} + \dfrac{\alpha + t}{\beta^2 + (\alpha + t)^2}$ $\qquad \text{Im}|\alpha| < \text{Re}(\beta),$

$$\mathcal{F}_c\left[f(t)\right] = \int_0^\infty \left[\frac{\alpha - t}{\beta^2 + (\alpha - t)^2} + \frac{\alpha + t}{\beta^2 + (\alpha + t)^2}\right] \cos\omega t\,dt$$
$$= \pi \sin\alpha\omega\; e^{-\beta\omega} \tag{3.2.43}$$

which can be considered as the imaginary part of the contour integral needed in (3.2.42) when α and β are real and positive.

3.2.3.2 FCT of Exponential and Logarithmic Functions

1. $f(t) = e^{-\alpha t}$ $\qquad \text{Re}(\alpha) > 0.$

$$\mathcal{F}_c\left[f(t)\right] = \int_0^\infty e^{-\alpha t} \cos\omega t\,dt = \frac{\alpha}{\alpha^2 + \omega^2} \tag{3.2.44}$$

which is identical to the Laplace transform of cos ωt.

2. $f(t)=\frac{1}{t}\left[e^{-\beta t}-e^{-\alpha t}\right]$ $\qquad$ $\mathrm{Re}(\alpha), \mathrm{Re}(\beta)>0$.

$$\begin{aligned}\mathcal{F}_c\left[f(t)\right] &= \int_0^\infty \frac{1}{t}\left[e^{-\beta t}-e^{-\alpha t}\right]\cos\omega t\, dt \\ &= \frac{1}{2}\ln\left(\frac{\alpha^2+\omega^2}{\beta^2+\omega^2}\right).\end{aligned} \tag{3.2.45}$$

The result is easily obtained using the integration property of the Laplace transform in the phase plane.

3. $f(t) = e^{-\alpha t2}$ $\qquad$ $\mathrm{Re}(\alpha) > 0$.

$$\begin{aligned}\mathcal{F}_c\left[f(t)\right] &= \int_0^\infty \frac{1}{t}e^{-\alpha t^2}\cos\omega t\, dt \\ &= \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\, e^{-\omega^2\ 4\alpha}\end{aligned} \tag{3.2.46}$$

This is easily seen as the result of the exponential Fourier transform of a Gaussian distribution.

4. $f(t) = \ln t[1 - U(t-1)]$

$$\begin{aligned}\mathcal{F}_c\left[f(t)\right] &= \int_0^1 \ln t\cos\omega t\, dt \\ &= -\frac{1}{\omega}\int_0^\omega \frac{\sin\tau}{\tau}d\tau = -\frac{1}{\omega}\mathrm{Si}(\omega).\end{aligned} \tag{3.2.47}$$

The result is obtained by integration by parts and a change of variables. The function $\mathrm{Si}(\omega)$ is defined as the sine integral function given by:

$$\mathrm{Si}(y) = \int_0^y \frac{\sin x}{x}dx. \tag{3.2.48}$$

5. $f(t)=\dfrac{\ln\beta t}{\left(t^2+\alpha^2\right)}$ $\qquad$ $\mathrm{Re}(\alpha)>0$

$$\begin{aligned}\mathcal{F}_c\left[f(t)\right] &= \int_0^\infty \frac{\ln\beta t}{\left(t^2+\alpha^2\right)}\cos\omega t\, dt \\ &= \frac{\pi}{4\alpha}\left\{2e^{-\alpha\omega}\ln(\alpha\beta)+e^{\alpha\omega}\mathrm{Ei}(-\alpha\omega)-e^{-\alpha\omega}\overline{\mathrm{Ei}}(\alpha\omega)\right\}\end{aligned} \tag{3.2.49}$$

where $\mathrm{Ei}(y)$ is the exponential integral function defined by,

$$\mathrm{Ei}(y) = -\int_{-y}^{\infty} \frac{e^{-t}}{t}\,dt, \quad |\arg(y)| < \pi,$$

and

$$\overline{\mathrm{Ei}}(y) = (1/2)\left[\mathrm{Ei}(y + j0) + \mathrm{Ei}(y - j0)\right]. \tag{3.2.50}$$

The integral in (3.2.49) is evaluated using contour integration.

6. $f(t) = \ln\left|\dfrac{t+a}{t-a}\right|, \qquad a > 0.$

$$\begin{aligned} \mathscr{F}_c\left[f(t)\right] &= \text{P.V.}\int_0^{\infty} \ln\left|\frac{t+a}{t-a}\right| \cos\omega t\,dt \\ &= \frac{2}{\omega}\left[\mathrm{si}(a\omega)\cos a\omega + \mathrm{ci}(a\omega)\sin a\omega\right] \end{aligned} \tag{3.2.51}$$

where $\mathrm{si}(y)$ and $\mathrm{ci}(y) = -\mathrm{Ci}(y)$ are defined in (3.2.39) and (3.2.36), respectively. The result is obtained through integration by parts, and manifests the shift property of the cosine transform.

3.2.3.3 FCT of Trigonometric Functions

1. $f(t) = \dfrac{\sin at}{t} \qquad a > 0.$

$$\begin{aligned} \mathscr{F}_c\left[f(t)\right] &= \int_0^{\infty} \frac{\sin at}{t} \cos\omega t\,dt \\ &= \pi/2 \quad \text{if } \omega < a, \\ &= \pi/4 \quad \text{if } \omega = a, \\ &= 0 \quad\;\;\; \text{if } \omega > a. \end{aligned} \tag{3.2.52}$$

The result is obtained easily after some algebraic manipulations. It is, however, better understood as the result of the inverse Fourier transform of a sinc function, which is simply a rectangular window function, as is evident in (3.2.52).

2. $f(t) = e^{-\beta t}\sin at, \qquad a, \mathrm{Re}(\beta) > 0.$

$$\begin{aligned} \mathscr{F}_c\left[f(t)\right] &= \int_0^{\infty} e^{-\beta t}\sin at\cos\omega t\,dt \\ &= \frac{1}{2}\left[\frac{a+\omega}{\beta^2 + (a+\omega)^2} + \frac{a-\omega}{\beta^2 + (a-\omega)^2}\right] \end{aligned} \tag{3.2.53}$$

The result can be easily understood as the Laplace transform of the function:

$$\frac{1}{2}\left[\sin\left(a+\omega\right)t+\sin\left(a-\omega\right)t\right].$$

3. $f(t) = e^{-\beta t}\cos \alpha t,\qquad \mathrm{Re}(\beta) > \left|\mathrm{Im}(\alpha)\right|.$

$$\mathscr{F}_c\left[f\left(t\right)\right]=\int_0^\infty e^{-\beta t}\cos\alpha t\cos\omega t\,dt$$

$$=\frac{\beta}{2}\left[\frac{1}{\beta^2+\left(\alpha-\omega\right)^2}+\frac{1}{\beta^2+\left(\alpha+\omega\right)^2}\right], \tag{3.2.54}$$

which is the Laplace transform of the function $\frac{1}{2}$ [cos $(\alpha + \omega)t$ + cos$(\alpha - \omega)t$].

4. $f\left(t\right)=\frac{t\sin at}{\left(t^2+\beta^2\right)}\qquad a,\ \mathrm{Re}\left(\beta\right)>0.$

$$\mathscr{F}_c\left[f\left(t\right)\right]=\int_0^\infty \frac{t\sin at}{\left(t^2+\beta^2\right)}\cos\omega t\,dt$$

$$=\frac{\pi}{2}e^{-a\beta}\cosh\beta\omega\quad \text{if } \omega<a \tag{3.2.55}$$

$$=-\frac{\pi}{2}e^{-\beta\omega}\sinh a\beta\quad \text{if } \omega>a.$$

The result is obtained by contour integration, as is the next cosine transform.

5. $f\left(t\right)=\frac{\cos at}{\left(t^2+\beta^2\right)}\qquad a,\ \mathrm{Re}\left(\beta\right)>0.$

$$\mathscr{F}_c\left[f\left(t\right)\right]=\int_0^\infty \frac{\cos at}{\left(t^2+\beta^2\right)}\cos\omega t\,dt$$

$$=\frac{\pi}{2\beta}e^{-a\beta}\cosh\beta\omega\quad \text{if } \omega<a, \tag{3.2.56}$$

$$=\frac{\pi}{2\beta}e^{-\beta\omega}\cosh a\beta\quad \text{if } \omega>a.$$

6. $f(t) = e^{-\beta t^2}\cos at,\qquad \mathrm{Re}(\beta) > 0.$

$$\mathscr{F}_c\left[f\left(t\right)\right]=\int_0^\infty e^{-\beta t^2}\cos at\cos\omega t\,dt$$

$$=\frac{1}{2}\sqrt{\frac{\pi}{\beta}}\,e^{-\left(a^2+\omega^2\right)\ 4\beta}\cosh\frac{a\omega}{2\beta}. \tag{3.2.57}$$

3.2.3.4 FCT of Orthogonal Polynomials

1. *Legendre polynomials*:

$$f(t) = P_n(1-2t^2) \qquad 0 < t < 1,$$
$$= 0 \qquad t > 1,$$

where the Legendre polynomial $P_n(x)$ is defined as

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n, \quad \text{for } |x| < 1 \text{ and } n = 0, 1, 2, \ldots$$

$$\mathcal{F}_c[f(t)] = \int_0^1 P_n(1-2t^2)\cos\omega t\, dt$$
$$= \frac{(-1)^n \pi}{2} J_{n+\frac{1}{2}}(\omega/2) J_{-n-\frac{1}{2}}(\omega/2), \qquad (3.2.58)$$

where $J_\upsilon(z)$ is the Bessel function of the first kind, and order υ, defined by

$$J_\upsilon(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{\upsilon+2m}}{\Gamma(m+1)\Gamma(\upsilon+m+1)}, \quad |z| < \infty, |\arg z| < \pi. \qquad (3.2.58')$$

2. *Chebyshev polynomials*:

$$f(t) = (a^2-t^2)^{-1/2} T_{2n}(t/a) \qquad 0 < t < a, n = 0, 1, 2, \ldots$$
$$= 0, \qquad t > a,$$

where the Chebyshev polynomial is defined by,

$$T_n(x) = \cos(n \cos^{-1} x), \quad n = 0, 1, 2, \ldots$$

$$\mathcal{F}_c[f(t)] = \int_0^a (a^2-t^2)^{-1/2} T_{2n}(t/a)\cos\omega t\, dt$$
$$= (-1)^n (\pi/2) J_{2n}(a\omega), \qquad (3.2.59)$$

where $J_{2n}(x)$ is the Bessel function defined in (3.2.58′) with $\upsilon = 2n$.

3. *Laguerre polynomial*:

$$f(t) = e^{-t^2/2} L_n(t^2)$$

where $L_n(x)$ is the Laguerre polynomial defined by,

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(x^n e^{-x}), \quad n = 0, 1, 2, \ldots$$

$$\mathcal{F}_c[f(t)] = \int_0^\infty e^{-t^2/2} L_n(t^2) \cos \omega t \, dt$$
$$= \sqrt{\frac{\pi}{2}} \frac{1}{n!} e^{-\omega^2/2} \{\mathrm{He}_n(\omega)\}^2, \tag{3.2.60}$$

where $He_n(x)$ is the Hermite polynomial given by,

$$\mathrm{He}_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}\left(e^{-x^2/2}\right), \quad n = 0, 1, 2, \ldots$$

4. *Hermite polynomials*:
(a) $f(t) = e^{-t^2/2}\, \mathrm{He}_{2n}(t) \quad n = 0, 1, 2, \ldots$

$$\mathcal{F}_c[f(t)] = \int_0^\infty e^{-t^2/2} \mathrm{He}_{2n}(t) \cos \omega t \, dt$$
$$= (-1)^n \sqrt{\frac{\pi}{2}} e^{-\omega^2/2} \omega^{2n} \tag{3.2.61}$$

which is obtained using the Rodriques formula for the Hermite polynomial given in (3) above.
(b) $f(t) = e^{-t^2/2}\, \{\mathrm{He}_n(t)\}^2$,

$$\mathcal{F}_c[f(t)] = \int_0^\infty e^{-t^2/2} \{\mathrm{He}_n(t)\}^2 \cos \omega t \, dt$$
$$= n! \sqrt{\frac{\pi}{2}} e^{-\omega^2/2} L_n(\omega^2), \tag{3.2.62}$$

which shows a rare symmetry with (3.2.60).

3.2.3.5 FCT of Some Special Functions

1. *The complementary error function*:

$$f(t) = t \,\mathrm{Erfc}(at) \qquad a > 0.$$

Here the complementary error function is defined as

$$\mathrm{Erfc}(x) = 1 - \mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt.$$

$$\mathcal{F}_c[f(t)] = \int_0^\infty t \,\mathrm{Erfc}(at) \cos \omega t \, dt$$
$$= \left[\frac{1}{2a^2} + \frac{1}{\omega^2}\right] e^{-\omega^2/4a^2} - \frac{1}{\omega^2}. \tag{3.2.63}$$

2. *The sine integral function*:

$$f(t) = \mathrm{si}(at) \qquad a > 0,$$

where si(x) is defined in (3.2.39).

$$\begin{aligned}\mathcal{F}_c\left[f(t)\right] &= \int_0^\infty \mathrm{si}(at)\cos\omega t\,dt \\ &= -(1/2\omega)\ln\left|\frac{\omega+a}{\omega-a}\right|, \quad \omega \neq a.\end{aligned} \tag{3.2.64}$$

Note certain amount of symmetry with (3.2.51).

3. *The cosine integral function*:

$$f(t) = \mathrm{Ci}(at) = -\mathrm{ci}(at) \qquad a > 0,$$

where ci(x) is defined in (3.2.36).

$$\begin{aligned}\mathcal{F}_c\left[f(t)\right] = \int_0^\infty \mathrm{Ci}(at)\cos\omega t\,dt &= 0 && \text{for } 0 < \omega < a, \\ &= -\pi/2\omega && \text{for } \omega > a.\end{aligned} \tag{3.2.65}$$

4. *The exponential integral function*:

$$f(t) = \mathrm{Ei}(-at) \qquad a > 0,$$

where Ei($-x$) is defined by

$$\mathrm{Ei}(-x) = -\int_x^\infty e^{-t}/t\,dt, \quad \left|\arg(x)\right| < \pi.$$

$$\mathcal{F}_c\left[f(t)\right] = \int_0^\infty \mathrm{Ei}(-at)\cos\omega t\,dt = -\frac{1}{\omega}\tan^{-1}(\omega/a). \tag{3.2.66}$$

5. *Bessel functions*: We list only a few here since a more comprehensive table is available in Chapter 9 on Henkel transforms:
   (a) $f(t) = J_0(at) \qquad a > 0$,
   where $J_n(x)$ is the Bessel function of the first kind defined in (3.2.58′).

$$\begin{aligned}\mathcal{F}_c\left[f(t)\right] &= \int_0^\infty J_0(at)\cos\omega t\,dt \\ &= \left(a^2-\omega^2\right)^{-1/2} && \text{for } 0 < \omega < a, \\ &= \infty, && \text{for } \omega = a, \\ &= 0, && \text{for } \omega > a.\end{aligned} \tag{3.2.67}$$

   (b) $f(t) = J_{2n}(at) \qquad a > 0.$

$$\mathcal{F}_c[f(t)] = \int_0^\infty J_{2n}(at)\cos\omega t\,dt$$

$$= (-1)^n (a^2-\omega^2)^{-1/2} T_{2n}(\omega/a) \quad \text{for } 0<\omega<a, \tag{3.2.68}$$

$$= \infty, \quad \text{for } \omega = a,$$

$$= 0, \quad \text{for } \omega > a.$$

Here, $T_{2n}(x)$ is the Chebyshev polynomial defined in (3.2.59). Note the symmetry between this and (3.2.29).

(c) $f(t) = t^{-n}J_n(at)$ $\quad a > 0$, and $n = 1, 2, \ldots$

$$\mathcal{F}_c[f(t)] = \int_0^\infty t^{-n} J_n(at)\cos\omega t\,dt$$

$$= \frac{\sqrt{\pi}}{\Gamma(n+1/2)}(2a)^{-n}(a^2-\omega^2)^{n-1/2}, \quad 0<\omega<a, \tag{3.2.69}$$

$$= 0, \quad \omega > a.$$

Here, $\Gamma(x)$ is the gamma function defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt. \tag{3.2.69'}$$

(d) $f(t) = Y_0(at)$ $\quad a > 0$,
where $Y_\upsilon(x)$ is the Bessel function of the second kind defined by:

$$Y_\upsilon(x) = \operatorname{cosec}(\upsilon\pi)[J_\upsilon(x)\cos(\upsilon\pi) - J_{-\upsilon}(x)] \tag{3.2.70}$$

$$\mathcal{F}_c[f(t)] = \int_0^\infty Y_0(at)\cos\omega t\,dt$$

$$= 0, \quad \text{for } 0<\omega<a, \tag{3.2.70'}$$

$$= -(\omega^2-a^2)^{-1/2} \quad \text{for } \omega > a.$$

(e) $f(t) = t^\upsilon Y_\upsilon(at)$ $\quad |\mathrm{Re}(\upsilon)| < 1/2,\ a > 0$,

$$\mathcal{F}_c[f(t)] = \int_0^\infty t^\upsilon Y_\upsilon(at)\cos\omega t\,dt$$

$$= -\sqrt{\pi}(2a)^\upsilon[\Gamma(1/2-\upsilon)]^{-1}(\omega^2-a^2)^{-\upsilon-1/2}, \quad \omega > a, \tag{3.2.71}$$

$$= 0, \quad \text{for } 0<\omega<a.$$

3.2.4 Examples on the Use of Some Operational Rules of FCT

In this section, some simple examples on the use of operational rules of the FCT are presented. The examples are based on very simple functions and are intended to illustrate the procedure and the features in the FCT operational rules that have been discussed in Section 3.2.2.

3.2.4.1 Differentiation-in-t

Let $f(t)$ be defined as $f(t) = e^{-\alpha t}$, where $\text{Re}(\alpha) > 0$. Then according to (3.2.44), its FCT is given by

$$F_c(\omega) = \frac{\alpha}{\alpha^2 + \omega^2}.$$

To obtain the FCT for $f''(t)$, we have, according to the differentiation-in-t property, (3.2.9)

$$\begin{aligned}\mathcal{F}_c\left[f''(t)\right] &= -\omega^2 F_c(\omega) - f'(0) = -\omega^2 \frac{\alpha}{\alpha^2+\omega^2} + \alpha \\ &= \frac{\alpha^3}{\alpha^2+\omega^2}\end{aligned} \tag{3.2.72}$$

This result is verified by noting that $f''(t) = \alpha^2 e^{-\alpha t}$, and that its FCT is given directly also by (3.2.72).

3.2.4.2 Differentiation-in-t of Functions with Jump Discontinuities

Consider the function $f(t) = tU(1 - t)$, which is sometimes called a ramp function. It has a jump discontinuity of $d = -1$ at $t = 1$. Its derivative is given by $f'(t) = U(1 - t)$, which also has a jump discontinuity at $t = 1$. Using the definition for FCT, we obtain

$$\mathcal{F}_c\left[f'(t)\right] = \mathcal{F}_c\left[U(1-t)\right] = \frac{\sin\omega}{\omega}. \tag{3.2.73}$$

The FCT rule of differentiation with jump discontinuity (3.2.14) can also be applied to get

$$\begin{aligned}\mathcal{F}_c\left[f'(t)\right] &= \omega F_s(\omega) - f(0) - d\cos(\omega t_0) \\ &= \omega\left[-\frac{\cos\omega}{\omega} + \frac{\sin\omega}{\omega^2}\right] - (-1)\cos\omega, \quad (\text{because } d = -1, \text{and } f(0) = 0.) \\ &= \frac{\sin\omega}{\omega}, \quad \text{as in } (3.2.73).\end{aligned}$$

3.2.4.3 Shift-in-t, Shift-in-ω, and the Kernel Product Property

Let $f(t) = e^{-\alpha t}$, where $\text{Re}(\alpha) > 0$. The FCT of a positive shift in the t-domain is easy to obtain,

$$\mathcal{F}_c\left[f(t+a)\right] = e^{-\alpha a}\frac{\alpha}{\alpha^2+\omega^2} \qquad a > 0. \tag{3.2.74}$$

To obtain the FCT of the function $f(|t - a|)$, one can apply the kernel product property in (3.2.18) to get:

$$\mathcal{F}_c[f(|t-a|)] = 2F_c(\omega)\cos a\omega - \mathcal{F}_c[f(t+a)].$$

Therefore,

$$\mathcal{F}_c\left[e^{-\alpha|t-a|}\right] = 2\frac{\alpha}{\alpha^2+\omega^2}\cos a\omega - e^{-\alpha a}\frac{\alpha}{\alpha^2+\omega^2}$$
$$= \frac{\alpha}{\alpha^2+\omega^2}\left[2\cos a\omega - e^{-\alpha a}\right] \tag{3.2.75}$$

which is much easier than direct evaluation.

Equation (3.2.21) typifies the shift-in-ω property and, when it is applied to the same function $f(t)$ above, we obtain,

$$\mathcal{F}_c\left(e^{-\alpha t}\cos\beta t\right) = \frac{1}{2}\left[\frac{\alpha}{\alpha^2+(\omega+\beta)^2} + \frac{\alpha}{\alpha^2+(\omega-\beta)^2}\right] \tag{3.2.76}$$

3.2.4.4 Differentiation-in-ω Property

This property, (3.2.23), can often be used to generate FCTs for functions that are not listed in the tables. As an example, consider again the function $f(t) = e^{-\alpha t}$, where $\mathrm{Re}(\alpha) > 0$. To obtain the FCT for the function $g(t) = t^2e^{-\alpha t}$, we can use (3.2.23) on $F_c(\omega)$ for $f(t) = e^{-\alpha t}$. Thus,

$$F_c''(\omega) = -2\alpha\frac{\alpha^2-3\omega^2}{\left(\alpha^2+\omega^2\right)^3}, \quad \text{because} \quad \mathcal{F}_c\left[e^{-\alpha t}\right] = \frac{\alpha}{\alpha^2+\omega^2},$$

and

$$\mathcal{F}_c\left[t^2e^{-\alpha t}\right] = 2\alpha\frac{\alpha^2-3\omega^2}{\left(\alpha^2+\omega^2\right)^3} \quad \text{using} \quad (3.2.23) \text{ with } n=1.$$

3.2.4.5 The Convolution Property

The convolution property for FCT is closely related to its kernel product property as illustrated by the following example.

Let $f(t) = e^{-\alpha t}$, $\mathrm{Re}(\alpha) > 0$, and $g(t) = U(t) - U(t-a)$, $a > 0$. The FCTs of these functions are given respectively by,

$$F_c(\omega) = \frac{\alpha}{\alpha^2+\omega^2}, \quad \text{and} \quad G_c(\omega) = \frac{\sin a\omega}{\omega}.$$

Thus, $2F_c(\omega)G_c(\omega) = 2\left[\frac{\alpha}{\alpha^2+\omega^2}\right]\left[\frac{\sin a\omega}{\omega}\right]$. According to the convolution property (3.2.20), this is the FCT of the convolution defined as:

$$\int_0^\infty \left[U(\tau) - U(\tau-a)\right]\left[e^{-\alpha(t+\tau)} + e^{-\alpha|t-\tau|}\right]d\tau. \tag{3.2.77}$$

Applying the operator $\mathcal{F}_c$ to (3.2.77) and integrating over t first, the kernel product property in the shift-in-t operation in (3.2.18) can be invoked to give,

$$\mathscr{F}_c\left\{\int_0^\infty \left[U(\tau)-U(\tau-a)\right]\left[e^{-\alpha(t+\tau)}+e^{-\alpha|t-\tau|}\right]d\tau\right\}$$

$$=2\int_0^\infty \left[U(\tau)-U(\tau-a)\right]\frac{\alpha}{\alpha^2+\omega^2}\cos\omega\tau\,d\tau = 2\left[\frac{\alpha}{\alpha^2+\omega^2}\right]\left[\frac{\sin a\omega}{\omega}\right],$$

as required.

3.3 The Fourier Sine Transform (FST)

3.3.1 Definitions and Relations to the Exponential Fourier Transforms

Similar to the Fourier cosine transform, the Fourier sine transform of a function $f(t)$, which is piecewise continuous and absolutely integrable over $[0, \infty)$, is defined by application of the operator $\mathscr{F}_s$ as:

$$F_s(\omega)=\mathscr{F}_s\left[f(t)\right]=\int_0^\infty f(t)\sin\omega t\,dt, \quad \omega>0. \tag{3.3.1}$$

The inverse operator $\mathscr{F}_s^{-1}$ is similarly defined:

$$f(t)=\mathscr{F}_s^{-1}\left[F_s(\omega)\right]=\frac{2}{\pi}\int_0^\infty F_s(\omega)\sin\omega t\,d\omega, \quad t\geq 0, \tag{3.3.2}$$

subject to the existence of the integral. Functions $f(t)$ and $F_s(\omega)$ defined by (3.3.2) and (3.3.1), respectively, are said to form a Fourier sine transform pair. It is noted in (3.2.3) and (3.3.2) for the inverse FCT and inverse FST that both transform operators have symmetric kernels and that they are involuntary or unitary up to a factor of $\sqrt{(2/\pi)}$.

Fourier sine transforms are also very closely related to the exponential Fourier transform defined in (3.2.6). Using the property that

$$\sin\omega t = \operatorname{Im}\left[e^{j\omega t}\right]=\frac{1}{2j}\left[e^{j\omega t}-e^{-j\omega t}\right], \tag{3.3.3}$$

one can consider the odd extension of the function $f(t)$ defined over $[0, \infty)$ as

$$\begin{aligned} f_o(t) &= f(t) && t\geq 0, \\ &= -f(-t) && t<0. \end{aligned}$$

Then the Fourier transform of $f_o(t)$ is

$$\mathscr{F}\left[f_o(t)\right]=\int_{-\infty}^\infty f_o(t)e^{-j\omega t}dt=-\int_0^\infty f(t)e^{j\omega t}dt+\int_0^\infty f(t)e^{-j\omega t}dt$$

$$=-2j\int_0^\infty f(t)\sin\omega t\,dt=-2j\mathscr{F}_s\left[f(t)\right],$$

and therefore,

$$\mathcal{F}_s\left[f(t)\right] = -\frac{1}{2j}\mathcal{F}\left[f_o(t)\right]. \tag{3.3.4}$$

Equation (3.3.4) provides the relation between the FST and the exponential Fourier transform. As in the case for cosine transforms, many properties of the sine transform can be related to those for the Fourier transform through this equation. We shall present some properties and operational rules for FST in the next section.

3.3.2 Basic Properties and Operational Rules

1. *Inverse Transformation*: The inverse transformation is exactly the same as the forward transformation except for the normalization constant. Combining the forward and inverse transformations leads to the Fourier sine integral formula, which states that,

$$f(t) = \frac{2}{\pi}\int_0^\infty F_s(\omega)\sin\omega t\, d\omega = \frac{2}{\pi}\int_0^\infty\left[\int_0^\infty f(\tau)\sin\omega\tau\, d\tau\right]\sin\omega t\, d\omega . \tag{3.3.5}$$

   The sufficient conditions for the inversion formula (3.3.2) are the same as for the cosine transform. Where $f(t)$ has a jump discontinuity at $t = t_0$, (3.3.5) converges to the mean of $f(t_0 + 0)$ and $f(t_0 - 0)$.
2. *Transforms of Derivatives*: Derivatives transform in a fashion similar to FCT, even orders involving sine transforms only and odd orders involving cosine transforms only. Thus, for example,

$$\mathcal{F}_s[f''(t)] = -\omega^2 F_s(\omega) + \omega f(0) \tag{3.3.6}$$

   and

$$\mathcal{F}_s[f'(t)] = -\omega F_c(\omega) , \tag{3.3.7}$$

   where $f(t)$ is assumed continuous to the first order.

   For the fourth-order derivative, we apply (3.3.6) twice to obtain,

$$\mathcal{F}_s[f^{(iv)}(t)] = \omega^4 F_s(\omega) - \omega^3 f(0) + \omega f''(0), \tag{3.3.8}$$

   if $f(t)$ is continuous at least to order three. When the function $f(t)$ and its derivatives have jump discontinuities at $t = t_0$, (3.3.8) is modified to become,

$$\begin{aligned}\mathcal{F}_s[f^{(iv)}(t)] = {} & \omega^4 F_s(\omega) - \omega^3 f(0) + \omega f''(0) - \omega^3 d \cos \omega t_0 \\ & + \omega^2 d' \sin \omega t_0 + \omega d'' \cos \omega t_0 - d''' \sin \omega t_0\end{aligned} \tag{3.3.9}$$

   where the jump discontinuities d, d', and d''' are as defined in (3.2.13). Similarly, for odd-order derivatives, when the function $f(t)$ has jump discontinuities, the operational rule must be modified. For example, (3.3.7) will become:

$$\mathcal{F}_s[f'(t)] = -\omega F_c(\omega) + d \sin \omega t_0. \tag{3.3.7$'$}$$

   Generalization to other orders and to more than one location for the jump discontinuities is straightforward.
3. *Scaling*: Scaling in the t-domain for the FST has exactly the same effect as in the case of FCT, giving,

$$\mathcal{F}_s\left[f(at)\right] = \frac{1}{a}F_s(\omega/a) \quad a > 0 . \tag{3.3.10}$$

4. *Shifting*:
   (a) Shift in the *t*-domain:
   As in the case of the Fourier cosine transform, we first define the even and odd extensions of the function $f(t)$ as,

$$f_e(t) = f(|t|), \quad \text{and } f_o(t) = \frac{t}{|t|} f(|t|). \tag{3.3.11}$$

   Then it can be shown that:

$$\mathcal{F}_s[f_o(t+a) + f_o(t-a)] = 2F_s(\omega)\cos a\omega \tag{3.3.12}$$

   and

$$\mathcal{F}_c[f_o(t+a) + f_o(t-a)] = 2F_s(\omega)\sin a\omega\,; \quad a > 0. \tag{3.3.13}$$

   These, together with (3.2.18) and (3.2.19), form a complete set of kernel-product relations for the cosine and the sine transforms.
   (b) Shift in the ω-domain:
   For a positive β shift in the ω-domain, it is easily shown that

$$\mathcal{F}_s[\omega + \beta] = F_s[f(t)\cos\beta t] + F_c[f(t)\sin\beta t] \tag{3.3.14}$$

   and combining with the result for a negative shift, we get:

$$\mathcal{F}_s[f(t)\cos\beta t] = (1/2)[F_s(\omega+\beta) + F_s(\omega-\beta)]. \tag{3.3.15}$$

   More generally, for $a, \beta > 0$, we have,

$$\mathcal{F}_s\left[f(at)\cos\beta t\right] = (1/2a)\left[F_s\left(\frac{\omega+\beta}{a}\right) + F_s\left(\frac{\omega-\beta}{a}\right)\right]. \tag{3.3.16}$$

   Similarly, we can easily show that

$$\mathcal{F}_s\left[f(at)\sin\beta t\right] = -(1/2a)\left[F_c\left(\frac{\omega+\beta}{a}\right) - F_c\left(\frac{\omega-\beta}{a}\right)\right]. \tag{3.3.17}$$

   The shift-in-ω properties are useful in deriving some FCTs and FSTs. As well, because the quantities being transformed are modulated sinusoids, these are useful in applications to communication problems.
5. *Differentiation in the ω-domain*: The sine transform behaves in a fashion similar to the cosine transform when it comes to differentiation in the ω-domain. Even-order derivatives involve only sine transforms and odd-order derivatives involve only cosine transforms. Thus,

$$F_s^{(2n)}(\omega) = \mathcal{F}_s\left[(-1)^n t^{2n} f(t)\right],$$

   and

$$F_s^{(2n+1)}(\omega) = \mathcal{F}_c\left[(-1)^n t^{2n+1} f(t)\right]. \tag{3.3.18}$$

It is again assumed that the integrals in (3.3.18) exist.

6. *Asymptotic behavior*: The Reimann-Lebesque theorem guarantees that any Fourier sine transform converges to zero as ω tends to infinity, that is,

$$\lim_{\omega \to \infty} F_s(\omega) = 0. \tag{3.3.19}$$

7. *Integration*:
   (a) Integration in the t-domain. In analogy to (3.2.26), we have

$$\mathcal{F}_s\left[\int_0^t f(\tau)d\tau\right] = (1/\omega)F_c(\omega) \tag{3.3.20}$$

   provided $f(t)$ is piece-wise smooth and absolutely integrable over $[0, \infty)$.
   (b) Integration in the ω-domain. As in the Fourier cosine transform, integration in the ω-domain results in division by t in the t-domain, giving,

$$\mathcal{F}_c^{-1}\left[\int_\omega^\infty F_s(\beta)d\beta\right] = (1/t)f(t) \tag{3.3.21}$$

   in parallel with (3.2.27).

8. *The convolution property*: If functions $f(t)$ and $g(t)$ are piece-wise continuous and absolutely integrable over $[0, \infty)$, a convolution property involving $F_s(\omega)$ and $G_c(\omega)$ is

$$2F_s(\omega)G_c(\omega) = \mathcal{F}_s\left\{\int_0^\infty f(\tau)\left[g(|t-\tau|) - g(t+\tau)\right]d\tau\right\}. \tag{3.3.22}$$

Equivalently,

$$2F_s(\omega)G_c(\omega) = \mathcal{F}_s\left\{\int_0^\infty g(\tau)\left[f(t+\tau) + f_o(t-\tau)\right]d\tau\right\} \tag{3.3.23}$$

where $f_o(x)$ is the odd extension of the function $f(x)$ defined as in (3.3.11).

One can establish a convolution theorem involving only sine transforms. This is obtained by imposing an additional condition on one of the functions, say, $g(t)$. We define the function $h(t)$ by,

$$h(t) = \int_t^\infty g(\tau)d\tau. \tag{3.3.24}$$

Then $g(t)$ must satisfy the condition that its integral $h(t)$ is absolutely integrable over $[0, \infty)$, so that the Fourier cosine transform of $h(t)$ exists. We note from (3.2.26) that

$$H_c(\omega) = (1/\omega)G_s(\omega) \tag{3.3.25}$$

Applying (3.3.22) to $f(t)$ and $h(t)$ yields immediately,

$$\left(2/\omega\right)F_s\left(\omega\right)G_s\left(\omega\right)=\mathscr{F}_s\left[\int_0^\infty f\left(\tau\right)\int_{|t-\tau|}^{t+\tau} g\left(\eta\right)d\eta\,d\tau\right] \tag{3.3.26}$$

noting that $g(t) = -h'(t)$.

Because the FSTs have properties and operation rules very similar to those for the FCTs, we refer the reader to Section 3.2.24 for simple examples on the use of these rules for FCTs.

3.3.3 Selected Fourier Sine Transforms

In this section, selected Fourier sine transforms are presented. These mostly correspond to those selected for the Fourier cosine transforms. It should be noted that because the sine and cosine transforms kernels are related through differentiation, many of the Fourier sine transforms can be derived without direct computation by using differentiation properties listed in Sections 3.2.2 and 3.3.2. As before, we present first the FST of algebraic functions.

3.3.3.1 FST of Algebraic Functions

1. *The unit rectangular function*:

$f(t) = U(t) - U(t-a)$, where $U(t)$ is the Heaviside unit step function.

$$\mathscr{F}_s\left[f\left(t\right)\right]=\int_0^a \sin\omega t\,dt=\left(1-\cos\omega a\right)/\omega\,. \tag{3.3.27}$$

2. *The unit height tent function*:

$$\begin{aligned} f\left(t\right)&=t/a\,, && 0<t<a\,,\\ &=\left(2a-t\right)/a && a<t<2a\,,\\ &=0 && \text{otherwise}\,. \end{aligned}$$

$$\begin{aligned} \mathscr{F}_s\left[f\left(t\right)\right]&=\int_0^a \left(t/a\right)\sin\omega t+\int_a^{2a}\left[\left(2a-t\right)/a\right]\sin\omega t\,dt\\ &=\frac{1}{a\omega^2}\left[2\sin a\omega-\sin 2a\omega\right]. \end{aligned} \tag{3.3.28}$$

3. *Delayed inverse*:

$$f(t) = (1/t)U(t-a).$$

$$\mathscr{F}_s\left[f\left(t\right)\right]=\int_a^\infty \left(1/t\right)\sin\omega t\,dt=\int_{a\omega}^\infty\left(1/\tau\right)\sin\tau\,d\tau=-\mathrm{si}\left(a\omega\right) \tag{3.3.29}$$

where $\mathrm{si}(x)$ is the sine integral function defined in (3.2.39).

4. *The inverse square root*:

$$f(t) = 1/\sqrt{t}.$$

$$\mathscr{F}_s\left[f(t)\right]=\int_0^{\infty}\frac{1}{\sqrt{t}}\sin\omega t\,dt=\sqrt{\frac{\pi}{2\omega}}\,. \tag{3.3.30}$$

5. *The inverse linear function*:

$$f(t) = (\alpha + t)^{-1},\ |\arg\ \alpha| < \pi.$$

$$\begin{aligned}\mathscr{F}_s\left[f(t)\right]&=\int_0^{\infty}\frac{1}{\alpha+t}\sin\omega t\,dt\\&=\sin\omega\alpha\,\mathrm{Ci}(\omega\alpha)-\cos\omega\alpha\,\mathrm{si}(\omega\alpha).\end{aligned} \tag{3.3.31}$$

Here $\mathrm{Ci}(x)$ is the cosine integral function defined in (3.2.36).

6. *Inverse quadratic functions*:

(a) $f(t) = (t^2 + a^2)^{-1} \quad a > 0.$

$$\begin{aligned}\mathscr{F}_s\left[f(t)\right]&=\int_0^{\infty}\frac{1}{a^2+t^2}\sin\omega t\,dt\\&=(1/2a)\left[e^{-a\omega}\,\overline{\mathrm{Ei}}\,(a\omega)-e^{a\omega}\mathrm{Ei}\,(-a\omega)\right]\end{aligned} \tag{3.3.32}$$

where $\mathrm{Ei}(x)$ and $\overline{\mathrm{Ei}}(x)$ are the exponential integral functions defined in (3.2.50). Here, we note that (3.3.32) is related to the FCT of the function,

$$f(t) = -t(t^2 + a^2)^{-1}$$

by considering the derivative of (3.3.32) with respect to ω. Thus,

$$\mathscr{F}_c\left[-t\left(t^2+a^2\right)^{-1}\right]=(1/2)\left[e^{-a\omega}\,\overline{\mathrm{Ei}}(a\omega)+e^{a\omega}\mathrm{Ei}(-a\omega)\right] \tag{3.3.33}$$

(b) $f(t) = (a^2 - t^2)^{-1} \quad a > 0.$

$$\begin{aligned}\mathscr{F}_s\left[f(t)\right]&=\mathrm{P.V.}\int_0^{\infty}\frac{1}{a^2-t^2}\sin\omega t\,dt\\&=\left[\sin a\omega\,\mathrm{Ci}(a\omega)-\cos a\omega\,\mathrm{Si}(a\omega)\right]/a\,,\end{aligned} \tag{3.3.34}$$

where $\mathrm{Ci}(x)$ and $\mathrm{Si}(x)$ are the cosine and sine integral functions defined in (3.2.36) and (3.2.39) and "P.V." denotes the principal value of the integral. Again, we note that (3.3.34) is related to the FCT of the function,

$$f(t) = -t(a^2 - t^2)^{-1}.$$

Thus,

$$\mathscr{F}_c[-t(a^2 - t^2)^{-1}] = \cos a\omega\ \mathrm{Ci}(a\omega) + \sin a\omega\ \mathrm{Si}(a\omega). \tag{3.3.35}$$

(c) $f(t)=\dfrac{\beta}{\beta^2+(a-t)^2}-\dfrac{\beta}{\beta^2+(a+t)^2}$ $\qquad \mathrm{Re}(\beta)>0$.

$$\mathcal{F}_s[f(t)]=\int_0^\infty\left[\frac{\beta}{\beta^2+(a-t)^2}-\frac{\beta}{\beta^2+(a+t)^2}\right]\sin\omega t\,dt \tag{3.3.36}$$
$$=\pi\sin a\omega\; e^{-\beta\omega}.$$

(d) $f(t)=\dfrac{a+t}{\beta^2+(a+t)^2}-\dfrac{a-t}{\beta^2+(a-t)^2}$ $\qquad \mathrm{Re}(\beta)>0$.

$$\mathcal{F}_s[f(t)]=\int_0^\infty\left[\frac{a+t}{\beta^2+(\alpha+t)^2}-\frac{a-t}{\beta^2+(\alpha-t)^2}\right]\sin\omega t\,dt \tag{3.3.37}$$
$$=\pi\sin a\omega\; e^{-\beta\omega}.$$

We note here the symmetry among the transforms in (3.3.36), (3.3.37), and those in (3.2.43) and (3.2.42).

3.3.3.2 FST of Exponential and Logarithmic Functions

1. $f(t) = e^{-\alpha t}$ $\qquad \mathrm{Re}(\alpha) > 0$.

$$\mathcal{F}_s[f(t)]=\int_0^\infty e^{-\alpha t}\sin\omega t\,dt=\frac{\omega}{\alpha^2+\omega^2} \tag{3.3.38}$$

which is also seen to be the Laplace transform of sin ωt.

2. $f(t)=\dfrac{e^{-\beta t}-e^{-\alpha t}}{t^2}$ $\qquad \mathrm{Re}(\beta), \mathrm{Re}(\alpha)>0$.

$$\mathcal{F}_s[f(t)]=\int_0^\infty \frac{e^{-\beta t}-e^{-\alpha t}}{t^2}\sin\omega t\,dt$$
$$=\frac{\omega}{2}\ln\left(\frac{\alpha^2+\omega^2}{\beta^2+\omega^2}\right)+\alpha\tan^{-1}\left(\frac{\omega}{\alpha}\right)-\beta\tan^{-1}\left(\frac{\omega}{\beta}\right). \tag{3.3.39}$$

Equation (3.3.39) is seen to be related to the result (3.2.45) through the differentiation-in-ω property of the sine transform as defined in (3.3.18).

3. $f(t) = te^{-\alpha t^2}$ $\qquad |\arg(\alpha)| < \pi/2$.

$$\mathcal{F}_s[f(t)]=\int_0^\infty te^{-\alpha t^2}\sin\omega t\,dt$$
$$=\frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}\,\omega e^{-\omega^2/4\alpha}, \tag{3.3.40}$$

which can also be related to the cosine transform in (3.2.46) using again the differentiation-in-ω property (3.3.18) of the sine transform.

4. $f(t) = \ln t[1 - U(t-1)]$

$$\mathcal{F}_s[f(t)] = \int_0^\infty \ln t[1-U(t-1)]\sin\omega t\,dt = -\frac{1}{\omega}\left[C+\ln\omega-\mathrm{Ci}(\omega)\right], \tag{3.3.41}$$

which is obtained easily through integration in parts. Here $C = 0.5772156649\ldots$ is the Euler constant and $\mathrm{Ci}(x)$ is the cosine integral function.

5. $f(t)=\dfrac{t\ln bt}{(t^2+a^2)}$ $\quad a, b>0$.

$$\mathcal{F}_s[f(t)] = \int_0^\infty \frac{t\ln bt}{(t^2+a^2)}\sin\omega t\,dt = \frac{\pi}{4}\left[2e^{-a\omega}\ln ab - e^{a\omega}\mathrm{Ei}(-a\omega) - e^{-a\omega}\,\overline{\mathrm{Ei}}(a\omega)\right] \tag{3.3.42}$$

Note that (3.3.42) is related to (3.2.49) through the differentiation-in-ω property of the Fourier cosine transform as defined in (3.2.24).

6. $f(t)=\ln\left|\dfrac{t+a}{t-a}\right|$ $\quad a>0$,

$$\mathcal{F}_s[f(t)] = \int_0^\infty \ln\left|\frac{t+a}{t-a}\right|\sin\omega t\,dt = \frac{\pi}{\omega}\sin a\omega. \tag{3.3.43}$$

The result is obtained using integration by parts and the shift-in-t properties (3.3.11) to (3.3.13) of the sine transform.

3.3.3.3 FST of Trigonometric Functions

1. $f(t)=\dfrac{\sin at}{t}$ $\quad a>0$,

$$\mathcal{F}_s[f(t)] = \int_0^\infty \frac{\sin at}{t}\sin\omega t\,dt = (1/2)\ln\left|\frac{\omega+a}{\omega-a}\right|. \tag{3.3.44}$$

This result is immediately understood when compared to (3.3.43), taking into account the normalization used in (3.3.1) and (3.3.2) for the definition of the Fourier sine transform.

2. $f(t) = \frac{e^{-\beta t}}{t} \sin \alpha t \qquad \text{Re}(\beta) > |\text{Im}(\alpha)|$

$$\mathcal{F}_s[f(t)] = \int_0^\infty \frac{e^{-\beta t}}{t} \sin \alpha t \sin \omega t \, dt$$
$$= (1/4) \ln\left(\frac{\beta^2 + (\omega + \alpha)^2}{\beta^2 + (\omega - \alpha)^2} \right). \qquad (3.3.45)$$

This result follows easily from the integration-in-ω property (3.2.27) as applied to the cosine transform in (3.2.53).

3. $f(t) = e^{-\beta t} \cos \alpha t \qquad \text{Re}(\beta) > |\text{Im}(\alpha)|$

$$\mathcal{F}_s[f(t)] = \int_0^\infty e^{-\beta t} \cos \alpha t \sin \omega t \, dt$$
$$= (1/2)\left[\frac{\omega - \alpha}{\beta^2 + (\omega - \alpha)^2} + \frac{\omega + \alpha}{\beta^2 + (\omega + \alpha)^2} \right], \qquad (3.3.46)$$

which is also recognized as the Laplace transform of the function cos αt sin ωt.

4. $f(t) = \frac{t \cos at}{(t^2 + \beta^2)} \qquad a, \text{Re}(\beta) > 0,$

$$\mathcal{F}_s[f(t)] = \int_0^\infty \frac{t \cos at}{(t^2 + \beta^2)} \sin \omega t \, dt$$
$$= -\frac{\pi}{2} e^{-a\beta} \sinh \beta\omega \quad \omega < a, \qquad (3.3.47)$$
$$= \frac{\pi}{2} e^{-\beta\omega} \cosh a\beta \quad \omega > a.$$

Note the symmetry of (3.3.47) with (3.2.55).

5. $f(t) = \frac{\sin at}{(t^2 + \beta^2)} \qquad a, \text{Re}(\beta) > 0,$

$$\mathcal{F}_s[f(t)] = \int_0^\infty \frac{\sin at}{(t^2 + \beta^2)} \sin \omega t \, dt$$
$$= \frac{\pi}{2\beta} e^{-a\beta} \sinh \beta\omega \quad \omega < a, \qquad (3.3.48)$$
$$= \frac{\pi}{2\beta} e^{-\beta\omega} \sinh a\beta \quad \omega > a.$$

The symmetry of (3.3.48) with (3.2.56) is apparent.

6. $f(t) = e^{-\beta t^2} \sin at \qquad \mathrm{Re}(\beta) > 0.$

$$\mathcal{F}_s[f(t)] = \int_0^\infty e^{-\beta t^2} \sin at \sin \omega t \, dt = \frac{1}{2}\sqrt{\frac{\pi}{\beta}} e^{-(\omega^2+a^2)/4\beta} \sinh \frac{a\omega}{2\beta} \tag{3.3.49}$$

similar to (3.2.57) for the cosine transform.

3.3.3.4 FST of Orthogonal Polynomials

1. *Legendre polynomial* (defined in [3.2.58]):

$$f(t) = P_n(1-2t^2)[1 - U(t-1)] \qquad n = 0, 1, 2, \ldots$$

$$\mathcal{F}_s[f(t)] = \int_0^1 P_n(1-2t^2) \sin \omega t \, dt = \frac{\pi}{2}\left[J_{n+1/2}\left(\frac{\omega}{2}\right)\right]^2 \tag{3.3.50}$$

where $J_\upsilon(x)$ is the Bessel function of the first kind defined in (3.2.58′).

2. *Chebyshev polynomial* (defined in [3.2.59]):

$$f(t) = (a^2 - t^2)^{-1/2} T_{2n+1}(t/a)[1 - U(t-a)], \quad n = 0, 1, 2, \ldots$$

$$\mathcal{F}_s[f(t)] = \int_0^a (a^2 - t^2)^{-1/2} T_{2n+1}(t/a) \sin \omega t \, dt = (-1)^n \frac{\pi}{2} J_{2n+1}(a\omega). \tag{3.3.51}$$

3. *Laguerre polynomials*:

$$f(t) = t^{2m} e^{-t^2/2} L_n^{2m+1}(t^2), \qquad m, n = 0, 1, 2, \ldots$$

$$\mathcal{F}_s[f(t)] = \int_0^\infty t^{2m} e^{-t^2/2} L_n^{2m+1}(t^2) \sin \omega t \, dt = \sqrt{\frac{\pi}{2}} (n!)^{-1} (-1)^m e^{-\omega^2/2} \mathrm{He}_n(\omega) \mathrm{He}_{n+2m+1}(\omega) \tag{3.3.52}$$

where $L_n^a(x) = \frac{e^x x^{-a}}{n!} \frac{d^n}{dx^n}\left(e^{-x} x^{n+a}\right)$, is a Laguerre polynomial ($L_n^0(x) = L_n(x)$ as defined in [3.2.60]). Here, $\mathrm{He}_n(x)$ is the Hermite polynomial defined in (3.2.61).

4. *Hermite polynomials* (defined in [3.2.62]):

$$f(t) = e^{-t^2/2}\ \mathrm{He}_{2n+1}(\sqrt{2}t)$$

$$\begin{aligned}\mathcal{F}_s\left[f(t)\right] &= \int_0^\infty e^{-t^2/2}\mathrm{He}_{2n+1}\left(\sqrt{2}t\right)\sin\omega t\,dt \\ &= \left(-1\right)^n\sqrt{\frac{\pi}{2}}e^{-\omega^2\ 2}\mathrm{He}_{2n+1}\left(\sqrt{2}\omega\right).\end{aligned} \tag{3.3.53}$$

3.3.3.5 FST of Some Special Functions

1. *The complementary error function* (defined in [3.2.63]):

$$f(t) = \mathrm{Erfc}(at) \qquad\qquad a > 0,$$

$$\begin{aligned}\mathcal{F}_s\left[f(t)\right] &= \int_0^\infty \mathrm{Erfc}\left(at\right)\sin\omega t\,dt \\ &= \frac{1}{\omega}\left[1 - e^{-\omega^2/4a^2}\right].\end{aligned} \tag{3.3.54}$$

2. *The sine integral function* (defined in [3.2.39]):

$$f(t) = \mathrm{si}(at) \qquad\qquad a > 0,$$

$$\begin{aligned}\mathcal{F}_s\left[f(t)\right] &= \int_0^\infty \mathrm{si}\left(at\right)\sin\omega t\,dt = 0 \qquad 0 \le \omega < a \\ &= -\frac{\pi}{2\omega} \qquad\qquad\qquad \omega > a.\end{aligned} \tag{3.3.55}$$

Note the symmetry of (3.3.55) with (3.2.65).

3. *The cosine integral function* (defined in [3.2.36]):

$$f(t) = \mathrm{Ci}(at) = -\mathrm{ci}(at) \qquad\qquad a > 0$$

$$\begin{aligned}\mathcal{F}_s\left[f(t)\right] &= \int_0^\infty \mathrm{Ci}\left(at\right)\sin\omega t\,dt \\ &= \frac{1}{2\omega}\ln\left|\frac{\omega^2}{a^2} - 1\right|.\end{aligned} \tag{3.3.56}$$

4. *The exponential integral function* (defined in [3.2.66]):

$$f(t) = \mathrm{Ei}(-at) \qquad\qquad a > 0$$

$$\mathcal{F}_s\left[f(t)\right] = \int_0^\infty \mathrm{Ei}\left(-at\right)\sin\omega t\,dt = -\frac{1}{2\omega}\ln\left(\frac{\omega^2}{a^2} + 1\right). \tag{3.3.57}$$

5. *Bessel functions* (defined in [3.2.58′]):

(a) $f(t) = J_0(at)$ $\quad a > 0$

$$\mathcal{F}_s\left[f(t)\right] = \int_0^\infty J_0(at)\sin\omega t\,dt = 0, \qquad 0 < \omega < a,$$
$$= \left(\omega^2 - a^2\right)^{-1/2} \qquad \omega > a. \tag{3.3.58}$$

(b) $f(t) = J_{2n+1}(at)$ $\quad a > 0$

$$\mathcal{F}_s\left[f(t)\right] = \int_0^\infty J_{2n+1}(at)\sin\omega t\,dt$$
$$= (-1)^n\left(a^2 - \omega^2\right)^{-1/2} T_{2n+1}(\omega/a) \quad 0 < \omega < a, \tag{3.3.59}$$
$$= 0 \qquad \omega > a,$$

where $T_n(x)$ is the Chebyshev polynomial defined in (3.2.59).

(c) $f(t) = t^{-n}J_{n+1}(at)$ $\quad a > 0$ and $n = 0, 1, 2, \ldots$

$$\mathcal{F}_s\left[f(t)\right] = \int_0^\infty t^{-n}J_{n+1}(at)\sin\omega t\,dt$$
$$= \frac{\sqrt{\pi}}{\Gamma(n+1/2)}\frac{1}{2^n a^{n+1}}\omega\left(a^2 - \omega^2\right)^{n-1/2}, \quad 0 < \omega < a \tag{3.3.60}$$
$$= 0 \qquad \omega > a,$$

where $\Gamma(x)$ is the gamma function defined in (3.2.69′).

(d) $f(t) = Y_0(at)$ $\quad a > 0.$

where $Y_\upsilon(x)$ is the Bessel function of the second kind (see [3.2.70]).

$$\mathcal{F}_s\left[f(t)\right] = \int_0^\infty Y_0(at)\sin\omega t\,dt$$
$$= \frac{2}{\pi}\left(a^2 - \omega^2\right)^{-1/2}\sin^{-1}\left(\frac{\omega}{a}\right), \qquad 0 < \omega < a \tag{3.3.61}$$
$$= \frac{2}{\pi}\left(\omega^2 - a^2\right)^{-1/2}\ln\left|\frac{\omega}{a} - \left(\frac{\omega^2}{a^2} - 1\right)^{1/2}\right|, \quad \omega > a.$$

(e) $f(t) = t^\upsilon Y_{\upsilon-1}(at)$ $\quad a > 0, |\mathrm{Re}(\upsilon)| < 1/2$

$$\mathscr{F}_s\left[f(t)\right] = \int_0^\infty t^\upsilon Y_{\upsilon-1}(at)\sin\omega t\, dt$$

$$= \frac{2^\upsilon a^{\upsilon-1}\sqrt{\pi}}{\Gamma(1/2-\upsilon)}\omega\left(\omega^2 - a^2\right)^{-\upsilon-1/2} \qquad \omega > a, \tag{3.3.62}$$

$$= 0 \qquad 0 < \omega < a.$$

As with the cosine transforms, more detailed results are found in the sections covering Henkel transforms.

3.4 The Discrete Sine and Cosine Transforms (DST and DCT)

In practical applications, the computations of the Fourier sine and cosine transforms are done with sampled data of finite duration. Because of the finite duration and the discrete nature of the data, much can be gained in theory and in ease of computation by formulating the corresponding discrete sine and cosine transforms (DST and DCT) directly. In what follows, we discuss the definitions and properties of the discrete sine and cosine transforms. It is possible to define four different types of each of the DCT and the DST (for details, see Rao and Yip, 1990). We shall concentrate on Type I, which can be defined by simply discretizing the FST and FCT, within a finite rectangular window of unit height.

3.4.1 Definitions of DCT and DST and Relations to FST and FCT

Consider the transform kernel of the FCT given by

$$K_c(\omega, t) = \cos \omega t. \tag{3.4.1}$$

Let $\omega_m = 2\pi m\Delta f$ and $t_n = n\Delta t$ be the sampled angular frequency and time, respectively. Here, Δf and Δt are the sample intervals for frequency and time, respectively. m and n are positive integers. The kernel in (3.4.1) can now be discretized as

$$K_c(m, n) = K_c(\omega_m, t_n) = \cos(2\pi\ mn\ \Delta f\ \Delta t). \tag{3.4.2}$$

If we further let $\Delta f\ \Delta t = 1/(2N)$, where N is a positive integer, we obtain the discrete cosine transform kernel:

$$K_c(m, n) = \cos(\pi\ mn/N) \tag{3.4.3}$$

where $m, n = 0, 1, \ldots, N$. The transform kernel in (3.4.3) is the DCT kernel of Type I. It represents the mnth element in an $(N + 1) \times (N + 1)$ transformation matrix, which, with the proper normalization, provides the definition for the DCT transformation matrix [C]. These elements are

$$\left[C\right]_{mn} = \sqrt{\frac{2}{N}}\left\{k_m k_n \cos\left(\frac{mn\pi}{N}\right)\right\}, \quad m, n = 0, 1, \ldots, N$$

where

$$\begin{aligned} k_i &= 1 && \text{for } i \neq 0 \text{ or } N \\ &= 1/\sqrt{2} && \text{for } i = 0 \text{ or } N. \end{aligned} \tag{3.4.4}$$

The discretization can be viewed as taking a finite time duration and dividing it into N intervals of Δt each. Including the boundary points, there are $N + 1$ sample points to be considered. If the discrete $N + 1$ sample points are represented by a vector $\mathbf{x}$, the DCT of this vector is a vector $\mathbf{X}_c$ given by,

$$\mathbf{X}_c = [C]\mathbf{x} \tag{3.4.5}$$

which, in an element-by-element form, means

$$\mathrm{X}_c(m) = \sqrt{\frac{2}{N}} \sum_{n=0}^{N} k_m k_n \cos\left(\frac{mn\pi}{N}\right) \mathrm{x}(n). \tag{3.4.6}$$

It can be shown that $[C]$ is a unitary matrix. Thus, the inverse transformation is given by

$$\mathrm{x}(n) = \sqrt{\frac{2}{N}} \sum_{m=0}^{N} k_m k_n \cos\left(\frac{mn\pi}{N}\right) \mathrm{X}_c(m). \tag{3.4.7}$$

Vectors $\mathbf{X}_c$ and $\mathbf{x}$ are said to be a DCT pair.

Similar consideration in discretizing the FST kernel

$$K_s(\omega, t) = \sin \omega t \tag{3.4.8}$$

will lead to the definition of the $(N - 1) \times (N - 1)$ DST transform matrix, whose elements are given by

$$[S]_{mn} = \sqrt{\frac{2}{N}} \sin\left(\frac{mn\pi}{N}\right) \quad m, n = 1, 2, \dots, N-1. \tag{3.4.9}$$

This matrix is also unitary and when it is applied to a data vector $\mathbf{x}$ of length $N - 1$, it produces a vector $\mathbf{X}_s$, whose elements are given by,

$$\mathrm{X}_s(m) = \sqrt{\frac{2}{N}} \sum_{n=1}^{N-1} \sin\left(\frac{mn\pi}{N}\right) \mathrm{x}(n). \tag{3.4.10}$$

The vectors $\mathbf{x}$ and $\mathbf{X}_s$ are said to form a DST pair. The inverse DST is given by

$$\mathrm{x}(n) = \sqrt{\frac{2}{N}} \sum_{m=1}^{N-1} \sin\left(\frac{mn\pi}{N}\right) \mathrm{X}_s(m). \tag{3.4.11}$$

It is evident in (3.4.7) and (3.4.11) that both DCT and DST are symmetric transforms. Both are obtained by discretizing a finite time duration into N equal intervals of Δt each, resulting in an $(N + 1) \times (N + 1)$ matrix for $[C]$ because the boundary elements are not zero, and resulting in an $(N - 1) \times (N - 1)$ matrix for $[S]$ because the boundary elements are zero.

3.4.2 Basic Properties and Operational Rules

3.4.2.1 The Unitarity Property

Let $\mathbf{c}_m$ denote the mth column vector in the matrix $[C]$. Consider the inner product of two such vectors:

$$\langle \mathbf{c}_m, \mathbf{c}_n \rangle = \sum_{p=0}^{N} k_m k_p \cos\left(\frac{mp\pi}{N}\right) k_p k_n \cos\left(\frac{pn\pi}{N}\right). \tag{3.4.12}$$

The summation can be carried out by defining the 2*N*th primitive root of unity as

$$W_{2N} = e^{-j\pi/N} = \cos\left(\frac{\pi}{N}\right) - j\sin\left(\frac{\pi}{N}\right), \tag{3.4.13}$$

and applying it to the summation in (3.4.12). This gives

$$\langle \mathbf{c}_m, \mathbf{c}_n \rangle = \left(\frac{k_m k_n}{N}\right) \mathrm{Re}\left[\sum_{p=0}^{N-1} \left(W_{2N}\right)^{-p(n-m)} + \sum_{p=1}^{N} \left(W_{2N}\right)^{-p(n+m)}\right] \tag{3.4.14}$$

where Re[·] denotes the real part of [·].

Considering the first summation in (3.4.14), and letting $\kappa = (n - m)$, the power series can be written as,

$$\begin{aligned} \sum_{p=0}^{N-1} \left(W_{2N}^{-\kappa}\right)^p &= \frac{\left(1 - W_{2N}^{-N\kappa}\right)}{\left(1 - W_{2N}^{-\kappa}\right)} \\ &= \left\{2\left[1 - \cos\left(\kappa\pi/N\right)\right]\right\}^{-1} \left\{1 - W_{2N}^{N\kappa} - W_{2N}^{\kappa} + W_{2N}^{-(N-1)\kappa}\right\}. \end{aligned} \tag{3.4.15}$$

Similarly, the second series in (3.4.14) can be summed by letting $\lambda = (n + m)$,

$$\sum_{p=1}^{N} \left(W_{2N}^{-\lambda}\right)^p = \left\{2\left[1 - \cos\left(\lambda\pi/N\right)\right]\right\}^{-1} \left\{W_{2N}^{-\lambda} - W_{2N}^{-(N+1)\lambda} - 1 + W_{2N}^{-N\lambda}\right\}. \tag{3.4.16}$$

Hence, for $m \neq n$, (i.e., $\kappa \neq 0$), the real part of (3.4.15) is

$$\mathrm{Re}\left[\sum_{p=0}^{N-1} \left(W_{2N}^{-\kappa}\right)^p\right] = \frac{\left[1 - (-1)^{\kappa}\right]\left[1 - \cos\left(\kappa\pi/N\right)\right]}{\left\{2\left[1 - \cos\left(\kappa\pi/N\right)\right]\right\}} = \left[1 - (-1)^{\kappa}\right] \Big/ 2,$$

and the real part of (3.4.16) is

$$\mathrm{Re}\left[\sum_{p=1}^{N} \left(W_{2N}^{-\lambda}\right)^p\right] = -\frac{\left[1 - (-1)^{\lambda}\right]\left[1 - \cos\left(\lambda\pi/N\right)\right]}{\left\{2\left[1 - \cos\left(\lambda\pi/N\right)\right]\right\}} = -\left[1 - (-1)^{\lambda}\right] \Big/ 2.$$

Combining these, and noting that κ and λ differ by $2m$, we obtain the orthogonality property for the inner product,

$$\langle \mathbf{c}_m, \mathbf{c}_n \rangle = 0 \quad \text{for } m \neq n. \tag{3.4.17}$$

For $m = n \neq 0$ or N, the inner product is,

$$\langle \mathbf{c}_m, \mathbf{c}_n \rangle = (1/N)\,\mathrm{Re}\left[\sum_{p=0}^{N-1} 1 + \sum_{p=1}^{N} \left(W_{2N}^{-2m}\right)^p\right] = 1,$$

and for $m = n = 0$ or N, the inner product is,

$$\langle \mathbf{c}_m, \mathbf{c}_n \rangle = (1/2N)\,\mathrm{Re}\left(\sum_{p=0}^{N-1} 1 + \sum_{p=1}^{N} 1\right) = 1.$$

Therefore, the inner product satisfies the orthonormality condition,

$$\langle \mathbf{c}_m, \mathbf{c}_n \rangle = \delta_{mn} \tag{3.4.18}$$

where δ_{mn} is the Kronecker delta and the DCT matrix $[C]$ is shown to be unitary.

Similar considerations can be applied to the DST matrix $[S]$ to show that it is also unitary.

3.4.2.2 Inverse Transformation

As alluded to in Section 3.4.1, the unitary matrices $[C]$ and $[S]$ are symmetric and, therefore, the inverse transformations are exactly the same as the forward transformations, based on the above unitarity properties. Therefore,

$$[C]^{-1} = [C] \quad \text{and} \quad [S]^{-1} = [S]. \tag{3.4.19}$$

3.4.2.3 Scaling

Recall that in the discretization of the FCT, the time and frequency intervals are related by

$$\Delta f\,\Delta t = 1/2N \quad \text{or } \Delta f = \frac{1}{2N\Delta t}. \tag{3.4.20}$$

Because the DCT and DST deal with discrete sample points, a scaling in time has no effect in the transform, except in changing the unit frequency interval in the transform domain. Thus, as Δt changes to $a\Delta t$, Δf changes to $\Delta f/a$, provided the number of divisions N remains the same. Hence, the properties (3.2.16) and (3.3.10) for the FCT and FST are retained, except for the $1/a$ factor, which is absent in the cases for DCT and DST.

Equation (3.4.20) may also be interpreted as giving the frequency resolution of a set of discrete data points, sampled at a time interval of Δt. Using $T = N\Delta t$ as the time duration of the sequence of data points, the frequency resolution for the transforms is

$$\Delta f = \frac{1}{2T}. \tag{3.4.21}$$

3.4.2.4 Shift-in-*t*

Because the data are sampled, we obtain the shift-in-time properties of DCT and DST by examining the time shifts in units of Δt. Thus, if $\mathbf{x} = [x(0), x(1), \ldots, x(N)]^T$, we define the right-shifted sequence as $\mathbf{x}^+ = [x(1), x(2), \ldots, x(N+1)]^T$. Their corresponding DCTs are given by

$$\mathbf{X}_c = [C]\mathbf{x} \quad \text{and} \quad \mathbf{X}_c^+ = [C]\mathbf{x}^+. \tag{3.4.22}$$

The shift-in-time property seeks to relate $\mathbf{X}_c^+$ with $\mathbf{X}_c$. It turns out that it relates not only to $\mathbf{X}_c$ but also to $\mathbf{X}_s$, the DST of $\mathbf{x}$. This is to be expected because the shift-in-time properties of FCT and FST are similarly related. It can be shown that the elements of $\mathbf{X}_c^+$ are given by

$$\begin{aligned} X_c^+(m) &= \cos\left(\frac{m\pi}{N}\right) X_c(m) + k_m \sin\left(\frac{m\pi}{N}\right) X_s(m) \\ &+ \sqrt{\frac{1}{N}}\, k_m \left[\left(-\frac{1}{\sqrt{2}}\right)\cos\left(\frac{m\pi}{N}\right) x(0) + \left(\frac{1}{\sqrt{2}} - 1\right) x(1) \right. \\ &\left. + (-1)^m \left(\frac{1}{\sqrt{2}} - 1\right) \cos\left(\frac{m\pi}{N}\right) x(N) + (-1)^m \frac{1}{\sqrt{2}} x(N+1)\right]. \end{aligned} \tag{3.4.23}$$

In (3.4.23), $X_c(m)$ and $X_s(m)$ are respectively the mth element of the DCT of the vector $[x(0), x(1), \ldots, x(N)]^T$ and the mth element of the DST of the vector $[x(1), x(2), \ldots, x(N + 1)]^T$. While properties analogous to the so-called kernel-product properties for FCT in Section 3.2.2 may be developed, (3.4.23) is more practical in that it provides for a way of updating a DCT of a given dimension without having to recompute all the components. The corresponding result of DST is

$$\begin{aligned} X_s^+(m) &= \cos\left(\frac{m\pi}{N}\right) X_s(m) - \sin\left(\frac{m\pi}{N}\right) X_c(m) \\ &+ \sqrt{\frac{2}{N}} \sin\left(\frac{m\pi}{N}\right) \left[\frac{1}{\sqrt{2}} x(0) - \left(1 - \frac{1}{\sqrt{2}}\right)(-1)^m x(N)\right]. \end{aligned} \tag{3.4.24}$$

Here, it is noted that $X_c(m)$ are the elements of the DCT of the vector $[x(0), \ldots, x(N)]^T$.

3.4.2.5 The Difference Property

For discrete sequences, the difference operator replaces the differential operator for continuous sequences. The FCT and the FST of a derivative, therefore, are analogous to the DCT and the DST of the difference operator. We can define a difference vector $\mathbf{d}$ as:

$$\mathbf{d} = \mathbf{x}^+ - \mathbf{x} \tag{3.4.25}$$

where $\mathbf{x}^+$ is the right-shifted version of $\mathbf{x}$. It is clear that the DCT and the DST of $\mathbf{d}$ are simply given by

$$\mathbf{D}_c = \mathbf{X}_c^+ - \mathbf{X}_c \quad \text{and} \quad \mathbf{D}_s = \mathbf{X}_s^+ - \mathbf{X}_s. \tag{3.4.26}$$

As we can see from (3.4.26), the main operational advantage of the FCT and FST, namely that in the differentiation properties, have not carried over to the discrete cases. As well, properties with both integration-in-t and integration-in-ω are also lost in the discrete cases.

We conclude this section by mentioning that no simple convolution properties exist in the cases of DCT and DST. For finite sequences, it is possible to define a circular convolution for two periodic sequences or a linear convolution of two nonperiodic sequences. With these, certain convolution properties for some of the discrete cosine transforms may be developed. (For more details, the reader is referred to Rao and Yip, 1990). The results, however, are neither simple nor easy to apply.

3.4.3 Relation to the Karhunen-Loeve Transform (KLT)

While the DCT and the DST discussed here are derived by discretizing the FCT and the FST, based on some unit time interval of Δt and some unit frequency interval of Δf, their forms are closely related to the Karhunen-Loeve transform (KLT) in digital signal processing. KLT is an optimal transform for digital signals in that it diagonalizes the auto-covariance matrix of a data vector. It completely decorrelates the signal in the transform domain, minimizes the mean squared errors (MSE) in data compression and packs the most energy (variance) in the fewest number of transform coefficients.

Consider a Markov-1 signal with correlation coefficient ρ. The $N \times N$ covariance matrix is a matrix $[A]$, which is real, symmetric, and Toeplitz. It is well known that a nonsingular symmetric Toeplitz matrix has an inverse of tri-diagonal form. In the case of the covariance matrix $[A]$ for a Markov-1 signal, we can write

$$[A]^{-1} = (1-\rho^2)^{-1}\begin{pmatrix} 1 & -\rho & 0 & 0 & \cdots & \cdots & \cdots \\ -\rho & 1+\rho^2 & -\rho & 0 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & 1+\rho^2 & -\rho \\ \cdots & \cdots & \cdots & \cdots & \cdots & -\rho & 1 \end{pmatrix}. \tag{3.4.27}$$

This matrix can be decomposed into a sum of two simpler matrices,

$$[A]^{-1} = [B] + [R]$$

where

$$[B] = (1-\rho^2)^{-1}\begin{pmatrix} 1+\rho^2 & -\sqrt{2}\rho & 0 & \cdots & \cdots \\ -\sqrt{2}\rho & 1+\rho^2 & -\rho & \cdots & \cdots \\ 0 & -\rho & 1+\rho^2 & -\rho & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & -\sqrt{2}\rho & 1+\rho^2 \end{pmatrix}$$

and

$$[R] = (1-\rho^2)^{-1}\begin{pmatrix} -\rho^2 & (\sqrt{2}-1)\rho & 0 & \cdots & \cdots \\ (\sqrt{2}-1)\rho & 0 & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & 0 & (\sqrt{2}-1)\rho \\ \cdots & \cdots & \cdots & (\sqrt{2}-1)\rho & -\rho^2 \end{pmatrix}. \tag{3.4.28}$$

We note that $[R]$ is almost a null matrix and can be considered so when N is very large. Thus, the diagonalization of the matrix $[B]$ is asymptotically equivalent to the diagonalization of the matrix $[A]^{-1}$. Furthermore, it is well known that the similarity transformation that diagonalizes $[A]^{-1}$ will also diagonalize $[A]$. From these arguments, it is concluded that the transformation that diagnalizes $[B]$ will, asymptotically, diagonalize $[A]$. The transformation that diagonalizes $[B]$ depends on a three-terms recurrence relation that is exactly satisfied by the Chebyshev polynomials. With these, it can be shown that the matrix $[V]$ that will diagonalize $[B]$ and, in turn, also $[A]$ asymptotically, is defined by

$$[V]_{mn} = k_n k_m \sqrt{\frac{2}{N-1}} \cos\left(\frac{mn\pi}{N-1}\right), \quad m, n = 0, 1, \ldots, N-1. \tag{3.4.29}$$

As can be seen in (3.4.29), these are the elements of the DCT matrix $[C]$, except that N has been replaced by $N - 1$. For large N, these are identical.

The foregoing has briefly demonstrated that for a Markov-1 signal, the diagonalization of the covariance matrix, which leads to the KLT, is provided by a transformation matrix $[V]$ which is almost identical to the DCT matrix $[C]$. This explains why the DCT performs so well in signal decorrelation, although it is signal independent. Similar arguments can be applied to the DST.

In Figure 3.1, the basis functions forming the KLT for $N = 16$ are shown. The signal is a Markov-1 signal with a correlation coefficient of $\rho = 0.95$. It is clear that the set of basis functions and, hence, the KLT is signal dependent, because they are the eigenvectors of the auto-covariance matrix of the signal vector.

In Figures 3.2 and 3.3, the basis functions for $N = 16$ of DCT and DST are shown. It is evident that they are very similar to the KLT basis functions. While it is true that the dimensions of the spaces spanned by the KLT and the DCT and DST are different, it can be shown that as N increases, both discrete transforms will asymptotically approach KLT.

However, it is true that the similarity of the basis functions does not guarantee the asymptotic behavior of the DCT and the DST, nor does it assure good performance. In applications, such as data compression and transform domain coding, the "variance distribution" of the transform coefficients is an important criterion of performance. The variance of a transform coefficient is basically a measure of the information content of that coefficient. Therefore, the higher the variances are in a few transform coefficients, the more room there is for data compression in that transform domain.

Let $[A]$ be the data covariance matrix and let $[T]$ be the transformation. Then, the covariance matrix in the transform domain, $[A]_T$, is given by,

$$[A]_T = [T][A][T]^{-1}. \tag{3.4.30}$$

The diagonal elements of $[A]_T$ are the variances of the transform coefficients. In Table 3.1, comparisons are shown for the variance distributions of the DCT, the DST, and the DFT, based on a Markov-1 signal of $\rho = 0.9$ and $N = 16$. It is clearly seen that both DCT and DST outperform DFT in using variance distribution as a performance criterion.

When the transformation $[T]$ in (3.4.30) is not the KLT, $[A]_T$ will not be diagonal. The nonzero off-diagonal elements in $[A]_T$ form a measure of the "residual correlation." The smaller the amount of residual correlation, the closer is the transform to being optimal. Figure 3.4 shows the residual correlation as a percentage of the total amount of correlation, for the transforms DCT, DST, and DFT, in a Markov-1 signal with $N = 16$. As can be seen, again DCT and DST outperform DFT generally.

There are other criteria of performance for a given transform, depending on what kind of signal processing is being done. However, using the KLT as a benchmark, DCT and DST are extremely good alternatives as signal independent, fast implementable transforms, because they are both asymptotic to the KLT. This asymptotic property of the discrete trigonometric transforms (particularly the DCT) has made them very important tools in digital signal processing. Although they are suboptimal, in the sense that they will not exactly diagonalize the data covariance matrix, they are signal independent and are computable using fast algorithms. KLT, though exactly optimal, is signal dependent and possesses no fast computational algorithm. Some typical applications are discussed in the next section.

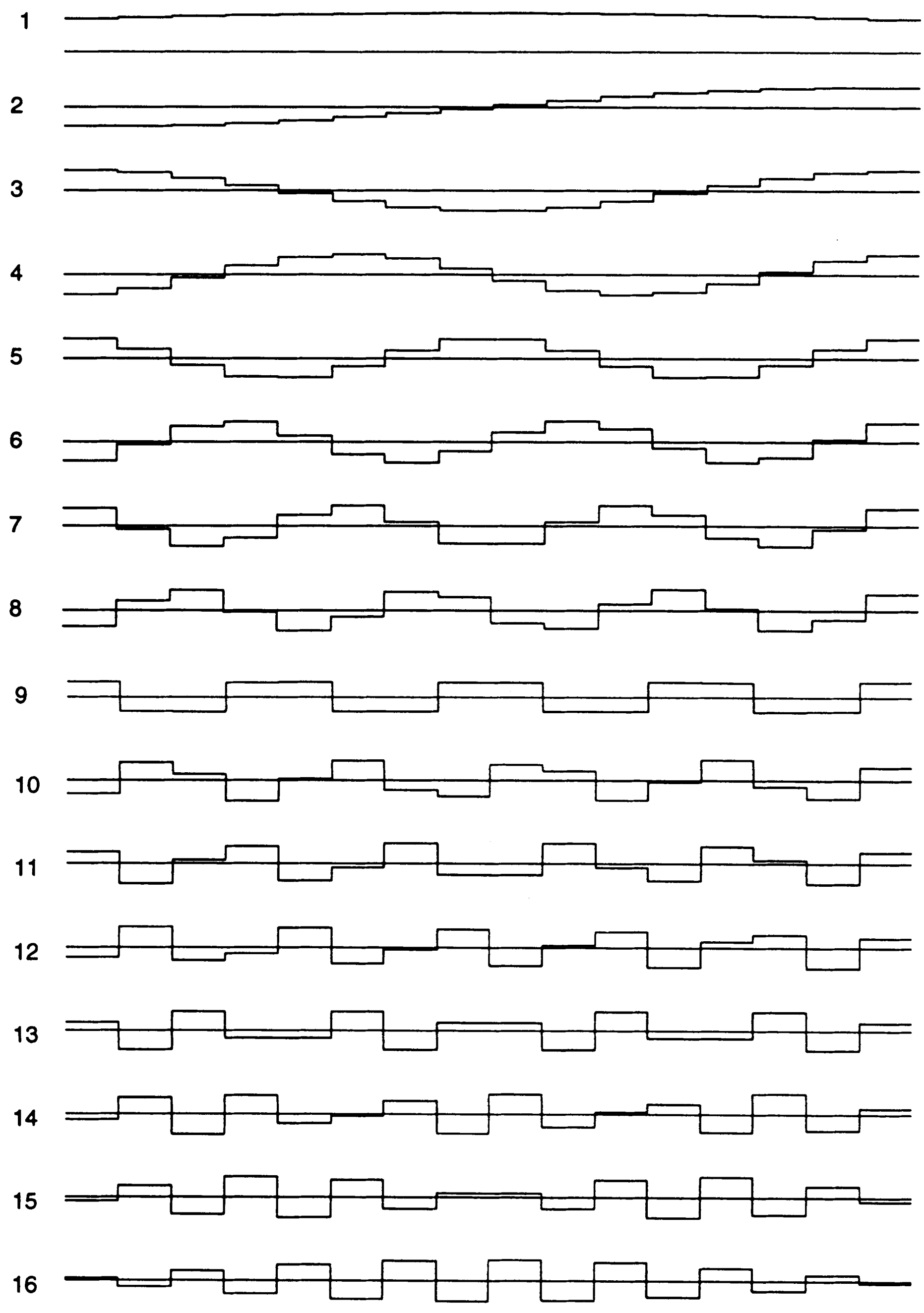

FIGURE 3.1 KLT Markov-1 signal $\rho = 0.95$, $N = 16$.

3.5 Selected Applications

This section contains some typical applications. We begin with fairly general applications to differential equations and conclude with quite specific applications in the area of data compression. (See Churchill, 1958 and Sneddon, 1972 for more applications.)

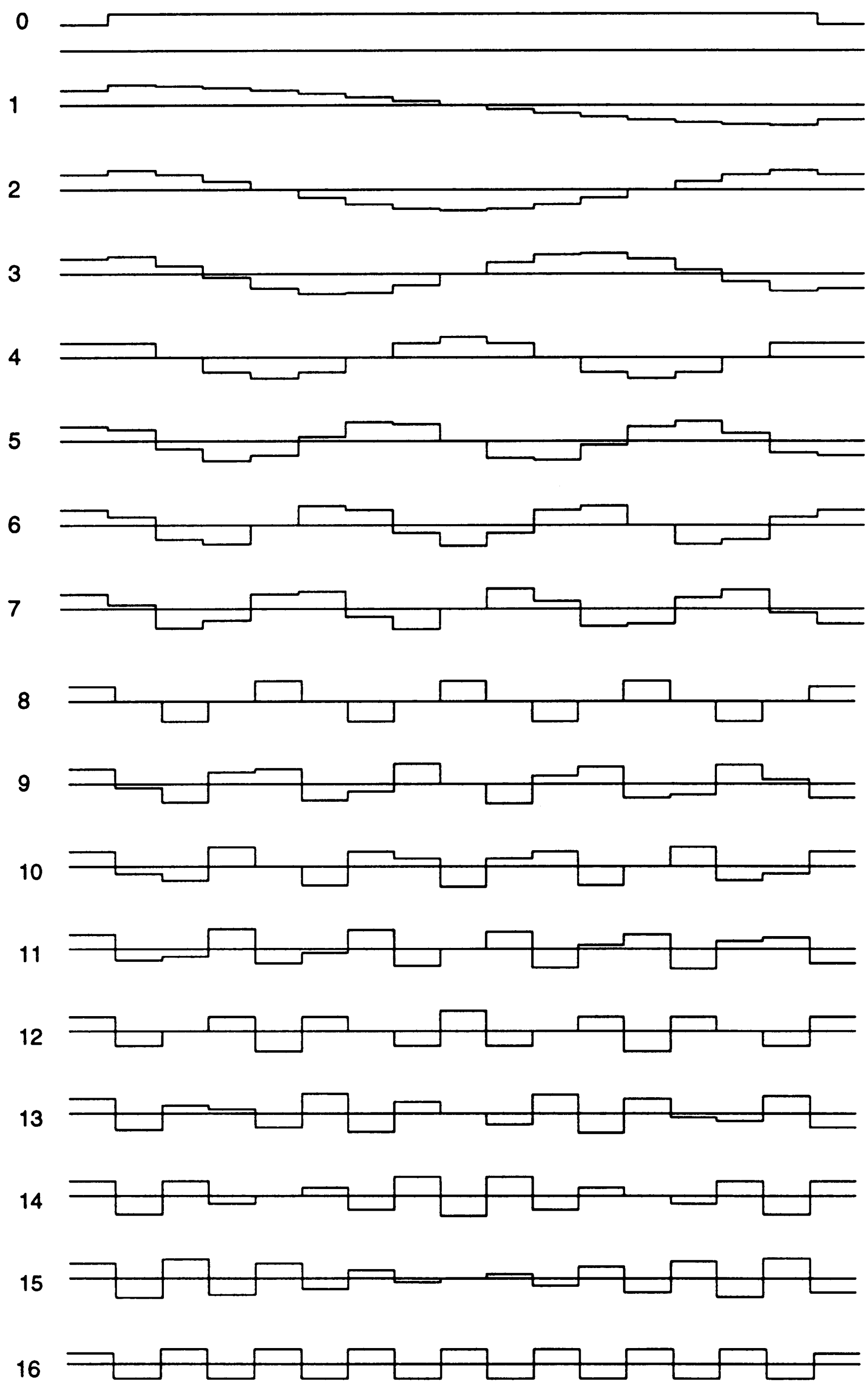

FIGURE 3.2 DCT $N = 16$.

3.5.1 Solution of Differential Equations

3.5.1.1 One-Dimensional Boundary Value Problem

Consider the second-order differential equation,

$$y''(t) - h^2 y(t) = F(t) \quad t \geq 0 \tag{3.5.1}$$

with boundary conditions: $y'(0)$ and 0 and $y(\infty) = 0$, and

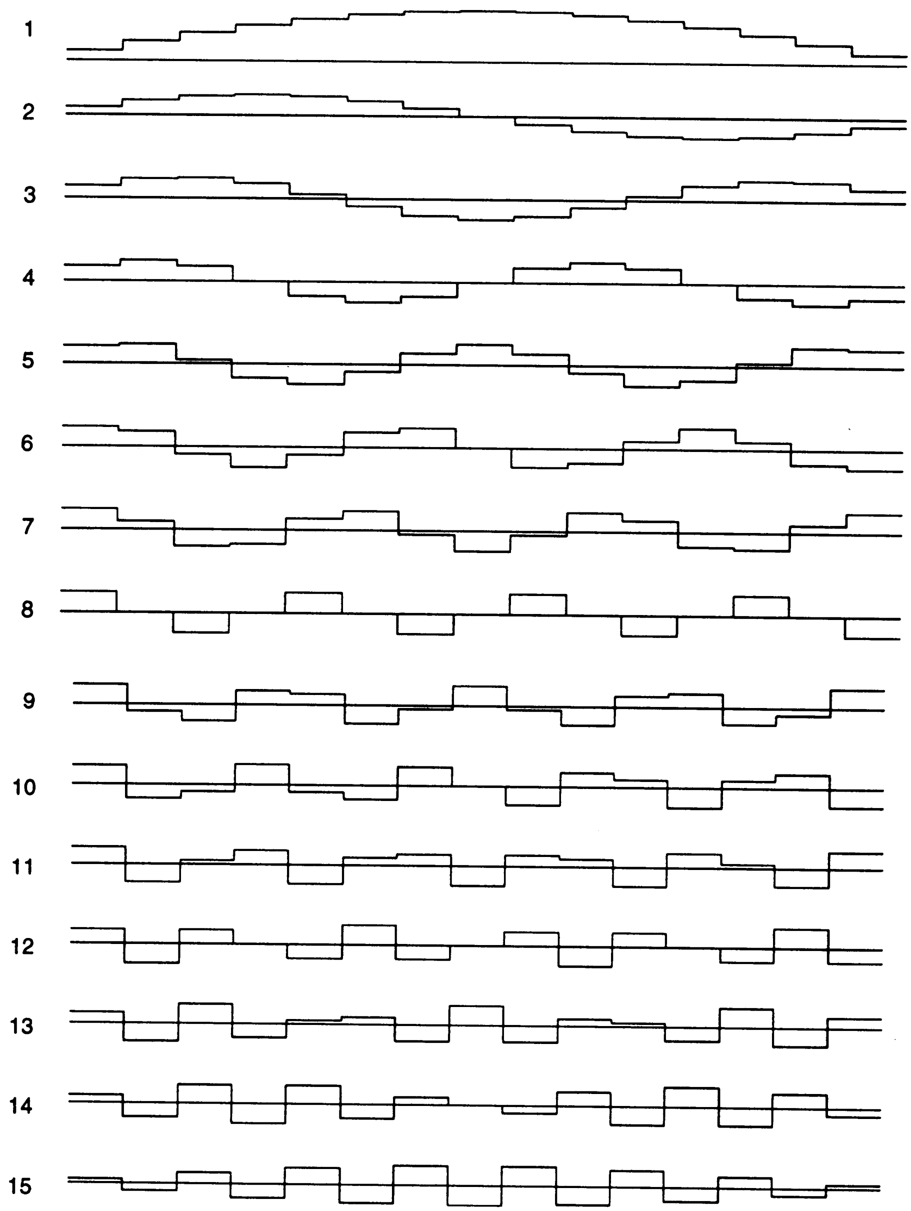

FIGURE 3.3 DST $N = 16$.

$$F(t) = A \quad \text{for } 0 < t < b$$
$$= 0 \quad \text{otherwise}.$$

We note that $F(t)$ can be expressed in terms of a Heaviside step function, Thus,

$$F(t) = A[1 - U(t - b)]. \tag{3.5.2}$$

Here, we assume h, A, and b to be constants. Applying the operator $\mathcal{F}_c$ to the differential equation and using the results in (3.2.9) and (3.2.34), we get

$$-\omega^2 Y_c - y'(0) - h^2 Y_c = \frac{A}{\omega} \sin \omega b. \tag{3.5.3}$$

TABLE 3.1 Variance Distributions for $N = 16$, $\rho = 0.9$

i	DCT*	DST	DFT
0	9.835	9.218	9.835
1	2.933	2.640	1.834
2	1.211	1.468	1.834
3	0.581	0.709	0.519
4	0.348	0.531	0.519
5	0.231	0.314	0.250
6	0.166	0.263	0.250
7	0.129	0.174	0.155
8	0.105	0.153	0.155
9	0.088	0.110	0.113
10	0.076	0.099	0.113
11	0.068	0.078	0.091
12	0.062	0.071	0.091
13	0.057	0.061	0.081
14	0.055	0.057	0.081
15	0.053	0.054	0.078

*DCT is DCT-II here.

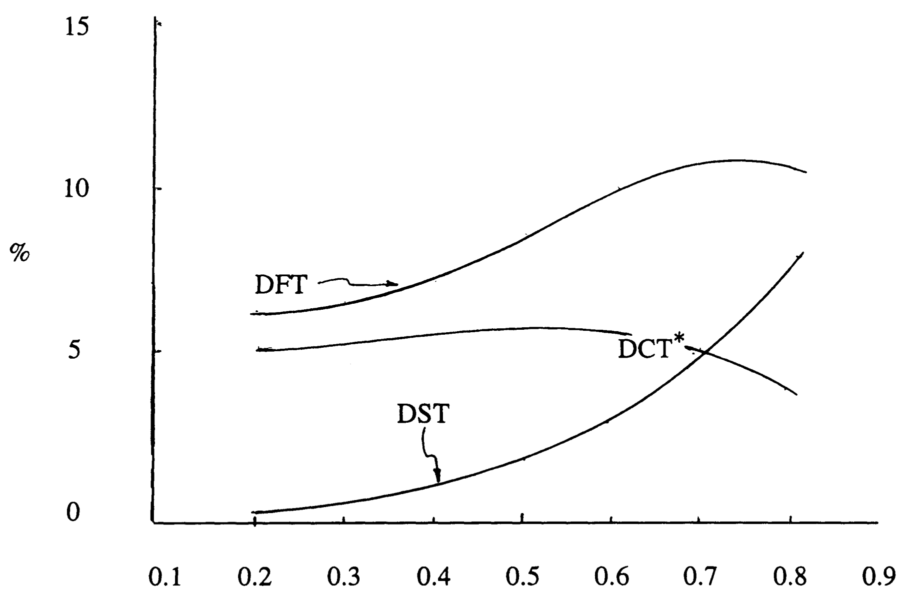

FIGURE 3.4 Percent Residual Correlation as a function of ρ, $N = 16$.

Applying the boundary condition and solving for Y_c, we obtain

$$\begin{aligned} Y_c &= -\frac{A}{\omega\left(\omega^2 + h^2\right)}\sin\omega b \\ &= -\frac{A}{h^2}\left(\frac{\sin\omega b}{\omega} - \frac{\omega\sin\omega b}{\omega^2 + h^2}\right). \end{aligned} \tag{3.5.4}$$

The inversion of Y_c can be accomplished with the use of (3.2.34), (3.2.55), and (3.2.3). Noting that the inverse FCT has a normalization factor of $2/\pi$, the solution for the original boundary value problem is given by

$$y(t) = -\frac{A}{h^2}\left[1 - U(t-b) - e^{-hb}\cosh ht\right] \quad t < b,$$

$$= -\frac{A}{h^2}\left[1 - U(t-b) + e^{-ht}\sinh hb\right] \quad \text{for } t > b.$$

These can be rewritten as

$$y(t) = \frac{A}{h^2}\left(e^{-hb}\cosh ht - 1\right) \quad \text{for } t < b,$$
$$= -\frac{A}{h^2} e^{-ht}\sinh hb \quad \text{for } t > b. \tag{3.5.5}$$

3.5.1.2 Two-Dimensional Boundary Value Problem

Consider a function $\upsilon(x, y)$, which is bounded for $x \geq 0$, $y \geq 0$. Let $\upsilon(x, y)$ satisfy the boundary value problem:

$$\frac{\partial^2 \upsilon}{\partial x^2} + \frac{\partial^2 \upsilon}{\partial y^2} = -h(x); \quad \left.\frac{\partial \upsilon}{\partial x}\right|_{x=0} = 0, \quad \upsilon(x,0) = f(x). \tag{3.5.6}$$

We further assume that $\int_0^\infty h(x)dx = 0$, and that the function

$$p(x) = \int_x^\infty \left[\int_0^r h(t)\,dt\right] dr \tag{3.5.7}$$

exists and that the functions $p(x)$ and $f(x)$ have FCTs. We note from (3.5.7) that

$$p''(x) = h(x) \quad \text{and} \quad p'(0) = 0,$$

leading to the following relation between their FCTs:

$$\omega^2 P_c(\omega) = H_c(\omega) \tag{3.5.8}$$

Applying $\mathscr{F}_c$ for the x variable in (3.5.6) reduces the partial differential equation to

$$-\omega^2 V_c(\omega, y) + \frac{\partial^2}{\partial y^2} V_c(\omega, y) = -\omega^2 P_c(\omega). \tag{3.5.9}$$

Because $V_c(\omega, y)$ is bounded for $y > 0$, (3.5.9) has the following solution,

$$V_c(\omega, y) = Ce^{-\omega y} + P_c(\omega) \tag{3.5.10}$$

where C is an arbitrary constant, to be determined by $\upsilon(x, 0) = f(x)$. In the ω-domain, this means

$$V_c(\omega, 0) = F_c(\omega). \tag{3.5.11}$$

Thus,

$$V_c(\omega, y) = [F_c(\omega) - P_c(\omega)]\, e^{-\omega y} + P_c(\omega). \tag{3.5.12}$$

This can be inverted and the solution in the (x, y) domain then is given by

$$\upsilon(x, y) = p(x) + \frac{1}{\pi}\int_0^\infty \left[f(t) - p(t)\right]\left[\frac{y}{(x+t)^2 + y^2} + \frac{y}{(x-t)^2 + y^2}\right] dt. \tag{3.5.13}$$

Here, we have made use of (3.2.44) and the convolution result of (3.2.20).

3.5.1.3 Time-Dependent One-Dimensional Boundary Value Problem

Consider the function $u(x, t)$, which is bounded for $x, t \geq 0$. Let this function satisfy the partial differential equation,

$$\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = h(x, t) \tag{3.5.14}$$

so that $u(x, 0) = f(x)$ and $u(0, t) = g(t)$ are the initial and boundary conditions.

Applying the FST for the variable x to (3.5.14) and assuming the existence of all the integrals involved, we obtain

$$\frac{\partial U_s}{\partial t} + \omega^2 U_s = \omega g(t) + H_s(\omega, t). \tag{3.5.15}$$

The solution for (3.5.15) is

$$U_s(\omega, t)\, e^{\omega^2 t} = \int_0^t \left[\omega g(\tau) + H_s(\omega, \tau)\right] e^{\omega^2 \tau} d\tau + C. \tag{3.5.16}$$

C is easily found to be $F_s(\omega)$ using the condition $U_s(\omega, 0) = F_s(\omega)$. With this, (3.5.16) can be inverse transformed by applying the operator $\mathscr{F}_s^{-1}$ to get

$$u(x, t) = \frac{2}{\pi}\int_0^\infty U_s(\omega, t) \sin \omega x \, d\omega. \tag{3.5.17}$$

We note that, depending on the forms of the functions F_s and H_s, the inverse FST may be obtained by table look-up.

3.5.2 Cepstral Analysis in Speech Processing

In cepstral analysis, a sequence is converted by a transform T, the logarithm of its absolute value is then taken and the cepstrum is then obtained by inverse transformation T^{-1}. Figure 3.5 shows the essential steps in cepstral analysis. Here, $\{x(n)\}$ is the input speech sequence, $\{X(k)\}$ is the transform sequence, and the output $\{x_R(n)\}$ is called the real cepstrum.

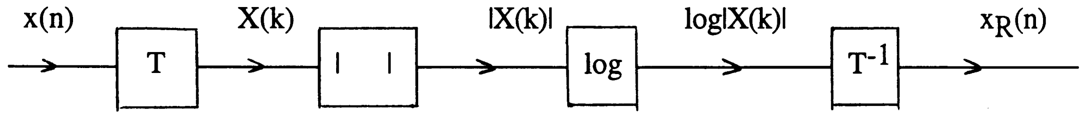

FIGURE 3.5 Block Diagram for Cepstral Analysis for $x(n)$.

The transform may be any invertible transform. When T is an N-point DFT, the scheme can be implemented using the DCT. In the computation to obtain the real cepstrum using the DFT, the input sequence has to be padded with trailing zeros to double its length. However, a simple relation between the DFT and the DCT for real even sequences reduces the DFT to a DCT.

Let $x(n)$, $n = 0, 1, 2, \ldots, M$ be the input speech sequence to be analyzed. To obtain the real cepstrum $x_R(n)$ using DFT, the sequences is padded with zeros so that $x(n) = 0$, for $n + M + 1, \ldots, 2M - 1$. If we consider a symmetric sequence $s(n)$ defined by

$$\begin{aligned} s(n) &= x(n) && 0 < n < M, \\ &= 2x(n) && n = 0, M \\ &= x(2M - n) && M < n \le 2M - 1, \end{aligned} \tag{3.5.18}$$

then the DFT of $s(n)$ can be obtained as

$$S_F(k) = 2\left[x(0) + (-1)^k x(M) + \sum_{n=1}^{M-1} x(n)\cos\left(\frac{nk\pi}{M}\right)\right]. \tag{3.5.19}$$

Equation (3.5.19) is clearly in the form of a DCT of the sequence $\{x(n)\}$ up to a constant factor of normalization. Now, because $\{s(n)\}$ is a symmetric real sequence, constructed out of $\{x(n)\}$, we have

$$S_F(k) = \mathrm{Re}[X_F(k)]$$

where $\{X_F(k)\}$is the 2M-point DFT of the zero-padded sequence. Combining this with (3.5.19) we see that

$$\mathrm{Re}[X_F(k)] = 2[X_c(k)] \tag{3.5.20}$$

where X_c is the $(M + 1)$-point DCT of the speech sequence $\{x(n)\}$. Equation (3.5.20) is valid up to a normalization constant. Because direct sparse matrix factorization of the $(M + 1) \times (M + 1)$ DCT matrix is possible, fast algorithms exist for the computation of the DCT. This means that in order to obtain the real cepstrum of $\{x(n)\}$, there is no need to pad the sequence with trailing zeros, and the computation for $x_R(k)$ can be achieved through the use of the DCT of the sequence $\{x(n)\}$.

Rather than using DCT as a means of computing the DFT, the transform T in the cepstral analysis can directly be a DCT or a DST. It has been found that the performance of speech cepstral analysis using DCT and DST is comparable to the traditional DFT cepstral analysis.

3.5.3 Data Compression

Data compression is an important application of transform coding when retrieval of a signal from a large database is required. Transform coefficients with large variances can be retained to represent significant features for pattern recognition, for example. Those with small variances, below a certain threshold, can be discarded. Such a scheme can be used in reducing the required bandwidth for purposes of transmission or storage.

The transforms used for these data compression purposes require maximal decorrelation of the data, with highest energy-packing efficiency possible (efficiency is defined as how much energy can be packed into the fewest number of transform coefficients). The ideal or optimal transform is the KLT, which will diagonalize the data covariance matrix and pack the most energy into the fewest transform coefficients. Unfortunately, KLT is data dependent, and has no known fast computational algorithm, and, therefore, is not practical. On the other hand, Markov models describe most of the data systems quite well, and suboptimal but asymptotically equivalent transforms such as the DCT and the DST are data independent, and implementable using fast algorithms. Therefore, in many applications, such as storage of electrocardiogram (ECG) or vector cardiogram (VCG) data, or video data transmission over telephone lines for video phones, suboptimal transforms such as the DCT are preferred over the optimal KLT. For such applications, depending upon the required fidelity of the reconstructed data, compression ratios of up to 10:1 have been reported, and compression ratios of 3:1 to 5:1 using DCT for both ECG (one-dimensional) and VCG (two-dimensional) are commonplace.

Figure 3.6a and 3.6b show the block diagrams for processing, storage, and retrieval of a one-dimensional ECG, using m:1 compression ratio.

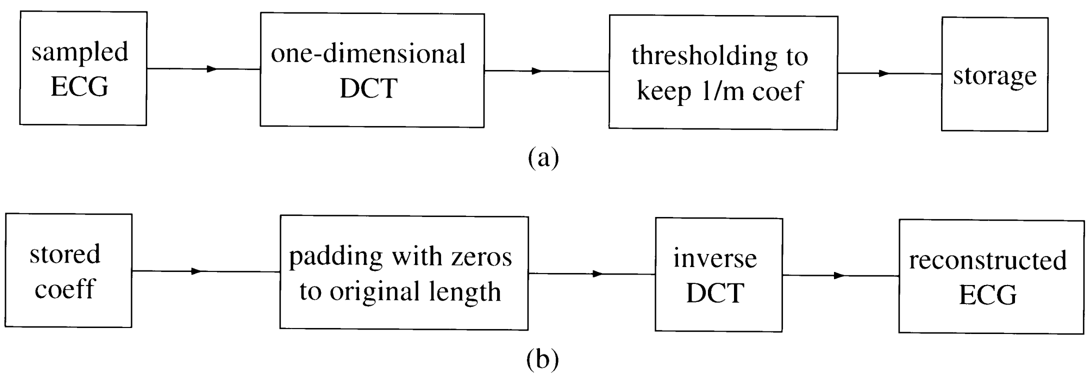

FIGURE 3.6 (a) Data compression for storage, (b) reconstruction from compressed data.

3.5.4 Transform Domain Processing

While discarding low variance coefficients in the DCT domain will provide data compression, certain details or desired features in the original data may be lost in the reconstruction. It is possible to remedy this partially by processing the transform coefficients before reconstruction. Adaptive processing can be applied based on some subjective criteria, such as in video phone applications. Coefficient quantization is another means of processing to minimize the effect of noise.

Other processing techniques such as subsampling (decimation) and up-sampling (interpolation) can also be performed in the DCT domain, effectively combining the operations of filtering and transform coding. Such processing techniques have been successfully employed to convert high definition TV signals to the standard NTSC TV signals.

One of the most popular digital signal processing tools is the adaptive least-mean-square (LMS) filtering. This can be done either in the time domain or in the transform domain. Figure 3.7 shows the block diagram for the adaptive DCT transform domain LMS filtering. Here a_{n0}, a_{n1}, …, $a_{n,N-1}$ are the adaptive weights for the transform domain filter. The desired response is $\{r(n)\}$ and $\{y(n)\}$ is the filtered output. It has been found that such transform domain filtering speeds up the convergence of the LMS algorithm for speech-related applications such as spectral analysis and echo cancellation.

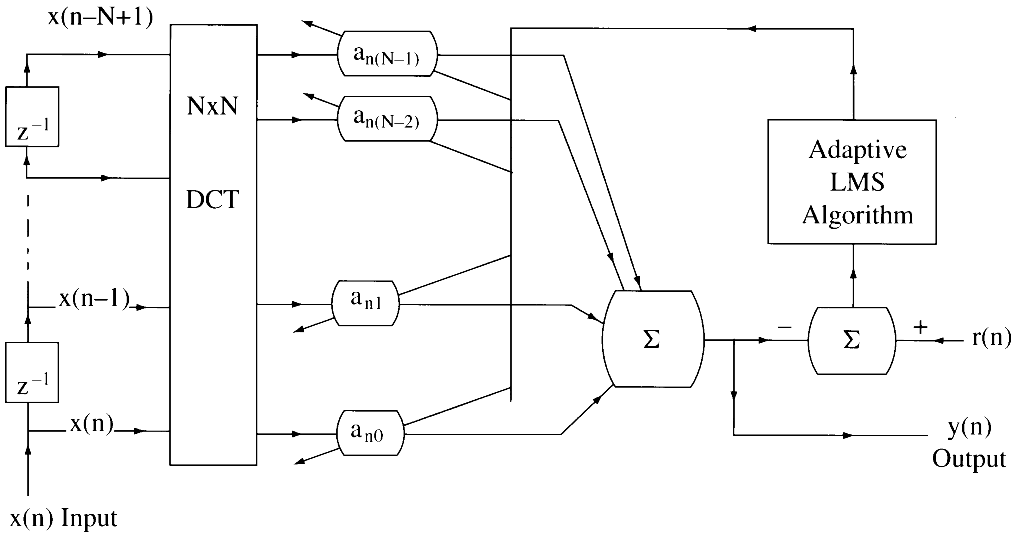

FIGURE 3.7 Adaptive transform domain LMS filtering.

3.5.5 Image Compression by the Discrete Local Sine Transform (DLS)

3.5.5.1 Introduction

Discrete Cosine Transform (DCT) has long been recognized as one of the best substitutes for the optimal, but data-dependent Karhunen-Loeve Transform (KLT), in image processing. Many standards, such as the JPEG (Joint Photographic Experts Group) and MPEG (Moving Pictures Experts Group) have adopted DCT as a standard transform technique for image compression. While both KLT and DCT satisfy the perfect reconstruction (PR) condition when no compression (or dropping of transform coefficients) takes place in the transform domain, both suffer from the artifact of "blocking" whenever compression is done. The severity of such an artifact depends on the amount of compression. In speech and audio processing, this appears as a clicking sound in the reconstructed speech. In image compression, it appears as "tiles" overlaying the reconstituted picture.

The blocking artifact can be attributed to the fact that two-dimensional image processing by transform generally takes place with blocks of pixels, the most common sizes being 8 × 8 and 16 × 16. When modification of the transform coefficients occurs in compression or other transform domain processing, the PR condition is violated. The mismatching of the edges in the reconstructed blocks produces this artifact.

Efforts to counter this compression artifact led to the development of lapped transforms (see Malvar, 1992). The transforms are based on basis functions with a wider support in the data domain than in the transform domain, leading to overlaps of the basis functions in the edge region of each block; hence, the name "lapped" transform. Many such lapped transforms can be constructed using different criteria. There are lapped orthogonal transforms (LOT), modulated lapped transforms (MLT), and hierarchical lapped transform (HLT). There are also lapped transforms based on the discrete sine or cosine basis functions.

In this subsection, one such lapped transform based on the discrete sine basis function is described. This is called the discrete local sine transform or DLS. The transform is applied in image compression at different compression ratios and the results are compared with other lapped transforms.

3.5.5.2 Elements of the Lapped Orthogonal Transform (LOT)

In general, a lapped transform will take N sample points in the data domain and transform these into M coefficients in the conjugate domain, where $N > M$. Very often, N can be as much as twice the size of M. In matrix vector notations, a data vector $\boldsymbol{x}_m$ of length N is transformed into a vector X_m of length M, and the transform is represented by the $M \times N$ matrix Φ^T in the equation

$$X_m = \Phi^{\mathrm{T}} \boldsymbol{x}_m \tag{3.5.21}$$

Here Φ is the lapped transform matrix of dimension $N \times M$. One might interpret such a matrix as an M-dimensional matrix spanned by M N-dimensional vectors. Specifically, if $M = 2$ and $N = 4$, then the two-dimensional vector space is spanned by two linearly independent four-dimensional vectors. As can be imagined, such a scheme will provide additional flexibility in the design of the transform basis functions.

When a data sequence is to be processed by a lapped transform, the basic block transform matrix Φ is of dimension $N \times M$, whereas the overall transform matrix Ψ will be in block diagonal form, given by

$$\Psi = \begin{pmatrix} \Phi & & O \\ & \Phi & \\ & & \ddots & \\ O & & \Phi \end{pmatrix} \tag{3.5.22}$$

If $\Phi^{\mathrm{T}} = \begin{bmatrix} a_{11} & a_{21} & a_{31} & a_{41} \\ a_{12} & a_{22} & a_{32} & a_{42} \end{bmatrix}$, the matrix Ψ will appear as

$$\Psi = \begin{bmatrix} \ddots & & & & & \\ & a_{11} & a_{12} & & & \\ & a_{21} & a_{22} & & & \\ & a_{31} & a_{32} & a_{11} & a_{12} & \\ & a_{41} & a_{42} & a_{21} & a_{22} & \\ & & & a_{31} & a_{32} & \\ & & & a_{41} & a_{42} & \\ & & & & & \ddots \end{bmatrix} \tag{3.5.23}$$

when the length of the overlap is 2.

For a data sequence $\boldsymbol{x}_m$ of dimension K, the lapped transformed sequence X_m is given by

$$X_m = \Psi^{\mathrm{T}} \boldsymbol{x}_m. \tag{3.5.24}$$

Evidently, in the segmented form of $\boldsymbol{x}_m$ (each segment of length N), the data points located at the ends, in the overlapped regions, will be processed in two consecutive block transforms. One can visualize this as a sliding window of size N moving over the data sequence in shifts of size M each.

When compression or other processing is not applied, all invertible transforms should satisfy the PR condition. In terms of the transformation matrix, this PR condition is stated simply as

$$\Psi\Psi^{\mathrm{T}} = I_{\mathrm{K}} \quad \text{and} \quad \Psi^{\mathrm{T}}\Psi = I_{\mathrm{K}} \tag{3.5.25}$$

where $\boldsymbol{I}_{\mathrm{K}}$ is a $K \times K$ identity matrix. From (3.5.25) conditions for the component block matrix Φ can be stated

$$\Phi^{\mathrm{T}}\Phi = \boldsymbol{I}_{\mathrm{M}} \tag{3.5.26}$$

and

$$\Phi^{\mathrm{T}} W \Phi = O_{\mathrm{M}}, \tag{3.5.27}$$

where W is an $M \times M$ "one block shift" matrix defined by

$$W = \begin{pmatrix} O_1 & I_L \\ O_2 & O_1 \end{pmatrix}.$$

Here, L is the length of the overlap region, O_1 is an $L \times (M\text{-}L)$ null matrix, O_2 is an $(M\text{-}L) \times (M\text{-}L)$ null matrix, and O_M is an $M \times M$ null matrix. Thus, in addition to the usual orthonormality condition (3.5.26), lapped transforms require the additional "lapped orthogonality" condition (3.5.27) to preserve the overall PR requirement.

3.5.5.3 The Discrete Local Sine Transform (DLS)

By properly choosing a "core" and a "lapped" region together with a specified function, a lapped transform basis set can be constructed to satisfy the PR condition. The DSL is just such a set, based on the continuous bases of Coifman and Meyer [See Coifman and Meyer, 1991.]

Let Φ_s be the DLS transform matrix, so that

$$\Phi_s = [\boldsymbol{\phi}_0, \boldsymbol{\phi}_1, \cdots, \boldsymbol{\phi}_{M-1}]. \tag{3.5.28}$$

Then the basis functions ϕ_r's are defined by

$$\phi_r(n) = \sqrt{(2/M)}\left\{b(n)\sin\left[\frac{2r+1}{2}\pi\left(\frac{n}{M}-\varepsilon\right)\right]\right\}; \quad n \in [0, M+L-1]; \quad r \in [0, M-1] \tag{3.5.29}$$

where n, r are respectively the index for the data sample and the index of the basis function; $\varepsilon = (L\text{-}1)/2M$; M is the number of basis functions in the set and L is the length of the lapped portion. $b(n)$ is called a bell function and it controls the roll-off over the lapped portion of the basis function. It is given by

$$b(n) = \begin{cases} S_\varepsilon(n) = \sin\left[\dfrac{n\pi}{2(L-1)} - \dfrac{1}{4}\sin\dfrac{2n\pi}{L-1}\right], & n = 0, \ldots, L-1, \\ 1, \qquad\qquad\qquad\qquad \text{for} & n = L, \ldots, M-1, \\ C_\varepsilon(n-M) = \cos\left[\dfrac{(n-M)\pi}{2(L-1)} - \dfrac{1}{4}\sin\dfrac{2(n-M)\pi}{L-1}\right], & n = M, \ldots M+L-1. \end{cases}$$

Figure 3.8 shows the DLS basis functions in time and frequency domains for $M = 8$, $L = 8$. These basis functions are very similar to those of the modulated lapped transform (MLT) developed by Malvar (1992).

3.5.5.4 Simulation Results (For details, see Li, 1997.)

The standard Lena image of 256 × 256 pixels is used in the simulations for image compression. The original image is represented by 8 bits/pixel or 8 bpp and is shown in Figure 3.9(a). Compressions based on a 16 × 16 block transform ($M = L = 16$ for lapped transforms) result in reconstructed images represented by 0.4 bpp, 0.24 bpp, and 0.16 bpp. A signal-to-noise ratio is calculated for the compressed image, based on the energy (variance) of the original image and the energy of the residual image. The residual image is defined as the difference between the original image and the compressed image. For lapped transforms, zeros are padded on the actual border of the image to enable the transform.

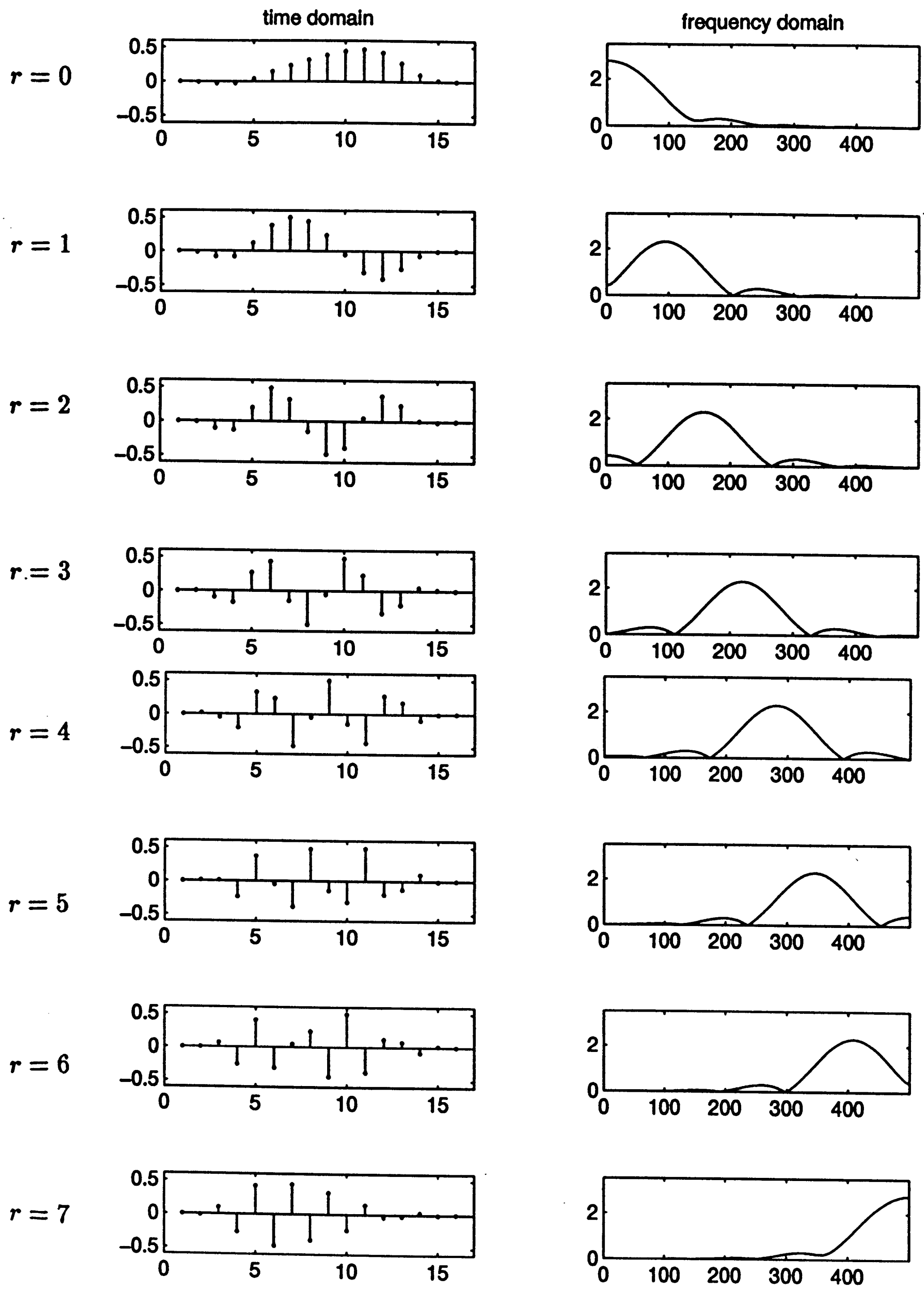

FIGURE 3.8 DLS basis functions in time and frequency domain, $L = M = 8$.

Table 3.2 shows a comparison of the final signal-to-noise ratios for the several lapped transforms against the more conventional DCT at different compression ratios. It is obvious that the lapped transforms are superior in performance compared to the DCT.

Figures 3.9, 3.10, and 3.11 depict the various reconstructed images using different lapped transforms at different compression ratios. It is seen that serious "block" artifacts are absent from the compressed images even at the very low bits per pixel rates. The performance of the DLS lies between those of the LOT and the MLT.

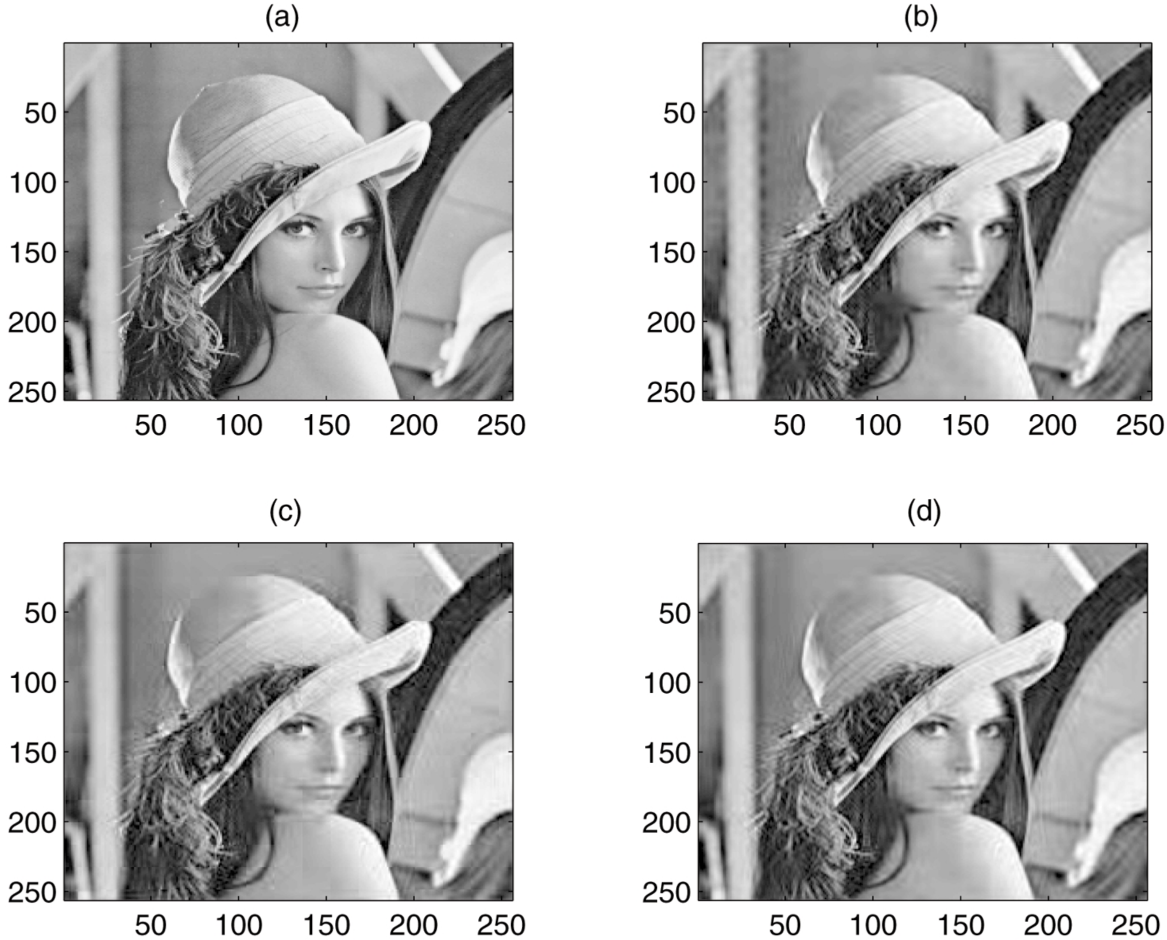

FIGURE 3.9 Comparison of original and reconstructed image, $M = L = 16$, at 0.4 bpp: (a) original at 8 bpp, (b) DLS, (c) LOT, (d) MLT.

TABLE 3.2 Comparison of Signal-to-Noise Ratio (dB)

	DLS	LOT	MLT	DCT
0.4 bpp	16.3	15.8	16.5	13.9
0.24 bpp	13.8	13.6	14.3	12.2
0.16 bpp	12.2	12.2	12.7	11.2

3.6 Computational Algorithms

In actual computations of FCT and FST, the basic integrations are performed with quadratures. Because the data are sampled and the duration is finite, most of the quadratures can be implemented via matrix computations. The fact that the FST and the FCT are closely related to the Fourier transform translates directly to the close relations between the computation of the DCT and the DST with that of the DFT. Many algorithms have been developed for the DFT. The most well known among them is the Cooley-Tukey fast Fourier transform (FFT), which is often regarded as the single most important development in modern digital signal processing. More recently, there have been other algorithms such as the Winograd algorithm, which are based on prime-factor decomposition and polynomial factorization.

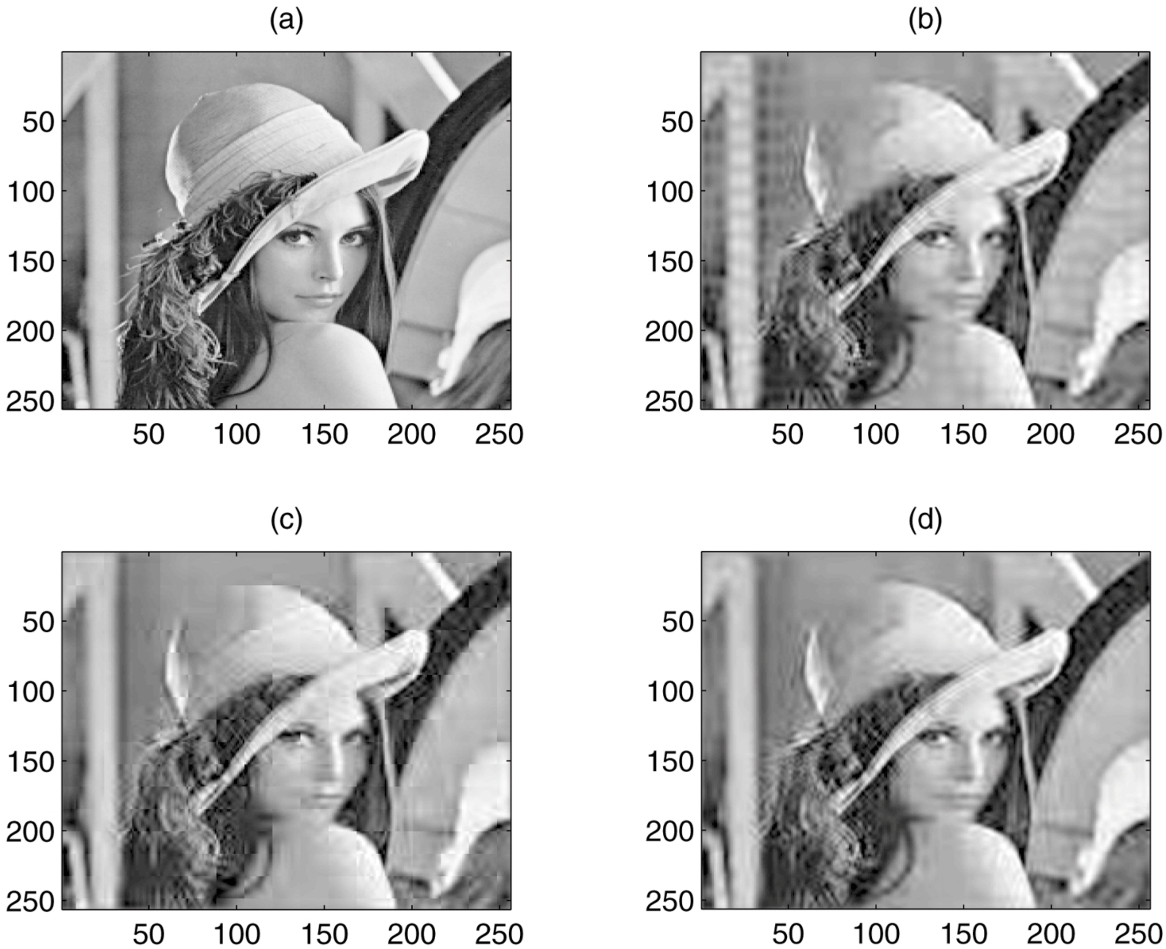

FIGURE 3.10 Comparisons for original and reconstructed image, $M = L = 16$, at 0.24 bpp: (a) original at 8 bpp, (b) DLS, (c) LOT, (d) MLT.

While DST and DCT can be computed using relations with DFT (thus, fast algorithms such as the Cookey-Tukey or the Winograd), the transform matrices have sufficient structure to be exploited directly, so that sparse factorizations can be applied to realize the transforms. The sparse factorization depends on the size of the transform, as well as the way permutations are applied to the data sequence. As a result, there are two distinct types of sparse factorizations, the decimation-in-time (DIT) algorithms and the decimation-in-frequency (DIF) algorithms. (DIT algorithms are of the Cooley-Tukey type while DIF algorithms are of the Sande-Tukey type).

In Section 3.6.1, the computations of FST and FCT using FFT are discussed. In Section 3.6.2, the direct fast computations of DCT and DST are presented. Both DIT and DIF algorithms are discussed. All algorithms discussed are radix-2 algorithms, where N, which is related to the sample size, is an integer power of two.

3.6.1 FCT and FST Algorithms Based on FFT

3.6.1.1 FCT of Real Data Sequence

Let $\{x(n), n = 0, 1, \ldots, N\}$ be an $(N + 1)$-point sequence. Its DCT as defined in (3.4.6) is given by

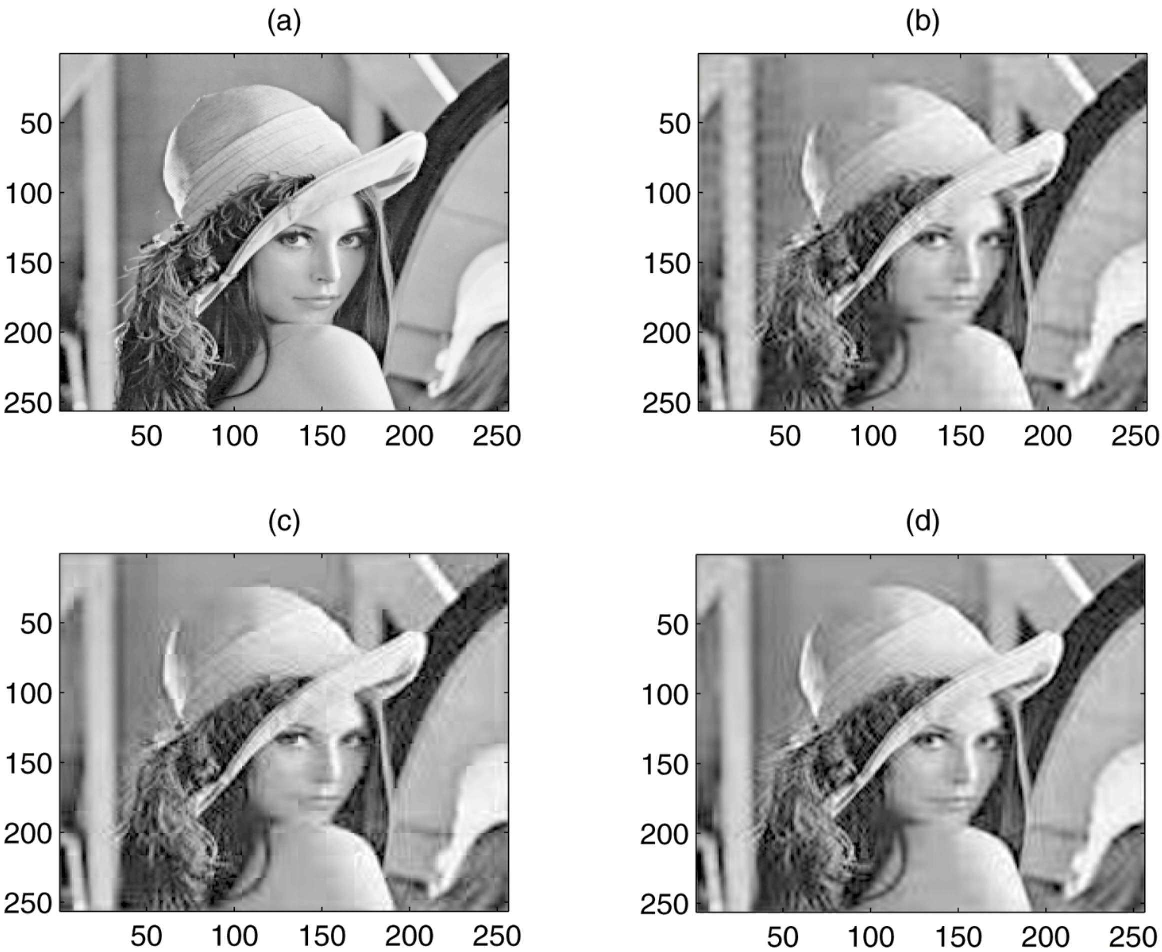

FIGURE 3.11 Comparisons of original and reconstructed image, $M = L = 16$, at 0.16 bpp: (a) original at 8 bpp, (b) DLS, (c) LOT, (d) MLT.

$$X_c(m) = \sqrt{\frac{2}{N}} \sum_{n=0}^{N} k_m k_n \cos\left(\frac{mn\pi}{N}\right) x(n),$$

where

$$\begin{aligned} k_n &= 1 \qquad \text{for } n \neq 0 \text{ or } N \\ &= 1/\sqrt{2} \qquad \text{for } n = 0 \text{ or } N. \end{aligned}$$

Construct an even or symmetric sequence using $\{x(n)\}$ in the following way,

$$\begin{aligned} s(n) &= x(n) \qquad 0 < n < N, \\ &= 2x(n) \qquad n = 0, N, \\ &= x(2N-n) \quad N < n \leq 2N-1. \end{aligned} \tag{3.6.1}$$

Based on the fact that the Fourier transform of a real symmetric sequence is real and is related to the cosine transform of the half-sequence, it can be shown that the DFT of $\{s(n)\}$ is given by

$$S_F(m) = 2\left[x(0) + (-1)^m x(N) + \sum_{n=1}^{N-1} \cos\left(\frac{mn\pi}{N}\right) x(n)\right]. \tag{3.6.2}$$

Thus, the $(N+1)$-point DCT of $\{x(n)\}$ is the same as the $2N$-point DFT of the sequence $\{s(n)\}$, up to a normalization constant as indicated by (3.4.6). This means that the DCT of $\{x(n)\}$ can be computed using a $2N$-point FFT of $\{s(n)\}$. We note here that

$$S_F(m) = \sum_{n=0}^{2N-1} s(n) W_{2N}^{mn}, \tag{3.6.3}$$

where $W_{2N} = e^{-j2\pi/2N}$, the principal $2N$th root of unity, is used for defining the DFT.

It should be pointed out that the direct $2N$-point DFT of a real even sequence may be considered inefficient, because inherent complex arithmetics are used to produce real coefficients in the transform. However, it is well known that a real $2N$-point DFT can be implemented using an N-point DFT for a complex sequence. For details, the reader is referred to Chapter 2 on Fourier transforms.

3.6.1.2 FST of Real Data Sequence

Let $\{x(n), n = 1, 2, \ldots, N-1\}$ be an $(N-1)$-point data sequence. Its DST as defined in (3.4.10) is given by

$$X_s(m) = \sqrt{\frac{2}{N}} \sum_{n=1}^{N-1} \sin\left(\frac{mn\pi}{N}\right) x(n).$$

Construct a $(2N-1)$-point odd or skew-symmetric sequence $\{s(n)\}$ using $\{x(n)\}$,

$$\begin{aligned} s(n) &= x(n) && 0 < n < N, \\ &= 0 && n = 0, N, \\ &= -x(2N-n) && N < n \le 2N-1. \end{aligned} \tag{3.6.4}$$

The Fourier transform of a real skew-symmetric sequence is purely imaginary and is related to the sine transform of the half-sequence. From this, it can be shown that the $2N$-point DFT of $\{s(n)\}$ in (3.6.4) is given by

$$S_F(m) = -2j \sum_{n=1}^{N-1} \sin\left(\frac{mn\pi}{N}\right) x(n). \tag{3.6.5}$$

Thus, the $2N$-point DFT of $\{s(n)\}$ is the same as the $(N-1)$-point DST of $\{x(n)\}$, up to a normalization constant. Again, $S_F(m)$ is as defined in (3.6.3) and the $2N$-point DFT for the real sequence can be implemented using an N-point DFT for a complex sequence.

3.6.2 Fast Algorithms for DST and DCT by Direct Matrix Factorization

3.6.2.1 Decimation-in-Time Algorithms

These are Cooley-Tukey-type algorithms, in which the time ordering of the input data sequence is permuted to allow for the sparse factorization of the transformation matrix. The essential idea is to reduce a size N transform matrix into a block diagonal form, in which each block is related to the same transform of size $N/2$. Recursively applying this procedure, one finally arrives at the basic 2×2 "butterfly." We present here the essential equations for this reduction and also the flow diagrams for the DIT computations of DCT and DST, in block form.

1. *DIT algorithm for the DCT*: Let

$$X_c(m) = \sum_{n=0}^{N} C_N^{mn} \tilde{x}(n), \quad m = 0, 1, 2, \ldots, N, \tag{3.6.6}$$

be the DCT of the sequence $\{x(n)\}$ (i.e., $\tilde{x}(n)$ is $x(n)$ scaled by the normalization constant and the factor k_n, while $X_c(m)$ is scaled by k_m, as in [3.4.6]). Here we have simplified the notations using the definition

$$C_N^{mn} = \cos\left(\frac{mn\pi}{N}\right). \tag{3.6.7}$$

Equation (3.6.6) can be reduced to:

$$\begin{aligned} X_c(m) &= g_c(m) + h_c(m), \\ X_c(N-m) &= g_c(m) - h_c(m), \quad \text{for } m = 0, 1, \ldots, N/2, \\ \text{and } X_c(N/2) &= g_c(N/2). \end{aligned} \tag{3.6.8}$$

Here, g_c and h_c are related to the DCT of size $N/2$, defined by the following equations:

$$\begin{aligned} g_c(m) &= \sum_{n=0}^{N/2} C_{N/2}^{mn} \tilde{x}(2n), \quad \text{for } m = 0, 1, \ldots, N/2, \\ h_c(m) &= \frac{1}{2C_N^m} \sum_{n=0}^{N/2} C_{N/2}^{mn} \left[\tilde{x}(2n+1) + \tilde{x}(2n-1)\right], \quad \text{for } m = 0, 1, \ldots, N/2 - 1, \end{aligned} \tag{3.6.9}$$

$$\text{and } h_c(N/2) = 0, \quad \text{and where } \tilde{x}(N+1) \text{ is set to zero}.$$

We note that both $g_c(m)$ and $h_c(m)$ are DCTs of half the original size. This way, the size of the transform can be reduced by a factor of two at each stage. Some combinations of inputs to the lower order DCT are required as shown by the definition for $h_c(m)$, as well as some scaling of the output of the DCT transform. Figure 3.12 shows a signal flow graph for an $N = 16$ DCT. Note the reduction into two $N = 8$ DCTs in the flow diagram.

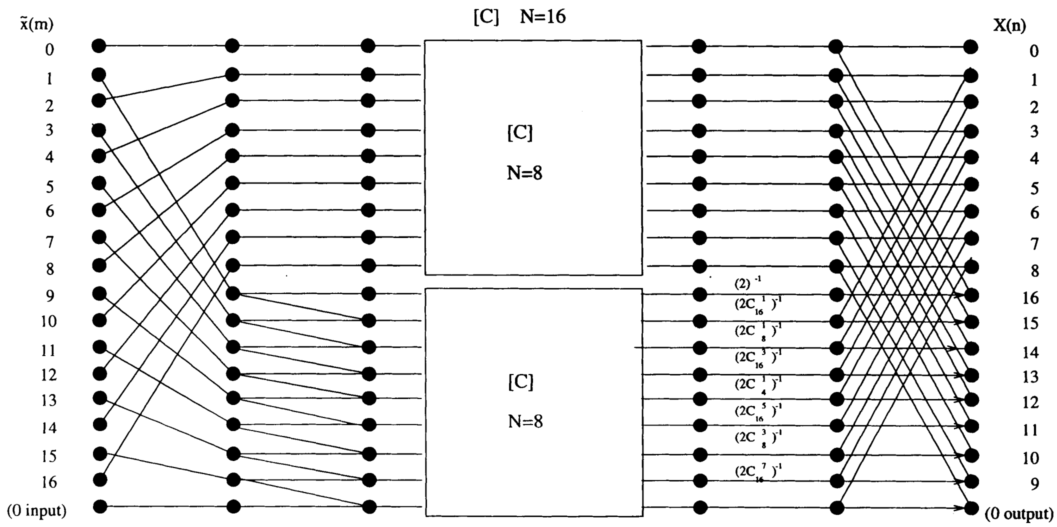

FIGURE 3.12 DIT DCT $N = 16$ flow graph → (–1).

2. *DIT algorithm for DST*: Let

$$X_s(m) = \sum_{n=1}^{N-1} S_N^{mn} \tilde{x}(n), \quad m = 1, 2, \ldots, N-1, \tag{3.6.10}$$

be the DST of the sequence $\{x(n)\}$, (i.e., $\tilde{x}(n)$ is $x(n)$ that has been scaled with the proper normalization constant as required in (3.4.10) and we have defined

$$S_N^{mn} = \sin\left(\frac{mn\pi}{N}\right). \tag{3.6.11}$$

Following the same reasoning for the DIT algorithm for DCT, (3.6.10) can be reduced to

$$\begin{aligned} X_s(m) &= g_s(m) + h_s(m), \\ X_s(N-m) &= g_c(m) - h_s(m), \quad \text{for } m = 1, 2, \ldots, N/2-1, \text{and} \\ X_s(N/2) &= \sum_{n=1}^{N/2-1} (-1)^n \tilde{x}(2N+1). \end{aligned} \tag{3.6.12}$$

Here, $g_s(m)$ and $h_s(m)$ are defined as:

$$\begin{aligned} g_s(m) &= \frac{1}{2C_N^m} \sum_{n=1}^{N/2-1} S_{N/2}^{mn} \left[\tilde{x}(2n+1) + \tilde{x}(2n-1)\right], \text{and} \\ h_s(m) &= \sum_{n=1}^{N/2-1} S_{N/2}^{mn} \tilde{x}(2n). \end{aligned} \tag{3.6.13}$$

As before, it can be seen that $g_s(m)$ and $h_s(m)$ are the DSTs of half the original size, one involving only the odd input samples, and the other involving only the even input samples. Figure 3.13 shows a DIT signal flow graph for the $N = 16$ DST. Note that it is reduced to two blocks of $N = 8$ DSTs.

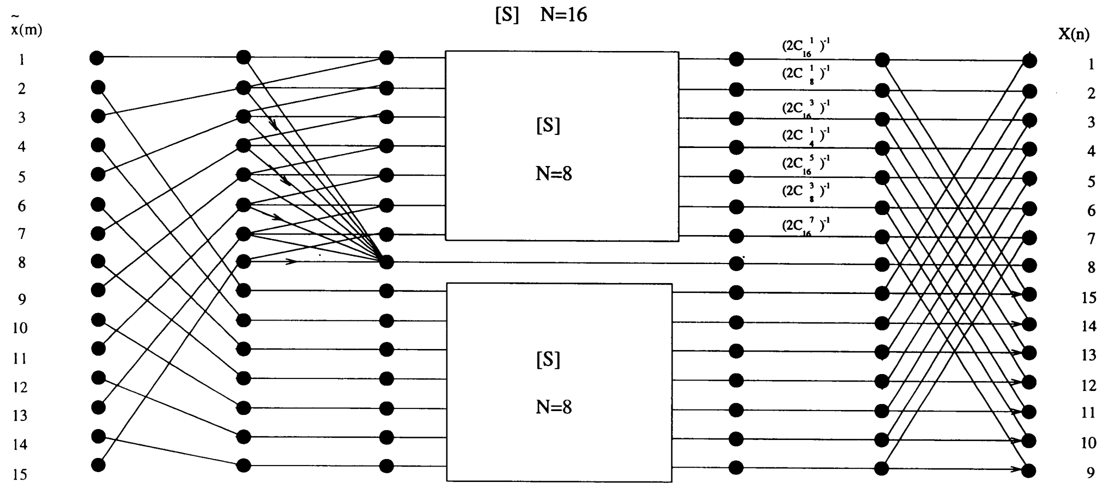

FIGURE 3.13 DIT DST $N = 16$ flow graph $\rightarrow$ (–1).

3.6.2.2 Decimation-in-Frequency Algorithms

These are Sande-Tukey-type algorithms in which the input sample sequence order is not permuted. Again, the basic principle is to reduce the size of the transform, at each stage of the computation, by a factor of two. It would be of no surprise that these algorithms are simply the conjugate versions of the DIT algorithms.

1. *The DIF algorithm for DCT*: In (3.6.6), consider the even ordered output points and the odd-ordered output points,

$$X_c(2m) = G_c(m), \quad \text{for } m = 0, 1, \ldots, N/2, \text{ and}$$
$$X_c(2m+1) = H_c(m) + H_c(m+1), \quad \text{for } m = 0, 1, \ldots, N/2 - 1. \qquad (3.6.14)$$

Here,

$$G_c(m) = \sum_{n=0}^{N/2-1} \left[\tilde{x}(n) + \tilde{x}(N-n)\right] C_{N/2}^{mn} + (-1)^m \tilde{x}(N/2), \text{ and}$$
$$H_c(m) = \sum_{n=0}^{N/2-1} \frac{1}{2C_N^n} \left[\tilde{x}(n) - \tilde{x}(N-n)\right] C_{N/2}^{mn}. \qquad (3.6.15)$$

As can be seen, both $G_c(m)$ and $H_c(m)$ are DCTs of size $N/2$. Therefore, at each stage of the computation, the size of the transform is reduced by a factor of two. The overall result is a sparse factorization of the original transform matrix. Figure 3.14 shows the signal flow graph for an $N = 16$ DIF type DCT.

2. *The DIF algorithm for DST*: The equation (3.6.11) can be split into even-ordered and odd-ordered output points, where

$$X_s(m) = G_s(m), \quad \text{for } m = 1, 2, \ldots, N/2 - 1,$$
$$X_s(2m-1) = H_s(m) + H_s(m-1) + (-1)^{m+1} \tilde{x}(N/2),$$
$$\text{for } m = 1, 2, \ldots, N/2 - 1, \quad \text{and} \qquad (3.6.16)$$
$$X_s(N-1) = H_s(N/2-1) + (-1)^{N/2+1} \tilde{x}(N/2).$$

Here, the outputs $G_s(m)$ and $H_s(m)$ are defined by DSTs of half the original size as

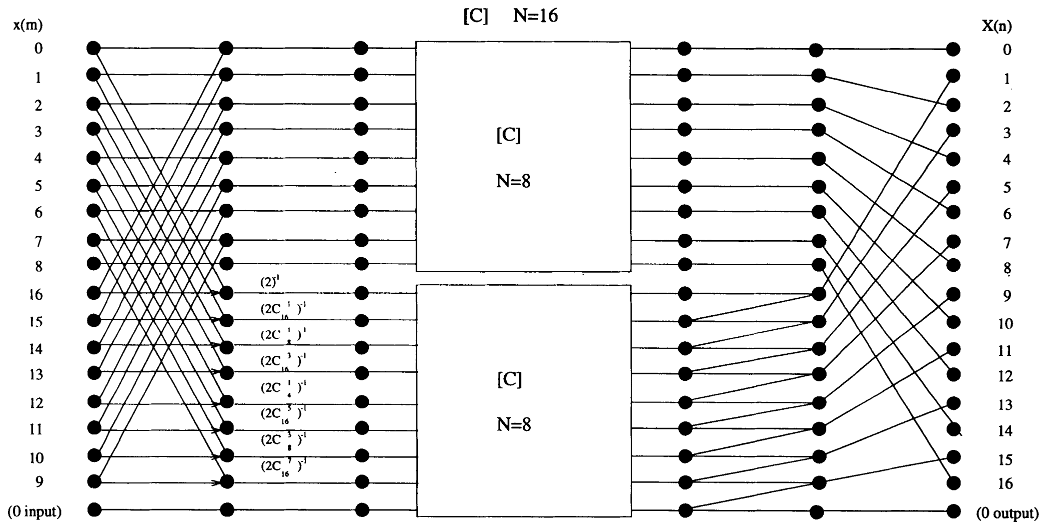

FIGURE 3.14 DIF DCT $N = 16$ flow graph $\rightarrow (-1)$.

$$G_s(m) = \sum_{n=1}^{N/2-1} \left[\tilde{x}(n) - \tilde{x}(N-n)\right] S_{N/2}^{mn}, \quad \text{and}$$

$$H_s(m) = \sum_{n=1}^{N/2-1} \frac{1}{2C_N^n} \left[\tilde{x}(n) - \tilde{x}(N-n)\right] S_{N/2}^{mn}. \tag{3.6.17}$$

Figure 3.15 shows the signal graph for an $N = 16$ DIF-type DST. Note that this flow graph is the conjugate of the flow graph shown in Figure 3.13.

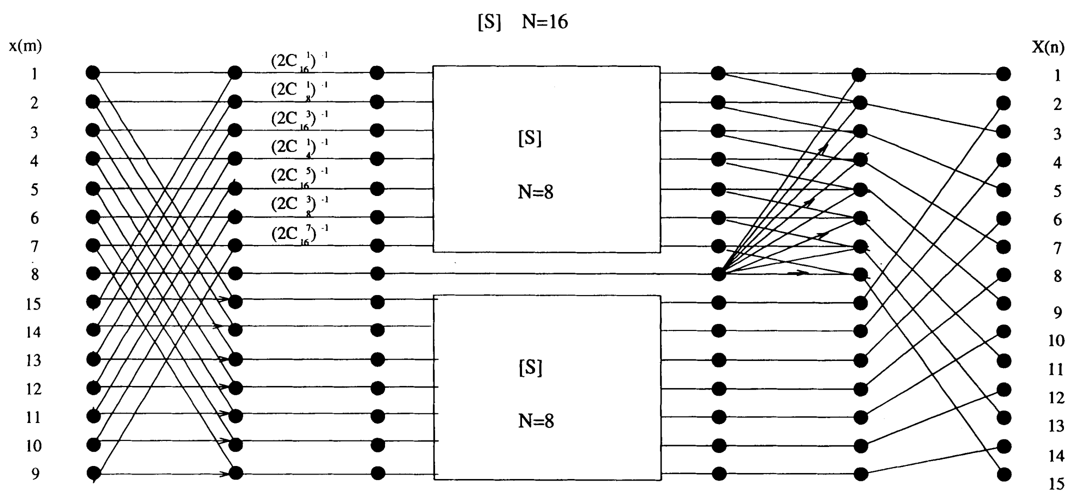

FIGURE 3.15 DIF DST $N = 16$ flow graph $\rightarrow (-1)$.

3.7 Tables of Transforms

This section contains tables of transforms for the FCT and the FST. They are not meant to be complete. For more details and a more complete listing of transforms, especially those of orthogonal and special functions, the reader is referred to the Bateman manuscripts (Erdelyi, 1954). Section 3.7.3 contains a list of conventions and definitions of some special functions that have been referred to in the tables.

3.7.1 Fourier Cosine Transforms

3.7.1.1 General Properties

	$f(t)$	$F_c(\omega) = \int_0^\infty f(t)\cos\omega t\,dt \qquad \omega > 0$
1	$F_c(t)$	$(\pi/2)f(\omega)$
2	$f(at) \qquad a > 0$	$(1/a)F_c(\omega/a)$
3	$f(at)\cos bt \qquad a, b > 0$	$(1/2a)\left[F_c\left(\frac{\omega+b}{a}\right) + F_c\left(\frac{\omega-b}{a}\right)\right]$
4	$f(at)\sin bt \qquad a, b > 0$	$(1/2a)\left[F_s\left(\frac{\omega+b}{a}\right) - F_s\left(\frac{\omega-b}{a}\right)\right]$
5	$t^{2n}f(t)$	$(-1)^n \frac{d^{2n}}{d\omega^{2n}} F_c(\omega)$
6	$t^{2n+1}f(t)$	$(-1)^n \frac{d^{2n+1}}{d\omega^{2n+1}} F_s(\omega)$
7	$\int_0^\infty f(r)[g(t+r) + g(\|t-r\|)]\,dr$	$2F_c(\omega)G_c(\omega)$
8	$\int_t^\infty f(r)\,dr$	$(1/\omega)F_s(\omega)$
9	$f(t+a) - f_o(t-a)$	$2F_s(\omega)\sin a\omega \qquad a > 0$
10	$\int_0^\infty f(r)[g(t+r) - g_o(t-r)]\,dr$	$2F_s(\omega)G_s(\omega)$

3.7.1.2 Algebraic Functions

	$f(t)$	$F_c(\omega)$
1	$(1/\sqrt{t})$	$\sqrt{(\pi/2)}(1/\omega)^{1/2}$
2	$(1/\sqrt{t})[1 - U(t-1)]$	$(2\pi/\omega)^{1/2}C(\omega)$
3	$(1/\sqrt{t})U(t-1)$	$(2\pi/\omega)^{1/2}[1/2 - C(\omega)]$
4	$(t+a)^{-1/2} \qquad \|\arg a\| < \pi$	$(\pi/2\omega)^{1/2}\{\cos a\omega[1 - 2C(a\omega)]$ $+ \sin a\omega[1 - 2S(a\omega)]\}$
5	$(t-a)^{-1/2}U(t-a)$	$(\pi/2\omega)^{1/2}[\cos a\omega - \sin a\omega]$
6	$a(t^2+a^2)^{-1} \qquad a > 0$	$(\pi/2)\exp(-a\omega)$
7	$t(t^2+a^2)^{-1} \qquad a > 0$	$-1/2[e^{-a\omega}\overline{\mathrm{Ei}}(a\omega) + e^{a\omega}\,\mathrm{Ei}(a\omega)]$
8	$(1-t^2)(1+t^2)^{-2}$	$(\pi/2)\omega\exp(-\omega)$
9	$-t(t^2-a^2)^{-1} \qquad a > 0$	$\cos a\omega\,\mathrm{Ci}(a\omega) + \sin a\omega\,\mathrm{Si}(a\omega)$

3.7.1.3 Exponential and Logarithmic Functions

	$f(t)$		$F_c(\omega)$
1	e^{-at}	$\operatorname{Re} a > 0$	$a(a^2+\omega^2)^{-1}$
2	$(1+t)e^{-t}$		$2(1+\omega^2)^{-2}$
3	$\sqrt{t}e^{-at}$	$\operatorname{Re} a > 0$	$\frac{\sqrt{\pi}}{2}(a^2+\omega^2)^{-3/4}\cos[3/2\tan^{-1}(\omega/a)]$
4	$e^{-at}/\sqrt{t}$	$\operatorname{Re} a > 0$	$\sqrt{(\pi/2)}(a^2+\omega^2)^{-1/2}$ $\bullet[(a^2+\omega^2)^{1/2}+a]^{1/2}$
5	$t^n e^{-at}$	$\operatorname{Re} a > 0$	$n![a/(a^2+\omega^2)]^{n+1}$ $\bullet\sum_{2m=0}^{n+1}(-1)^m \binom{n+1}{2m}\left(\frac{\omega}{a}\right)^{2m}$
6	$\exp(-at^2)/\sqrt{t}$	$\operatorname{Re} a > 0$	$\pi(\omega/8a)^{1/2}\exp(-\omega^2/8a)$ $\bullet I_{-1/4}(-\omega^2/8a)$
7	$t^{2n}\exp(-a^2t^2)$	$\lvert\arg a\rvert < \pi/4$	$(-1)^n\sqrt{\pi}2^{-n-1}a^{-2n-1}$ $\bullet\exp[-(\omega/2a)^2]\,\mathrm{He}_{2n}(2^{-1/2}\omega/a)$
8	$t^{-3/2}\exp(-a/t)$	$\operatorname{Re} a > 0$	$(\pi/a)^{1/2}\exp[-(2a\omega)^{1/2}]\cos(2a\omega)^{1/2}$
9	$t^{-1/2}\exp(-a/\sqrt{t})$	$\operatorname{Re} a > 0$	$(\pi/2\omega)^{1/2}[\cos(2a\sqrt{\omega})-\sin(2a\sqrt{\omega})]$
10	$t^{-1/2}\ln t$		$-(\pi/2\omega)^{1/2}[\ln(4\omega)+C+\pi/2]$
11	$(t^2-a^2)^{-1}\ln t$	$a > 0$	$(\pi/2\omega)\{\sin(a\omega)[\mathrm{ci}(a\omega)-\ln a]$ $-\cos(a\omega)[\mathrm{si}(a\omega)-\pi/2]\}$
12	$t^{-1}\ln(1+t)$		$(1/2)\{[\mathrm{ci}(\omega)]^2+[\mathrm{si}(\omega)]^2\}$
13	$\exp(-t/\sqrt{2})\sin(\pi/4+t/\sqrt{2})$		$(1+\omega^4)^{-1}$
14	$\exp(-t/\sqrt{2})\cos(\pi/4+t/\sqrt{2})$		$\omega^2(1+\omega^4)^{-1}$
15	$\ln\frac{a^2+t^2}{1+t^2}$	$a > 0$	$(\pi/\omega)[\exp(-\omega)-\exp(-a\omega)]$
16	$\ln[1+(a/t)^2]$	$a > 0$	$(\pi/\omega)[1-\exp(-a\omega)]$

3.7.1.4 Trigonometric Functions

	$f(t)$	$F_c(\omega)$
1	$t^{-1}e^{-t}\sin t$	$(1/2)\tan^{-1}(2\omega^{-2})$
2	$t^{-2}\sin^2(at) \qquad a>0$	$(\pi/2)(a-\omega/2) \quad \omega<2a$ $0 \quad \omega>2a$
3	$\left(\dfrac{\sin t}{t}\right)^n \qquad n=2,3,\ldots$	$\dfrac{n\pi}{2^n}\displaystyle\sum_{r>0}^{r<(\omega+n)/2}\dfrac{(-1)^r(\omega+n-2r)^{n-1}}{r!(n-r)!}, \quad 0<\omega<n$ $0 \quad n\le\omega$
4	$\exp(-\beta t^2)\cos at \qquad \operatorname{Re}\beta>0$	$(1/2)(\pi/\beta)^{1/2}\exp\left(-\dfrac{a^2+\omega^2}{4\beta}\right)\cosh\left(\dfrac{a\omega}{2\beta}\right)$
5	$(a^2+t^2)^{-1}(1-2\beta\cos t+\beta^2)^{-1}$ $\operatorname{Re}a>0, \lvert\beta\rvert<1$	$(1/2)(\pi/a)(1-\beta^2)^{-1}(e^a-\beta)^{-1}$ $\bullet(e^{a-a\omega}+\beta e^{a\omega}) \quad 0\le\omega<1$
6	$\sin(at^2) \qquad a>0$	$(1/4)(2\pi/a)^{1/2}\left[\cos\left(\dfrac{\omega^2}{4a}\right)-\sin\left(\dfrac{\omega^2}{4a}\right)\right]$
7	$\sin[a(1-t^2)] \qquad a>0$	$-(1/2)(\pi/a)^{1/2}\cos[a+\pi/4+\omega^2/(4a)]$
8	$\cos(at^2) \qquad a>0$	$(1/4)(2\pi/a)^{1/2}\left[\cos\left(\dfrac{\omega^2}{4a}\right)+\sin\left(\dfrac{\omega^2}{4a}\right)\right]$
9	$\cos[a(1-t^2)] \qquad a>0$	$(1/2)(\pi/a)^{1/2}\sin[a+\pi/4+\omega^2/(4a)]$
10	$\tan^{-1}(a/t) \qquad a>0$	$(2\omega)^{-1}[e^{-a\omega}\operatorname{Ei}(a\omega)-e^{a\omega}\operatorname{Ei}(-a\omega)]$

3.7.2 Fourier Sine Transforms

3.7.2.1 General Properties

	$f(t)$	$F_s(\omega)=\int_0^\infty f(t)\sin\omega t\,dt \qquad \omega>0$
1	$F_s(t)$	$(\pi/2)f(\omega)$
2	$f(at) \qquad a>0$	$(1/a)F_s(\omega/a)$
3	$f(at)\cos bt \qquad a,b>0$	$(1/2a)\left[F_s\left(\dfrac{\omega+b}{a}\right)+F_s\left(\dfrac{\omega-b}{a}\right)\right]$
4	$f(at)\sin bt \qquad a,b>0$	$-(1/2a)\left[F_c\left(\dfrac{\omega+b}{a}\right)-F_c\left(\dfrac{\omega-b}{a}\right)\right]$

General Properties (Continued)

	$f(t)$	$F_s(\omega) = \int_0^\infty f(t)\sin\omega t\,dt \qquad \omega > 0$
5	$t^{2n} f(t)$	$(-1)^n \dfrac{d^{2n}}{d\omega^{2n}} F_s(\omega)$
6	$t^{2n+1} f(t)$	$(-1)^{n+1} \dfrac{d^{2n+1}}{d\omega^{2n+1}} F_c(\omega)$
7	$\int_0^\infty f(r) \int_{\|t-r\|}^{t+r} g(s)\,ds\,dr$	$(2/\omega) F_s(\omega) G_s(\omega)$
8	$f_o(t+a) + f_o(t-a)$	$2F_s(\omega)\cos a\omega$
9	$f_e(t-a) - f_e(t+a)$	$2F_c(\omega)\sin a\omega$
10	$\int_0^\infty f(r)[g(\|t-r\|) - g(t+r)]\,dr$	$2F_s(\omega)G_c(\omega)$

3.7.2.2 Algebraic Functions

	$f(t)$	$F_s(\omega)$
1	$1/t$	$\pi/2$
2	$1/\sqrt{t}$	$(\pi/2\omega)^{1/2}$
3	$1/\sqrt{t}[1 - U(t-1)]$	$(2\pi/\omega)^{1/2} S(\omega)$
4	$(1/\sqrt{t})U(t-1)$	$(2\pi/\omega)^{1/2}[1/2 - S(\omega)]$
5	$(t+a)^{-1/2} \qquad \|\arg a\| < \pi$	$(\pi/2\omega)^{1/2}\{\cos a\omega[1 - 2S(a\omega)] - \sin a\omega[1 - 2C(a\omega)]\}$
6	$(t-a)^{-1/2} U(t-a)$	$(\pi/2\omega)^{1/2}(\sin a\omega + \cos a\omega)$
7	$t(t^2+a^2)^{-1} \qquad a > 0$	$(\pi/2)\exp(-a\omega)$
8	$t(a^2-t^2)^{-1} \qquad a > 0$	$-(\pi/2)\cos a\omega$
9	$t(a^2+t^2)^{-2} \qquad a > 0$	$(\pi\omega/4a)\exp(-a\omega)$
10	$a^2[t(a^2+t^2)]^{-1} \qquad a > 0$	$(\pi/2)[1 - \exp(-a\omega)]$
11	$t(4+t^4)^{-1}$	$(\pi/4)\exp(-\omega)\sin\omega$

3.7.2.3 Exponential and Logarithmic Functions

	$f(t)$	$F_s(\omega)$
1	e^{-at} Re $a > 0$	$\omega(a^2+\omega^2)^{-1}$
2	te^{-at} Re $a > 0$	$(2a\omega)(a^2+\omega^2)^{-2}$
3	$t(1+at)e^{-at}$ Re $a > 0$	$(8a^3\omega)(a^2+\omega^2)^{-3}$
4	$e^{-at}\sqrt{t}$ Re $a > 0$	$\sqrt{(\pi/2)}(a^2+\omega^2)^{-1/2} \bullet [(a^2+\omega^2)^{1/2}-a]^{1/2}$
5	$t^{-3/2}e^{-at}$ Re $a > 0$	$(2\pi)^{1/2}[(a^2+\omega^2)^{1/2}-a]^{1/2}$
6	$\exp(-at^2)$ Re $a > 0$	$-j(1/2)(\pi/a)^{1/2}\exp(-\omega^2/4a)\,\mathrm{Erf}\left(\frac{j\omega}{2\sqrt{a}}\right)$
7	$t\exp(-t^2/4a)$ Re $a > 0$	$2a\omega\sqrt{(\pi a)}\exp(-a\omega^2)$
8	$t^{-3/2}\exp(-a/t)$ $\lvert\arg a\rvert < \pi/2$	$(\pi/a)^{1/2}\exp[-(2a\omega)^{1/2}]\sin(2a\omega)^{1/2}$
9	$t^{-3/4}\exp(-a\sqrt{t})$ $\lvert\arg a\rvert < \pi/2$	$-(\pi/2)(a/\omega)^{1/2}[J_{1/4}(a^2/8\omega) \bullet \cos(\pi/8+a^2/8\omega)+Y_{1/4}(a^2/8\omega) \bullet \sin(\pi/8+a^2/8\omega)]$
10	$t^{-1}\ln t$	$-(\pi/2)[C+\ln\omega]$
11	$t(t^2-a^2)^{-1}\ln t$ $a > 0$	$-(\pi/2)\{\cos a\omega[\mathrm{Ci}(a\omega)-\ln a] + \sin a\omega[\mathrm{Si}(a\omega)-\pi/2]\}$
12	$t^{-1}\ln(1+a^2t^2)$ $a > 0$	$-\pi\,\mathrm{Ei}(-\omega/a)$
13	$\ln\frac{t+a}{\lvert t-a\rvert}$ $a > 0$	$(\pi/\omega)\sin a\omega$

3.7.2.4 Trigonometric Functions

	$f(t)$	$F_s(\omega)$
1	$t^{-1}\sin^2(at) \quad a>0$	$\pi/4 \quad 0<\omega<2a$ $\pi/8 \quad \omega=2a$ $0 \quad \omega>2a$
2	$t^{-2}\sin^2(at) \quad a>0$	$(1/4)(\omega+2a)\ln\lvert\omega+2a\rvert$ $+(1/4)(\omega-2a)\ln\lvert\omega-2a\rvert-(1/2)\omega\ln\omega$
3	$t^{-2}[1-\cos at] \quad a>0$	$(\omega/2)\ln\lvert(\omega^2-a^2)/\omega^2\rvert$ $+(a/2)\ln\lvert(\omega+a)/(\omega-a)\rvert$
4	$\sin(at^2) \quad a>0$	$(\pi/2a)^{1/2}\{\cos(\omega^2/4a)C[\omega/(2\pi a)^{1/2}]$ $+\sin(\omega^2/4a)S[\omega/(2\pi a)^{1/2}]$
5	$\cos(at^2) \quad a>0$	$(\pi/2a)^{1/2}\{\sin(\omega^2/4a)C[\omega/(2\pi a)^{1/2}]$ $-\cos(\omega^2/4a)S[\omega/(2\pi a)^{1/2}]$
6	$\tan^{-1}(a/t) \quad a>0$	$(\pi/2\omega)[1-\exp(-a\omega)]$

3.7.3 Notations and Definitions

1. $f(t)$: Piece-wise smooth and absolutely integrable function on the positive real line.
2. $F_c(\omega)$: The Fourier cosine transform of $f(t)$.
3. $F_s(\omega)$: The Fourier sine transform of $f(t)$.
4. $f_o(t)$: The odd extension of the function f over the entire real line.
5. $f_e(t)$: The even extension of the function f over the entire real line.
6. $C(\omega)$ is defined as the integral:

$$(2\pi)^{-1/2}\int_0^{\omega} t^{-1/2}\cos t\,dt.$$

7. $S(\omega)$ is defined as the integral:

$$(2\pi)^{-1/2}\int_0^{\omega} t^{-1/2}\sin t\,dt.$$

8. $\mathrm{Ei}(x)$ is the exponential integral function defined as

$$-\int_{-x}^{\infty} t^{-1}e^{-t}\,dt, \quad \lvert\arg(x)\rvert<\pi.$$

9. $\overline{\mathrm{Ei}}(x)$ is defined as $(1/2)[\mathrm{Ei}(x+j0)+\mathrm{Ei}(x-j0)]$.
10. $\mathrm{Ci}(x)$ is the cosine integral function defined as

$$-\int_{-x}^{\infty} t^{-1}\cos t\,dt.$$

11. Si(x) is the sine integral function defined as

$$\int_0^x t^{-1} \sin t \, dt .$$

12. $I_\upsilon(z)$ is the modified Bessel function of the first kind defined as

$$\sum_{m=0}^{\infty} \frac{(z/2)^{\upsilon+2m}}{m!\Gamma(\upsilon+m+1)}, \quad |z| < \infty, |\arg(x)| < \pi .$$

13. He$_n(x)$ is the Hermite polynomial function defined as

$$(-1)^n \exp(x^2/2) \frac{d^n}{dx^n}\left[\exp(-x^2/2)\right].$$

14. C is the Euler constant defined as

$$\lim_{m\to\infty}\left[\sum_{n=1}^{m}(1/n) - \ln m\right] = 0.5772156649\ldots$$

15. ci(x) and si(x) are related to Ci(x) and Si(x) by the equations:

$$\text{ci}(x) = -\text{Ci}(x), \quad \text{si}(x) = \text{Si}(x) - \pi/2.$$

16. Erf(x) is the error function defined by

$$(2/\sqrt{\pi})\int_0^x \exp(-t^2)\, dt .$$

17. $J_\upsilon(x)$ and $Y_\upsilon(x)$ are the Bessel functions for the first and second kind, respectively,

$$J_\upsilon(x) = \sum_{m=0}^{\infty} (-1)^m \frac{(x/2)^{\upsilon+2m}}{m!\Gamma(\upsilon+m+1)}$$

and

$$Y_\upsilon(x) = \text{cosec}\{\upsilon\pi[J_\upsilon(x) \cos \upsilon\pi - J_{-\upsilon}(x)]\}.$$

18. $U(t)$: is the Heaviside step function defined as

$$U(t) = 0 \quad t < 0,$$

$$= 1 \quad t > 0 .$$

19. $\binom{m}{n}$ is the binomial coefficient defined as $\frac{m!}{n!(m-n)!}$.

20. $\Gamma(x)$: is the Gamma function defined as

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt .$$

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